CHAPTER 1

1.1 Let

$$r_u(k) = E[u(n)u^*(n-k)] \quad (1)$$

$$r_y(k) = E[y(n)y^*(n-k)] \quad (2)$$

We are given that

$$y(n) = u(n+a) - u(n-a) \quad (3)$$

Hence, substituting Eq. (3) into (2), and then using Eq. (1), we get

$$r_y(k) = E[(u(n+a) - u(n-a))(u^*(n+a-k) - u^*(n-a-k))]$$

$$= 2r_u(k) - r_u(2a+k) - r_u(-2a+k)$$

1.2 We know that the correlation matrix **R** is Hermitian; that is

$$\mathbf{R}^H = \mathbf{R}$$

Given that the inverse matrix $\mathbf{R}^{-1}$ exists, we may write

$$\mathbf{R}^{-1}\mathbf{R}^H = \mathbf{I}$$

where **I** is the identity matrix. Taking the Hermitian transpose of both sides:

$$\mathbf{R}\mathbf{R}^{-H} = \mathbf{I}$$

Hence,

$$\mathbf{R}^{-H} = \mathbf{R}^{-1}$$

That is, the inverse matrix $\mathbf{R}^{-1}$ is Hermitian.

1.3 For the case of a two-by-two matrix, we may

$$\mathbf{R}_u = \mathbf{R}_s + \mathbf{R}_\nu$$

$$= \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} + \sigma^2 & r_{12} \\ r_{21} & r_{22} + \sigma^2 \end{bmatrix}$$

For $\mathbf{R}_u$ to be nonsingular, we require

$$\det(\mathbf{R}_u) = (r_{11} + \sigma^2)(r_{22} + \sigma^2) - r_{12}r_{21} > 0$$

With $r_{12} = r_{21}$ for real data, this condition reduces to

$$(r_{11} + \sigma^2)(r_{22} + \sigma^2) - r_{12}r_{21} > 0$$

Since this is quadratic in σ^2, we may impose the following condition on σ^2 for nonsingularity of $\mathbf{R}_u$:

$$\sigma^2 > \frac{1}{2}(r_{11} + r_{22})\left(\sqrt{1 - \frac{4\Delta_r}{(r_{11} + r_{22})^2 - 1}}\right)$$

where $\Delta_r = r_{11}r_{22} - r_{12}^2$

1.4 We are given

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This matrix is positive definite because

$$a^T\mathbf{Ra} = [a_1, a_2]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= a_1^2 + 2a_1a_2 + a_2^2$$

$$= (a_1 + a_2)^2 > 0 \text{ for all nonzero values of } a_1 \text{ and } a_2$$

(Positive definiteness is stronger than nonnegative definiteness.)

But the matrix $\mathbf{R}$ is singular because

$$\det(\mathbf{R}) = (1)^2 - (1)^2 = 0$$

Hence, it is possible for a matrix to be positive definite and yet it can be singular.

1.5 (a)

$$\mathbf{R}_{M+1} = \left[\begin{array}{c|c} r(0) & \mathbf{r}^H \\ \hline \mathbf{r} & \mathbf{R}_M \end{array}\right] \tag{1}$$

Let

$$\mathbf{R}_{M+1}^{-1} = \left[\begin{array}{c|c} a & \mathbf{b}^H \\ \hline \mathbf{b} & \mathbf{C} \end{array}\right] \tag{2}$$

where a, $\mathbf{b}$ and $\mathbf{C}$ are to be determined. Multiplying (1) by (2):

$$\mathbf{I}_{M+1} = \left[\begin{array}{c|c} r(0) & \mathbf{r}^H \\ \hline \mathbf{r} & \mathbf{R}_M \end{array}\right] \left[\begin{array}{c|c} a & \mathbf{b}^H \\ \hline \mathbf{b} & \mathbf{C} \end{array}\right]$$

where $\mathbf{I}_{M+1}$ is the identity matrix. Therefore,

$$r(0)a + \mathbf{r}^H\mathbf{b} = 1 \tag{3}$$

$$\mathbf{r}a + \mathbf{R}_M\mathbf{b} = \mathbf{0} \tag{4}$$

$$\mathbf{r}\mathbf{b}^H + \mathbf{R}_M\mathbf{C} = \mathbf{I}_M \tag{5}$$

$$r(0)\mathbf{b}^H + \mathbf{r}^H\mathbf{C} = \mathbf{0}^T \tag{6}$$

From Eq. (4):

$$\mathbf{b} = -\mathbf{R}_M^{-1}\mathbf{r}a \tag{7}$$

Hence, from (3) and (7):

$$a = \frac{1}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} \tag{8}$$

Correspondingly,

$$\mathbf{b} = -\frac{\mathbf{R}_M^{-1}\mathbf{r}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} \tag{9}$$

