

SECOND EDITION

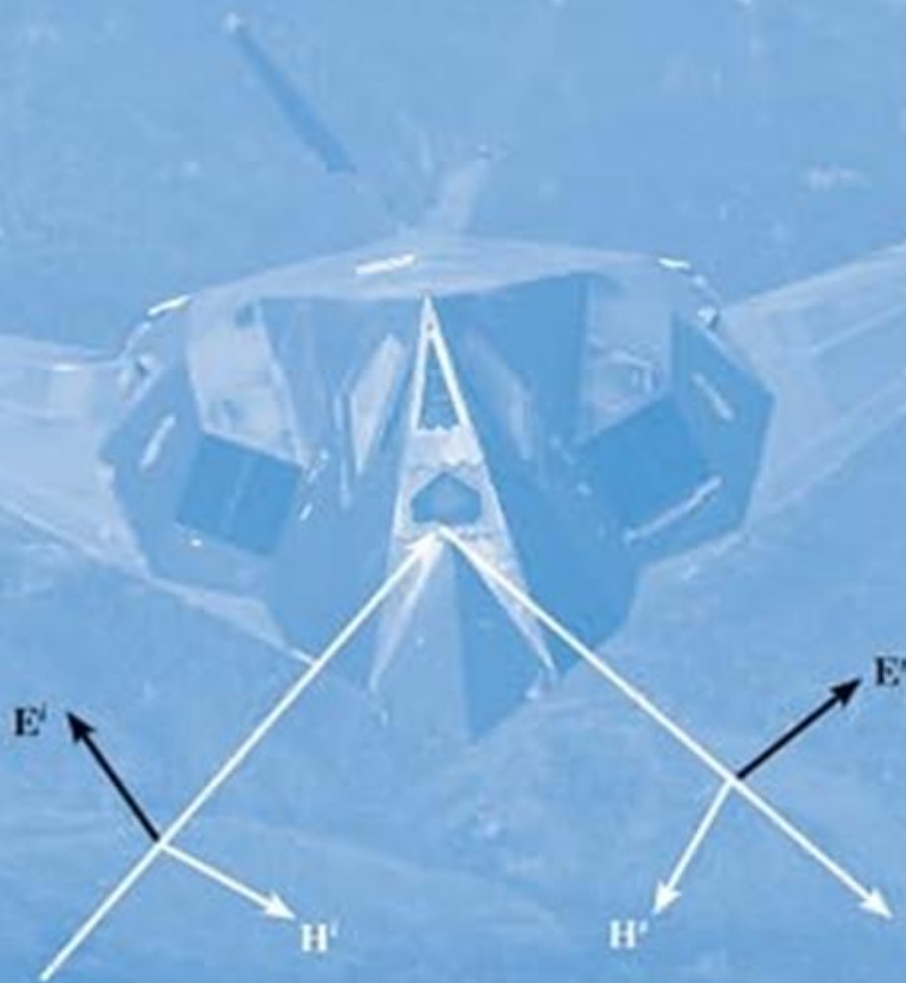
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ADVANCED ENGINEERING ELECTROMAGNETICS

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SOLUTIONS MANUAL



CHAPTER 1

$$\boxed{1.1} \quad \nabla \times \underline{H} = \underline{J}_{ic} + \frac{\partial \underline{D}}{\partial t}$$

Taking the divergence of both sides

$$\nabla \cdot (\nabla \times \underline{H}) = \nabla \cdot \underline{J}_{ic} + \nabla \cdot \frac{\partial \underline{D}}{\partial t} = \nabla \cdot \underline{J}_{ic} + \frac{\partial}{\partial t} \nabla \cdot \underline{D}$$

Using the vector identity of

$\nabla \cdot (\nabla \times \underline{A}) = 0$ and (1-3) we can write that

$$0 = \nabla \cdot \underline{J}_{ic} + \frac{\partial}{\partial t} (q_{ve}) \Rightarrow \boxed{\nabla \cdot \underline{J}_{ic} = - \frac{\partial q_{ve}}{\partial t}}$$

$$\boxed{1.2} \quad \nabla \times \underline{E} = - \underline{M}_i - \frac{\partial \underline{B}}{\partial t}$$

Taking a surface integral of both sides, we can write that

$$\iint_S (\nabla \times \underline{E}) \cdot d\underline{s} = - \iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Applying Stokes' theorem of (1-7) to the left side of the equation above leads to

$$\oint_C \underline{E} \cdot d\underline{l} = - \iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Using the same procedure, we can write that

$$\oint_C \underline{H} \cdot d\underline{l} = \iint_S \underline{J}_{ic} \cdot d\underline{s} + \iint_S \underline{J}_{ec} \cdot d\underline{s} + \frac{\partial}{\partial t} \iint_S \underline{D} \cdot d\underline{s}$$

For the remaining three equations of Table 1-1 we proceed as follows:

$$\nabla \cdot \underline{D} = q_{ve}$$

Taking a volume integral of both sides, we can write that

$$\iiint_V \nabla \cdot \underline{D} \, dv = \iiint_V q_{ve} \, dv = Q_e$$

1.2 cont'd

Applying the divergence theorem of (1-8) on the left side of the equation above leads to

$$\oint_S \underline{D} \cdot d\underline{s} = Q_e$$

Using the same procedure, we can write that

$$\oint_S \underline{B} \cdot d\underline{s} = Q_m$$

and

$$\oint_S \underline{J}_{lc} \cdot d\underline{s} = -\frac{\partial}{\partial t} \iiint_V \rho_{ve} dv = -\frac{\partial Q_e}{\partial t}$$

1.3

(a) $\underline{D} = \hat{a}_x(3+x)$

$$Q_e = \oint_S \underline{D} \cdot d\underline{s} = \int_0^1 \int_0^1 \hat{a}_x(3+x) \cdot (-\hat{a}_x dy dz) + \int_0^1 \int_0^1 \hat{a}_x(3+x) \cdot \hat{a}_x dy dz = -3+4=1$$

(b) $\underline{D} = \hat{a}_y(4+y^2)$

$$Q_e = \oint_S \underline{D} \cdot d\underline{s} = \int_0^1 \int_0^1 \hat{a}_y(4+y^2) \cdot (-\hat{a}_y dx dz) + \int_0^1 \int_0^1 \hat{a}_y(4+y^2) \cdot \hat{a}_y dx dz = -4+5=1$$

1.4

$$\underline{D}_2 = 6\hat{a}_x + 3\hat{a}_z, \quad \chi_{e2} = \epsilon_{r2} - 1 = 2.56 - 1 = 1.56$$

$$(a) \underline{E}_2 = \frac{\underline{D}_2}{\epsilon_{r2}} = \frac{1}{2.56\epsilon_0} (6\hat{a}_x + 3\hat{a}_z) = \frac{1}{\epsilon_0} \left(\frac{6}{2.56} \hat{a}_x + \frac{3}{2.56} \hat{a}_z \right)$$

$$\underline{E}_2 = \frac{1}{\epsilon_0} (2.34\hat{a}_x + 1.1718\hat{a}_z)$$

$$(b) \underline{P}_2 = \epsilon_0 \chi_e \underline{E}_2 = \epsilon_0 \left[1.56 \frac{1}{\epsilon_0} (2.34\hat{a}_x + 1.1718\hat{a}_z) \right]$$

$$\underline{P}_2 = 3.65\hat{a}_x + 1.828\hat{a}_z$$

(c) $\underline{E}_{1x} = \underline{E}_{2x}$ Continuity of tangential components of $\underline{E}$ -field

$$\frac{D_{1x}}{\epsilon_0} = \frac{D_{2x}}{\epsilon_{r2}\epsilon_0} \Rightarrow D_{1x} = \frac{D_{2x}}{\epsilon_{r2}} = \frac{6}{2.56} = 2.344$$

$$\hat{n} \cdot (\underline{D}_2 - \underline{D}_1) = \rho_{es} \quad \text{Discontinuity of normal components of } \underline{D} \text{ density.}$$

$$\hat{n} = \hat{a}_z: \quad D_{2z} - D_{1z} = \rho_{es} = 0.2 \Rightarrow D_{1z} = D_{2z} - \rho_{es} = 3 - 0.2 = 2.8$$

$$\underline{D}_1 = 2.344\hat{a}_x + 2.8\hat{a}_z$$

Cont'd

1.4 cont'd

$$(d) \quad \epsilon_1 E_1 = D_1 \Rightarrow \epsilon_0 E_1 = D_1 \Rightarrow E_1 = \frac{1}{\epsilon_0} D_1 = \frac{1}{\epsilon_0} (2.34 \hat{a}_x + 2.8 \hat{a}_z)$$

$$E_1 = \frac{1}{\epsilon_0} (2.34 \hat{a}_x + 2.8 \hat{a}_z)$$

$$(e) \quad \chi_{e1} = \epsilon_{sr1} - 1 = 1 - 1 = 0$$

$$P_1 = \epsilon_0 \chi_{e1} E_1 = 0$$

1.5

$$H_1 = 3\hat{a}_x + \hat{a}_z \text{ A}, \quad \mu_2 = 4\mu_0$$

$$(a) \quad B_1 = \mu_0 H_1 = \mu_0 (3\hat{a}_x + \hat{a}_z)$$

$$(b) \quad M_1 = \chi_{m1} H_1, \quad \chi_{m1} = \mu_{sr} - 1 = 1 - 1 = 0$$

$$M_1 = 0$$

$$(c) \quad H_{2x} = H_{1x} = 3 \quad \text{Continuity of tangential } H\text{-field.}$$

Continuity of normal B density

$$B_{2z} = B_{1z} = 9\mu_0 \Rightarrow \mu_2 H_{2z} = 9\mu_0 \Rightarrow 4\mu_0 H_{2z} = 9\mu_0$$

$$H_{2z} = \frac{9}{4} = 2.25$$

$$H_2 = 3\hat{a}_x + 2.25\hat{a}_z$$

$$(d) \quad B_2 = \mu_2 H_2 = 4\mu_0 (3\hat{a}_x + \frac{9}{4}\hat{a}_z) = \mu_0 (12\hat{a}_x + 9\hat{a}_z)$$

$$(e) \quad M_2 = \chi_{m2} H_2 \quad \chi_{m2} = \mu_{2r} - 1 = 4 - 1 = 3$$

$$M_2 = 3 (3\hat{a}_x + 2.25\hat{a}_z) = 9\hat{a}_x + 6.75\hat{a}_z$$

1.6 $D_0 = \epsilon_0 E_0$

(a) $E_{0n} = E_0 \cos 30^\circ = 0.866 E_0$

$E_{0t} = E_0 \sin 30^\circ = 0.5 E_0$

$D_{0n} = \epsilon_0 E_{0n} = 0.866 \epsilon_0 E_0$

$D_{0t} = \epsilon_0 E_{0t} = 0.5 \epsilon_0 E_0$

From B.C.s.

$E_{1t} = E_{0t} = 0.5 E_0$

$D_{1t} = \epsilon_1 E_{1t} = 0.5 (4) \epsilon_0 E_0$

$D_{1t} = 2 \epsilon_0 E_0$

$D_{1n} = D_{0n} = 0.866 \epsilon_0 E_0$

$E_{1n} = \frac{D_{1n}}{\epsilon_1} = \frac{0.866 \epsilon_0 E_0}{4 \epsilon_0}$

$E_{1n} = 0.2165 E_0$

$E_1 = (0.5 \hat{a}_t + 0.2165 \hat{a}_n) E_0$

$E_1 = \sqrt{(E_{1t})^2 + (E_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2} E_0 = \sqrt{0.25 + 0.046875} E_0$

$E_1 = \sqrt{0.296875} E_0 = 0.54486 E_0$

$E_1 = 0.54486 E_0$

$D_1 = (2 \hat{a}_t + 0.866 \hat{a}_n) \epsilon_0 E_0$

$D_1 = \sqrt{(D_{1t})^2 + (D_{1n})^2} = \sqrt{(2)^2 + (0.866)^2} \epsilon_0 E_0 = \sqrt{4 + 0.749956} \epsilon_0 E_0$

$D_1 = \sqrt{4.749956} \epsilon_0 E_0 = 2.17944 \epsilon_0 E_0 = 0.54486 (4 \epsilon_0) E_0$

$D_1 = 2.17944 \epsilon_0 E_0 = 0.54486 (4 \epsilon_0) E_0$

(b) $\theta_1 = \tan^{-1} \left(\frac{E_{1t}}{E_{1n}} \right) = \tan^{-1} \left(\frac{0.5 E_0}{0.2165 E_0} \right) = \tan^{-1} (2.309) = 66.587^\circ$

$\theta_1 = \tan^{-1} \left(\frac{D_{1t}}{D_{1n}} \right) = \tan^{-1} \left(\frac{2 \epsilon_0 E_0}{0.866 \epsilon_0 E_0} \right) = \tan^{-1} \left(\frac{2}{0.866} \right) = \tan^{-1} (2.308) = 66.571^\circ$

1.7 $B_o = \mu_o H_o$

$$H_{on} = H_o \cos 30^\circ = 0.866 H_o$$

$$H_{ot} = H_o \sin 30^\circ = 0.5 H_o$$

$$B_{on} = \mu_o H_{on} = 0.866 \mu_o H_o$$

$$B_{ot} = \mu_o H_{ot} = 0.5 \mu_o H_o$$

From B.Cs.

$$H_{1t} = H_{ot} = 0.5 H_o$$

$$B_{1t} = \mu_1 H_{1t} = 0.54 \mu_o H_o$$

$$B_{1t} = 2 \mu_o H_o$$

$$B_{1n} = B_{on} = 0.866 \mu_o H_o$$

$$H_{1n} = \frac{B_{1n}}{\mu_1} = \frac{0.866 \mu_o H_o}{4 \mu_o}$$

$$H_{1n} = 0.2165 H_o$$

$$H_1 = (0.5 \hat{a}_t + 0.2165 \hat{a}_n) H_o$$

$$H_1 = \sqrt{(H_{1t})^2 + (H_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2} H_o = \sqrt{0.296875} H_o$$

$$H_1 = 0.54486 H_o$$

$$B_1 = (2 \hat{a}_t + 0.866 \hat{a}_n) \mu_o H_o$$

$$B_1 = \sqrt{(B_{1t})^2 + (B_{1n})^2} = \sqrt{(2)^2 + (0.866)^2} \mu_o H_o = \sqrt{4.751} \mu_o H_o$$

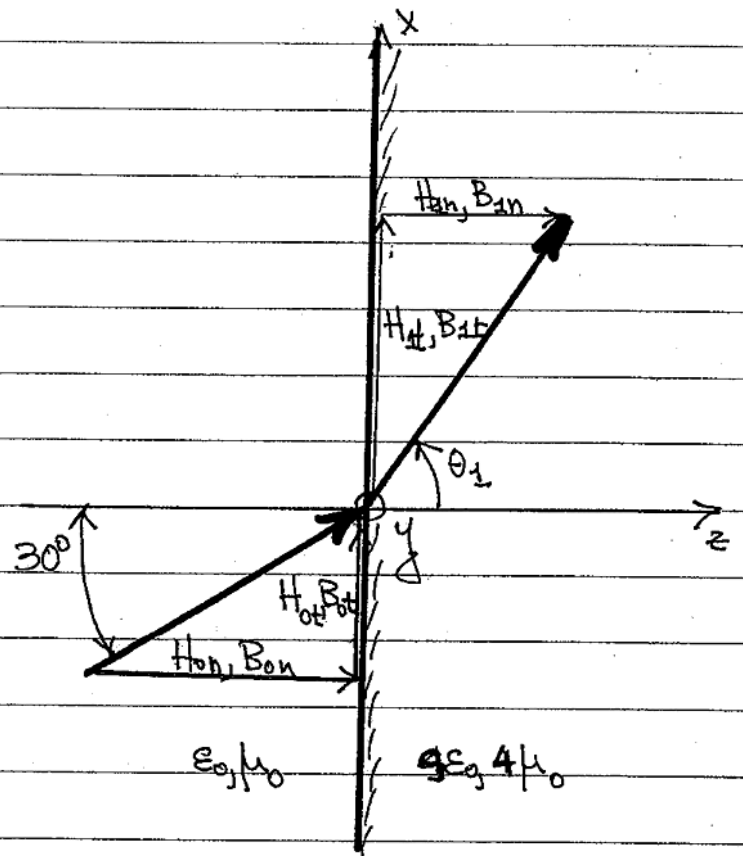
$$B_1 = 2.18 \mu_o H_o = 0.545 (4 \mu_o) H_o$$

b. $\theta_1 = \tan^{-1} \left(\frac{H_{1t}}{H_{1n}} \right) = \tan^{-1} \left(\frac{0.5 H_o}{0.2165 H_o} \right) = \tan^{-1} (2.309) = 66.587^\circ$

$$\theta_1 = 66.587^\circ$$

$$\theta_1 = \tan^{-1} \left(\frac{B_{1t}}{B_{1n}} \right) = \tan^{-1} \left(\frac{2 \mu_o H_o}{0.866 \mu_o H_o} \right) = \tan^{-1} (2.308) = 66.571^\circ$$

$$\theta_1 = 66.571^\circ$$



1.8 Snell's Law of Refraction $\epsilon_n = 1$

$$(a) \quad \beta_1 \sin \theta_1 = \beta_2 \sin \theta_2$$

$$w\sqrt{\mu_1 \epsilon_1} \sin \theta_1 = w\sqrt{\mu_2 \epsilon_2} \sin \theta_2$$

$$\text{Since } \mu_1 = \mu_2 = \mu_0$$

$$\sqrt{\epsilon_1} \sin \theta_1 = \sqrt{\epsilon_2} \sin \theta_2$$

$$\sin \theta_2 = \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin \theta_1$$

$$= \sqrt{\frac{1}{\epsilon_2/\epsilon_1}} \sin \theta_1$$

$$\sin \theta_2 = \sqrt{\frac{1}{4}} \sin \theta_1 = \frac{1}{2} \sin(30^\circ) = \frac{1}{2} \left(\frac{1}{2} \right) = \frac{1}{4}$$

$$\theta_2 = \sin^{-1}\left(\frac{1}{4}\right) = 14.4775^\circ = \theta_4$$

$$\beta_2 \sin \theta_4 = \beta_3 \sin \alpha \Rightarrow w\sqrt{\mu_2 \epsilon_2} \sin \theta_4 = w\sqrt{\mu_3 \epsilon_3} \sin \alpha$$

$$\text{Since } \mu_2 = \mu_3 = \mu_0:$$

$$\sqrt{\epsilon_3} \sin \alpha = \sqrt{\epsilon_2} \sin \theta_4$$

$$\alpha = \sin^{-1} \left[\sqrt{\frac{\epsilon_2}{\epsilon_3}} \sin \theta_4 \right] = \sin^{-1} \left(\sqrt{\frac{4}{9}} \sin \theta_4 \right) = \sin^{-1} \left(\frac{2}{3} \sin(14.4775^\circ) \right)$$

$$\alpha = \sin^{-1} \left(\frac{2}{3} (0.25) \right) = \sin^{-1}(0.1667)$$

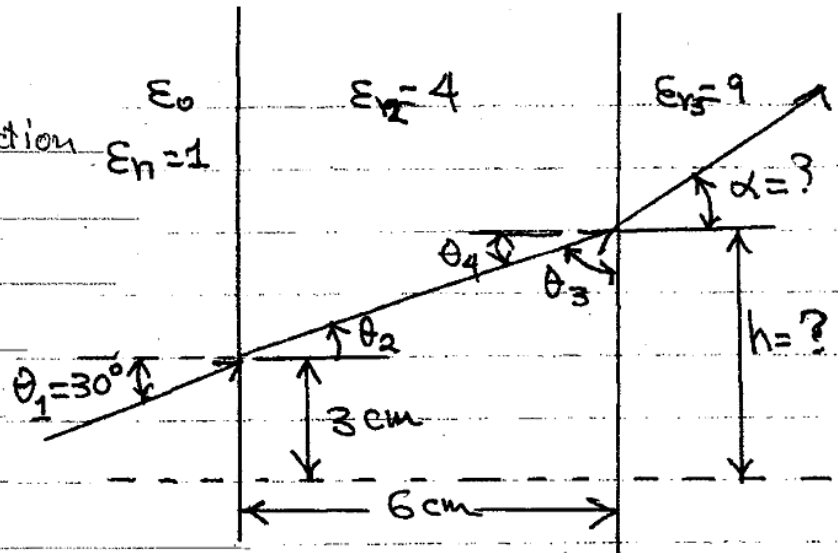
$$\boxed{\alpha = 9.594^\circ}$$

$$(b) \quad \tan \theta_2 = \left(\frac{h-3}{6} \right) = \tan(14.4775^\circ)$$

$$\frac{h-3}{6} = (0.2582) \Rightarrow h-3 = 6(0.2582)$$

$$h = 3 + 6(0.2582) = 4.5492 \text{ cm}$$

$$\boxed{h = 4.5492 \text{ cm}}$$



1.9 $\underline{D} = \epsilon_0 \underline{E}$, $Q_e = \oint_S \underline{D} \cdot d\underline{s}$

$$Q_e = \oint_S \underline{D} \cdot d\underline{s} = \epsilon_0 \int_S \hat{a}_z E_z \cdot (-\hat{a}_z d\underline{s}) + \epsilon_0 \int_S \hat{a}_z E_z \cdot \hat{a}_z d\underline{s}$$

$$Q_e = -\epsilon_0 \int_S \left[-\frac{c}{h} - \frac{bh^2}{6\epsilon_0} \right] d\underline{s} + \epsilon_0 \int_S \left[-\frac{c}{h} + 2\frac{bh^2}{6\epsilon_0} \right] d\underline{s} = \frac{bh^2}{6} (\pi a^2) = \frac{\pi}{2} b(ha)^2$$

1.10 $\epsilon_r = 4, \mu_r = 9, a = 4 \text{ cm}$

$$\underline{H} = 3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z$$

(a) $\underline{B} = \mu \underline{H} = \mu \mu_0 \underline{H} = 9 \mu_0 (3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z) = \mu_0 (27\hat{a}_\rho + 54\hat{a}_\phi + 72\hat{a}_z)$

$$\underline{B} = \mu_0 (27\hat{a}_\rho + 54\hat{a}_\phi + 72\hat{a}_z)$$

(b) $\underline{H}_0 = \hat{a}_\rho H_{\rho 0} + \hat{a}_\phi H_{\phi 0} + \hat{a}_z H_{z 0}$

$$H_{\phi 0} = H_\phi = 6$$

$$H_{z 0} = H_z = 8$$

$$B_{\rho 0} = B_\rho = \mu_0 H_{\rho 0} = \mu H_\rho = \mu_r \mu_0 H_\rho$$

$$H_{\rho 0} = \mu_r H_\rho = 9(3) = 27$$

$$\underline{H}_0 = (27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

(c)

$$\underline{B}_0 = \mu_0 \underline{H}_0 = \mu_0 (27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

1.11 $\nabla \cdot \underline{E} = 0$ for a source-free and homogeneous medium.

