

Chapter 2

Problems

1. (a) $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), (b, g), (b, b)\}$
 (b) $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$
2. $S = \{(n, x_1, \dots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \dots, n-1\}$, with the interpretation that the outcome is $(n, x_1, \dots, x_{n-1})$ if the first 6 appears on roll n , and x_i appears on roll $i, i = 1, \dots, n-1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.
3. $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.
 $E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$.
 EF^c is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.
4. $A = \{1,0001,0000001, \dots\}$ $B = \{01, 00001, 00000001, \dots\}$
 $(A \cup B)^c = \{00000 \dots, 001, 000001, \dots\}$
5. (a) $2^5 = 32$
 (b) $W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0), (1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1), (0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$
 (c) 8
 (d) $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$
6. (a) $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$
 (b) $A = \{(1, s), (0, s)\}$
 (c) $B = \{(0, g), (0, f), (0, s)\}$
 (d) $\{(1, s), (0, s), (1, g), (1, f)\}$
7. (a) 6^{15}
 (b) $6^{15} - 3^{15}$
 (c) 4^{15}
8. (a) .8
 (b) .3
 (c) 0

9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

$$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10. Let R and N denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$

Thus, $P(RN) = .1$

11. Let A be the event that a randomly chosen person is a cigarette smoker and let B be the event that she or he is a cigar smoker.

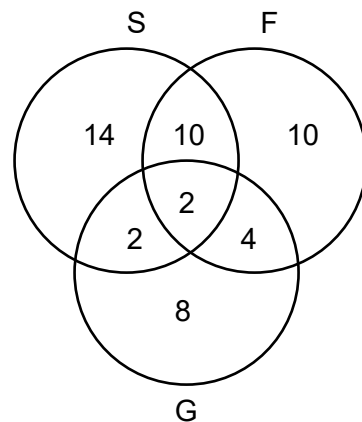
(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^cB) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

12. (a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$

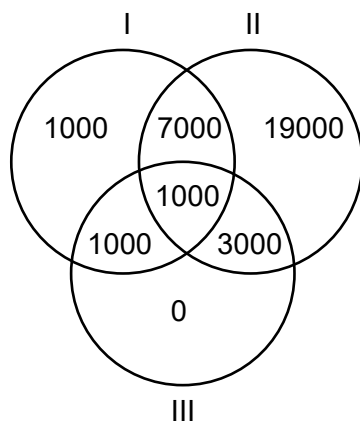
The desired probability is $1 - 1/2 = 1/2$.

- (b) Use the Venn diagram below to obtain the answer $32/100$.



- (c) Since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\frac{\binom{50}{2}}{\binom{100}{2}} = 49/98$ and so the probability that at least one is taking a course is $149/98$.

13.



- (a) 20,000
 (b) 12,000
 (c) 11,000
 (d) 68,000
 (e) 10,000

14. $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15.

(a) $4 \frac{\binom{13}{5}}{\binom{52}{5}}$

(b) $13 \frac{\binom{4}{2} \binom{12}{3} \binom{4}{1} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}}$

(c) $\frac{\binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{44}{1}}{\binom{52}{5}}$

(d) $13 \frac{\binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1}}{\binom{52}{5}}$

(e) $13 \frac{\binom{4}{4} \binom{48}{1}}{\binom{52}{5}}$

16.

(a) $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5}$

(b)

$$\frac{6 \binom{5}{2} 5 \cdot 4 \cdot 3}{6^5}$$

(c)

$$\frac{\binom{6}{2} 4 \binom{5}{2} \binom{3}{2}}{6^5}$$

(d) $\frac{6 \cdot 5 \cdot 4 \binom{5}{3}}{21}$

(e)

$$\frac{6 \cdot 5 \binom{5}{3}}{6^5}$$

(f)

$$\frac{6 \cdot 5 \binom{5}{4}}{6^5}$$

(g) $\frac{6}{6^5}$

$$17. \frac{\binom{15}{8}\binom{10}{8}\binom{7}{1}}{\binom{25}{16}\binom{9}{1}} = .1102$$

$$18. \frac{2 \cdot 4 \cdot 16}{52 \cdot 51}$$

$$19. 4/36 + 4/36 + 1/36 + 1/36 = 5/18$$

20. Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \frac{4 \cdot 4 \cdot 16}{52 \cdot 51} + \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49} \\ &= .0983 \end{aligned}$$

where the preceding used that $P(A) = P(B) = 2 \times \frac{4 \cdot 16}{52 \cdot 51}$. Hence, the probability that neither is dealt blackjack is .9017.

$$21. (a) p_1 = 4/20, p_2 = 8/20, p_3 = 5/20, p_4 = 2/20, p_5 = 1/20$$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48, q_2 = 16/48, q_3 = 15/48, q_4 = 8/48, q_5 = 5/48$$