From (5):

$$\mathbf{C} = \mathbf{R}_M^{-1} - \mathbf{R}_M^{-1}\mathbf{r}\mathbf{b}^H$$

$$= \mathbf{R}_M^{-1} + \frac{\mathbf{R}_M^{-1}\mathbf{r}\mathbf{r}^H\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} \tag{10}$$

As a check, the results of Eqs. (9) and (10) should satisfy Eq. (6).

$$r(0)\mathbf{b}^H + \mathbf{r}^H\mathbf{C} = -\frac{r(0)\mathbf{r}^H\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}} + \mathbf{r}^H\mathbf{R}_M^{-1} + \frac{\mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}\mathbf{r}^H\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^H\mathbf{R}_M^{-1}\mathbf{r}}$$

$$= \mathbf{0}^T$$

We have thus shown that

$$\mathbf{R}_{M+1}^{-1} = \left[\begin{array}{c:c} 0 & \mathbf{0}^T \\ \hdashline \mathbf{0} & \mathbf{R}_M^{-1} \end{array}\right] + a\left[\begin{array}{c:c} 1 & -\mathbf{r}^H\mathbf{R}_M^{-1} \\ \hdashline -\mathbf{R}_M^{-1}\mathbf{r} & \mathbf{R}_M^{-1}\mathbf{r}\mathbf{r}^H\mathbf{R}_M^{-1} \end{array}\right]$$

$$= \left[\begin{array}{c:c} 0 & \mathbf{0}^T \\ \hdashline \mathbf{0} & \mathbf{R}_M^{-1} \end{array}\right] + a\left[\begin{array}{c} 1 \\ \hdashline -\mathbf{R}_M^{-1}\mathbf{r} \end{array}\right] \left[\begin{array}{c:c} 1 & -\mathbf{r}^H\mathbf{R}_M^{-1} \end{array}\right]$$

where the scalar a is defined by Eq. (8):

(b) $$\mathbf{R}_{M+1} = \left[\begin{array}{c:c} \mathbf{R}_M & \mathbf{r}^{B*} \\ \hdashline \mathbf{r}^{BT} & r(0) \end{array}\right] \qquad (11)$$

Let

$$\mathbf{R}_{M+1}^{-1} = \left[\begin{array}{c:c} \mathbf{D} & \mathbf{e} \\ \hdashline \mathbf{e}^{H} & f \end{array}\right] \qquad (12)$$

where **D**, **e** and f are to be determined. Multiplying (11) by (12):

$$\mathbf{I}_{M+1} = \left[\begin{array}{c:c} \mathbf{R}_M & \mathbf{r}^{B*} \\ \hdashline \mathbf{r}^{BT} & r(0) \end{array}\right] \left[\begin{array}{c:c} \mathbf{D} & \mathbf{e} \\ \hdashline \mathbf{e}^{H} & f \end{array}\right]$$

Therefore

$$\mathbf{R}_M\mathbf{D} + \mathbf{r}^{B*}\mathbf{e}^{H} = \mathbf{I} \qquad (13)$$

$$\mathbf{R}_M\mathbf{e} + \mathbf{r}^{B*}f = \mathbf{0} \qquad (14)$$

$$\mathbf{r}^{BT}\mathbf{e} + r(0)f = 1 \qquad (15)$$

$$\mathbf{r}^{BT}\mathbf{D} + r(0)\mathbf{e}^{H} = \mathbf{0}^{T} \qquad (16)$$

From (14):

$$\mathbf{e} = -\mathbf{R}_M^{-1}\mathbf{r}^{B*}f \qquad (17)$$

Hence, from (15) and (17):

$$f = \frac{1}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \qquad (18)$$

Correspondingly,

$$\mathbf{e} = -\frac{\mathbf{R}_M^{-1}\mathbf{r}^{B*}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{19}$$