Thus

$$\nabla \cdot \underline{E} = \left[\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \right] \cdot \left[\hat{a}_x A(x+y) + \hat{a}_y B(x-y) \right] \cos(\omega t) = 0$$

$$A \frac{\partial}{\partial x} (x+y) + B \frac{\partial}{\partial y} (x-y) = A(1) + B(-1) = 0 \Rightarrow A = B$$

Also

$$\underline{E} = \hat{a}_x A(x+y) + \hat{a}_y B(x-y)$$

$$\nabla \times \underline{E} = -j\omega\mu \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(x+y) & B(x-y) & 0 \end{vmatrix}$$

$$\underline{H} = -\frac{1}{j\omega\mu} \left[\hat{a}_x(0) + \hat{a}_y(0) + \hat{a}_z(B-A) \right]$$

$$\nabla \times \underline{H} = j\omega\epsilon \underline{E} \Rightarrow \underline{E} = 0 \Rightarrow A = B = 0$$

1.12

$$\underline{B} = \hat{a}_z \frac{10^{-12}}{1+25\rho} \cos(1500\pi t)$$

$$\begin{aligned} \text{a. } \Psi_m &= \iint_{S_0} \underline{B} \cdot d\underline{s} = \int_0^{2\pi} \int_0^a \hat{a}_z B_z \cdot \hat{a}_z \rho d\rho d\phi = \int_0^{2\pi} \int_0^a B_z \rho d\rho d\phi \\ &= 2\pi \int_0^a B_z \rho d\rho = 2\pi \times 10^{-12} \cos(1500\pi t) \int_0^a \frac{\rho}{1+25\rho} d\rho \end{aligned}$$

Using the integral

$$\int \frac{u}{a+bu} du = \frac{1}{b^2} [a+bu - a \ln(a+bu)]$$

we can write the flux as

$$\begin{aligned} \Psi_m &= 2\pi \times 10^{-12} \cos(1500\pi t) \left\{ \frac{1}{(25)^2} [1+25\rho - \ln(1+25\rho)] \right\}_0^a \\ &= 2\pi \times 10^{-12} \cos(1500\pi t) \left\{ \frac{1}{625} [25a - \ln(1+25a)] \right\} \\ \Psi_m &= 2\pi \times 10^{-12} \cos(1500\pi t) \left\{ \frac{1}{625} [2.5 - \ln(3.5)] \right\} = 1.2539 \times 10^{-14} \cos(1500\pi t) \end{aligned}$$

$$\text{b. } \oint \underline{E} \cdot d\underline{l} = -\frac{\partial \Psi}{\partial t}$$

$$\begin{aligned} \int_0^{2\pi} (\hat{a}_\phi E_\phi) \cdot \hat{a}_\phi \rho d\phi &= -\frac{\partial}{\partial t} \int_0^{2\pi} \int_0^a \underline{B} \cdot d\underline{s} = -\frac{\partial}{\partial t} \int_0^{2\pi} \int_0^a (\hat{a}_z B_z) \cdot \hat{a}_z \rho d\rho d\phi \\ 2\pi \rho E_\phi &= -2\pi \frac{\partial}{\partial t} \int_0^a B_z \rho d\rho = 2\pi \times 10^{-12} (1500\pi) \sin(1500\pi t) \int_0^a \frac{\rho}{1+25\rho} d\rho \\ \rho E_\phi &= 1500\pi \times 10^{-12} \sin(1500\pi t) \left\{ \frac{1}{(25)^2} [1+25\rho - \ln(1+25\rho)] \right\}_0^a \end{aligned}$$

$$E_\phi = \frac{7.5398 \times 10^{-12}}{\rho} [25\rho - \ln(1+25\rho)] \sin(1500\pi t)$$

To check: Use Maxwell's equation $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

$$\hat{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\phi) = -\frac{\partial}{\partial t} \left[\hat{a}_z \frac{10^{-12}}{1+25\rho} \cos(1500\pi t) \right]$$

$$\begin{aligned} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left\{ 7.5398 \times 10^{-12} [25\rho - \ln(1+25\rho)] \sin(1500\pi t) \right\} &= 1500\pi \times 10^{-12} \sin(1500\pi t) \frac{1}{1+25\rho} \\ \frac{4.712 \times 10^{-9}}{1+25\rho} \sin(1500\pi t) &= \frac{4.712 \times 10^{-9}}{1+25\rho} \sin(1500\pi t) \quad \text{QED} \end{aligned}$$

$$1.13 \quad \underline{B} = \hat{a}_x B_x \cos(2y) \sin(\omega t - \pi z) + \hat{a}_y B_y \cos(2x) \cos(\omega t - \pi z)$$

$$\nabla \times \underline{H} = \underline{J}_i + \underline{J}_c + \underline{J}_d = \underline{J}_d = \frac{\partial \underline{D}}{\partial t}$$

$$\begin{aligned} \underline{J}_d &= \nabla \times \underline{H} = \frac{1}{\mu_0} \nabla \times \underline{B} = \frac{1}{\mu_0} \left\{ -\hat{a}_x \frac{\partial B_y}{\partial z} + \hat{a}_y \frac{\partial B_x}{\partial z} + \hat{a}_z \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] \right\} \\ &= \frac{1}{\mu_0} \left\{ -\hat{a}_x \pi B_y \cos(2x) \sin(\omega t - \pi z) - \hat{a}_y \pi B_x \cos(2y) \cos(\omega t - \pi z) \right. \\ &\quad \left. + \hat{a}_z [-2B_y \sin(2x) \cos(\omega t - \pi z) + 2B_x \sin(2y) \sin(\omega t - \pi z)] \right\} \end{aligned}$$

$$\begin{aligned} \underline{J}_d &= -\hat{a}_x 2.5 \times 10^6 B_y \cos(2x) \sin(\omega t - \pi z) - \hat{a}_y 2.5 \times 10^6 B_x \cos(2y) \cos(\omega t - \pi z) \\ &\quad + \hat{a}_z [-1.59 \times 10^6 B_y \sin(2x) \cos(\omega t - \pi z) + 1.59 \times 10^6 B_x \sin(2y) \sin(\omega t - \pi z)] \end{aligned}$$

$$1.14 \quad \underline{J}_d = \hat{a}_x yz + \hat{a}_y y^2 + \hat{a}_z xyz, \quad I_d = \oint_S \underline{J}_d \cdot d\underline{s}$$

$$\begin{aligned} I_d &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_x yz) \cdot (-\hat{a}_x dy dz) + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_x yz) \cdot (\hat{a}_x dy dz) \\ &\quad + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_y y^2) \cdot (-\hat{a}_y dx dz) + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_y y^2) \cdot (\hat{a}_y dx dz) \\ &\quad + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_z xyz) \cdot (-\hat{a}_z dx dy) + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_z xyz) \cdot (\hat{a}_z dx dy) \end{aligned}$$

$$I_d = 0 + 0 + 0 = 0$$

$$1.15 \quad \underline{E} = \hat{a}_r \frac{10^{-9}}{4\pi} \frac{1}{r^2} \cos(\omega t - \beta r) = \text{Re} \left[\hat{a}_r \frac{10^{-9}}{4\pi r^2} e^{j(\omega t - \beta r)} \right] = \text{Re} \left[\underline{D} e^{j\omega t} \right]$$

$$\text{where } \underline{D} = \hat{a}_r \frac{10^{-9}}{4\pi r^2} e^{-j\beta r}, \quad \beta = \omega \sqrt{\mu_0 \epsilon_0}$$

$$Q_e = \oint_S \underline{D} \cdot d\underline{s} = \int_0^{2\pi} \int_0^\pi (\hat{a}_r D_r) \cdot \hat{a}_r r^2 \sin\theta d\theta d\phi = \frac{10^{-9}}{4\pi} e^{-j\beta r} \int_0^{2\pi} \int_0^\pi \sin\theta d\theta d\phi$$

$$Q_e = \frac{10^{-9}}{4\pi} e^{-j\beta r} 2\pi (-\cos\theta)_0^\pi = 10^{-9} e^{-j\beta r}$$

$$|Q_e| = 10^{-9} \text{ Coulombs}$$

$$\boxed{1.16} \quad \underline{E} = \text{Re} [\underline{E}(r, \theta) e^{j\omega t}] = \text{Re} [\hat{a}_\phi E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r} e^{j\omega t}] = \hat{a}_\phi E_0 \sin\theta \frac{\cos(\omega t - \beta_0 r)}{r}$$

where $\underline{E}(r, \theta) = \hat{a}_\phi E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r}$

Using Maxwell's equation

$$\underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left[\hat{a}_r \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (E_\phi \sin\theta) - \hat{a}_\theta \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \right]$$

$$= -\hat{a}_r \frac{2E_0}{j\omega\mu_0} \cos\theta \frac{e^{-j\beta_0 r}}{r^2} - \hat{a}_\theta \sqrt{\frac{E_0}{\mu_0}} E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r}$$

$$\underline{H} \approx -\hat{a}_\theta \sqrt{\frac{E_0}{\mu_0}} E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r} \Rightarrow \underline{H} = \text{Re} [\underline{H} e^{j\omega t}] \approx -\hat{a}_\theta \sqrt{\frac{E_0}{\mu_0}} E_0 \sin\theta \frac{\cos(\omega t - \beta_0 r)}{r}$$

$$\boxed{1.17} \quad v(t) = 10 \cos(\omega t), \quad \underline{J}_c = \sigma \underline{E}, \quad \underline{J}_d = \frac{\partial \underline{D}}{\partial t} = \epsilon \frac{\partial \underline{E}}{\partial t}$$

$$\underline{E}(t) = \frac{10}{2 \times 10^{-2}} \cos(\omega t) = 500 \cos(\omega t), \quad \frac{\partial \underline{E}}{\partial t} = -500\omega \sin(\omega t)$$

a. $f = 1 \text{ MHz}$

$$|\underline{J}_c|_{\max} = |\sigma \underline{E}|_{\max} = (3.7 \times 10^{-4}) 500 = 0.185$$

$$|\underline{J}_d|_{\max} = \left| \epsilon \frac{\partial \underline{E}}{\partial t} \right|_{\max} = \left| -500\omega \epsilon \sin(\omega t) \right|_{\max} = \frac{2.56}{36} = 0.0711$$

b. $f = 100 \text{ MHz}$

$$|\underline{J}_c|_{\max} = |\sigma \underline{E}|_{\max} = (3.7 \times 10^{-4}) 500 = 0.185$$

$$|\underline{J}_d|_{\max} = \left| \epsilon \frac{\partial \underline{E}}{\partial t} \right|_{\max} = \frac{2.56(100)}{36} = 7.111$$

$$\boxed{1.18} \quad \underline{E} = [\hat{a}_y 5 + \hat{a}_z 10] \cos(\omega t - \beta x) = \text{Re} [(\hat{a}_y 5 + \hat{a}_z 10) e^{j(\omega t - \beta x)}] = \text{Re} [\underline{E} e^{j\omega t}]$$

where $\underline{E} = (\hat{a}_y 5 + \hat{a}_z 10) e^{-j\beta x}$

a. $\underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = \hat{a}_y \left(-\frac{\partial E_z}{\partial x} \right) + \hat{a}_z \left(\frac{\partial E_y}{\partial x} \right) = 4.244 \times 10^{-3} (-\hat{a}_y 10 + \hat{a}_z 5) e^{-j\beta x}$

$$\underline{H} = \text{Re} [\underline{H} e^{j\omega t}] = 4.244 \times 10^{-3} (-\hat{a}_y 10 + \hat{a}_z 5) \cos(\omega t - \beta x)$$

(continued)

1.18 cont'd.

The tangential components of the electric and magnetic fields must be continuous across the boundaries.

The normal components of the electric and magnetic fields must be discontinuous across the boundaries.

b. $\underline{E}_z(y=h^+) = 10 \cos(\omega t - \beta x); \underline{E}_y(y=h^+) = \frac{\epsilon}{\epsilon_0}(5) \cos(\omega t - \beta x) = 12.8 \cos(\omega t - \beta x)$

$$\underline{E}(y=h^+) = (\hat{a}_y 12.8 + \hat{a}_z 10) \cos(\omega t - \beta x)$$

$$\underline{H}_z(y=h^+) = 4.244 \times 10^{-3}(5) \cos(\omega t - \beta x); \underline{H}_y(y=h^+) = 4.244 \times 10^{-3}(-10) \cos(\omega t - \beta x)$$

$$\underline{H}(y=h^+) = 4.244 \times 10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5) \cos(\omega t - \beta x)$$

In a similar manner

$$\underline{E}(y=-h^+) = (\hat{a}_y 12.8 + \hat{a}_z 10) \cos(\omega t - \beta x)$$

$$\underline{H}(y=-h^+) = 4.244 \times 10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5) \cos(\omega t - \beta x)$$

1.19

$$\underline{J} = \hat{a}_z 10^4 e^{-10^6 y} \cos(2\pi \times 10^9 t)$$

$$\underline{J}(y=0.25 \times 10^{-3}) = \hat{a}_z 10^4 e^{-10^6(2.5 \times 10^{-4})} \cos(2\pi \times 10^9 t) = \hat{a}_z 10^4 e^{-250} \cos(2\pi \times 10^9 t) \approx 0$$

$$I = \iint_S \underline{J} \cdot d\underline{s} \approx 2 \int_0^{2.5 \times 10^{-4}} \int_0^{5 \times 10^{-3}} [\hat{a}_z 10^4 e^{-10^6 y} \cos(2\pi \times 10^9 t)] \cdot \hat{a}_z dx dy$$

$$I \approx 2(5 \times 10^{-3})(10^4) \cos(2\pi \times 10^9 t) \int_0^{2.5 \times 10^{-4}} e^{-10^6 y} dy = 10^{-4} \cos(2\pi \times 10^9 t)$$

1.20

a. $\underline{E} = \hat{a}_y E_0 \sin(\frac{\pi}{a} x) \cos(\omega t - \beta_z z) = \text{Re} [\hat{a}_y E_0 \sin(\frac{\pi}{a} x) e^{j(\omega t - \beta_z z)}] = \text{Re} [\underline{E} e^{j\omega t}]$

where $\underline{E} = \hat{a}_y E_0 \sin(\frac{\pi}{a} x) e^{-j\beta_z z}$

$$\underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = \hat{a}_x \frac{1}{j\omega\mu_0} \frac{\partial E_y}{\partial z} - \hat{a}_z \frac{1}{j\omega\mu_0} \frac{\partial E_y}{\partial x}$$

$$= -\hat{a}_x \frac{\beta_z}{\omega\mu_0} E_0 \sin(\frac{\pi}{a} x) e^{-j\beta_z z} + \hat{a}_z j \frac{E_0}{\omega\mu_0} (\frac{\pi}{a}) \cos(\frac{\pi}{a} x) e^{-j\beta_z z}$$

$$\underline{H} = \text{Re} [\underline{H} e^{j\omega t}] = -\hat{a}_x \frac{\beta_z}{\omega\mu_0} E_0 \sin(\frac{\pi}{a} x) \cos(\omega t - \beta_z z) + \hat{a}_z \frac{E_0}{\omega\mu_0} (\frac{\pi}{a}) \cos(\frac{\pi}{a} x) \cos(\omega t - \beta_z z)$$

b. Using $\nabla \times \underline{H} = j\omega\epsilon_0 \underline{E} \Rightarrow \hat{a}_y (\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}) = j\omega\epsilon_0 E_0 \sin(\frac{\pi}{a} x) e^{-j\beta_z z}$

$$\hat{a}_y j \frac{E_0}{\omega\mu_0} [\beta_z^2 + (\frac{\pi}{a})^2] \sin(\frac{\pi}{a} x) e^{-j\beta_z z} = j\omega\epsilon_0 E_0 \sin(\frac{\pi}{a} x) e^{-j\beta_z z}$$

$$\omega\epsilon_0 = \frac{1}{\omega\mu_0} [\beta_z^2 + (\frac{\pi}{a})^2] \Rightarrow \beta_z = \pm \sqrt{\omega^2\mu_0\epsilon_0 - (\frac{\pi}{a})^2}$$

$$1.21 \quad \underline{E} = \hat{a}_\rho \left(\frac{100}{\rho} \right) \cos(10^8 t - \beta z) = \text{Re} \left[\hat{a}_\rho \frac{100}{\rho} e^{j(10^8 t - \beta z)} \right] = \text{Re} \left[\underline{E} e^{j\omega t} \right]$$

$$\text{where } \underline{E} = \hat{a}_\rho \frac{100}{\rho} e^{-j\beta z}$$

$$a. \quad \underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left(\hat{a}_\phi \frac{\partial E_\rho}{\partial z} \right) = \hat{a}_\phi \frac{\beta}{\omega\mu_0} \frac{100}{\rho} e^{-j\beta z}$$

$$\underline{H} = \text{Re} \left[\underline{H} e^{j\omega t} \right] = \text{Re} \left[\underline{H} e^{j10^8 t} \right] = \hat{a}_\phi \frac{\beta}{\omega\mu_0} \left(\frac{100}{\rho} \right) \cos(10^8 t - \beta z)$$

$$b. \quad \nabla \times \underline{H} = j\omega\epsilon \underline{E} \Rightarrow -\hat{a}_\rho \frac{\partial H_\phi}{\partial z} = \hat{a}_\rho j\omega\epsilon \frac{100}{\rho} e^{-j\beta z}$$

$$\hat{a}_\rho j \frac{\beta^2}{\omega\mu_0} \left(\frac{100}{\rho} \right) e^{-j\beta z} = \hat{a}_\rho j\omega\epsilon \left(\frac{100}{\rho} \right) e^{-j\beta z}$$

$$\omega\epsilon = \frac{\beta^2}{\omega\mu_0} \Rightarrow \beta^2 = \omega^2 \mu_0 \epsilon$$

$$c. \quad \underline{D} = \epsilon \frac{\partial \underline{E}}{\partial t} = \epsilon \frac{\partial}{\partial t} \left[\hat{a}_\rho \frac{100}{\rho} \cos(10^8 t - \beta z) \right] = -\hat{a}_\rho \frac{10^8 \epsilon}{\rho} \sin(10^8 t - \beta z)$$

$$= -\hat{a}_\rho \frac{8.854 \times 10^{-12} \times 10^8}{\rho} \sin(10^8 t - \beta z) = -\hat{a}_\rho \frac{2.25(8.854 \times 10^{-2})}{\rho} \sin(10^8 t - \beta z)$$

$$\underline{D} = -\hat{a}_\rho \frac{0.1992}{\rho} \sin(10^8 t - \beta z)$$

$$1.22 \quad \underline{H} = \hat{a}_\phi \frac{2}{\rho} \cos\left(\frac{\pi}{2} z\right) \cos(4\pi \times 10^8 t) = \text{Re} \left[\hat{a}_\phi \frac{2}{\rho} \cos\left(\frac{\pi}{2} z\right) e^{j4\pi \times 10^8 t} \right] = \text{Re} \left[\underline{H} e^{j\omega t} \right]$$

$$\text{where } \underline{H} = \hat{a}_\phi \frac{2}{\rho} \cos\left(\frac{\pi}{2} z\right), \quad \omega = 4\pi \times 10^8$$

$$a. \quad \underline{E} = \frac{1}{j\omega\epsilon_0} \nabla \times \underline{H} = -\hat{a}_\rho \frac{1}{j\omega\epsilon_0} \frac{\partial H_\phi}{\partial z} = -\hat{a}_\rho j \frac{1}{\omega\epsilon_0} \left(\frac{\pi}{2} \right) \frac{2}{\rho} \sin\left(\frac{\pi}{2} z\right)$$

$$\underline{E} = \text{Re} \left[\underline{E} e^{j\omega t} \right] = \hat{a}_\rho \frac{1}{\omega\epsilon_0} \left(\frac{\pi}{2} \right) \frac{2}{\rho} \sin\left(\frac{\pi}{2} z\right) \sin(4\pi \times 10^8 t)$$

$$b. \quad \underline{D}_s = \hat{n} \times (\underline{H}_2 - \underline{H}_1): \quad \text{At } \rho = a: \underline{D}_s = \hat{a}_\rho \times \left(\hat{a}_\phi \underline{H}_\phi \right) \Big|_{\rho=a} = \hat{a}_z \frac{2}{a} \cos\left(\frac{\pi}{2} z\right) \cos(4\pi \times 10^8 t)$$

$$\text{At } \rho = b: \underline{D}_s = -\hat{a}_\rho \times \left(\hat{a}_\phi \underline{H}_\phi \right) \Big|_{\rho=b} = -\hat{a}_z \frac{2}{b} \cos\left(\frac{\pi}{2} z\right) \cos(4\pi \times 10^8 t)$$

$$c. \quad \underline{D}_d = \epsilon_0 \frac{\partial \underline{E}_\phi}{\partial t} = \hat{a}_\rho \frac{4\pi \times 10^8}{\omega} \left(\frac{\pi}{2} \right) \frac{2}{\rho} \sin\left(\frac{\pi}{2} z\right) \cos(4\pi \times 10^8 t) = \hat{a}_\rho \frac{\pi}{2} \left(\frac{2}{\rho} \right) \sin\left(\frac{\pi}{2} z\right) \cos(4\pi \times 10^8 t)$$

$$d. \quad I_d = \int_0^l \int_0^{2\pi} \underline{D}_d \cdot d\underline{s} = \int_0^l \int_0^{2\pi} \left(\hat{a}_\rho \underline{D}_d \right) \cdot \hat{a}_\rho \rho d\phi dz = \frac{\pi}{2} (4\pi) \left[-\frac{l}{\pi} \cos\left(\frac{\pi}{2} z\right) \right]_0^l \cos(4\pi \times 10^8 t)$$

$$I_d = 8\pi \cos(4\pi \times 10^8 t)$$

$$\boxed{1.23} \quad \nabla \times \underline{E} = -\underline{M}_i - \frac{\partial \underline{B}}{\partial t}$$