22. The ordering will be unchanged if for some k , $0 \leq k \leq n$, the first k coin tosses land heads and the last $n - k$ land tails. Hence, the desired probability is $(n + 1)/2^n$

23. The answer is $5/12$, which can be seen as follows:

$$\begin{aligned} 1 &= P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + p\{\text{same}\} \\ &= 2P\{\text{second higher}\} + 1/6 \end{aligned}$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

$$25. P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{6}{36}, \quad \sum_{n=1}^{\infty} P(E_n) = \frac{2}{5}$$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

$$28. \quad P\{\text{same}\} = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

$$P\{\text{different}\} = \frac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$\begin{aligned} P\{\text{different}\} &= P(RBG) + P(BRG) + P(RGB) + \dots + P(GBR) \\ &= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3} \end{aligned}$$

$$29. \quad (a) \quad \frac{n(n-1) + m(m-1)}{(n+m)(n+m-1)}$$

- (b) Putting all terms over the common denominator $(n+m)^2(n+m-1)$ shows that we must prove that

$$n^2(n+m-1) + m^2(n+m-1) \geq n(n-1)(n+m) + m(m-1)(n+m)$$

which is immediate upon multiplying through and simplifying.

$$30. \quad (a) \quad \frac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$$

$$(b) \quad \frac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} - 1/18 = 1/6$$

$$(c) \quad \frac{\binom{7}{3}\binom{8}{4} + \binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$$

$$31. \quad P(\{\text{complete}\}) =$$

$$P\{\text{same}\} =$$

$$32. \quad \frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$$

$$33. \quad \frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \frac{70}{323}$$

$$34. \quad \binom{32}{13} \bigg/ \binom{52}{13}$$

$$35. \quad (a) \quad \frac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}}$$

$$(b) \quad 1 - \frac{\binom{34}{7}}{\binom{46}{7}} - \frac{\binom{12}{1}\binom{34}{6}}{\binom{46}{7}}$$

$$(c) \quad \frac{\binom{12}{7} + \binom{16}{7} + \binom{18}{7}}{\binom{46}{7}}$$

$$(d) \quad P(R_3 \cup B_3) = P(R_3) + P(B_3) - P(R_3 B_3) = \frac{\binom{12}{3}\binom{34}{4}}{\binom{46}{7}} + \frac{\binom{16}{3}\binom{30}{4}}{\binom{46}{7}} - \frac{\binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}}$$

$$36. \quad (a) \quad \binom{4}{2} \bigg/ \binom{52}{2} \approx .0045,$$

$$(b) \quad 13 \binom{4}{2} \bigg/ \binom{52}{2} = 1/17 \approx .0588$$

$$37. \quad (a) \quad \binom{7}{5} / \binom{10}{5} = 1/12 \approx .0833$$

$$(b) \quad \binom{7}{4} \binom{3}{1} / \binom{10}{5} + 1/12 = 1/2$$

$$38. \quad 1/2 = \binom{3}{2} / \binom{n}{2} \text{ or } n(n-1) = 12 \text{ or } n = 4.$$

$$39. \quad \frac{5 \cdot 4 \cdot 3}{5 \cdot 5 \cdot 5} = \frac{12}{25}$$

$$40. \quad .8134; .1148$$

$$41. \quad 1 - \frac{5^4}{6^4}$$

$$42. \quad 1 - \left(\frac{35}{36}\right)^n$$

$$43. \quad \frac{2(n-1)(n-2)}{n!} = \frac{2}{n} \text{ in a line}$$

$$\frac{2n(n-2)!}{n!} = \frac{2}{n-1} \text{ if in a circle, } n \geq 2$$

$$44. \quad (a) \text{ If } A \text{ is first, then } A \text{ can be in any one of 3 places and } B\text{'s place is determined, and the others can be arranged in any of } 3! \text{ ways. As a similar result is true, when } B \text{ is first, we see that the probability in this case is } 2 \cdot 3 \cdot 3!/5! = 3/10$$

$$(b) \quad 2 \cdot 2 \cdot 3!/5! = 1/5$$

$$(c) \quad 2 \cdot 3!/5! = 1/10$$

$$45. \quad 1/n \text{ if discard, } \frac{(n-1)^{k-1}}{n^k} \text{ if do not discard}$$

$$46. \quad \text{If } n \text{ in the room,}$$

$$P\{\text{all different}\} = \frac{12 \cdot 11 \cdot \dots \cdot (13-n)}{12 \cdot 12 \cdot \dots \cdot 12}$$

When $n = 5$ this falls below $1/2$. (Its value when $n = 5$ is .3819)

$$47. \quad \frac{\binom{8}{2}\binom{5}{2}}{\binom{14}{5}} = .1399$$

$$48. \quad \binom{12}{4}\binom{8}{4} \frac{(20)!}{(3!)^4(2!)^4} / (12)^{20}$$

$$49. \quad \binom{6}{3}\binom{6}{3} / \binom{12}{6}$$

$$50. \quad \binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5} / \binom{52}{13}\binom{39}{13}$$

$$51. \quad \binom{n}{m} (n-1)^{n-m} / N^n$$

$$52. \quad (a) \quad \frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$$

$$(b) \quad \frac{\binom{10}{1}\binom{9}{6} \frac{8!}{2!} 2^6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$$