From (13):

$$\mathbf{D} = \mathbf{R}_M^{-1} - \mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{e}^H$$

$$= \mathbf{R}_M^{-1} + \frac{\mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} \tag{20}$$

As a check, the results of Eqs. (19) and (20) must satisfy Eq. (16). Thus

$$\mathbf{r}^{BT}\mathbf{D} + r(0)\mathbf{e}^H = \mathbf{r}^{BT}\mathbf{R}_M^{-1} + \frac{\mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}} - \frac{r(0)\mathbf{r}^{BT}\mathbf{R}_M^{-1}}{r(0) - \mathbf{r}^{BT}\mathbf{R}_M^{-1}\mathbf{r}^{B*}}$$

$$= \mathbf{0}^T$$

We have thus shown that

$$\mathbf{R}_{M+1}^{-1} = \left[\begin{array}{c:c} \mathbf{R}_M^{-1} & \mathbf{0} \\ \hdashline \mathbf{0}^T & 0 \end{array}\right] + f\left[\begin{array}{c:c} \mathbf{R}_M^{-1}\mathbf{r}^{B*}\mathbf{r}^{BT}\mathbf{R}_M^{-1} & -\mathbf{R}_M^{-1}\mathbf{r}^{B*} \\ \hdashline -\mathbf{r}^{BT}\mathbf{R}_M^{-1} & 1 \end{array}\right]$$

$$= \left[\begin{array}{c:c} \mathbf{R}_M^{-1} & \mathbf{0} \\ \hdashline \mathbf{0}^T & 0 \end{array}\right] + f\left[\begin{array}{c} -\mathbf{R}_M^{-1}\mathbf{r}^{B*} \\ \hdashline 1 \end{array}\right] \left[\begin{array}{c:c} -\mathbf{r}^{BT}\mathbf{R}_M^{-1} & 1 \end{array}\right]$$

where the scalar f is defined by Eq. (18).

1.6 (a) We express the difference equation describing the first-order AR process $u(n)$ as

$$u(n) = v(n) + w_1 u(n-1)$$

where $w_1 = -a_1$. Solving this equation by repeated substitution, we get

$$u(n) = v(n) + w_1 v(n-1) + w_1 u(n-2)$$

$$= \dots$$

$$= v(n) + w_1 v(n-1) + w_1^2 v(n-2) + \dots + w_1^{n-1} v(1) \tag{1}$$

Here we have used the initial condition

$$u(0) = 0$$

or equivalently

$$u(1) = v(1)$$

Taking the expected value of both sides of Eq. (1) and using

$$E[v(n)] = \mu \qquad \text{for all } n,$$

we get the geometric series

$$E[u(n)] = \mu + w_1\mu + w_1^2\mu + \dots + w_1^{n-1}\mu$$

$$= \begin{cases} \mu\left(\dfrac{1-w_1^n}{1-w_1}\right), & w_1 \neq 1 \\ \mu n, & w_1 = 1 \end{cases}$$

This result shows that if $\mu \neq 0$, then $E[u(n)]$ is a function of time n. Accordingly, the AR process $u(n)$ is not stationary. If, however, the AR parameter satisfies the condition:

$$|a_1| < 1 \quad \text{or} \quad |w_1| < 1$$

then

$$E[(n)] \rightarrow \frac{\mu}{1-w_1} \quad \text{as} \quad n \rightarrow \infty$$

Under this condition, we say that the AR process is asymptotically stationary to order one.