Defining

$$\underline{E} = \text{Re}[\underline{E} e^{j\omega t}]$$

$$\underline{M}_i = \text{Re}[\underline{M}_i e^{j\omega t}]$$

$$\underline{B} = \text{Re}[\underline{B} e^{j\omega t}]$$

Thus

$$\nabla \times (\text{Re}[\underline{E} e^{j\omega t}]) = -\text{Re}[\underline{M}_i e^{j\omega t}] - \frac{\partial}{\partial t} (\text{Re}[\underline{B} e^{j\omega t}])$$

Interchanging differentiation with the Real part leads to

$$\text{Re}[\nabla \times (\underline{E} e^{j\omega t})] = \text{Re}[-\underline{M}_i e^{j\omega t}] + \text{Re}\left[-\frac{\partial}{\partial t} (\underline{B} e^{j\omega t})\right]$$

or

$$\text{Re}[(\nabla \times \underline{E}) e^{j\omega t}] = \text{Re}[-\underline{M}_i e^{j\omega t}] + \text{Re}[-j\omega \underline{B} e^{j\omega t}]$$

Lemma: If $\underline{A}$ and $\underline{B}$ are complex quantities and

$$\text{Re}[\underline{A} e^{j\omega t}] = \text{Re}[\underline{B} e^{j\omega t}] \quad \text{for all } t$$

then

$$\underline{A} = \underline{B}$$

Using this lemma, we can write that

$$\nabla \times \underline{E} = -\underline{M}_i - j\omega \underline{B}$$

The same procedure can be used for all the other differential form equations.

For the integral form

$$\oint_C \underline{E} \cdot d\underline{\ell} = \frac{1}{S} \iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Using the above definitions, we can write the integral form as

$$\oint_C \text{Re}[\underline{E} e^{j\omega t}] \cdot d\underline{\ell} = \frac{1}{S} \iint_S \text{Re}[\underline{M}_i e^{j\omega t}] \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \text{Re}[\underline{B} e^{j\omega t}] \cdot d\underline{s}$$

$$\text{Re}\left\{ \left[\oint_C \underline{E} \cdot d\underline{\ell} \right] e^{j\omega t} \right\} = \text{Re}\left\{ \left[\frac{1}{S} \iint_S \underline{M}_i \cdot d\underline{s} \right] e^{j\omega t} \right\} + \text{Re}\left\{ \left[-j\omega \iint_S \underline{B} \cdot d\underline{s} \right] e^{j\omega t} \right\}$$

Using the above lemma leads to

$$\oint_C \underline{E} \cdot d\underline{\ell} = \frac{1}{S} \iint_S \underline{M}_i \cdot d\underline{s} - j\omega \iint_S \underline{B} \cdot d\underline{s}$$

The same procedure can be used for all the other integral form equations.

1.24

$$\begin{aligned}\underline{E} &= \text{Re}[\underline{E} e^{j\omega t}] = \text{Re}[(\underline{E}_R + j\underline{E}_X)(\cos\omega t + j\sin\omega t)] \\ &= \text{Re}[(\underline{E}_R \cos\omega t - \underline{E}_X \sin\omega t) + j(\underline{E}_R \sin\omega t + \underline{E}_X \cos\omega t)] \\ \underline{E} &= (\underline{E}_R \cos\omega t - \underline{E}_X \sin\omega t)\end{aligned}$$

Similarly

$$\begin{aligned}\underline{E} &= \frac{1}{2}[\underline{E} e^{j\omega t} + (\underline{E} e^{j\omega t})^*] = \frac{1}{2}[\underline{E} e^{j\omega t} + \underline{E}^* e^{-j\omega t}] \\ &= \frac{1}{2}[(\underline{E}_R + j\underline{E}_X)(\cos\omega t + j\sin\omega t) + (\underline{E}_R - j\underline{E}_X)(\cos\omega t - j\sin\omega t)] \\ \underline{E} &= \frac{1}{2}[2(\underline{E}_R \cos\omega t - \underline{E}_X \sin\omega t)] = (\underline{E}_R \cos\omega t - \underline{E}_X \sin\omega t)\end{aligned}$$

1.25

$$\begin{aligned}(a) \quad \underline{E}(z, t) &= \hat{a}_x E_0 \sin[(\omega t - \beta_0 z) - \frac{\pi}{2}] \\ &= \hat{a}_x E_0 \left[\sin(\omega t - \beta_0 z) \cos(-\frac{\pi}{2}) + \cos(\omega t - \beta_0 z) \sin(-\frac{\pi}{2}) \right] \\ &= -\hat{a}_x E_0 \cos(\omega t - \beta_0 z) = -\hat{a}_x E_0 \text{Re}[e^{j(\omega t - \beta_0 z)}] \\ \underline{E}(z, t) &= -\hat{a}_x E_0 \text{Re}[e^{j\omega t} e^{-j\beta_0 z}] = \text{Re}[\underbrace{-\hat{a}_x E_0 e^{-j\beta_0 z}}_{\underline{E}_X(z)} e^{j\omega t}]\end{aligned}$$

$$\underline{E}(z) = -\hat{a}_x E_0 e^{-j\beta_0 z}$$

$$(b) \quad \nabla \times \underline{E} = -j\omega\mu_0 \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \nabla \times [-\hat{a}_x E_0 e^{-j\beta_0 z}]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{1}{j\omega\mu_0} \left[\hat{a}_x(0) + \hat{a}_y \left(\frac{\partial E_x}{\partial z} \right) + \hat{a}_z \left(-\frac{\partial E_x}{\partial y} \right) \right]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0} \left[\hat{a}_y \left(\frac{\partial E_x}{\partial z} \right) \right] = -\hat{a}_y \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} [-E_0 e^{-j\beta_0 z}] = -\hat{a}_y E_0 \frac{\beta_0}{\omega\mu_0} e^{-j\beta_0 z}$$

$$\underline{H} = -\hat{a}_y E_0 \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\mu_0} e^{-j\beta_0 z} = -\hat{a}_y E_0 \sqrt{\frac{\epsilon_0}{\mu_0}} e^{-j\beta_0 z} = -\hat{a}_y \frac{E_0}{\eta_0} e^{-j\beta_0 z}, \quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

(c)

$$S_{\text{ave}} = \frac{1}{2} \text{Re}(\underline{E} \times \underline{H}^*) = \frac{1}{2} \text{Re} \left[(-\hat{a}_x E_0 e^{-j\beta_0 z}) \times \left(-\hat{a}_y E_0 \sqrt{\frac{\epsilon_0}{\mu_0}} e^{j\beta_0 z} \right)^* \right]$$

$$S_{\text{ave}} = \hat{a}_z \frac{1}{2} |E_0|^2 \sqrt{\frac{\epsilon_0}{\mu_0}} = \hat{a}_z \frac{1}{2\eta_0} |E_0|^2$$

1.26

$$\underline{H} = \hat{a}_\phi H_0 \frac{e^{-j\beta_0 \rho}}{\sqrt{\rho}}$$

$$\begin{aligned} \underline{E} &= \frac{1}{j\omega\epsilon_0} \nabla \times \underline{H} = \frac{1}{j\omega\epsilon_0} \left[\hat{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) \right] = \frac{H_0}{j\omega\epsilon_0} \hat{a}_z \frac{\partial}{\partial \rho} [\rho^{1/2} e^{-j\beta_0 \rho}] \\ &= \hat{a}_z \frac{H_0}{j\omega\epsilon_0} \frac{1}{\rho} \left[\rho^{1/2} (-j\beta_0 e^{-j\beta_0 \rho}) + \frac{1}{2} \frac{1}{\rho^{1/2}} e^{-j\beta_0 \rho} \right] \end{aligned}$$

$$= \hat{a}_z H_0 \left[-\frac{\beta_0}{\omega\epsilon_0} \frac{e^{-j\beta_0 \rho}}{\sqrt{\rho}} + \frac{e^{-j\beta_0 \rho}}{j2\omega\epsilon_0 (\rho)^{3/2}} \right] \xrightarrow{\rho \rightarrow \text{large}} -\hat{a}_z H_0 \frac{\beta_0}{\omega\epsilon_0} \frac{e^{-j\beta_0 \rho}}{\sqrt{\rho}}$$

$$\underline{E} = -\hat{a}_z H_0 \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\epsilon_0} \frac{e^{-j\beta_0 \rho}}{\sqrt{\rho}} = -\hat{a}_z H_0 \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{e^{-j\beta_0 \rho}}{\sqrt{\rho}}$$

1.27

$$\underline{E} = \hat{a}_r E_r(r, \theta) + \hat{a}_\theta E_\theta(r, \theta), \quad E_r = E_0 \frac{\cos\theta}{r^2} \left(1 + \frac{1}{j\beta_0 r} \right) e^{j\beta_0 r}, \quad E_\theta = jE_0 \frac{\beta_0 \sin\theta}{2r} \left[1 + \frac{1}{j\beta_0 r} - \frac{1}{(\beta_0 r)^2} \right] e^{j\beta_0 r}$$

$$\underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left\{ \frac{\hat{a}_\phi}{r} \left[\frac{\partial}{\partial r} (r E_\theta) - \frac{\partial E_r}{\partial \theta} \right] \right\}$$

Expanding using the above electric field components leads to

$$\underline{H} = \hat{a}_\phi j\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\beta_0 \sin\theta}{2r} \left(1 + \frac{1}{j\beta_0 r} \right) e^{-j\beta_0 r}$$

or $H_r = H_\theta = 0$

$$H_\phi = j\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\beta_0 \sin\theta}{2r} \left(1 + \frac{1}{j\beta_0 r} \right) e^{-j\beta_0 r}$$

$$1.28 \quad \underline{E} = \hat{a}_\phi E_0 \frac{\sin \theta}{r} \left(1 + \frac{1}{j\beta_0 r}\right) e^{-j\beta_0 r} = \hat{a}_\phi E_\phi(r, \theta)$$

$$\underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left\{ \hat{a}_r \frac{\partial}{\partial \theta} (E_\phi \sin \theta) + \frac{\hat{a}_\theta}{r} \left[\frac{\partial}{\partial r} (r E_\phi) \right] \right\}$$

Using the above electric field component leads to

$$\underline{H} = \hat{a}_r H_r + \hat{a}_\theta H_\theta \quad \text{where} \quad H_r = j \frac{2E_0 \cos \theta}{\omega\mu_0 r^2} \left(1 + \frac{1}{j\beta_0 r}\right) e^{-j\beta_0 r}$$

$$H_\theta = -\frac{E_0 \sin \theta}{\eta r} \left[1 + \frac{1}{j\beta_0 r} - \frac{1}{(\beta_0 r)^2}\right] e^{-j\beta_0 r}$$

$$1.29 \quad \underline{E} = \hat{a}_z (1+j) \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right)$$

Using Maxwell's equation of $\nabla \times \underline{E} = -j\omega\mu_0 \underline{H}$ leads to

$$\underline{H} = -\frac{1}{j\omega\mu_0} \left[\hat{a}_x \frac{\partial E_z}{\partial y} - \hat{a}_y \frac{\partial E_z}{\partial x} \right] = -\frac{(1+j)}{j\omega\mu_0} \left[\hat{a}_x \left(\frac{\pi}{b}\right) \cos\left(\frac{\pi}{b}y\right) \sin\left(\frac{\pi}{a}x\right) - \hat{a}_y \left(\frac{\pi}{a}\right) \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{b}y\right) \right]$$

Now using Maxwell's equation of $\nabla \times \underline{H} = \underline{J}_c + j\omega\epsilon \underline{E} = \sigma \underline{E} + j\omega\epsilon \underline{E}$ leads to

$$\nabla \times \underline{H} = -\hat{a}_z \frac{(1+j)}{j\omega\mu_0} \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \right] \sin\left(\frac{\pi}{b}y\right) \sin\left(\frac{\pi}{a}x\right) = \hat{a}_z (\sigma + j\omega\epsilon) (1+j) \sin\left(\frac{\pi}{b}y\right) \sin\left(\frac{\pi}{a}x\right)$$

Equating both sides, we can write that

$$(\sigma + j\omega\epsilon) = -\frac{1}{j\omega\mu_0} \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \right] \Rightarrow \sigma = 0$$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = \frac{1}{\omega^2 \mu_0 \epsilon_0} \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 \right]$$

$$1.20 \quad \underline{E}^i = \hat{a}_x e^{-j\beta_0 z} \Rightarrow \underline{H}^i = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E}^i = -\hat{a}_y \frac{1}{j\omega\mu_0} \frac{\partial E_x^i}{\partial z} = \hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} e^{-j\beta_0 z}$$

$$\underline{E}^r = -\hat{a}_x e^{+j\beta_0 z} \Rightarrow \underline{H}^r = -\hat{a}_y \frac{1}{j\omega\mu_0} \frac{\partial E_x^r}{\partial z} = \hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} e^{+j\beta_0 z}$$

$$\underline{J}_s = \hat{n} \times (\underline{H}_2 - \underline{H}_1) = \hat{n} \times \underline{H}_2 = -\hat{a}_z \times \hat{a}_y (H^i + H^r) = \hat{a}_x (H^i + H^r) = \hat{a}_x \sqrt{\frac{\epsilon_0}{\mu_0}} (e^{-j\beta_0 z} + e^{+j\beta_0 z})$$

$$= \hat{a}_x 2 \sqrt{\frac{\epsilon_0}{\mu_0}} = \hat{a}_x \frac{2}{377} = \hat{a}_x 5.3 \times 10^{-3} \text{ A/m}$$

1.31 $\underline{E}^i = \hat{a}_y E_o e^{-j\beta_o(x \sin \theta_i + z \cos \theta_i)}$, $\underline{E}^r = \hat{a}_y E_o \Gamma_h e^{-j\beta_o(x \sin \theta_i - z \cos \theta_i)}$
 Along the interface ($z=0$) $(\underline{E}^i + \underline{E}^r)_{z=0}^{\tan} = 0 = \hat{a}_y E_o (1 + \Gamma_h) e^{-j\beta_o x \sin \theta_i}$
 which is satisfied provided $(1 + \Gamma_h) = 0 \Rightarrow \Gamma_h = -1$

1.32 Using Maxwell's equation of $\nabla \times \underline{E} = -j\omega \mu_o \underline{H}$, the magnetic field components corresponding to the electric fields of Problem 1.31 can be written as

$$\underline{H}^i = \frac{E_o}{\sqrt{\mu_o/\epsilon_o}} (-\hat{a}_x \cos \theta_i + \hat{a}_z \sin \theta_i) e^{-j\beta_o(x \sin \theta_i + z \cos \theta_i)}$$

$$\underline{H}^r = -\frac{E_o}{\sqrt{\mu_o/\epsilon_o}} (\hat{a}_x \cos \theta_i + \hat{a}_z \sin \theta_i) e^{-j\beta_o(x \sin \theta_i - z \cos \theta_i)}$$

b. $\underline{J}_s = \hat{n} \times (\underline{H}_2 - \underline{H}_1)_{z=0} = -\hat{a}_z \times (\underline{H}^i + \underline{H}^r)_{z=0} = -\hat{a}_z \times [\hat{a}_x (H^i + H^r) + \hat{a}_z (H^i + H^r)]_{z=0}$
 $\underline{J}_s = -\hat{a}_z \times \hat{a}_x (H^i + H^r) = \hat{a}_y \frac{2E_o}{\sqrt{\mu_o/\epsilon_o}} \cos \theta_i e^{-j\beta_o x \sin \theta_i}$

1.33 $\underline{E}^i = (\hat{a}_x \cos \theta_i - \hat{a}_z \sin \theta_i) E_o e^{-j\beta_o(x \sin \theta_i + z \cos \theta_i)}$

$$\underline{E}^r = (\hat{a}_x \cos \theta_i + \hat{a}_z \sin \theta_i) \Gamma_e e^{-j\beta_o(x \sin \theta_i - z \cos \theta_i)}$$

Along the interface ($z=0$) $(\underline{E}^i + \underline{E}^r)_{z=0}^{\tan} = 0 = \hat{a}_x \cos \theta_i E_o (1 + \Gamma_e) e^{-j\beta_o x \sin \theta_i}$

which is satisfied provided $(1 + \Gamma_e) = 0 \Rightarrow \Gamma_e = -1$

1.34 Along the interface the normal components of the electric flux density must be continuous; that is

$$\hat{n} \cdot (\underline{D}_2 - \underline{D}_1) = \hat{n} \cdot \underline{D}_2 = -\hat{n} \cdot \hat{a}_z \epsilon_o E_o \sin \theta_i (1 + \Gamma_e) e^{-j\beta_o x \sin \theta_i} = 0$$

$$= -\hat{a}_z \cdot \hat{a}_z \epsilon_o E_o \sin \theta_i (1 + \Gamma_e) e^{-j\beta_o x \sin \theta_i} = 0$$

which is satisfied provided

$$(1 + \Gamma_e) = 0 \Rightarrow \Gamma_e = -1$$

1.35

Given the electric fields of Problem 1.33, the corresponding magnetic field components can be found using Maxwell's equation of

$$\nabla \times \underline{E} = -j\omega\mu_0 \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left[\hat{a}_x \frac{\partial E_z}{\partial y} + \hat{a}_y \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \hat{a}_z \frac{\partial E_y}{\partial x} \right]$$

For the incident field

$$E_x^i = \cos\theta_i E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$E_y^i = 0$$

$$E_z^i = -\sin\theta_i E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

Using these

$$\frac{\partial E_x^i}{\partial y} = 0, \quad \frac{\partial E_z^i}{\partial y} = 0$$

However

$$\frac{\partial E_x^i}{\partial z} = -j\beta_0 \cos^2\theta_i E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$\frac{\partial E_z^i}{\partial x} = +j\beta_0 \sin^2\theta_i E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

Thus we can write the incident magnetic field as

$$\begin{aligned} \underline{H}^i &= -\frac{E_0}{j\omega\mu_0} \hat{a}_y (-j\beta_0) (\cos^2\theta_i + \sin^2\theta_i) e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)} \\ &= \hat{a}_y \frac{\beta_0}{\omega\mu_0} E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)} = \hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)} \end{aligned}$$

Using the same procedure we can write the reflected magnetic field as

$$\underline{H}^r = -\hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 \Gamma_e e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)} = \hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$$

$$b. \quad \underline{J}_s = \hat{n} \times (\underline{H}_2 - \underline{H}_1) = \hat{n} \times \underline{H}_2 = -\hat{a}_z \times \hat{a}_y (H_z^i + H_z^r)_{z=0} = \hat{a}_x 2E_0 \sqrt{\frac{\epsilon_0}{\mu_0}} e^{-j\beta_0 x \sin\theta_i}$$

1.36

To determine the coefficients Γ_0 and T_0 we apply the boundary conditions along the interface at $z=0$. To do this we first find the corresponding magnetic field components. This is accomplished using Maxwell's equation of $\nabla \times \underline{E} = -j\omega\mu_0 \underline{H} \Rightarrow \underline{H}_0 = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E}$. Doing (continued)

1.36 cont'd this for each component leads to

$$\underline{H}^i = \hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 e^{-j\beta_0 z}, \quad \underline{H}^r = -\hat{a}_y \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 \Gamma_0 e^{+j\beta_0 z}, \quad \underline{H}^t = \hat{a}_y \sqrt{\frac{\epsilon}{\mu_0}} E_0 T_0 e^{-j\beta z}$$

Applying the boundary conditions on the continuity of the tangential electric and magnetic fields along the interface at $z=0$ leads to

$$1 + \Gamma_0 = T_0 \quad \text{from continuity of the electric fields}$$

$$1 - \Gamma_0 = \sqrt{\epsilon_r} T_0 = \sqrt{81} T_0 \quad \text{from continuity of the magnetic fields}$$

Solving these two equations, we find that

$$T_0 = \frac{2}{1 + \sqrt{\epsilon_r}} = \frac{2}{1 + \sqrt{81}} = \frac{2}{1 + 9} = \frac{1}{5} = 0.2$$

$$\Gamma_0 = T_0 - 1 = \frac{1}{5} - 1 = -\frac{4}{5} = -0.8$$

1.37 The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

Electric Fields: $(\underline{E}^i + \underline{E}^r)_{z=0}^{\tan} = (\underline{E}^t)_{z=0}^{\tan}$

$$1 + \Gamma_h = T_h$$

Magnetic Fields: $(\underline{H}^i + \underline{H}^r)_{z=0}^{\tan} = (\underline{H}^t)_{z=0}^{\tan}$

$$\cos \theta_i (-1 + \Gamma_h) \sqrt{\epsilon_0} = -\sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2 \theta_i} \sqrt{\epsilon} T_h$$

Solving these two equations leads to

$$\Gamma_h = \frac{\cos \theta_i - \sqrt{\frac{\epsilon}{\epsilon_0}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2 \theta_i}}{\cos \theta_i + \sqrt{\frac{\epsilon}{\epsilon_0}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2 \theta_i}}$$

$$T_h = \frac{2 \cos \theta_i}{\cos \theta_i + \sqrt{\frac{\epsilon}{\epsilon_0}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2 \theta_i}}$$

1.38 The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

Electric Flux Density: $(\underline{D}^i + \underline{D}^r)_{z=0}^{nor} = (\underline{D}^t)_{z=0}^{nor}$

Since there are no normal components, this boundary condition is automatically satisfied.