53. Let A_i be the event that couple i sit next to each other. Then

$$P(\cup_{i=1}^4 A_i) = 4 \frac{2 \cdot 7!}{8!} - 6 \frac{2^2 \cdot 6!}{8!} + 4 \frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!}$$

and the desired probability is 1 minus the preceding.

$$54. \quad P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(C) - P(SH) - \dots - P(SHDC)$$

$$\begin{aligned} &= \frac{4 \binom{39}{13}}{\binom{52}{13}} - \frac{6 \binom{26}{13}}{\binom{52}{13}} + \frac{4 \binom{13}{13}}{\binom{52}{13}} \\ &= \frac{4 \binom{39}{13} - 6 \binom{26}{13} + 4}{\binom{52}{13}} \end{aligned}$$

55. (a) $P(S \cup H \cup D \cup C) = P(S) + \dots - P(SHDC)$

$$= \frac{4 \binom{2}{2} - 6 \binom{2}{2} \binom{2}{2} \binom{48}{9} + 4 \binom{2}{2}^3 \binom{46}{7} - \binom{2}{2}^4 \binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4 \binom{50}{11} - 6 \binom{48}{9} + 4 \binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

(b) $P(1 \cup 2 \cup \dots \cup 13) = \frac{13 \binom{48}{9} - \binom{13}{2} \binom{44}{5} + \binom{13}{3} \binom{40}{1}}{\binom{52}{13}}$

56. Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability $5/9$.

Theoretical Exercises

$$5. \quad F_i = E_i \bigcap_{j=1}^{i-1} E_j^c$$

$$6. \quad (a) \quad EF^cG^c$$

$$(b) \quad EF^cG$$

$$(c) \quad E \cup F \cup G$$

$$(d) \quad EF \cup EG \cup FG$$

$$(e) \quad EFG$$

$$(f) \quad E^cF^cG^c$$

$$(g) \quad E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$$

$$(h) \quad (EFG)^c$$

$$(i) \quad EFG^c \cup EF^cG \cup E^cFG$$

$$(j) \quad S$$

8. The number of partitions that has $n + 1$ and a fixed set of i of the elements $1, 2, \dots, n$ as a subset is T_{n-i} . Hence, (where $T_0 = 1$). Hence, as there are $\binom{n}{i}$ such subsets.

$$T_{n+1} = \sum_{i=0}^n \binom{n}{i} T_{n-i} = 1 + \sum_{i=0}^{n-1} \binom{n}{i} T_{n-i} = 1 + \sum_{k=1}^n \binom{n}{k} T_k.$$

$$11. \quad 1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$$

$$12. \quad \begin{aligned} P(EF^c \cup E^cF) &= P(EF^c) + P(E^cF) \\ &= P(E) - P(EF) + P(F) - P(EF) \end{aligned}$$

$$13. \quad E = EF \cup EF^c$$

$$15. \quad \frac{\binom{M}{k} \binom{N}{r-k}}{\binom{M+N}{r}}$$

16. $P(E_1 \dots E_n) \geq P(E_1 \dots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Ineq.

$$\geq \sum_{i=1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

19.
$$\frac{\binom{n}{r-1} \binom{m}{k-r} (n-r+1)}{\binom{n+m}{k-1} (n+m-k+1)}$$

21. Let $y_1, y_2, \dots, y_k$ denote the successive runs of losses and $x_1, \dots, x_k$ the successive runs of wins. There will be $2k$ runs if the outcome is either of the form $y_1, x_1, \dots, y_k, x_k$ or $x_1, y_1, \dots, x_k, y_k$ where all x_i, y_i are positive, with $x_1 + \dots + x_k = n, y_1 + \dots + y_k = m$. By Proposition 6.1 there are $2 \binom{n-1}{k-1} \binom{m-1}{k-1}$ number of outcomes and so

$$P\{2k \text{ runs}\} = 2 \binom{n-1}{k-1} \binom{m-1}{k-1} / \binom{m+n}{n}.$$

There will be $2k+1$ runs if the outcome is either of the form $x_1, y_1, \dots, x_k, y_k, x_{k+1}$ or $y_1, x_1, \dots, y_k, x_k, y_{k+1}$ where all are positive and $\sum x_i = n, \sum y_i = m$. By Proposition 6.1 there are

$$\binom{n-1}{k} \binom{m-1}{k-1} \text{ outcomes of the first type and } \binom{n-1}{k-1} \binom{m-1}{k} \text{ of the second.}$$