(b) When the white noise process $v(n)$ has zero mean, the AR process $u(n)$ will likewise have zero mean. Then

$$\mathrm{var}[v(n)] = \sigma_v^2$$

$$\mathrm{var}[u(n)] = E[u^2(n)]. \tag{2}$$

Substituting Eq. (1) into (2), and recognizing that for the white noise process

$$E[v(n)v(k)] = \begin{cases} \sigma_v^2 & n = k \\ 0, & n \neq k \end{cases} \tag{3}$$

we get the geometric series

$$\mathrm{var}[u(n)] = \sigma_v^2(1 + w_1^2 + w_1^4 + \cdots + w_1^{2n-2})$$

$$= \begin{cases} \sigma_v^2\left(\dfrac{1 - w_1^{2n}}{1 - w_1^2}\right), & w_1 \neq 1 \\ \sigma_v^2 n, & w_1 = 1 \end{cases}$$

When $|a_1| < 1$ or $|w_1| < 1$, then

$$\mathrm{var}[u(n)] \approx \frac{\sigma_v^2}{1 - w_1^2} = \frac{\sigma_v^2}{1 - a_1^2} \text{ for large } n$$

(c) The autocorrelation function of the AR process $u(n)$ equals $E[u(n)u(n-k)]$. Substituting Eq. (1) into this formula, and using Eq. (3), we get

$$E[u(n)u(n-k)] = \sigma_v^2(w_1^k + w_1^{k+2} + \cdots + w_1^{k+2n-2})$$

$$= \begin{cases} \sigma_v^2 w_1^k\left(\dfrac{1 - w_1^{2n}}{1 - w_1^2}\right), & w_1 \neq 1 \\ \sigma_v^2 n, & w_1 = 1 \end{cases}$$

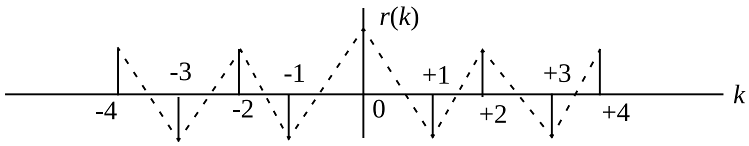
For $|a_1| < 1$ or $|w_1| < 1$, we may therefore express this autocorrelation function as

$$r(k) = E[u(n)u(n-k)]$$

$$\approx \frac{\sigma_v^2 w_1^k}{1 - w_1^2} \quad \text{for large } n$$

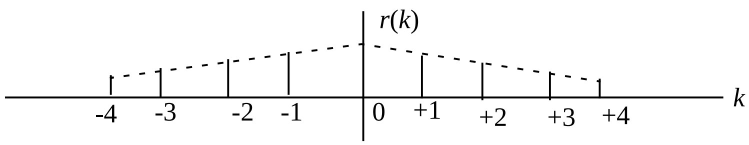
Case 1: $0 < a_1 < 1$

In this case, $w_1 = -a_1$ is negative, and $r(k)$ varies with k as follows:



Case 2: $-1 < a_1 < 0$

In this case, $w_1 = -a_1$ is positive and $r(k)$ varies with k as follows:



1.7 (a) The second-order AR process $u(n)$ is described by the difference equation:

$$u(n) = u(n-1) - 0.5u(n-2) + v(n)$$

Hence

$$w_1 = 1$$

$$w_2 = -0.5$$

and the AR parameters equal

$$a_1 = -1$$

$$a_2 = 0.5$$

Accordingly, we write the Yule-Walker equations as

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} r(1) \\ r(2) \end{bmatrix}$$

(b) Writing the Yule-Walker equations in expanded form:

$$r(0) - 0.5r(1) = r(1)$$

$$r(1) - 0.5r(0) = r(2)$$

Solving the first relation for $r(1)$:

$$r(1) = \frac{2}{3}r(0) \tag{1}$$

Solving the second relation for $r(2)$:

$$r(2) = \frac{1}{6}r(0) \tag{2}$$

(c) Since the noise $v(n)$ has zero mean, so will the AR process $u(n)$. Hence,

$$\begin{aligned} \mathrm{var}[u(n)] &= E[(u^2 n)] \\ &= r(0). \end{aligned}$$

We know that

$$\begin{aligned} \sigma_v^2 &= \sum_{k=0}^{2} a_k r(k) \\ &= r(0) + a_1 r(1) + a_2 r(2) \end{aligned} \tag{3}$$