Magnetic Flux Density: $(\underline{B}^i + \underline{B}^r)_{z=0}^{nor} = (\underline{B}^t)_{z=0}^{nor}$

or $\sin\theta_i \mu_0 \sqrt{\epsilon_0} (1 + \Gamma_h) = \sqrt{\frac{\epsilon_0}{\epsilon}} \sin\theta_i \mu_0 \sqrt{\epsilon} T_h$

$$1 + \Gamma_h = T_h$$

This is identical to one of the equations for the solution of Problem 1.37. However we do not have another equation from the normal components of the electric field. Therefore we can not solve for Γ_h and T_h using only the normal components of Problem 1.37.

1.39 The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

Electric Fields: $(\underline{E}^i + \underline{E}^r)_{z=0}^{tan} = (\underline{E}^t)_{z=0}^{tan}$

$$\cos\theta_i (1 + \Gamma_e) = \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2\theta_i} T_e$$

Magnetic Fields: $(\underline{H}^i + \underline{H}^r)_{z=0}^{tan} = (\underline{H}^t)_{z=0}^{tan}$

$$\sqrt{\epsilon_0} (1 - \Gamma_e) = \sqrt{\epsilon} T_e$$

Solving these two equations leads to

$$\Gamma_e = \frac{-\cos\theta_i + \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2\theta_i}}$$

$$T_e = \frac{2 \sqrt{\frac{\epsilon_0}{\epsilon}} \cos\theta_i}{\cos\theta_i + \sqrt{\frac{\epsilon_0}{\epsilon}} \sqrt{1 - \frac{\epsilon_0}{\epsilon} \sin^2\theta_i}}$$

1.40 The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

Electric Flux Density: $(\underline{D}^i + \underline{D}^r)_{z=0}^{\text{nor}} = (\underline{D}^t)_{z=0}^{\text{nor}}$

$$\sin\theta_i \epsilon_0 (-1 + \Gamma_e) = -\sqrt{\frac{\epsilon_0}{\mu_0}} \epsilon \sin\theta_t T_e$$

or

$$\sqrt{\epsilon_0} (1 - \Gamma_e) = \sqrt{\epsilon} T_e \quad \text{This equation is identical to one of the equations for the solution of Problem 1.37.}$$

Magnetic Flux Density: $(\underline{B}^i + \underline{B}^r)_{z=0}^{\text{nor}} = (\underline{B}^t)_{z=0}^{\text{nor}}$

Since there are no normal components, this boundary condition is automatically satisfied. However we only have one equation and two unknowns; therefore we can not solve for Γ_e and T_e using only the normal components of Problem 1.37.

1.41 From Problem 1.16 and at large distances

$$\underline{E} = \hat{a}_\phi E_0 \sin\theta \frac{\cos(\omega t - \beta_0 r)}{r}, \quad \underline{E} = \text{Re}[\underline{E} e^{j\omega t}] \Rightarrow \underline{E} = E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r} \hat{a}_\phi$$

$$\underline{H} = -\hat{a}_\theta \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 \sin\theta \frac{\cos(\omega t - \beta_0 r)}{r}, \quad \underline{H} = \text{Re}[\underline{H} e^{j\omega t}] \Rightarrow \underline{H} = -\hat{a}_\theta \sqrt{\frac{\epsilon_0}{\mu_0}} E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r}$$

$$\begin{aligned} \text{a. } \underline{S}_{av} = \underline{S} &= \frac{1}{2} \text{Re}(\underline{E} \times \underline{H}^*) = \frac{1}{2} \text{Re}(\hat{a}_\phi E_\phi \times \hat{a}_\theta H_\theta^*) = \frac{1}{2} \text{Re}(\hat{a}_r E_\phi H_\theta^*) \\ &= \hat{a}_r \frac{|E_0|^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{\sin^2\theta}{r^2} \end{aligned}$$

$$\begin{aligned} \text{b. } P_{av} &= \oint\oint_{S_a} \underline{S}_{av} \cdot d\underline{s} = \oint\oint_{S_a} \underline{S}_{av} \cdot \hat{a}_r r^2 \sin\theta d\theta d\phi = \frac{|E_0|^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \int_0^{2\pi} \int_0^\pi \sin^3\theta d\theta d\phi = \frac{|E_0|^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} (2\pi) \left(\frac{4}{3}\right) \\ &= |E_0|^2 \frac{4\pi}{3} \sqrt{\frac{\epsilon_0}{\mu_0}} \end{aligned}$$

$$\begin{aligned} \text{1.42 } \underline{H}_{\text{total}}^{\text{inc}} &= \frac{1}{377} (-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i), \quad \underline{H}^{\text{ref}} = \frac{1}{377} (-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) \\ \underline{H} &= \underline{H}^{\text{inc}} + \underline{H}^{\text{ref}} = -\hat{a}_x \frac{2}{377} \cos\theta_i = -\hat{a}_x 5.31 \times 10^{-3} \cos\theta_i \\ \underline{I} &= \hat{n} \times \underline{H}^{\text{total}} = \hat{a}_y \times (-\hat{a}_x 5.31 \times 10^{-3} \cos\theta_i) = \hat{a}_z 5.31 \times 10^{-3} \cos\theta_i \end{aligned}$$

1.43 $\underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a} x\right) e^{-j\beta_z z}$

$$\text{a. } \underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left(-\hat{a}_x \frac{\partial E_y}{\partial z} + \hat{a}_z \frac{\partial E_y}{\partial x} \right) = -\hat{a}_x \frac{\beta_z}{\omega\mu_0} E_0 \sin\left(\frac{\pi}{a} x\right) e^{-j\beta_z z} - \hat{a}_z \frac{E_0}{j\omega\mu_0} \left(\frac{\pi}{a}\right) \cos\left(\frac{\pi}{a} x\right) e^{-j\beta_z z}$$

$$\text{b. } P_s = -\frac{1}{2} \iiint_V (\underline{H}^* \cdot \underline{M}^{\text{O}} + \underline{E} \cdot \underline{J}^{\text{O}}) dv = 0$$

$$\text{c. } P_e = \oint\oint_S \left(\frac{1}{2} \underline{E} \times \underline{H}^* \right) \cdot d\underline{s}$$

Cont'd

1.43 cont'd

$$\frac{1}{2} \underline{E} \times \underline{H}^* = \frac{1}{2} \hat{a}_y E_y \times (-\hat{a}_x H_x^* - \hat{a}_z H_z^*) = \frac{1}{2} (\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*)$$

$$\underline{S} = \frac{1}{2} \underline{E} \times \underline{H}^* = \hat{a}_z \frac{\beta_z}{2\omega\mu_0} |E_0|^2 \sin^2\left(\frac{\pi}{a}x\right) + \hat{a}_x \frac{|E_0|^2}{2\omega\mu_0} \left(\frac{\pi}{a}\right) \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{a}x\right)$$

From the two side walls at $x=0$ and $x=a$:

$$\int_0^1 \int_0^b (\hat{a}_x S_x) \big|_{x=0} \cdot (-\hat{a}_x dy dz) + \int_0^1 \int_0^b (\hat{a}_x S_x) \big|_{x=a} \cdot (\hat{a}_x dy dz) = 0 + 0 = 0$$

From the front and back cross sections at $z=0$ and $z=1$:

$$\int_0^b \int_0^a (\hat{a}_z S_z) \big|_{z=0} \cdot (\hat{a}_z dx dy) = \frac{\beta_z}{2\omega\mu_0} |E_0|^2 \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx dy = \frac{\beta_z}{2\omega\mu_0} \frac{ab}{2} |E_0|^2$$

$$\int_0^b \int_0^a (\hat{a}_z S_z) \big|_{z=1} \cdot (-\hat{a}_z dx dy) = -\frac{\beta_z}{2\omega\mu_0} |E_0|^2 \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx dy = -\frac{\beta_z}{2\omega\mu_0} \frac{ab}{2} |E_0|^2$$

From the top and bottom walls at $y=0$ and $y=b$:

Since there are no y components of the power density, there is no contribution from the top and bottom walls.

Therefore

$$P_e = 0 + \frac{\beta_z}{2\omega\mu_0} \frac{ab}{2} |E_0|^2 - \frac{\beta_z}{2\omega\mu_0} \frac{ab}{2} |E_0|^2 = 0$$

d. $P_d = \frac{1}{2} \iiint_V \sigma |\underline{E}|^2 dv = 0$

e. $\bar{W}_m = \iiint_V \frac{1}{4} \mu_0 |\underline{H}|^2 dv = \frac{\mu_0}{4} |E_0|^2 \left\{ \int_0^1 \int_0^b \int_0^a \left(\frac{\beta_z}{\omega\mu_0}\right)^2 \sin^2\left(\frac{\pi}{a}x\right) dx dy dz + \left(\frac{1}{\omega\mu_0}\right)^2 \left(\frac{\pi}{a}\right)^2 \int_0^1 \int_0^b \int_0^a \cos^2\left(\frac{\pi}{a}x\right) dx dy dz \right\}$

$$= \frac{\mu_0}{4} |E_0|^2 \left\{ \left(\frac{\beta_z}{\omega\mu_0}\right)^2 \frac{ab}{2} + \left(\frac{1}{\omega\mu_0}\right)^2 \left(\frac{\pi}{a}\right)^2 \left(\frac{ab}{2}\right) \right\} = \frac{\mu_0}{4} |E_0|^2 \left\{ \frac{\epsilon_0}{\mu_0} \frac{ab}{2} - \left(\frac{1}{\omega\mu_0}\right)^2 \left(\frac{\pi}{a}\right)^2 \frac{ab}{2} + \left(\frac{1}{\omega\mu_0}\right)^2 \left(\frac{\pi}{a}\right)^2 \frac{ab}{2} \right\}$$

$$\bar{W}_m = \frac{\epsilon_0}{8} ab |E_0|^2$$

f. $\bar{W}_e = \iiint_V \frac{1}{4} \epsilon_0 |\underline{E}|^2 dv = \frac{\epsilon_0}{4} |E_0|^2 \int_0^1 \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx dy dz = \frac{\epsilon_0}{4} |E_0|^2 \frac{ab}{2} = \frac{\epsilon_0}{8} ab |E_0|^2$

$$\bar{W}_e = \frac{\epsilon_0}{8} ab |E_0|^2$$

Ultimately $P_s = P_e + P_d + j2\omega(\bar{W}_m - \bar{W}_e)$

$$0 = 0 + 0 + j2\omega\left(\frac{\epsilon_0}{8} ab |E_0|^2 - \frac{\epsilon_0}{8} ab |E_0|^2\right) = 0 + 0 + 0 = 0$$

1.44

$$-\nabla \cdot \left(\frac{1}{2} \underline{E} \times \underline{H}^* \right) = \frac{1}{2} \underline{H}^* \cdot \nabla \underline{E} + \frac{1}{2} \underline{E} \cdot \nabla \underline{H}^* + j 2 \omega \left(\frac{1}{4} \mu_0 |\underline{H}|^2 - \frac{1}{4} \epsilon_0 |\underline{E}|^2 \right)$$

$$\begin{aligned} \frac{1}{2} (\underline{E} \times \underline{H}^*) &= \hat{a}_x \frac{|\underline{E}_0|^2}{j 2 \omega \mu_0} \left(\frac{\pi}{a} \right) \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{a} x\right) + \hat{a}_z \frac{\beta_z}{2 \omega \mu_0} |\underline{E}_0|^2 \sin^2\left(\frac{\pi}{a} x\right) \\ &= \hat{a}_x \frac{|\underline{E}_0|^2}{j 4 \omega \mu_0} \left(\frac{\pi}{a} \right) \sin\left(\frac{2\pi}{a} x\right) + \hat{a}_z \frac{\beta_z}{2 \omega \mu_0} |\underline{E}_0|^2 \sin^2\left(\frac{\pi}{a} x\right) \end{aligned}$$

$$-\nabla \cdot \left(\frac{1}{2} \underline{E} \times \underline{H}^* \right) = - \frac{|\underline{E}_0|^2}{j 4 \omega \mu_0} \left(\frac{\pi}{a} \right) \left(\frac{2\pi}{a} \right) \cos\left(\frac{2\pi}{a} x\right) = j \frac{|\underline{E}_0|^2}{2 \omega \mu_0} \left(\frac{\pi}{a} \right)^2 \cos\left(\frac{2\pi}{a} x\right)$$

$$\begin{aligned} \frac{1}{4} \mu_0 |\underline{H}|^2 &= \frac{\mu_0 |\underline{E}_0|^2}{4 (\omega \mu_0)^2} \left[\beta_z^2 \sin^2\left(\frac{\pi}{a} x\right) + \left(\frac{\pi}{a}\right)^2 \cos^2\left(\frac{\pi}{a} x\right) \right] \\ &= \frac{\mu_0 |\underline{E}_0|^2}{4 (\omega \mu_0)^2} \left[\beta_z^2 \sin^2\left(\frac{\pi}{a} x\right) + \left(\frac{\pi}{a}\right)^2 \left(\cos^2\frac{\pi}{a} x - \sin^2\frac{\pi}{a} x \right) \right] \\ &= \frac{\mu_0 |\underline{E}_0|^2}{4 (\omega \mu_0)^2} \left[\beta_z^2 \sin^2\left(\frac{\pi}{a} x\right) + \left(\frac{\pi}{a}\right)^2 \left(\frac{1 + \cos\left(\frac{2\pi}{a} x\right)}{2} - \frac{1 - \cos\left(\frac{2\pi}{a} x\right)}{2} \right) \right] \end{aligned}$$

$$\frac{1}{4} \mu_0 |\underline{H}|^2 = \frac{|\underline{E}_0|^2}{4} \epsilon_0 \sin^2\left(\frac{\pi}{a} x\right) + \frac{|\underline{E}_0|^2}{4} \frac{1}{\omega^2 \mu_0} \left(\frac{\pi}{a} \right)^2 \cos\left(\frac{2\pi}{a} x\right)$$

$$\frac{1}{4} \epsilon_0 |\underline{E}|^2 = \frac{|\underline{E}_0|^2}{4} \epsilon_0 \sin^2\left(\frac{\pi}{a} x\right)$$

Therefore the conservation of energy equation in differential form can be written as

$$j \frac{|\underline{E}_0|^2}{2 \omega \mu_0} \left(\frac{\pi}{a} \right)^2 \cos\left(\frac{2\pi}{a} x\right) = 0 + 0 + j 2 \omega \left[\frac{|\underline{E}_0|^2}{4} \epsilon_0 \sin^2\left(\frac{\pi}{a} x\right) + \frac{|\underline{E}_0|^2}{4} \frac{1}{\omega^2 \mu_0} \left(\frac{\pi}{a} \right)^2 \cos\left(\frac{2\pi}{a} x\right) - \frac{|\underline{E}_0|^2}{4} \epsilon_0 \sin^2\left(\frac{\pi}{a} x\right) \right]$$

$$j \frac{|\underline{E}_0|^2}{2 \omega \mu_0} \left(\frac{\pi}{a} \right)^2 \cos\left(\frac{2\pi}{a} x\right) = j \frac{|\underline{E}_0|^2}{2 \omega \mu_0} \left(\frac{\pi}{a} \right)^2 \cos\left(\frac{2\pi}{a} x\right) \quad QED$$

1.45

Boundary Conditions on PEC:

$$(a) \quad E_x(0 \leq x \leq a, y=0, 0 \leq z \leq c) = E_x(\text{bottom wall}) = 0$$

$$E_x = \cos(\beta_x x) \sin(\beta_y y) \sin(\beta_z z) \Big|_{y=0} = 0$$

$$E_x(0 \leq x \leq a, y=b, 0 \leq z \leq c) = E_x(\text{top wall}) = 0$$

$$E_x = \cos(\beta_x x) \sin(\beta_y b) \sin(\beta_z z) = 0$$

$$\sin(\beta_y b) = 0 \Rightarrow \beta_y b = \sin^{-1}(0) = n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\boxed{\beta_y = (n\pi/b), \quad n = 0, \pm 1, \pm 2, \dots}$$

[Contd.]

1.45 (cont)

$$(b) E_y(x=0, 0 \leq y \leq b, 0 \leq z \leq c) = E_y(\text{left wall}) = 0$$

$$E_y = \sin(0) \cos(\beta_y y) \sin(\beta_z z) = 0$$

$$E_y(x=a, 0 \leq y \leq b, 0 \leq z \leq c) = E_y(\text{right wall}) = 0$$

$$E_y = \sin(\beta_x a) \cos(\beta_y y) \sin(\beta_z z) = 0$$

$$\sin(\beta_x a) = 0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi, m=0, \pm 1, \pm 2, \dots$$

$$\boxed{\beta_x = (m\pi/a), m=0, \pm 1, \pm 2, \dots}$$

$$c. E_y(0 \leq x \leq a, 0 \leq y \leq b, z=0) = E_y(\text{front wall}) = 0$$

$$E_y = \sin(\beta_x x) \cos(\beta_y y) \sin(0) = 0$$

$$E_y(0 \leq x \leq a, 0 \leq y \leq b, z=c) = E_y(\text{back wall}) = 0$$

$$E_y = \sin(\beta_x x) \cos(\beta_y y) \sin(\beta_z c) = 0$$

$$\sin(\beta_z c) = 0 \Rightarrow \beta_z c = \sin^{-1}(0) = p\pi, p=0, \pm 1, \pm 2, \dots$$

$$\boxed{\beta_z = (p\pi/c) = (p\pi/c), p=\pm 1, \pm 2, \dots}$$

($m=n=0$ simultaneously)
trivial solution
E-fields vanish

$p \neq 0$, because $p=0$ leads to
trivial solution; E-fields vanish

$$\boxed{1.46} \quad \underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{c} z\right), \quad \omega_r = \omega = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2}$$

$$a. \underline{H} = -\frac{1}{j\omega\mu_0} \nabla \times \underline{E} = -\frac{1}{j\omega\mu_0} \left[-\hat{a}_x \frac{\partial E_y}{\partial z} + \hat{a}_z \frac{\partial E_y}{\partial x} \right]$$

$$= \hat{a}_x \frac{E_0}{j\omega\mu_0} \left(\frac{\pi}{c}\right) \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{c} z\right) - \hat{a}_z \frac{E_0}{j\omega\mu_0} \left(\frac{\pi}{a}\right) \cos\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{c} z\right)$$

$$b. P_s = -\frac{1}{2} \iiint_V \left[\underline{H}^* \cdot \underline{M}_i + \underline{E} \cdot \underline{J}_i^* \right] dv = 0$$

(continued)

1.46 cont'd.

$$c. P_e = \oint \left(\frac{1}{2} \underline{E} \times \underline{H}^* \right) \cdot d\underline{s} = \oint_{S_a} \underline{S} \cdot d\underline{s}, \quad \underline{S} = \frac{1}{2} \underline{E} \times \underline{H}^*$$

$$\underline{S} = \frac{1}{2} \underline{E} \times \underline{H}^* = \frac{1}{2} \hat{a}_y E_y \times (\hat{a}_x H_x^* - \hat{a}_z H_z^*) = \frac{1}{2} (-\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*)$$

$$= \frac{1}{2} \left[\hat{a}_x \frac{E_0}{j\omega\mu_0} \left(\frac{\pi}{a} \right) \sin\left(\frac{\pi}{a}x\right) \cos\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{c}z\right) + \hat{a}_z \frac{|E_0|^2}{j\omega\mu_0} \left(\frac{\pi}{c} \right) \sin^2\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{c}z\right) \cos\left(\frac{\pi}{c}z\right) \right]$$

Contributions to P_e from the different walls:

Left and right walls:

$$\int_0^c \int_0^b (\hat{a}_x S_x) \cdot (-\hat{a}_x dy dz) = 0; \quad \int_0^c \int_0^b (\hat{a}_x S_x) \cdot (\hat{a}_x dy dz) = 0$$

Back and front walls:

$$\int_0^b \int_0^a (\hat{a}_z S_z) \cdot (-\hat{a}_z dx dy) = 0; \quad \int_0^b \int_0^a (\hat{a}_z S_z) \cdot (\hat{a}_z dx dy) = 0$$

Top and bottom walls:

Since there are no y components of the power density, there are no contributions from the top and bottom walls.