Substituting (1) and (2) into (3), and solving for $r(0)$, we get

$$r(0) = \frac{\sigma_v^2}{1 + \frac{2}{3}a_1 \frac{1}{6}a_2} = 1.2$$

1.8 By definition,

P_0 = average power of the AR process $u(n)$

$$= E[|u(n)|^2]$$

$$= r(0) \qquad (1)$$

where $r(0)$ is the autocorrelation function of $u(n)$ for zero lag. We note that

$$\{a_1, a_2, \cdots, a_M\} \rightleftharpoons \left\{\frac{r(1)}{r(0)}, \frac{r(2)}{r(0)}, \cdots, \frac{r(M)}{r(0)}\right\}$$

Equivalently, except for the scaling factor r(0),

$$\{a_1, a_2, \cdots, a_M\} \rightleftharpoons \{r(1), r(2), \cdots, r(M)\} \qquad (2)$$

Combining Eqs. (1) and (2):

$$\{P_0, a_1, a_2, \cdots, a_M\} \rightleftharpoons \{r(0), r(1), r(2), \cdots, r(M)\} \qquad (3)$$

1.9 (a) The transfer function of the MA model of Fig. 2.3 is

$$H(z) = 1 + b_1^* z^{-1} + b_2^* z^{-2} + \dots + b_K^* z^{-K}$$

(b) The transfer function of the ARMA model of Fig. 2.4 is

$$H(z) = \frac{b_0^* + b_1^* z^{-1} + b_2^* z^{-2} + \dots + b_K^* z^{-K}}{1 + a_1^* z^{-1} + a_2^* z^{-2} + \dots + a_M^* z^{-M}}$$

(c) The ARMA model reduces to an AR model when

$$b_0 = b_1 = \dots = b_K = 0$$

It reduces to an MA model when

$$a_1 = a_2 = \dots = a_M = 0$$

1.10 We are given

$$x(n) = \upsilon(n) + 0.75\upsilon(n-1) + 0.25\upsilon(n-2)$$

Taking the z-transforms of both sides:

$$X(z) = (1 + 0.75z^{-1} + 0.25z^{-2})V(z)$$

Hence, the transfer function of the MA model is

$$\frac{X(z)}{V(z)} = 1 + 0.75z^{-1} + 0.25z^{-2}$$

$$= \frac{1}{(1 + 0.75z^{-1} + 0.25z^{-2})^{-1}} \qquad (1)$$

Using long division, we may perform the following expansion of the denominator in Eq. (1):

$$(1 + 0.75z^{-1} + 0.25z^{-2})^{-1}$$

$$= 1 - \frac{3}{4}z^{-1} + \frac{5}{16}z^{-2} - \frac{3}{64}z^{-3} - \frac{11}{256}z^{-4} + \frac{45}{1024}z^{-5}$$

$$- \frac{91}{4096}z^{-6} + \frac{93}{16283}z^{-7} + \frac{85}{65536}z^{-8} - \frac{627}{262144}z^{-9} + \frac{1541}{1048576}z^{-10} + \dots$$

$$\approx 1 - 0.75z^{-1} + 0.3125z^{-2} - 0.0469z^{-3} - 0.043z^{-4} + 0.0439z^{-5}$$

$$- 0.0222z^{-6} + 0.0057z^{-7} + 0.0013z^{-8} - 0.0024z^{-9} + 0.0015z^{-10} \qquad (2)$$

(a) $M = 2$

Retaining terms in Eq. (2) up to z^{-2}, we may approximate the MA model with an AR model of order two as follows:

$$\frac{X(z)}{V(z)} \approx \frac{1}{1 - 0.75z^{-1} + 0.3125z^{-2}}$$

(b) $M = 5$

Retaining terms in Eq. (2) up to z^{-5}, we obtain the following approximation in the forms of an AR model of order five:

$$\frac{X(z)}{V(z)} \approx \frac{1}{1 - 0.75z^{-1} + 0.3125z^{-2} - 0.0469z^{-3} - 0.043z^{-4} + 0.0439z^{-5}}$$

(c) $M = 10$

Finally, retaining terms in Eq. (2) up to z^{-10}, we obtain the following approximation in the form of an AR model of order ten:

$$\frac{X(z)}{V(z)} \approx \frac{1}{D(z)}$$

where $D(z)$ is given by the polynomial on the right-hand side of Eq. (2).

1.11 (a) The filter output is

$$x(n) = \mathbf{w}^H \mathbf{u}(n)$$

where $\mathbf{u}(n)$ is the tap-input vector. The average power of the filter output is therefore

$$\begin{aligned} E[|x(n)|^2] &= E[\mathbf{w}^H \mathbf{u}(n) \mathbf{u}^H(n) \mathbf{w}] \\ &= \mathbf{w}^H E[\mathbf{u}(n) \mathbf{u}^H(n)] \mathbf{w} \\ &= \mathbf{w}^H \mathbf{R} \mathbf{w} \end{aligned}$$

(b) If $\mathbf{u}(n)$ is extracted from a zero mean white noise of variance σ^2, we have

$$\mathbf{R} = \sigma^2 \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. Hence,

$$E[|x(n)|^2] = \sigma^2 \mathbf{w}^H \mathbf{w}$$

1.12 (a) The process $u(n)$ is a linear combination of Gaussian samples. Hence, $u(n)$ is Gaussian.

(b) From inverse filtering, we recognize that $v(n)$ may also be expressed as a linear combination of samples represented by $u(n)$. Hence, if $u(n)$ is Gaussian, then $v(n)$ is also Gaussian.

1.13 (a) From the Gaussian moment factoring theorem:

$$E\left[(u_1^* u_2)^k\right] = E[u_1^* \cdots u_1^* u_2 \cdots u_2]$$

$$= k! \; E[u_1^* u_2] \cdots E[u_1^* u_2]$$

$$= k! \; (E[u_1^* u_2])^k \qquad (1)$$

(b) Putting $u_2 = u_1 = u$, Eq. (1) reduces to

$$E[|u|^{2k}] = k! \; (E[|u|^2])^k$$

1.14 It is not permissible to interchange the order of expectation and limiting operations in Eq. (1.113). The reason is that the expectation is a linear operation, whereas the limiting operation with respect to the number of samples N is nonlinear.

1.15 The filter output is

$$y(n) = \sum_i h(i) u(n-i)$$

Similarly, we may write

$$y(m) = \sum_k h(k) u(m-k)$$

Hence,

$$r_y(n, m) = E[y(n) y^*(m)]$$

$$= E\left[\sum_i h(i) u(n-i) \sum_k h^*(k) u^*(m-k)\right]$$

$$= \sum_i \sum_k h(i) h^*(k) E[u(n-i) u^*(m-k)]$$

$$= \sum_i \sum_k h(i) h^*(k) r_u(n-i, m-k)$$

1.16 The mean-square value of the filter output in response to white noise input is

$$P_o = \frac{2\sigma^2 \Delta\omega}{\pi}$$

The value P_o is linearly proportional to the filter bandwidth $\Delta\omega$. This relation holds irrespective of how small $\Delta\omega$ is, compared to the mid-band frequency of the filter.

1.17 (a) The variance of the filter output is

$$\sigma_y^2 = \frac{2\sigma^2\Delta\omega}{\pi}$$

We are given

$$\sigma^2 = 0.1 \text{ volt}^2$$

$$\Delta\omega = 2\pi \times 1 \text{ radians/sec.}$$

Hence,

$$\sigma_y^2 = \frac{2 \times 0.1 \times 2}{\pi} = 0.4 \text{ volt}^2$$

(b) The pdf of the filter output y is

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-y^2/2\sigma_y^2}$$

$$= \frac{1}{0.63\sqrt{2\pi}} e^{-y^2/0.8}$$

1.18 (a) We are given

$$U_k = \sum_{n=\infty}^{N\text{-}1} u(n)\exp(-jn\omega_k), \qquad k = 0,1,...,N\text{-}1$$

where $u(n)$ is real valued and

$$\omega_k = \frac{2\pi}{N}k$$

Hence,

$$E[U_k U_l^*] = E\left[\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} u(n)u(m)\exp(-jn\omega_k + jm\omega_l)\right]$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \exp(-jn\omega_k + jm\omega_l)E[u(n)u(m)]$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \exp(-jn\omega_k + jm\omega_l)r(n-m)$$

$$= \sum_{m=0}^{N-1} \exp(jn\omega_k)\sum_{n=0}^{N-1} r(n-m)\exp(-jn\omega_k) \qquad (1)$$