Therefore $P_e = 0 + 0 + 0 + 0 + 0 = 0$

$$d. P_d = \frac{1}{2} \iiint \sigma |\underline{E}|^2 dv = 0$$

$$e. \bar{W}_m = \frac{\mu_0}{4} \iiint |\underline{H}|^2 dv = |E_0|^2 \frac{\mu_0}{4} \frac{1}{(\omega\mu_0)^2} \left[\left(\frac{\pi}{c} \right)^2 \int_0^c \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) \cos^2\left(\frac{\pi}{c}z\right) dx dy dz + \left(\frac{\pi}{a} \right)^2 \int_0^c \int_0^b \int_0^a \cos^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{c}z\right) dx dy dz \right]$$

$$\bar{W}_m = |E_0|^2 \frac{\mu_0}{4} \frac{1}{(\omega\mu_0)^2} \left[\left(\frac{\pi}{c} \right)^2 \frac{abc}{4} + \left(\frac{\pi}{a} \right)^2 \frac{abc}{4} \right] = |E_0|^2 \frac{abc}{16} \frac{\epsilon_0}{\omega^2 \mu_0 \epsilon_0} \left[\left(\frac{\pi}{a} \right)^2 + \left(\frac{\pi}{c} \right)^2 \right] = |E_0|^2 \frac{abc}{16} \epsilon_0$$

$$f. \bar{W}_e = \frac{\epsilon_0}{4} \iiint |\underline{E}|^2 dv = |E_0|^2 \frac{\epsilon_0}{4} \int_0^c \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) \sin^2\left(\frac{\pi}{c}z\right) dx dy dz = |E_0|^2 \frac{\epsilon_0}{4} \frac{abc}{4} = |E_0|^2 \frac{abc}{16} \epsilon_0$$

Ultimately $P_s = P_e + P_d + j2\omega(\bar{W}_m - \bar{W}_e)$

$$0 = 0 + 0 + j2\omega \left(|E_0|^2 \frac{abc}{16} \epsilon_0 - |E_0|^2 \frac{abc}{16} \epsilon_0 \right)$$

CHAPTER 2

2.1 $\underline{P} = \hat{a}_y 2.762 \times 10^{-11} \text{ C/m}^2$, $\underline{E} = \hat{a}_y 2 \text{ V/m}$

a. $q_{sb} = |\underline{P}| = 2.762 \times 10^{-11} \text{ C/m}^2$

$q_{sb} = +2.762 \times 10^{-11} \text{ C/m}^2$ at $y = 2 \text{ cm}$ face

$q_{sb} = -2.762 \times 10^{-11} \text{ C/m}^2$ at $y = 0 \text{ cm}$ face

b. $Q_p = \oint_S \underline{P} \cdot d\underline{s} = \underbrace{\int_0^{0.04} \int_0^{0.04} \hat{a}_y P_y \cdot \hat{a}_y dx dz}_{y=2 \text{ cm face}} + \underbrace{\int_0^{0.04} \int_0^{0.04} \hat{a}_y P_y \cdot (-\hat{a}_y dx dz)}_{y=0 \text{ cm face}} = \int_0^{0.04} \int_0^{0.04} P_y dx dz - \int_0^{0.04} \int_0^{0.04} P_y dx dz = 0$

c. $\oint_S \underline{P} \cdot d\underline{s} = \iiint_V \nabla \cdot \underline{P} dv = \iiint_V q_{vb} dv = Q_b$

$q_{vb} = \nabla \cdot \underline{P} = \frac{\partial P_y}{\partial y} = 0$

d. $\epsilon_r = \frac{\epsilon}{\epsilon_0} = \frac{(1 + \chi_e) \epsilon_0}{\epsilon_0} = 1 + \chi_e$; $\underline{D} = \epsilon_0 \underline{E}_a + \underline{P} = (1 + \chi_e) \epsilon_0 \underline{E}_a$

$\chi_e = \frac{P}{\epsilon_0 E_a} = \frac{2.762 \times 10^{-11}}{8.854 \times 10^{-12} (2)} = \frac{2.762}{1.7708} = 1.5597$

$\epsilon_r = 1 + 1.5597 = 2.5597 \approx 2.56$

2.2 $\underline{P} = \hat{a}_\rho \frac{2}{\rho} \times 10^{-10} \text{ C/m}^2$, $\underline{E} = \hat{a}_\rho \frac{7.53}{\rho} \text{ V/m}$; $a \leq \rho \leq b$

a. $q_{sb}|_{\rho=a} = -|\underline{P}| = -\frac{2 \times 10^{-10}}{2 \times 10^{-2}} = -10^{-8} \text{ C/m}^2$

$q_{sb}|_{\rho=b} = +|\underline{P}| = \frac{2 \times 10^{-10}}{6 \times 10^{-2}} = \frac{1}{3} \times 10^{-8} \text{ C/m}^2$

$q_{sb} = 0$ elsewhere

b. $Q_p|_{\rho=a} = q_{sb} A_a = -\frac{2}{\rho} \times 10^{-10} (2\pi \rho \ell) = -4\pi \ell \times 10^{-10} = -1.256 \times 10^{-10}$

$Q_p|_{\rho=b} = q_{sb} A_b = \frac{2}{\rho} \times 10^{-10} (2\pi \rho \ell) = 4\pi \ell \times 10^{-10} = +1.256 \times 10^{-10}$

$Q_{\text{total}} = Q_p|_{\rho=a} + Q_p|_{\rho=b} = 0$

(continued)

2.2 cont'd.

$$c. Q_p = \oint_S \underline{P} \cdot d\underline{s} = \iiint_V \nabla \cdot \underline{P} dv = \iiint_V q_{vb} dv$$

$$q_{vb} = \nabla \cdot \underline{P} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho P_\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{2}{\rho} \times 10^{-10}) = 0$$

$$d. \underline{D} = \epsilon \underline{E}_a = \epsilon_0 \underline{E}_a + \underline{P} = \epsilon_0 \underline{E}_a + \chi_e \epsilon_0 \underline{E}_a = \epsilon_0 (1 + \chi_e) \underline{E}_a$$

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = (1 + \chi_e)$$

$$\chi_e = \frac{P}{\epsilon_0 E_a} = \frac{\frac{2}{\rho} \times 10^{-10}}{\epsilon_0 \frac{7.53}{\rho}} = \frac{2 \times 10^{-10}}{7.53 \epsilon_0} = 2.9998$$

$$\epsilon_r = 1 + \chi_e = 1 + 2.9998 = 3.9998 \approx 4$$

2.3 $\underline{P} = \hat{a}_r \frac{31.87}{r^2} \times 10^{-12} \text{ C/m}^2, \underline{E} = \hat{a}_r \frac{0.45}{r^2} \text{ V/m}; \quad a \leq r \leq b$

$$a. q_{sb}|_{r=a} = -|\underline{P}| = -\frac{31.87 \times 10^{-12}}{(2 \times 10^{-2})^2} = -\frac{31.87 \times 10^{-12}}{4 \times 10^{-4}} = -7.9675 \times 10^{-9} \text{ C/m}^2$$

$$q_{sb}|_{r=b} = +|\underline{P}| = \frac{31.87 \times 10^{-12}}{(4 \times 10^{-2})^2} = \frac{31.87 \times 10^{-12}}{16 \times 10^{-4}} = 1.9919 \times 10^{-8} \text{ C/m}^2$$

$$b. Q_p|_{r=a} = q_{sb} A_a = -\frac{31.87 \times 10^{-12}}{r^2} (4\pi r^2) = -31.87 (4\pi \times 10^{-12}) = -4.005 \times 10^{-10}$$

$$Q_p|_{r=b} = q_{sb} A_b = \frac{31.87 \times 10^{-12}}{r^2} (4\pi r^2) = 31.87 (4\pi \times 10^{-12}) = 4.005 \times 10^{-10}$$

$$Q_{\text{total}} = Q_p|_{r=a} + Q_p|_{r=b} = 0$$

$$c. Q_p = \oint_S \underline{P} \cdot d\underline{s} = \iiint_V \nabla \cdot \underline{P} dv = \iiint_V q_{vb} dv$$

$$q_{vb} = \nabla \cdot \underline{P} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{31.87}{r^2} \times 10^{-12}) = 0$$

$$d. \chi_e = \frac{P}{\epsilon_0 E_a} = \frac{\frac{31.87 \times 10^{-12}}{r^2}}{\epsilon_0 \frac{0.45}{r^2}} = \frac{31.87 \times 10^{-12}}{0.45 \epsilon_0} = 7.9989$$

$$\epsilon_r = 1 + \chi_e = 1 + 7.9989 = 8.9989 \approx 9$$

2.4

a. $E_o(0.25 \times 10^{-3}) + E_1(1 \times 10^{-3}) = 100 \text{ Volts}$

Also $\epsilon_o E_o = \epsilon_1 E_1 \Rightarrow E_1 = \frac{\epsilon_o}{\epsilon_1} E_o$

Thus $E_o(0.25 \times 10^{-3}) + E_o(\frac{10^{-3}}{5}) = E_o(0.25 + 0.2) \times 10^{-3} = 0.45 \times 10^{-3} E_o = 100$

$E_o = \frac{100 \times 10^3}{0.45} = 2.22 \times 10^5 \text{ V/m}$

$E_1 = \frac{\epsilon_o}{\epsilon_1} E_o = \frac{2.22 \times 10^5}{5} = 0.444 \times 10^5 \text{ V/m}$

b. $D_o = \epsilon_o E_o = 8.854 \times 10^{-12} (2.22 \times 10^5) = 1.96755 \times 10^{-6} \text{ C/m}^2$

$D_1 = \epsilon_1 E_1 = 5 (8.854 \times 10^{-12}) (0.444 \times 10^5) = 1.96755 \times 10^{-6} = D_o \text{ C/m}^2$

c. $q_s = 1.96755 \times 10^{-6} \text{ C/m}^2$ in upper plate where voltage is positive

$q_s = -1.96755 \times 10^{-6} \text{ C/m}^2$ in lower plate where voltage is negative

d. $Q = q_s A = 1.96755 \times 10^{-6} (2 \times 10^{-2}) = 3.9351 \times 10^{-8} \text{ C}$ in upper plate

$Q = q_s A = -1.96755 \times 10^{-6} (2 \times 10^{-2}) = -3.9351 \times 10^{-8} \text{ C}$ in lower plate

e. $C_o = \frac{Q}{V_o}, V_o = E_o(0.25 \times 10^{-3}) = 2.22 \times 10^5 (0.25 \times 10^{-3}) = 55.5556 \text{ Volts}$

$C_o = \frac{3.9351 \times 10^{-8}}{55.5556} = 7.0832 \times 10^{-10}$

$C_1 = \frac{Q}{V_1}, V_1 = E_1(1 \times 10^{-3}) = 44.4444 \text{ Volts}$

$C_1 = \frac{3.9351 \times 10^{-8}}{44.4444} = 8.854 \times 10^{-10}$

$C_t = \frac{Q}{V} = \frac{3.9351 \times 10^{-8}}{100} = 3.9351 \times 10^{-10} = \frac{C_o C_1}{C_o + C_1} = \frac{7.0775 (8.8628)}{7.0775 + 8.8628} \times 10^{-10}$

f. $W_{eo} = \frac{1}{2} C_o V_o^2 = \frac{1}{2} (7.0832 \times 10^{-10}) (55.5556)^2 = 1.0931 \times 10^{-6} \text{ Joules}$

$W_{e1} = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} (8.854 \times 10^{-10}) (44.4444)^2 = 0.87445 \times 10^{-6} \text{ Joules}$

$W_{et} = W_{eo} + W_{e1} = 1.96755 \times 10^{-6} \text{ Joules}$

Also $W_{et} = \frac{1}{2} C_t V^2 = \frac{1}{2} (3.9351 \times 10^{-10}) (100)^2 = 1.96755 \times 10^{-6} \text{ Joules}$

- 2.5 Before the insertion of the slab the electric flux density between the plates was equal to

$$D = \epsilon_0 E_0 = 8.854 \times 10^{-12} \left(\frac{100}{1.25 \times 10^{-3}} \right) = 7.0832 \times 10^{-7} \text{ C/m}^2$$

Thus the surface charge density was equal to $|q_s| = D = 7.0832 \times 10^{-7} \text{ C/m}^2$
 a, b. After the removal of the battery and the insertion of the dielectric sheet, the charge density on the plates remains the same as before the insertion of the dielectric sheet. Thus

$$q_s = +7.0832 \times 10^{-7} \text{ C/m}^2 \text{ on the upper plate where the voltage is positive.}$$

$$Q = q_s A = 7.0832 \times 10^{-7} (2 \times 10^{-2}) = 14.1664 \times 10^{-9} \text{ C}$$

$$q_s = -7.0832 \times 10^{-7} \text{ C/m}^2 \text{ on the lower plate where the voltage is negative.}$$

$$Q = q_s A = -7.0832 \times 10^{-7} (2 \times 10^{-2}) = -14.1664 \times 10^{-9} \text{ C}$$

- c. Using the boundary conditions on the normal components of the electric flux density

$$\hat{n} \cdot (D_2 - D_1) = q_s \text{ or } D_{2n} - D_{1n} = D_{2n} = q_s$$

Thus the electric flux density in the dielectric slab and free space is equal to

$$D_0 = D_1 = 7.0832 \times 10^{-7} \text{ C/m}^2$$

$$d. E_0 = \frac{D_0}{\epsilon_0} = \frac{7.0832 \times 10^{-7}}{8.854 \times 10^{-12}} = 80 \times 10^3 \text{ V/m}$$

$$E_1 = \frac{D_1}{\epsilon_1} = \frac{7.0832 \times 10^{-7}}{5 \epsilon_0} = 16 \times 10^3 \text{ V/m}$$

$$e. V_0 = E_0 (0.25 \times 10^{-3}) = 20 \text{ Volts}$$

$$V_1 = E_1 (1 \times 10^{-3}) = 16 \text{ Volts}$$

$$V_t = V_0 + V_1 = 20 + 16 = 36 \text{ Volts}$$

$$f. C_0 = \frac{Q}{V_0} = \frac{14.1664 \times 10^{-9}}{20} = 0.70832 \times 10^{-9} = 7.0832 \times 10^{-10}$$

$$C_1 = \frac{Q}{V_1} = \frac{14.1664 \times 10^{-9}}{16} = 0.8854 \times 10^{-9} = 8.854 \times 10^{-10}$$

$$C_t = \frac{Q}{V_t} = \frac{14.1664 \times 10^{-9}}{36} = 0.39351 \times 10^{-9} = 3.9351 \times 10^{-10} = \frac{C_0 C_1}{C_0 + C_1} = \frac{7.0832(8.854) \times 10^{-10}}{7.0832 + 8.854}$$

$$g. W_{e0} = \frac{1}{2} C_0 V_0^2 = \frac{1}{2} (7.0832 \times 10^{-10}) (20)^2 = 0.14166 \times 10^{-6} \text{ Joules}$$

$$W_{e1} = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} (8.854 \times 10^{-10}) (16)^2 = 0.11333 \times 10^{-6} \text{ Joules}$$

$$W_{et} = \frac{1}{2} C_t V_t^2 = \frac{1}{2} (3.9351 \times 10^{-10}) (36)^2 = 0.25499 \times 10^{-6} = W_{e0} + W_{e1} \text{ Joules}$$

2.6 Before the insertion of the slab

$$E = \frac{V}{d} = \frac{8}{4 \times 10^{-2}} = 200 \text{ V/m}$$

$$D = \epsilon_0 E = 8.854 \times 10^{-12} (200) = 1.7708 \times 10^{-9} \text{ C/m}^2$$

a. $q_s = \pm D = \pm 1.7708 \times 10^{-9} \text{ C/m}^2$

$$Q = \pm q_s A = \pm 1.7708 \times 10^{-9} (64 \times 10^{-4}) = \pm 1.133312 \times 10^{-11} \text{ C}$$

b. $E = 200 \text{ V/m}$

c. $D = 1.7708 \times 10^{-9} \text{ C/m}^2$

d. $C = \epsilon_0 \frac{A}{d} = 8.854 \times 10^{-12} \frac{64 \times 10^{-4}}{4 \times 10^{-2}} = 141.664 \times 10^{-14} = 1.41664 \times 10^{-12}$

e. $W_e = \frac{1}{2} CV^2 = \frac{1}{2} (1.41664 \times 10^{-12}) (8)^2 = 45.3326 \times 10^{-12} \text{ Joules}$

After the insertion of the slab the electric field intensity remained the same; that is $E_0 = E_1 = 200 \text{ Volts/m}$

$$D_0 = \epsilon_0 E_0 = 8.854 \times 10^{-12} (200) = 1.7708 \times 10^{-9} \Rightarrow q_{s0} = 1.7708 \times 10^{-9}$$

$$D_1 = \epsilon_1 E_1 = 2.56 D_0 = 2.56 (1.7708 \times 10^{-9}) = 4.5332 \times 10^{-9} \Rightarrow q_{s1} = 4.5332 \times 10^{-9}$$

f. $Q_0 = \pm q_{s0} A = \pm 1.7708 \times 10^{-9} (32 \times 10^{-4}) = \pm 5.66656 \times 10^{-12} \text{ C (in free space)}$

$$Q_1 = \pm q_{s1} A = \pm 4.5332 \times 10^{-9} (32 \times 10^{-4}) = \pm 14.50639 \times 10^{-12} \text{ C (in dielectric)}$$

g. $E_0 = E_1 = 200 \text{ V/m}$

h. $D_0 = 1.7708 \times 10^{-9} \text{ C/m}^2 \text{ (in free space)}$

$$D_1 = 4.5332 \times 10^{-9} \text{ C/m}^2 \text{ (in dielectric)}$$

i. $C_0 = \frac{Q_0}{V} = \frac{5.66656 \times 10^{-12}}{8} = 0.70832 \times 10^{-12} \text{ (in free space)}$

$$C_1 = \frac{Q_1}{V} = \frac{14.50639 \times 10^{-12}}{8} = 1.813299 \times 10^{-12} \text{ (in dielectric)}$$

j. $C_t = \frac{Q_t}{V} = \frac{(5.66656 + 14.50639) \times 10^{-12}}{8} = \frac{20.17295 \times 10^{-12}}{8} = 2.521619 \times 10^{-12} = C_0 + C_1$

k. $W_{e0} = \frac{1}{2} C_0 V^2 = \frac{1}{2} (0.70832 \times 10^{-12}) (8)^2 = 22.66624 \times 10^{-12} \text{ (in free space)}$

$$W_{e1} = \frac{1}{2} C_1 V^2 = \frac{1}{2} (1.813299 \times 10^{-12}) (8)^2 = 58.02528 \times 10^{-12} \text{ (in dielectric)}$$

l. $W_{et} = \frac{1}{2} C_t V^2 = \frac{1}{2} (2.521619 \times 10^{-12}) (8)^2 = 80.6918 \times 10^{-12} \text{ Joules}$

$$W_{et} = W_{e0} + W_{e1}$$

2.7 Parts (a)-(e) are the same as in the solution of Problem 2.6.

- f. After the removal of the voltage source and the insertion of the slab, the total charge on the plates stays the same as before the removal of the source. Therefore from part a of Problem 2.6

$$Q_0 + Q_1 = Q_t = 1.133312 \times 10^{-11}$$

Also along the interface between the free space and the dielectric slab the tangential components of the electric field must be continuous. Therefore

$$E_0 = E_1 \Rightarrow \frac{D_0}{\epsilon_0} = \frac{D_1}{\epsilon_1} \Rightarrow \frac{q_{s0}}{\epsilon_0} = \frac{q_{s1}}{\epsilon_1} \Rightarrow \frac{Q_0/A_0}{\epsilon_0} = \frac{Q_1/A_1}{\epsilon_1} \Rightarrow Q_1 = \frac{\epsilon_1 A_1}{\epsilon_0 A_0} Q_0$$

Since the two areas are the same $A_0 = A_1$. Thus

$$Q_1 = \frac{\epsilon_1}{\epsilon_0} Q_0 = \epsilon_r Q_0 \quad \text{where } \epsilon_r = \frac{\epsilon_1}{\epsilon_0}$$