By definition, we also have

$$\sum_{n=0}^{N-1} r(n)\exp(-jn\omega_k) = S_k$$

Moreover, since $r(n)$ is periodic with period N, we may invoke the time-shifting property of the discrete Fourier transform to write

$$\sum_{n=0}^{N-1} r(n-m)\exp(-jn\omega_k) = \exp(-jm\omega_k)S_k$$

Thus, recognizing that $\omega_k = (2\pi/N)k$, Eq. (1) reduces to

$$E[U_k U_l^*] = S_k \sum_{m=0}^{N-1} \exp(jm(\omega_l - \omega_k))$$

$$= \begin{cases} S_k, & l = k \\ 0, & \text{otherwise} \end{cases}$$

(b) Part (a) shows that the complex spectral samples U_k are uncorrelated. If they are Gaussian, then they will also be statistically independent. Hence,

$$f_{\mathbf{U}}(U_0, U_1, \ldots, U_{N\text{-}1}) = \frac{1}{(2\pi)^N \det(\Lambda)} \exp\left(-\frac{1}{2}\mathbf{U}^H \Lambda \mathbf{U}\right)$$

where

$$\mathbf{U} = [U_0, U_1, \ldots, U_{N\text{-}1}]^T$$

$$\Lambda = \frac{1}{2}E[\mathbf{U}\mathbf{U}^H]$$

$$= \frac{1}{2}\text{diag}(S_0, S_1, \ldots, S_{N-1})$$

$$\det(\Lambda) = \frac{1}{2^N} \prod_{k=0}^{N\text{-}1} S_k$$

Therefore,

$$f_{\mathbf{U}}(U_0, U_1, \ldots, U_{N\text{-}1}) = \frac{1}{(2\pi)^N 2^{-N} \prod_{k=0}^{N\text{-}1} S_k} \exp\left(-\frac{1}{2}\sum_{k=0}^{N\text{-}1} \frac{|U_k|^2}{\frac{1}{2}S_k}\right)$$

$$= \pi^{-N} \exp\left(\sum_{k=0}^{N\text{-}1} - \left(\frac{|U_k|^2}{S_k}\right) - \ln S_k\right)$$

1.19 The mean square value of the increment process $dz(\omega)$ is

$$E[|dz(\omega)|^2] = S(\omega)d\omega$$

Hence $E[|dz(\omega)|^2]$ is measured in watts.

1.20 The third-order cumulant of a process $u(n)$ is

$$c_3(\tau_1, \tau_2) = E[u(n)u(n + \tau_1)u(n + \tau_2)]$$
$$= \text{third-order moment.}$$

All odd-order moments of a Gaussian process are known to be zero; hence,

$$c_3(\tau_1, \tau_2) = 0$$

The fourth-order cumulant is

$$c_4(\tau_1, \tau_2, \tau_3) = E[u(n)u(n+\tau_1)u(n+\tau_2)u(n+\tau_3)]$$

$$- E[u(n)u(n+\tau_1)]E[u(n+\tau_2)u(n+\tau_3)]$$

$$- E[u(n)u(n+\tau_2)]E[u(n+\tau_1)u(n+\tau_3)]$$

$$- E[u(n)u(n+\tau_3)]E[u(n+\tau_1)u(n+\tau_2)]$$

For the special case of $\tau = \tau_1 = \tau_2 = \tau_3$, the fourth-order moment of a zero-mean Gaussian process of variance σ^2 is $3\sigma^4$, and its second-order moment of σ^2. Hence, the fourth-order cumulant is zero. Indeed, all cumulants higher than order two are zero.

1.21 The trispectrum is

$$C_4(\omega_1, \omega_2, \omega_3) = \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} \sum_{\tau_3=-\infty}^{\infty} c_4(\tau_1, \tau_2, \tau_3) e^{-j(\omega_1\tau_1 + \omega_2\tau_2 + \omega_3\tau_3)}$$

Let the process be passed through a three-dimensional band-pass filter centered on ω_1, ω_2, and ω_3. We assume that the bandwidth (along each dimension) is small compared to the respective center frequency. The average power of the filter output is proportional to the trispectrum, $C_4(\omega_1, \omega_2, \omega_3)$.