Using these two equations

$$Q_0 + Q_1 = Q_t$$

$$Q_1 = \epsilon_r Q_0$$

we can write that

$$Q_0 = \frac{Q_t}{1 + \epsilon_r} = \frac{1.133312 \times 10^{-11}}{1 + 2.56} = 0.31835 \times 10^{-11} = 3.1835 \times 10^{-12} \text{ C (in free space)}$$

$$Q_1 = \epsilon_r Q_0 = 2.56 (3.1835 \times 10^{-12}) = 8.14966 \times 10^{-12} \text{ C (in dielectric)}$$

$$g. \quad q_{s0} = \frac{Q_0}{A_0} = \frac{3.1835 \times 10^{-12}}{32 \times 10^{-4}} = 0.9948 \times 10^{-9} = D_0 \Rightarrow E_0 = \frac{0.9948 \times 10^{-9}}{8.854 \times 10^{-12}} = 112.36 \text{ V/m (in free space)}$$

$$q_{s1} = \frac{Q_1}{A_1} = \frac{8.14966 \times 10^{-12}}{32 \times 10^{-4}} = 2.5468 \times 10^{-9} = D_1 \Rightarrow E_1 = \frac{2.5468 \times 10^{-9}}{2.56(8.854 \times 10^{-12})} = 112.36 \text{ V/m (in dielectric)}$$

$$h. \quad D_0 = 0.9948 \times 10^{-9} \text{ C/m}^2 \text{ (in free space); } D_1 = 2.5468 \times 10^{-9} \text{ C/m}^2 \text{ (in dielectric)}$$

$$i. \quad C_0 = \frac{Q_0}{V_0} = \frac{3.1835 \times 10^{-12}}{112.36(4 \times 10^{-2})} = 0.708326 \times 10^{-12} \text{ farads} = \epsilon_0 \frac{A_0}{d}$$

$$C_1 = \frac{Q_1}{V_1} = \frac{8.14966 \times 10^{-12}}{112.36(4 \times 10^{-2})} = 1.81329 \times 10^{-12} \text{ farads} = \epsilon_1 \frac{A_1}{d}$$

$$j. \quad C_t = C_0 + C_1 = 2.521616 \times 10^{-12} \text{ farads}$$

$$k. \quad W_0 = \frac{1}{2} C_0 V_0^2 = \frac{1}{2} (0.708326 \times 10^{-12}) [112.36(4 \times 10^{-2})]^2 = 7.15396 \times 10^{-12} \text{ Joules}$$

$$l. \quad W_1 = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} (1.81329 \times 10^{-12}) [112.36(4 \times 10^{-2})]^2 = 18.31389 \text{ Joules}$$

$$W_t = W_0 + W_1 = 25.46785 \times 10^{-12} \text{ Joules}$$

2.8 $V = 100 \text{ Volts}$, $A = 2 \text{ cm}^2 = 2 \text{ cm}^2 \left(\frac{\text{m}}{100 \text{ cm}} \right)^2 = 2 \times 10^{-4}$

(a) Before the insertion of the slab, the electric flux density between the plates is equal to

$$D = \epsilon_0 E_0 = 8.854 \times 10^{-12} \left(\frac{100}{1.25 \times 10^{-3}} \right) = 7.0832 \times 10^{-7} \text{ C/m}^2$$

Thus the surface charge density is equal to $|q_s| = D = 7.0832 \times 10^{-7} \text{ C/m}^2$

After removal of the battery and the insertion of the dielectric slab, the charge density remains the same. Thus,

$$Q_u = Q(\text{upper}) = q_s(\text{upper}) A = +7.0832 \times 10^{-7} (2 \times 10^{-4}) = +14.1664 \times 10^{-11} \text{ C}$$

$$Q_l = Q(\text{lower}) = q_s(\text{lower}) A = -7.0832 \times 10^{-7} (2 \times 10^{-4}) = -14.1664 \times 10^{-11} \text{ C}$$

(b) $q_s(\text{upper}) = +7.0832 \times 10^{-7} \text{ C/m}^2$

$$q_s(\text{lower}) = -7.0832 \times 10^{-7} \text{ C/m}^2$$

(c) The electric flux density in the dielectric slab and free space is

$$\hat{n} \cdot (D_2 - D_1) = q_s \Rightarrow D_{2n} - D_{1n} = D_{2n} = q_s$$

$$D_1 = 7.0832 \times 10^{-7} \text{ C/m}^2$$

$$D_0 = 7.0832 \times 10^{-7} \text{ C/m}^2$$

(d) $\epsilon_d E_d = D_d \Rightarrow E_d = \frac{D_d}{\epsilon_d} = \frac{7.0832 \times 10^{-7}}{5(8.854 \times 10^{-12})} = 0.8 \times 10^5 = 16 \times 10^3 \text{ V/m}$

$$\epsilon_0 E_0 = D_0 \Rightarrow E_0 = \frac{D_0}{\epsilon_0} = \frac{7.0832 \times 10^{-7}}{8.854 \times 10^{-12}} = 0.8 \times 10^5 = 80 \times 10^3 \text{ V/m}$$

(e) $V_d = E_d d_d = 16 \times 10^3 (1 \times 10^{-3}) = 16 \text{ Volts}$

$$V_0 = E_0 d_0 = 80 \times 10^3 (0.25 \times 10^{-3}) = 20 \text{ Volts}$$

$$V_t = V_0 + V_d = 20 + 16 = 36 \text{ Volts}$$

(f) $C_d = \frac{Q}{V_d} = \frac{14.1664 \times 10^{-11}}{16} = 0.8854 \times 10^{-10} = 8.854 \times 10^{-12} \text{ farads}$

$$C_0 = \frac{Q}{V_0} = \frac{14.1664 \times 10^{-11}}{20} = 0.70832 \times 10^{-10} = 7.0832 \times 10^{-12} \text{ farads}$$

$$C_t = \frac{Q}{V_t} = \frac{14.1664 \times 10^{-11}}{36} = 3.9351 \times 10^{-12} = \frac{C_0 C_1}{C_0 + C_1} = \frac{7.0832(8.854) \times 10^{-24}}{(7.0832 + 8.854) \times 10^{-12}}$$

2.9 $A = 1 \text{ cm}^2$, $V = 10 \text{ volt}$, $\epsilon_1 = 2$, $\epsilon_2 = 6$, $d_1 = d_2 = 1 \text{ cm}$

(a) $E_1 d_1 + E_2 d_2 = 10 \Rightarrow E_1 + E_2 = \frac{10}{10^{-2}} = 10^3 \text{ V/m}$

$D_1 = D_2 \Rightarrow \epsilon_1 E_1 = \epsilon_2 E_2 \Rightarrow E_1 = \frac{\epsilon_2}{\epsilon_1} E_2 = \frac{6}{2} E_2 = 3 E_2$

$E_1 + E_2 = 3 E_2 + E_2 = 4 E_2 = 10^3 \Rightarrow E_2 = 0.25 \times 10^3 \text{ V/m}$, $E_1 = 3 E_2 = 0.75 \times 10^3 \text{ V/m}$

$E_1 = 0.75 \times 10^3 \text{ V/m}$, $E_2 = 0.25 \times 10^3 \text{ V/m}$

(b) $D_1 = \epsilon_1 E_1 = 2 \epsilon_0 (0.75 \times 10^3) = 1.5 \epsilon_0 \times 10^3 \text{ C/m}^2$

$D_2 = \epsilon_2 E_2 = 6 \epsilon_0 (0.25 \times 10^3) = 1.5 \epsilon_0 \times 10^3 \text{ C/m}^2$

(c) $Q_{\text{upper}} = A q_{s1} = 10^{-4} \epsilon_0 (1.5 \times 10^3) = 0.15 \epsilon_0 \text{ C}$

$Q_{\text{lower}} = A q_{s2} = 10^{-4} \epsilon_0 (1.5 \times 10^3) = 0.15 \epsilon_0 \text{ C}$

(d) $C_1 = \frac{Q}{V_1} = \frac{Q_1}{E_1 d_1} = \frac{0.15 \epsilon_0}{(0.75 \times 10^3)(10^{-2})} = 0.02 \epsilon_0 \text{ F}$

$C_2 = \frac{Q_2}{V_2} = \frac{0.15 \epsilon_0}{E_2 d_2} = \frac{0.15 \epsilon_0}{(0.25 \times 10^3)(10^{-2})} = 0.06 \epsilon_0 \text{ F}$

$C_T = \frac{Q}{V} = \frac{0.15 \epsilon_0}{10} = 0.015 \epsilon_0 \text{ F}$

(e)

$C_1 = \epsilon_1 \frac{A_1}{d_1} = 2 \epsilon_0 \frac{10^{-4}}{10^{-2}} = 0.02 \epsilon_0$

$C_2 = \epsilon_2 \frac{A_2}{d_2} = 6 \epsilon_0 \frac{10^{-4}}{10^{-2}} = 0.06 \epsilon_0$

$C_T = \frac{C_1 C_2}{C_1 + C_2} = \frac{(0.02 \epsilon_0)(0.06 \epsilon_0)}{(0.02 \epsilon_0 + 0.06 \epsilon_0)} = \frac{0.02(0.06)}{0.08} \epsilon_0$

$C_T = \frac{2(6)}{8} \times 10^{-2} \epsilon_0 = 0.015 \epsilon_0$

(f) They are the same, as they should be.

2.10

$$(a) \quad E_1 = \frac{10}{1 \text{ cm}} \left(\frac{100 \text{ cm}}{1 \text{ m}} \right) = 10^{+3} \text{ V/m}$$

$$E_2 = \frac{10}{1 \text{ cm}} \left(\frac{100 \text{ cm}}{1 \text{ m}} \right) = 10^{+3} \text{ V/m}$$

$$(b) \quad D_1 = E_1 \epsilon_1 = 10^{+3} (2 \times 8.854 \times 10^{-12}) = 17.708 \times 10^{-9} \text{ C/m}^2$$

$$D_2 = E_2 \epsilon_2 = 10^{+3} (6 \times 8.854 \times 10^{-12}) = 53.124 \times 10^{-9} \text{ C/m}^2$$

$$(c) \quad q_1 = 17.708 \times 10^{-9} \text{ C/m}^2$$

$$q_2 = 53.124 \times 10^{-9} \text{ C/m}^2$$

$$(d) \quad Q_1 = q_1 A = 17.708 \times 10^{-9} \frac{\text{C}}{\text{m}^2} (1 \times 10^{-4} \text{ m}^2) = 17.708 \times 10^{-13} = 1.7708 \times 10^{-12} \text{ C}$$

$$Q_2 = q_2 A = 53.124 \times 10^{-9} \frac{\text{C}}{\text{m}^2} (1 \times 10^{-4} \text{ m}^2) = 53.124 \times 10^{-13} = 5.3124 \times 10^{-12} \text{ C}$$

$$(e) \quad C_1 = \frac{Q_1}{V_1} = \frac{1.7708 \times 10^{-12}}{10} = 1.7708 \times 10^{-13} \text{ farads}$$

$$C_2 = \frac{Q_2}{V_2} = \frac{5.3124 \times 10^{-12}}{10} = 5.3124 \times 10^{-13} \text{ farads}$$

$$(f) \quad C_T = C_1 + C_2 = (1.7708 + 5.3124) \times 10^{-13} = 7.0832 \times 10^{-13} = 0.70832 \times 10^{-12}$$

$$(g) \quad C_1 = \epsilon_1 \frac{A_1}{S_1} = 2 (8.854 \times 10^{-12}) \frac{10^{-4}}{10^{-2}} = 17.708 \times 10^{-14} = 1.7708 \times 10^{-13} \text{ farads}$$

$$C_2 = \epsilon_2 \frac{A_2}{S_2} = 6 (8.854 \times 10^{-12}) \frac{10^{-4}}{10^{-2}} = 53.124 \times 10^{-14} = 5.3124 \times 10^{-13} \text{ farad}$$

$$(h) \quad C_T = C_1 + C_2 = (1.7708 + 5.3124) \times 10^{-13} = 7.0824 \times 10^{-13} = 0.70824 \times 10^{-12} \text{ farads}$$

(i) They are the same as they should. Both are derived based on basic and related principles.

2.11

Before the insertion of the slabs and before the power supply is disconnected

$$E_1 = E_2 = \frac{10}{2 \times 10^{-2}} = 1,000 \text{ V/m} = 10^3 \text{ V/m}$$

$$D_1 = D_2 = \epsilon_0 E_1 = \epsilon_0 E_2 = 8.854 \times 10^{-12} (10^3) = 8.854 \times 10^{-9} \text{ C/m}^2$$

$$q_{s1} = q_{s2} = D_1 = D_2 = 8.854 \times 10^{-9} \text{ C/m}^2$$

$$Q_1 = Q_2 = q_{s1} A_1 = q_{s2} A_2 = 8.854 \times 10^{-9} (10^{-4}) = 8.854 \times 10^{-13}$$

After the removal of the power supply, the charge density (C/m²) in each plate remains the same before the insertion of the slabs.

(a) $q_{s1} = q_{s2} = 8.854 \times 10^{-9} \text{ C/m}^2$

(b) $D_1 = D_2 = |q_{s1}| = |q_{s2}| = 8.854 \times 10^{-9} \text{ C/m}^2$

(c) $Q_1 = -Q_2 = q_{s1} A_1 = q_{s2} A_2 = 8.854 \times 10^{-13} \text{ C}$

(d) $D_1 = \epsilon_1 E_1 \Rightarrow E_1 = \frac{D_1}{\epsilon_1} = \frac{8.854 \times 10^{-9}}{2(8.854 \times 10^{-12})} = \frac{10^3}{2} = 500 \text{ V/m}$

$D_2 = \epsilon_2 E_2 \Rightarrow E_2 = \frac{D_2}{\epsilon_2} = \frac{8.854 \times 10^{-9}}{6(8.854 \times 10^{-12})} = \frac{10^3}{6} = 166.67 \text{ V/m}$

(e) $V_1 = E_1 d_1 = 500(10^{-2}) = 5 \text{ Volts}$

$V_2 = E_2 d_2 = 166.67(10^{-2}) = 1.6667 \text{ Volts}$

(f) $C_1 = \epsilon_1 \frac{A_1}{d_1} = 2(8.854 \times 10^{-12}) \frac{10^{-4}}{10^{-2}} = 17.708 \times 10^{-14} \text{ F}$

$C_2 = \epsilon_2 \frac{A_2}{d_2} = 6(8.854 \times 10^{-12}) \frac{10^{-4}}{10^{-2}} = 53.124 \times 10^{-14} \text{ F}$

(g) $C_1 = \frac{Q_1}{V_1} = \frac{8.854 \times 10^{-13}}{5} = 1.7708 \times 10^{-13} = 17.708 \times 10^{-14} \text{ F}$

$C_2 = \frac{Q_2}{V_2} = \frac{8.854 \times 10^{-13}}{1.6667} = 5.3124 \times 10^{-13} = 53.124 \times 10^{-14} \text{ F}$

(h) They should be the same. Same definition.

2.12 Using Coulomb's Law

$$a. \oint_S \underline{D} \cdot d\underline{s} = \int_0^l \int_0^{2\pi} (\hat{a}_\rho D_\rho) \cdot \hat{a}_\rho \rho d\phi dz = \int_0^l \int_0^{2\pi} \epsilon_0 E_\rho \rho d\phi dz = 2\pi \epsilon_0 l \rho E_\rho = Q$$

or

$$E_\rho = \frac{Q}{2\pi \epsilon_0 l} \frac{1}{\rho} = \frac{Q}{\rho} (3 \times 10^{11}) = 3 \times 10^{11} \frac{Q}{\rho}$$

Since the voltage is 10 Volts, then

$$V = \int_a^b \underline{E} \cdot d\underline{l} = \int_a^b (\hat{a}_\rho E_\rho) \cdot \hat{a}_\rho \rho d\rho = 3 \times 10^{11} Q \int_a^b \frac{d\rho}{\rho} = 3 \times 10^{11} Q \ln(\rho) \Big|_a^b$$

$$V = 10 = 3 \times 10^{11} Q \ln\left(\frac{b}{a}\right) = 3 \times 10^{11} Q \ln(2) = 3 \times 10^{11} Q (0.69315)$$

$$Q = \frac{10}{3 \times 10^{11} (0.69315)} = 48.0898 \times 10^{-12} \text{ C}$$

$$E_\rho = 3 \times 10^{11} \frac{Q}{\rho} = 3 \times 10^{11} \frac{48.0898 \times 10^{-12}}{\rho} = \frac{14.42695}{\rho}$$

$$b. Q = 48.0898 \times 10^{-12} \text{ C}$$

$$c. q_{sa} = -\frac{Q}{A_a} = -\frac{Q}{2\pi a l} = -\frac{48.0898 \times 10^{-12}}{2\pi (2 \times 10^{-2}) (6 \times 10^{-2})} = -6.378 \times 10^{-9} \text{ C/m}^2$$

$$q_{sb} = \frac{Q}{A_b} = \frac{Q}{2\pi b l} = \frac{48.0898 \times 10^{-12}}{2\pi (4 \times 10^{-2}) (6 \times 10^{-2})} = 3.189 \times 10^{-9} \text{ C/m}^2$$

$$d. D = \epsilon_0 E_\rho = 8.854 \times 10^{-12} \left(\frac{14.42695}{\rho} \right) = \frac{1.27736}{\rho} \times 10^{-10} \text{ C/m}^2$$

$$e. C = \frac{Q}{V} = \frac{48.0898 \times 10^{-12}}{10} = 4.80898 \times 10^{-12} \text{ farads}$$

$$f. W_e = \frac{1}{2} CV^2 = \frac{1}{2} (4.80898 \times 10^{-12}) (100) = 2.4045 \times 10^{-10} \text{ Joules}$$

2.13 After the removal of the voltage source, the total charge stays the same. Using this we can answer the questions.

a. Same as in part b of Problem 2.12

$$Q = 48.0898 \times 10^{-12} \text{ C}$$

b. Same as in part c of Problem 2.12

$$q_{sa} = -6.378 \times 10^{-9} \text{ C/m}^2$$

$$q_{sb} = 3.189 \times 10^{-9} \text{ C/m}^2 \quad (\text{continued})$$

2.13 cont'd

Same as in part d of Problem 2.12 (in free space and dielectric)

c. $D_{p0} = D_{p1} = \frac{1.27736 \times 10^{-10}}{\rho}$

d. $E_{p0} = \frac{D_{p0}}{\epsilon_0} = \frac{1.27736 \times 10^{-10}}{8.854 \times 10^{-12} \rho} = \frac{14.4269}{\rho}$

$E_{p1} = \frac{D_{p1}}{\epsilon_1} = \frac{1.27736 \times 10^{-10}}{2.56(8.854 \times 10^{-12}) \rho} = \frac{5.6355}{\rho}$

e. $V = \int_a^c \underline{E}_1 \cdot d\underline{\ell} + \int_c^b \underline{E}_0 \cdot d\underline{\ell} = 5.6355 \int_a^c \frac{d\rho}{\rho} + 14.4269 \int_c^b \frac{d\rho}{\rho} = 5.6355 \ln\left(\frac{c}{a}\right) + 14.4269 \ln\left(\frac{b}{c}\right)$
 $= 5.6355 \ln\left(\frac{3}{2}\right) + 14.4269 \ln\left(\frac{4}{3}\right) = 5.6355(0.405465) + 14.4269(0.28768)$

$V = 2.285 + 4.1504 = 6.4354 \text{ Volts}$

f. $C = \frac{Q}{V} = \frac{48.0898 \times 10^{-12}}{6.4354} = 7.472698 \times 10^{-12} \text{ farads}$

g. $W_e = \frac{1}{2} C V^2 = \frac{1}{2} (7.472698) (6.4354)^2 \times 10^{-12} = 1.54738 \times 10^{-10} \text{ Joules}$

2.14 Since the voltage source is connected at all times, the total voltage is maintained at 10 Volts. Also the normal components of the electric flux density is continuous along the interface.

a. $V = \int_a^c \underline{E}_1 \cdot d\underline{\ell} + \int_c^b \underline{E}_0 \cdot d\underline{\ell} = 10 \text{ Volts} = \int_a^c \hat{a}_\rho E_{p1} \cdot \hat{a}_\rho d\rho + \int_c^b \hat{a}_\rho E_{p0} \cdot \hat{a}_\rho d\rho$
 $V = \int_a^c E_{p1} d\rho + \int_c^b E_{p0} d\rho = \int_a^c \frac{D_{p1}}{\epsilon_1} d\rho + \int_c^b \frac{D_{p0}}{\epsilon_0} d\rho = \frac{1}{\epsilon_1} \int_a^c D_{p1} d\rho + \int_c^b D_{p0} d\rho$

However $D_{p1} = D_{p0} = \frac{Q}{2\pi l} \frac{1}{\rho}$

Thus

$V = \frac{1}{\epsilon_1} \frac{Q}{2\pi l} \ln\left(\frac{c}{a}\right) + \frac{1}{\epsilon_0} \frac{Q}{2\pi l} \ln\left(\frac{b}{c}\right) = \frac{Q}{\epsilon_0(2\pi l)} \left[\frac{\ln(c/a)}{\epsilon_r} + \ln(b/c) \right] = 10$

$Q = \frac{10(2\pi \epsilon_0 l)}{\frac{\ln(3/2)}{2.56} + \ln(4/3)} = \frac{10^{-10}}{3 \left[\frac{0.405465}{2.56} + 0.28768 \right]} = 7.47276 \times 10^{-9} \text{ C}$

$E_{p0} = \frac{Q}{2\pi \epsilon_0 l} \frac{1}{\rho} = \frac{7.47276 \times 10^{-9}}{2\pi \epsilon_0 (6 \times 10^{-2}) \rho} = \frac{2,241.827}{\rho} \text{ (in free space)}$
 (continued)

2.14 cont'd. $E_{p1} = \frac{Q}{2\pi\epsilon_1 l} \frac{1}{r} = \frac{E_{p0}}{\epsilon_r} = \frac{875.7136}{9}$ (in dielectric)

b. $D_{p0} = \epsilon_0 E_{p0} = \frac{19.849 \times 10^{-9}}{9} \text{ C/m}^2$

$D_{p1} = \epsilon_1 E_{p1} = \frac{19.849 \times 10^{-9}}{9} \text{ C/m}^2$

c. $q_{sa} = D_{p1} \big|_{r=2 \times 10^{-2}} = \frac{19.849 \times 10^{-9}}{2 \times 10^{-2}} = 0.99246 \times 10^{-6} \text{ C/m}^2$

$q_{sb} = D_{p0} \big|_{r=4 \times 10^{-2}} = \frac{q_{sa}}{2} = 0.49623 \times 10^{-6} \text{ C/m}^2$

d. $Q_a = q_{sa} (2\pi a l) = 0.99246 \times 10^{-6} (2\pi \times 2 \times 10^{-2} \times 6 \times 10^{-2}) = 74.8295 \times 10^{-10} \text{ C}$

$Q_b = q_{sb} (2\pi b l) = 0.49623 \times 10^{-6} (2\pi \times 4 \times 10^{-2} \times 6 \times 10^{-2}) = 74.8295 \times 10^{-10} \text{ C}$

e. $C = \frac{Q}{V} = \frac{74.8295 \times 10^{-10}}{10} = 7.48295 \times 10^{-10} = 0.748295 \times 10^{-9} \text{ farads}$

f. $W_e = \frac{1}{2} C V^2 = \frac{1}{2} (0.748295 \times 10^{-9}) (100) = 37.41475 \times 10^{-9} \text{ Joules}$

2.15 The voltage source is connected across at all times.

a. $E_0 = \frac{100}{4 \times 10^{-2}} = 2,500 \text{ V/m} = E_1$

b. $D_0 = \epsilon_0 E_0 = 22.135 \times 10^{-9} \text{ C/m}^2$ (in free space)

$D_1 = \epsilon_1 E_1 = 2.56 D_0 = 56.6656 \times 10^{-9} \text{ C/m}^2$ (in dielectric)

c. $Q_0 = 22.135 \times 10^{-9} (2 \times 10^{-2}) = 0.4427 \times 10^{-9} \text{ C}$ (in free space)

$Q_1 = 56.6656 \times 10^{-9} (2 \times 10^{-2}) = 1.13331 \times 10^{-9} \text{ C}$ (in dielectric)

d. $W_{e0} = \frac{1}{2} C_0 V_0^2 = \frac{1}{2} \left(\frac{Q_0}{V_0} \right) V_0^2 = \frac{1}{2} Q_0 V_0 = \frac{1}{2} (0.4427 \times 10^{-9}) (100) = 22.135 \times 10^{-9} \text{ J (free space)}$

$W_{e1} = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} \left(\frac{Q_1}{V_1} \right) V_1^2 = \frac{1}{2} Q_1 V_1 = \frac{1}{2} (1.13331 \times 10^{-9}) (100) = 56.6655 \times 10^{-9} \text{ J (dielectric)}$

e. $Q_t = Q_0 + Q_1 = 1.57601 \times 10^{-9}$

f. $C_t = \frac{Q_t}{V} = \frac{1.57601 \times 10^{-9}}{100} = 1.57601 \times 10^{-11} = 15.7601 \times 10^{-12} = C_0 + C_1 \text{ farads}$

g. $W_{et} = \frac{1}{2} C_t V^2 = \frac{1}{2} (15.7601 \times 10^{-12}) (100)^2 = 78.8005 \times 10^{-9} \text{ J} = W_{e0} + W_{e1}$

2.16

$$(a) \oint_S \underline{D} \cdot d\underline{S} = \int_0^l \int_0^{2\pi} \hat{a}_\rho D_{\rho 0} \cdot \hat{a}_\rho dz d\phi + \int_0^l \int_\pi^{2\pi} \hat{a}_\rho D_{\rho 1} \cdot \hat{a}_\rho dz d\phi$$

$$\oint_S \underline{D} \cdot d\underline{S} = \pi l (D_{\rho 0} + D_{\rho 1}) \rho = Q_t \Rightarrow D_{\rho 0} + D_{\rho 1} = \frac{Q_t}{2\pi l} \left(\frac{1}{\rho} \right)$$

Also, because the tangential electric field components are continuous along the radial interfaces between the dielectric free space

$$E_{\rho 0} = E_{\rho 1} \Rightarrow \frac{D_{\rho 0}}{\epsilon_0} = \frac{D_{\rho 1}}{\epsilon_1} \Rightarrow D_{\rho 1} = \frac{\epsilon_1}{\epsilon_0} D_{\rho 0} = \epsilon_r D_{\rho 0}, \quad \epsilon_r = \frac{\epsilon_1}{\epsilon_0}$$

Thus

$$D_{\rho 0} (1 + \epsilon_r) = \frac{Q_t}{2\pi l} \frac{1}{\rho} \Rightarrow D_{\rho 0} = \frac{Q_t}{2\pi l (1 + \epsilon_r)} \frac{1}{\rho}, \quad D_{\rho 1} = \frac{Q_t}{2\pi l} \frac{\epsilon_r}{1 + \epsilon_r} \frac{1}{\rho}$$

$$E_{\rho 0} = \frac{D_{\rho 0}}{\epsilon_0} = \frac{Q_t}{\pi \epsilon_0 l (1 + \epsilon_r)} \frac{1}{\rho}, \quad \text{in free space}$$

$$E_{\rho 1} = \frac{D_{\rho 1}}{\epsilon_1} = \frac{Q_t}{\pi \epsilon_1 l} \frac{\epsilon_r}{1 + \epsilon_r} \frac{1}{\rho}, \quad \text{in dielectric}$$

$$(b) \int_a^b \underline{E} \cdot d\underline{l} = \int_a^b \hat{a}_\rho E_{\rho 0} \cdot \hat{a}_\rho d\rho = \int_a^b E_{\rho 0} d\rho = \frac{Q_t \ln(b/a)}{2\pi \epsilon_0 l (1 + \epsilon_r)} = V = 10$$

$$Q_t = \frac{10 (\pi \epsilon_0 l) (1 + \epsilon_r)}{\ln(2)} = \frac{(1 + 2.56) \times 10^{-10}}{6 \ln(2)} = 0.856 \times 10^{-10} \text{ C}$$

$$(c, d) D_{\rho 0} = \epsilon_0 E_{\rho 0} = \frac{Q_t}{\pi l (1 + \epsilon_r)} \frac{1}{\rho} = \frac{0.856 \times 10^{-10}}{2\pi (6 \times 10^{-2}) (1 + 2.56)} \frac{1}{\rho} = \frac{1.2756 \times 10^{-10}}{\rho} \text{ C/m}^2$$

$$D_{\rho 0}|_{\rho=a} = \frac{1.2756 \times 10^{-10}}{2 \times 10^{-2}} = 0.6378 \times 10^{-8} \text{ C/m}^2 \Rightarrow q_{\rho 0}|_{\rho=a} = 0.6378 \times 10^{-8} \text{ C/m}^2$$

$$D_{\rho 0}|_{\rho=b} = \frac{1.2756 \times 10^{-10}}{4 \times 10^{-2}} = 0.3189 \times 10^{-8} \text{ C/m}^2 \Rightarrow q_{\rho 0}|_{\rho=b} = 0.3189 \times 10^{-8} \text{ C/m}^2$$

$$D_{\rho 1} = \epsilon_1 E_{\rho 1} = \frac{Q_t}{\pi l} \frac{\epsilon_r}{1 + \epsilon_r} \frac{1}{\rho} = \epsilon_r D_{\rho 0} = \frac{3.2655 \times 10^{-10}}{\rho} \text{ C/m}^2$$

$$D_{\rho 1}|_{\rho=a} = \frac{3.2655 \times 10^{-10}}{2 \times 10^{-2}} = 1.63277 \times 10^{-8} \text{ C/m}^2 \Rightarrow q_{\rho 1}|_{\rho=a} = 1.63277 \times 10^{-8} \text{ C/m}^2$$

$$D_{\rho 1}|_{\rho=b} = \frac{3.2655 \times 10^{-10}}{4 \times 10^{-2}} = 0.81638 \times 10^{-8} \text{ C/m}^2 \Rightarrow q_{\rho 1}|_{\rho=b} = 0.81638 \times 10^{-8} \text{ C/m}^2$$

(e)

$$C_0 = \frac{Q_0}{V} = \frac{0.6378 \times 10^{-8} (\pi \times 2 \times 10^{-2} \times 6 \times 10^{-2})}{10} = 2.40445 \times 10^{-12} \text{ farads (in free space)}$$

$$C_1 = \frac{Q_1}{V} = \frac{1.63277 \times 10^{-8} (\pi \times 2 \times 10^{-2} \times 6 \times 10^{-2})}{10} = 6.155398 \times 10^{-12} \text{ farads (in dielectric)}$$

$$C_t = C_0 + C_1 = 8.559847 \times 10^{-12} \text{ farads}$$

$$(f) W_{e0} = \frac{1}{2} C_0 V^2 = \frac{1}{2} (2.40445 \times 10^{-12}) (100) = 1.2022 \times 10^{-10} \text{ Joules (in free space)}$$

$$W_{e1} = \frac{1}{2} C_1 V^2 = \frac{1}{2} (6.155398 \times 10^{-12}) (100) = 3.077699 \times 10^{-10} \text{ Joules (in dielectric)}$$

$$W_t = W_{e0} + W_{e1} = 4.279949 \times 10^{-10} \text{ Joules (total)}$$

2.17 From Problem 2.16 $E_p = \frac{Q}{2\pi\epsilon_0 r^2}$ where Q is the total charge.

- a. Since the charge stays the same at all times and is the same as that of Problem 2.16, or $Q_t = 85.5985 \times 10^{-12} \text{ C}$. Also to satisfy the boundary conditions along the interface between the two media

$$E_{p0} = E_{p1}$$

b. $D_{p0} = \epsilon_0 E_{p0} = \frac{1.2756 \times 10^{-10}}{r} \text{ C/m}^2$

$$D_{p1} = \epsilon_1 E_{p1} = \epsilon_r E_{p0} = \frac{3.2655 \times 10^{-10}}{r} \text{ C/m}^2$$

c. $Q_{s0}|_{r=a} = -D_{p0}|_{r=a} = -\frac{1.2756 \times 10^{-10}}{2 \times 10^{-2}} = -0.6378 \times 10^{-8} \text{ C/m}^2$

$$Q_{s0}|_{r=b} = D_{p0}|_{r=b} = \frac{1.2756 \times 10^{-10}}{4 \times 10^{-2}} = 0.3189 \times 10^{-8} \text{ C/m}^2$$

$$Q_{s1}|_{r=a} = -D_{p1}|_{r=a} = -\frac{3.2655 \times 10^{-10}}{2 \times 10^{-2}} = -1.63277 \times 10^{-8} \text{ C/m}^2$$

$$Q_{s1}|_{r=b} = D_{p1}|_{r=b} = \frac{3.2655 \times 10^{-10}}{4 \times 10^{-2}} = 0.81638 \times 10^{-8} \text{ C/m}^2$$

d. $Q_0|_{r=a} = Q_{s0}|_{r=a} (\pi a l) = -0.6378 \times 10^{-8} (\pi \times 2 \times 10^{-2} \times 6 \times 10^{-2}) = -24.0445 \times 10^{-12} \text{ C}$

$$Q_0|_{r=b} = Q_{s0}|_{r=b} (\pi b l) = 0.3189 \times 10^{-8} (\pi \times 4 \times 10^{-2} \times 6 \times 10^{-2}) = 24.0445 \times 10^{-12} \text{ C}$$

$$Q_1|_{r=a} = Q_{s1}|_{r=a} (\pi a l) = -1.63277 \times 10^{-8} (\pi \times 2 \times 10^{-2} \times 6 \times 10^{-2}) = -61.5540 \times 10^{-12} \text{ C}$$

$$Q_1|_{r=b} = Q_{s1}|_{r=b} (\pi b l) = 1.63277 \times 10^{-8} (\pi \times 4 \times 10^{-2} \times 6 \times 10^{-2}) = 61.5540 \times 10^{-12} \text{ C}$$

e. $C_0 = \frac{Q_0}{V_0} = \frac{24.0445 \times 10^{-12}}{10} = 2.40445 \times 10^{-12} \text{ farads}$ in free space

$$C_1 = \frac{Q_1}{V_1} = \frac{61.5540 \times 10^{-12}}{10} = 6.15540 \times 10^{-12} \text{ farads}$$
 in polystyrene

$$C_t = C_0 + C_1 = (2.40445 + 6.15540) \times 10^{-12} = 8.55985 \times 10^{-12}$$

f. $W_{e0} = \frac{1}{2} C_0 V_0^2 = \frac{1}{2} (2.40445 \times 10^{-12}) (100) = 1.2022 \times 10^{-10} \text{ Joules}$ (in free space)

$$W_{e1} = \frac{1}{2} C_1 V_1^2 = \frac{1}{2} (6.15540 \times 10^{-12}) (100) = 3.0777 \times 10^{-10} \text{ Joules}$$
 (in polystyrene)

$$W_{et} = \frac{1}{2} C_t V_0^2 = \frac{1}{2} C_t V_1^2 = \frac{1}{2} (8.55985 \times 10^{-12}) (100) = 4.2799 \times 10^{-10} \text{ Joules}$$

$$= W_{e0} + W_{e1}$$

2.18

$$\underline{\underline{\underline{E}}} = \hat{a}_z 10^{-3} \sin(2\pi \times 10^7 t), \quad \epsilon_{sr} = 2.56$$

$$(a) \quad 1 + \chi_e = \frac{\epsilon_s}{\epsilon_0} = \epsilon_{sr} \Rightarrow \chi_e = \epsilon_{sr} - 1 = 2.56 - 1 = 1.56$$

$$(b) \quad \underline{\underline{\underline{D}}} = \epsilon_s \underline{\underline{\underline{E}}} = \hat{a}_z 2.56 (\epsilon_0) 10^{-3} \sin(2\pi \times 10^7 t) = \hat{a}_z 2.56 \times 10^{-3} \epsilon_0 \sin(2\pi \times 10^7 t)$$

$$\underline{\underline{\underline{D}}} = \hat{a}_z 2.56 \times 10^{-3} \epsilon_0 \sin(2\pi \times 10^7 t)$$

$$(c) \quad \underline{\underline{\underline{P}}} = \epsilon_0 \chi_e \underline{\underline{\underline{E}}} = \hat{a}_z \epsilon_0 (1.56 \times 10^{-3} \epsilon_0) \sin(2\pi \times 10^7 t) = \hat{a}_z 1.56 \times 10^{-3} \epsilon_0 \sin(2\pi \times 10^7 t)$$

$$\underline{\underline{\underline{P}}} = \hat{a}_z 1.56 \times 10^{-3} \epsilon_0 \sin(2\pi \times 10^7 t)$$

$$(d) \quad \underline{\underline{\underline{J_d}}} = \frac{\partial \underline{\underline{\underline{D}}}}{\partial t} = \hat{a}_z 2.56 \times 10^{-3} \epsilon_0 \frac{\partial}{\partial t} \sin(2\pi \times 10^7 t)$$

$$= \hat{a}_z (2.56 \times 10^{-3} \times 2\pi \times 10^7) \epsilon_0 \cos(2\pi \times 10^7 t)$$

$$\underline{\underline{\underline{J_d}}} = \hat{a}_z 16.085 \times 10^4 \epsilon_0 \cos(2\pi \times 10^7 t)$$

$$(e) \quad \underline{\underline{\underline{J_p}}} = \frac{\partial \underline{\underline{\underline{P}}}}{\partial t} = \hat{a}_z \frac{\partial}{\partial t} [1.56 \times 10^{-3} \epsilon_0 \sin(2\pi \times 10^7 t)]$$

$$= \hat{a}_z 1.56 \times 10^{-3} (2\pi \times 10^7) \epsilon_0 \cos(2\pi \times 10^7 t)$$

$$\underline{\underline{\underline{J_p}}} = \hat{a}_z 9.802 \times 10^4 \epsilon_0 \cos(2\pi \times 10^7 t)$$

2.19 $\underline{M} = \hat{a}_z 1.245 \times 10^6 \text{ A/m}, \underline{H} = \hat{a}_z 5 \times 10^3 \text{ A/m}$

a. $\underline{J}_{ms} = \underline{M} \times \hat{n}$

$y=0$: $\underline{J}_{ms} = \hat{a}_z M_z \times (-\hat{a}_y) = \hat{a}_x M_z = \hat{a}_x 1.245 \times 10^6$

$x=4 \text{ cm}$: $\underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_x = \hat{a}_y M_z = \hat{a}_y 1.245 \times 10^6$

$y=6 \text{ cm}$: $\underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_y = -\hat{a}_x M_z = -\hat{a}_x 1.245 \times 10^6$

$x=0$: $\underline{J}_{ms} = \hat{a}_z M_z \times (-\hat{a}_x) = -\hat{a}_y M_z = -\hat{a}_y 1.245 \times 10^6$

$z=0$: $\underline{J}_{ms} = \hat{a}_z M_z \times (-\hat{a}_z) = 0$

$z=1 \text{ cm}$: $\underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_z = 0$

b. $\underline{J}_m = \nabla \times \underline{M} = 0$

c. $\underline{I}_m = \oint_S \underline{J}_m \cdot d\underline{s} = \iiint_V (\nabla \cdot \underline{J}_m) dV = 0$

d. $\chi_m = \frac{M}{H} = \frac{1.245 \times 10^6}{5 \times 10^3} = 249$

$\mu_r = 1 + \chi_m = 1 + 249 = 250$

2.20 $\underline{H} = \hat{a}_\phi \frac{0.3183}{\rho} \text{ A/m}, \underline{M} = \hat{a}_\phi \frac{190.67}{\rho} \text{ A/m}$

a. $\underline{J}_{ms} = \underline{M} \times \hat{n}$

$\rho=a$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times (-\hat{a}_\rho) \Big|_{\rho=a} = \hat{a}_z M_\phi \Big|_{\rho=a} = \hat{a}_z \frac{190.67}{1 \times 10^{-2}} = \hat{a}_z 19.067 \times 10^3$

$\rho=b$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times \hat{a}_\rho \Big|_{\rho=b} = -\hat{a}_z M_\phi \Big|_{\rho=b} = -\hat{a}_z \frac{190.67}{3 \times 10^{-2}} = -\hat{a}_z 6.3557 \times 10^3$

b. $\underline{J}_m = \nabla \times \underline{M} = \hat{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho M_\phi) = \hat{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} (190.67) = 0$

c. $\underline{I}_m = \oint_S \underline{J}_m \cdot d\underline{s} = \iiint_V (\nabla \cdot \underline{J}_m) dV = 0$

d. $\chi_m = \frac{M}{H} = \frac{190.67/\rho}{0.3183/\rho} = \frac{190.67}{0.3183} = 599.03 \approx 599$

$\mu_r = 1 + \chi_m = 1 + 599 = 600$

$z=0$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times \hat{a}_z = \hat{a}_\rho M_\phi$
 $\underline{J}_{ms} = \hat{a}_\rho \frac{160.67}{\rho}$

$z=-l$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times (-\hat{a}_z)$
 $= -\hat{a}_\rho M_\phi$
 $\underline{J}_{ms} = -\hat{a}_\rho \frac{160.67}{\rho}$

2.21 $\underline{M} = \hat{a}_\phi 10 \frac{A}{m}$

a. $\underline{J}_{ms} = \underline{M} \times \hat{n} \Big|_{\rho=1} = (\hat{a}_\phi M_\phi \times \hat{a}_\rho) = -\hat{a}_z M_\phi \Big|_{\rho=1} = -\hat{a}_z 10 \text{ A/m}$

b. $\underline{J}_m = \nabla \times \underline{M} = \hat{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho M_\phi) = \hat{a}_z \frac{1}{\rho} \frac{\partial}{\partial \rho} (10\rho) = \hat{a}_z \frac{10}{\rho} \text{ A/m}^2$

c. $I_m = \iint_S \underline{J}_m \cdot d\underline{s} = \int_0^{2\pi} \int_0^1 \hat{a}_z J_{m_z} \cdot \hat{a}_z \rho d\rho d\phi = \int_0^{2\pi} \int_0^1 10 d\rho d\phi = 2\pi(10) = 62.83185 \text{ A}$