1.22 (a) Starting with the formula

$$c_k(\tau_1, \tau_2, \cdots, \tau_{k-1}) = \gamma_k \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} \cdots h_{i+\tau_{k-1}}$$

the third-order cumulant of the filter output is

$$c_3(\tau_1, \tau_2) = \gamma_3 \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} h_{i+\tau_2}$$

where γ_3 is the third-order cumulant of the filter input. The bispectrum is

$$C_3(\omega_1, \omega_2) = \gamma_3 \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty} c_3(\tau_1, \tau_2) e^{-j(\omega_1\tau_1 + \omega_2\tau_2)}$$

$$= \gamma_3 \sum_{i=-\infty}^{\infty} \sum_{\tau_1=-\infty}^{\infty} \sum_{\tau_2=-\infty}^{\infty} h_i h_{i+\tau_1} h_{i+\tau_2} e^{-j(\omega_1\tau_1 + \omega_2\tau_2)}$$

Hence,

$$C_3(\omega_1, \omega_2) = \gamma_3 H\left(e^{j\omega_1}\right) H\left(e^{j\omega_2}\right) H^*\left(e^{j(\omega_1 + \omega_2)}\right)$$

(b) From this formula, we immediately deduce that

$$\arg[C_3(\omega_1, \omega_2)] = \arg\left[H\left(e^{j\omega_1}\right)\right] + \arg\left[H\left(e^{j\omega_2}\right)\right] - \arg\left[H\left(e^{j(\omega_1 + \omega_2)}\right)\right]$$

1.23 The output of a filter of impulse response h_i due to an input $u(i)$ is given by the convolution sum

$$y(n) = \sum_i h_i u(n-i)$$

The third-order cumulant of the filter output is, for example,

$$\begin{aligned} C_3(\tau_1, \tau_2) &= E[y(n)y(n+\tau_1)t(n+\tau_2)] \\ &= E\left[\sum_i h_i u(n-i) \sum_k h_k u(n+\tau_1-k) \sum_l h_l u(n+\tau_2-l)\right] \\ &= E\left[\sum_i h_i u(n-i) \sum_k h_{k+\tau_1} u(n-k) \sum_l h_{l+\tau_2} u(n-l)\right] \\ &= \sum_i \sum_k \sum_l h_i h_{k+\tau_1} h_{l+\tau_2} E[u(n-i)u(n-k)u(n-l)] \end{aligned}$$

For an input sequence of independent and identically distributed random variables, we note that

$$E[u(n-i)u(n-k)u(n-l)] = \begin{cases} \gamma_3 & i = k = l \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$C_3(\tau_1, \tau_2) = \gamma_3 \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} h_{i+\tau_2}$$

In general, we may thus write

$$C_3(\tau_1, \tau_2, \ldots, \tau_{k-1}) = \gamma_k \sum_{i=-\infty}^{\infty} h_i h_{i+\tau_1} \ldots h_{i+\tau_{k-1}}$$

1.24 By definition:

$$r^{(\alpha)}(k) = \frac{1}{N} \sum_{n=0}^{N-1} E[u(n)u^*(n-k)e^{-j2\pi\alpha n}]e^{j\pi\alpha k}$$

Hence,

$$r^{(\alpha)}(-k) = \frac{1}{N} \sum_{n=0}^{N-1} E[u(n)u^*(n+k)e^{-j2\pi\alpha n}]e^{-j\pi\alpha k}$$

$$r^{(\alpha)*}(k) = \frac{1}{N} \sum_{n=0}^{N-1} E[u^*(n)u(n-k)e^{j2\pi\alpha n}]e^{-j\pi\alpha k}$$

We are told that the process $u(n)$ is cyclostationary, which means that

$$E[u(n)u^*(n+k)e^{-j2\pi\alpha n}] = E[u^*(n)u(n-k)e^{j2\pi\alpha n}]$$

It follows therefore that

$$r^{(\alpha)}(-k) = r^{(\alpha)*}(k)$$

1.25 For $\alpha = 0$, the input to the time-average cross-correlator reduces to the squared amplitude of a narrow-band filter with mid-band frequency ω. Correspondingly, the time-average cross-correlator reduces to an average power meter. Thus, for $\alpha = 0$, the instrumentation of Fig. 1.16 reduces to that of Fig. 1.13.