2.22 $\underline{J} = \hat{a}_z J_0 e^{-10^2[(a-|x|)+(a-|y|)]} \text{ A/m}^2$

$$I = \iint_{S_0} \underline{J} \cdot d\underline{s} = 4 \int_0^a \int_0^a \hat{a}_z J_z \cdot \hat{a}_z dx dy = 4 J_0 \left[\int_0^a e^{-100(a-x)} dx \int_0^a e^{-100(a-y)} dy \right]$$

$$= 4 J_0 \left[\int_0^a e^{-100(a-x)} dx \right]^2 = 4 J_0 \left\{ \left[\frac{1}{100} e^{-100(a-x)} \right]_0^a \right\}^2 = 4 J_0 \left\{ \frac{1}{100} [1 - e^{-100a}] \right\}^2$$

$$I = 4 J_0 \left\{ \frac{1}{100} [1 - e^{-1}] \right\}^2 = 4 J_0 \left[\frac{1}{100} (1 - 0.36788) \right]^2 = (7.9915 \times 10^{-5}) J_0 \text{ A}$$

$$= 15.983 \times 10^{-5} J_0 \text{ A}$$

2.23 $\underline{J} = \hat{a}_z J_0 e^{-10^4(a-\rho)} \text{ A/m}^2$

a. $I = \iint_{S_0} \underline{J} \cdot \hat{a}_z d\underline{s} = \int_0^{2\pi} \int_0^a \hat{a}_z J_z \cdot \hat{a}_z \rho d\rho d\phi = \int_0^{2\pi} \int_0^a J_0 e^{-10^4(a-\rho)} \rho d\rho d\phi$

$$= 2\pi J_0 \int_0^a e^{-10^4(a-\rho)} \rho d\rho = 2\pi J_0 \int_0^a e^{-10^4 a} e^{10^4 \rho} \rho d\rho$$

Let $u = 10^4 \rho \Rightarrow du = 10^4 d\rho \Rightarrow d\rho = 10^{-4} du, \rho = 10^{-4} u$

$$I = 2\pi \times 10^{-8} J_0 \int_0^{10^4 a} e^{-10^4 a} e^u u du = 2\pi \times 10^{-8} J_0 e^{-10^4 a} \int_0^{10^4 a} e^u u du$$

Using the integral $\int_0^c e^u u du = e^u (u-1) \Big|_0^c$ we can write that

$$I = 2\pi J_0 \times 10^{-8} e^{-10^4 a} \left[e^{10^4 a} (10^4 a - 1) - e^0 (0-1) \right]_{a=0.01} = 0.622 \times 10^{-5} J_0 = 10 \text{ A}$$

$$I = 10 \approx 0.622 \times 10^{-5} J_0 \Rightarrow J_0 = 10 / 0.622 \times 10^{-5} = 1.6077 \times 10^6 \text{ A/m}^2$$

b. $J = J_0 e^{-10^4(a-\rho)} = e^{-1} J_0 = 0.368 J_0 \Rightarrow 10^4(a-\rho) = 1$

$$a - \rho = 10^{-4} \Rightarrow \rho = a - 10^{-4}$$

Distance from surface = 10^{-4} m (very small; almost on the surface)

2.24 $\sigma = 5.76 \times 10^7 \text{ S/m}, \epsilon = 8.854 \times 10^{-12} \text{ f/m}$

$$T_r = \frac{\epsilon}{\sigma} = \frac{8.854 \times 10^{-12}}{5.76 \times 10^7} = 1.5372 \times 10^{-19} \text{ secs.}$$

In the microwave region ($f = 1-10 \text{ GHz}$) the period is

$$f = 1 \text{ GHz: } T = \frac{1}{f} = \frac{1}{10^9} = 10^{-9} \gg T_r = 1.5372 \times 10^{-19}$$

$$f = 2 \text{ GHz: } T = \frac{1}{f} = \frac{1}{10^9} = 10^{-9} \gg T_r = 1.5372 \times 10^{-19}$$

For x-rays [$\lambda = (1-10) \times 10^{-8} \text{ cm}$] the period is

$$\lambda = 1 \times 10^{-8} \text{ cm: } f = \frac{v}{\lambda} = \frac{3 \times 10^{10}}{10^{-8}} = 3 \times 10^{18}$$

$$T = \frac{1}{f} = \frac{1}{3 \times 10^{18}} = 0.333 \times 10^{-18} = 3.33 \times 10^{-19}$$

$$\lambda = 10 \times 10^{-8} \text{ cm: } f = \frac{v}{\lambda} = \frac{3 \times 10^{10}}{10 \times 10^{-8}} = 3 \times 10^{17}$$

$$T = \frac{1}{f} = \frac{1}{3 \times 10^{17}} = 0.333 \times 10^{-17} = 3.33 \times 10^{-18}$$

which are comparable to the relaxation time constant.

2.25 $\sigma = 3.96 \times 10^7 \text{ S/m}, \mu_e = 2.2 \times 10^{-3} \text{ m}^2/(\text{V-s}), \underline{E} = \hat{a}_x 2 \text{ V/m}$

a. $\sigma = -\mu_e q_{ve} \Rightarrow q_{ve} = -\frac{\sigma}{\mu_e} = -\frac{3.96 \times 10^7}{2.2 \times 10^{-3}} = -1.8 \times 10^{10} \text{ C/m}^3$

b. $\underline{v}_e = -\mu_e \underline{E} = -2.2 \times 10^{-3} (\hat{a}_x E_x) = -\hat{a}_x (2.2 \times 2 \times 10^{-3}) = -\hat{a}_x 4.4 \times 10^{-3} \text{ m/sec.}$

c. $\underline{J}_c = \sigma \underline{E} = 3.96 \times 10^7 (\hat{a}_x 2) = \hat{a}_x 7.92 \times 10^7 \text{ A/m}^2$

d. $I = \iint_{S_0} \underline{J}_c \cdot d\underline{s} = \iint_{S_0} \hat{a}_x J_{cx} \cdot \hat{a}_x ds = \iint_{S_0} J_{cx} ds = 7.92 \times 10^7 (S_0) = 7.92 \times 10^7 (10^{-3})$

$$I = 7.92 \times 10^4 \text{ A}$$

e. $N_e = \frac{q_{ve}}{q_e} = \frac{-1.8 \times 10^{10}}{-1.602 \times 10^{-19}} = 1.1236 \times 10^{29} \frac{\text{electrons}}{\text{m}^3}$

CHAPTER 3

3.1 $\nabla \times \underline{E} = -\underline{M}_i - j\omega\mu \underline{H}$, $\nabla \times \underline{H} = \underline{J}_i + \sigma \underline{E} + j\omega\epsilon \underline{E}$

$$\nabla \times \nabla \times \underline{E} = -\nabla \times \underline{M}_i - j\omega\mu \nabla \times \underline{H}$$

$$\nabla \times \nabla \times \underline{H} = \nabla \times \underline{J}_i + \sigma \nabla \times \underline{E} + j\omega\epsilon \nabla \times \underline{E}$$

Using Maxwell's equations from above and the vector identity of

$$\nabla \times \nabla \times \underline{E} = \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E}$$

we can write

$$\begin{aligned} \nabla(\nabla \cdot \underline{E}) - \nabla^2 \underline{E} &= -\nabla \times \underline{M}_i - j\omega\mu [\underline{J}_i + \sigma \underline{E} + j\omega\epsilon \underline{E}] \\ &= -\nabla \times \underline{M}_i - j\omega\mu \underline{J}_i - j\omega\mu \sigma \underline{E} + \omega^2 \mu \epsilon \underline{E} \end{aligned}$$

Since $\nabla \cdot \underline{D} = \nabla \cdot (\epsilon \underline{E}) = \epsilon \nabla \cdot \underline{E} = \rho_{ve} \Rightarrow \nabla \cdot \underline{E} = \frac{\rho_{ve}}{\epsilon}$

Now we can write that

$$\nabla^2 \underline{E} = \nabla \times \underline{M}_i + j\omega\mu \underline{J}_i + \frac{1}{\epsilon} \nabla \rho_{ve} + j\omega\mu \sigma \underline{E} - \omega^2 \mu \epsilon \underline{E}$$

which is an uncoupled second-order differential equation.

Using the equation for the magnetic field from above along with Maxwell's equations and the vector identity, we can write

$$\nabla(\nabla \cdot \underline{H}) - \nabla^2 \underline{H} = \nabla \times \underline{J}_i + (\sigma + j\omega\epsilon) \nabla \times \underline{E} = \nabla \times \underline{J}_i + (\sigma + j\omega\epsilon)(-\underline{M}_i - j\omega\mu \underline{H})$$

Since $\nabla \cdot \underline{B} = \nabla \cdot (\mu \underline{H}) = \mu \nabla \cdot \underline{H} = \rho_{vm} \Rightarrow \nabla \cdot \underline{H} = \frac{1}{\mu} \rho_{vm}$

then

$$\nabla^2 \underline{H} = -\nabla \times \underline{J}_i + \sigma \underline{M}_i + \frac{1}{\mu} \nabla \rho_{vm} + j\omega\epsilon \underline{M}_i + j\omega\mu \sigma \underline{H} - \omega^2 \mu \epsilon \underline{H}$$

which also is an uncoupled second order differential equation.

3.2 $\frac{d^2 f}{dx^2} = -\beta_x^2 f$, $f = f_1 = A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}$

Using $f = f_1 = A_1 e^{-j\beta_x x}$, then

$$(-j\beta_x)^2 A_1 e^{-j\beta_x x} = -\beta_x^2 A_1 e^{-j\beta_x x} = -\beta_x^2 A_1 e^{-j\beta_x x}$$

(continued)

Q.E.D.

3.2 cont'd. The same can be shown by letting $f = f_1 = B_1 e^{+j\beta_x x}$

Now let us try the cosinusoidal solutions.

$$\text{Let } f = f_2 = C_1 \cos(\beta_x x)$$

Substituting this into the differential equation leads to

$$-\beta_x^2 C_1 \cos(\beta_x x) = -\beta_x^2 C_1 \cos(\beta_x x) \quad \text{Q.E.D.}$$

The same can be shown by letting $f = f_2 = D_1 \sin(\beta_x x)$

3.3 $E_x(x, y, z, t) = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)] [C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] B_3 \cos(\omega t + \beta_z z)$

To follow a point z_p for different values of t we must maintain constant the amplitude of the cosine term. This is accomplished by letting

$$\omega t + \beta_z z = C_1 = \text{constant}$$

Taking a derivative of both sides with respect to time, we can write

$$\omega(1) + \beta_z \frac{dz}{dt} = 0 \Rightarrow \omega + \beta_z v_p = 0 \Rightarrow v_p = -\frac{\omega}{\beta_z}$$

which indicates that the wave is moving in the $-z$ direction.

3.4 $\nabla^2 E_x - \gamma^2 E_x = 0 = \frac{d^2 E_x}{dx^2} + \frac{d^2 E_x}{dy^2} + \frac{d^2 E_x}{dz^2} - \gamma^2 E_x$

Letting $E_x(x, y, z) = f(x)g(y)h(z)$ and substituting above leads to

$$gh \frac{d^2 f}{dx^2} + fh \frac{d^2 g}{dy^2} + fg \frac{d^2 h}{dz^2} - \gamma^2 fgh = 0$$

Dividing both sides by fgh , we can write

$$\frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} + \frac{1}{h} \frac{d^2 h}{dz^2} = \gamma^2$$

By letting each term on the left equal to a constant leads to

$$\frac{1}{f} \frac{d^2 f}{dx^2} = \gamma_x^2 \Rightarrow f = f_1 = A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x}; f = f_2 = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x)$$

$$\frac{1}{g} \frac{d^2 g}{dy^2} = \gamma_y^2 \Rightarrow g = g_1 = A_2 e^{-\gamma_y y} + B_2 e^{+\gamma_y y}; g = g_2 = C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y)$$

$$\frac{1}{h} \frac{d^2 h}{dz^2} = -\gamma_z^2 \Rightarrow h = h_1 = A_3 e^{-\gamma_z z} + B_3 e^{+\gamma_z z}; h = h_2 = C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z)$$

provided that $\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2$

3.5 $\nabla(\nabla \cdot \underline{E}) - \nabla \times \nabla \times \underline{E} = -\beta^2 \underline{E}$, $\underline{E} = \hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z$

Using cylindrical coordinates

$$\nabla \cdot \underline{E} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

$$\begin{aligned} \nabla(\nabla \cdot \underline{E}) &= \nabla \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right] \\ &= \hat{a}_\rho \left\{ \frac{\partial}{\partial \rho} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right] \right\} + \hat{a}_\phi \frac{1}{\rho} \left\{ \frac{\partial}{\partial \phi} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right] \right\} \\ &\quad + \hat{a}_z \left\{ \frac{\partial}{\partial z} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right] \right\} \end{aligned}$$

$$\begin{aligned} \nabla(\nabla \cdot \underline{E}) &= \hat{a}_\rho \left[\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\rho}{\partial \rho} - \frac{E_\rho}{\rho^2} + \frac{1}{\rho} \frac{\partial^2 E_\phi}{\partial \rho \partial \phi} - \frac{1}{\rho^2} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial^2 E_z}{\partial \rho \partial z} \right] \\ &\quad + \hat{a}_\phi \left[\frac{1}{\rho} \frac{\partial^2 E_\rho}{\partial \rho \partial \phi} + \frac{1}{\rho^2} \frac{\partial E_\rho}{\partial \phi} + \frac{1}{\rho^2} \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 E_z}{\partial \phi \partial z} \right] + \hat{a}_z \left[\frac{\partial^2 E_\rho}{\partial \rho \partial z} + \frac{1}{\rho} \frac{\partial E_\rho}{\partial z} + \frac{1}{\rho} \frac{\partial^2 E_\phi}{\partial \phi \partial z} + \frac{\partial^2 E_z}{\partial z^2} \right] \end{aligned}$$

$$\nabla \times \underline{E} = \hat{a}_\rho \left[\frac{1}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right] + \hat{a}_\phi \left[\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \right] + \hat{a}_z \left[\frac{\partial E_\phi}{\partial \rho} + \frac{E_\phi}{\rho} - \frac{1}{\rho} \frac{\partial E_\rho}{\partial \phi} \right]$$

$$\begin{aligned} \nabla \times \nabla \times \underline{E} &= \hat{a}_\rho \left\{ \frac{1}{\rho} \frac{\partial^2 E_\phi}{\partial \rho \partial \phi} + \frac{1}{\rho^2} \frac{\partial E_\phi}{\partial \phi} - \frac{1}{\rho^2} \frac{\partial^2 E_\rho}{\partial \phi^2} - \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial^2 E_z}{\partial \rho \partial z} \right\} \\ &\quad + \hat{a}_\phi \left\{ \frac{1}{\rho} \frac{\partial^2 E_z}{\partial \phi \partial z} - \frac{\partial^2 E_\phi}{\partial z^2} - \frac{\partial^2 E_\phi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial E_\phi}{\partial \rho} + \frac{E_\phi}{\rho^2} + \frac{1}{\rho} \frac{\partial^2 E_\rho}{\partial \rho \partial \phi} - \frac{1}{\rho^2} \frac{\partial E_\rho}{\partial \phi} \right\} \\ &\quad + \hat{a}_z \left\{ \frac{\partial^2 E_\rho}{\partial \rho \partial z} + \frac{1}{\rho} \frac{\partial E_\rho}{\partial z} - \frac{\partial^2 E_z}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial^2 E_\phi}{\partial \phi \partial z} \right\} \end{aligned}$$

Substituting all these into the wave equation and equating identical components from the left and right sides, it leads to

ρ component:

$$\begin{aligned} &\left[\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial^2 E_\rho}{\partial z^2} \right] - \frac{E_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} = -\beta^2 E_\rho \\ \text{or } \nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} \right) &= -\beta^2 E_\rho \end{aligned}$$

ϕ component:

$$\begin{aligned} &\left[\frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial z^2} \right] - \frac{1}{\rho^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} = -\beta^2 E_\phi \\ \text{or } \nabla^2 E_\phi + \left[-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi} \right] &= -\beta^2 E_\phi \end{aligned}$$

z component:

$$\begin{aligned} &\left[\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} \right] = -\beta^2 E_z \\ \text{or } \nabla^2 E_z &= -\beta^2 E_z \end{aligned}$$

3.6 $\underline{E} = \hat{a}_\rho E_\rho(\rho, \phi, z) + \hat{a}_\phi E_\phi(\rho, \phi, z) + \hat{a}_z E_z$

Using the rectangular to cylindrical coordinate transformation, we can write

$$\hat{a}_\rho = \hat{a}_x \cos \phi + \hat{a}_y \sin \phi$$

$$\hat{a}_\phi = -\hat{a}_x \sin \phi + \hat{a}_y \cos \phi$$

Therefore

$$\frac{\partial \hat{a}_\rho}{\partial \rho} = 0 ; \quad \frac{\partial \hat{a}_\phi}{\partial \rho} = 0 ; \quad \frac{\partial \hat{a}_z}{\partial \rho} = 0$$

$$\frac{\partial \hat{a}_\rho}{\partial \phi} = -\hat{a}_x \sin \phi + \hat{a}_y \cos \phi = \hat{a}_\phi ; \quad \frac{\partial^2 \hat{a}_\rho}{\partial \phi^2} = -\hat{a}_x \cos \phi - \hat{a}_y \sin \phi = -\hat{a}_\rho$$

$$\frac{\partial \hat{a}_\phi}{\partial \phi} = -\hat{a}_x \cos \phi - \hat{a}_y \sin \phi = -\hat{a}_\rho ; \quad \frac{\partial^2 \hat{a}_\phi}{\partial \phi^2} = \hat{a}_x \sin \phi - \hat{a}_y \cos \phi = -\hat{a}_\phi$$

$$\frac{\partial \hat{a}_z}{\partial \phi} = 0$$

$$\frac{\partial \hat{a}_\rho}{\partial z} = 0 ; \quad \frac{\partial \hat{a}_\phi}{\partial z} = 0 ; \quad \frac{\partial \hat{a}_z}{\partial z} = 0$$

$$\nabla^2 \underline{E} = \nabla^2(\hat{a}_\rho E_\rho) + \nabla^2(\hat{a}_\phi E_\phi) + \nabla^2(\hat{a}_z E_z) = \nabla^2(\hat{a}_\rho E_\rho) + \nabla^2(\hat{a}_\phi E_\phi) + \hat{a}_z \nabla^2 E_z$$

$$\nabla^2(\hat{a}_\rho E_\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial}{\partial \rho} (\hat{a}_\rho E_\rho) \right] + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} (\hat{a}_\rho E_\rho) + \frac{\partial^2 (\hat{a}_\rho E_\rho)}{\partial z^2}$$

$$\frac{\partial^2}{\partial \phi^2} (\hat{a}_\rho E_\rho) = \frac{\partial}{\partial \phi} \left[\frac{\partial}{\partial \phi} (\hat{a}_\rho E_\rho) \right] = \frac{\partial}{\partial \phi} \left[\hat{a}_\rho \frac{\partial E_\rho}{\partial \phi} + E_\rho \frac{\partial \hat{a}_\rho}{\partial \phi} \right] = \frac{\partial}{\partial \phi} \left[\hat{a}_\rho \frac{\partial E_\rho}{\partial \phi} + \hat{a}_\phi E_\rho \right]$$

$$= \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial E_\rho}{\partial \phi} \frac{\partial \hat{a}_\rho}{\partial \phi} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} + E_\rho \frac{\partial \hat{a}_\phi}{\partial \phi}$$

$$= \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} - \hat{a}_\rho E_\rho = -\hat{a}_\rho E_\rho + \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \hat{a}_\phi 2 \frac{\partial E_\rho}{\partial \phi}$$

$$\frac{\partial^2}{\partial z^2} (\hat{a}_\rho E_\rho) = \frac{\partial}{\partial z} \left[\frac{\partial}{\partial z} (\hat{a}_\rho E_\rho) \right] = \frac{\partial}{\partial z} \left[\hat{a}_\rho \frac{\partial E_\rho}{\partial z} + E_\rho \frac{\partial \hat{a}_\rho}{\partial z} \right] = \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial E_\rho}{\partial z} \frac{\partial \hat{a}_\rho}{\partial z}$$

Therefore

$$\nabla^2(\hat{a}_\rho E_\rho) = \hat{a}_\rho \left[\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho^2} \left(\frac{\partial^2 E_\rho}{\partial \phi^2} - E_\rho \right) + \frac{\partial^2 E_\rho}{\partial z^2} \right] + \hat{a}_\phi \frac{2}{\rho^2} \frac{\partial E_\rho}{\partial \phi}$$

Similarly

$$\nabla^2(\hat{a}_\phi E_\phi) = \frac{\partial^2}{\partial \rho^2} (\hat{a}_\phi E_\phi) + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\hat{a}_\phi E_\phi) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} (\hat{a}_\phi E_\phi) + \hat{a}_\phi \frac{\partial^2 E_\phi}{\partial z^2}$$

$$= -\hat{a}_\phi \frac{2}{\rho^2} \frac{\partial E_\phi}{\partial \phi} + \hat{a}_\phi \left[\frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_\phi}{\partial \phi^2} - \frac{E_\phi}{\rho^2} + \frac{\partial^2 E_\phi}{\partial z^2} \right]$$

$$\nabla^2(\hat{a}_z E_z) = \hat{a}_z \nabla^2 E_z = \hat{a}_z \left[\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} \right] \quad (\text{continued})$$