SOLUTIONS TO CHAPTER 1

Problem 1.1

(a) Since the growth rate of a variable equals the time derivative of its log, as shown by equation (1.10) in the text, we can write

$$(1)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)Y(t)]}{dt}.$$

Since the log of the product of two variables equals the sum of their logs, we have

$$(2)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) + \ln Y(t)]}{dt} = \frac{d\ln X(t)}{dt} + \frac{d\ln Y(t)}{dt},$$

or simply

$$(3)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) Again, since the growth rate of a variable equals the time derivative of its log, we can write

$$(4)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)/Y(t)]}{dt}.$$

Since the log of the ratio of two variables equals the difference in their logs, we have

$$(5)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) - \ln Y(t)]}{dt} = \frac{d\ln X(t)}{dt} - \frac{d\ln Y(t)}{dt},$$

or simply

$$(6)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}.$$

(c) We have

$$(7)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)^{\alpha}]}{dt}.$$

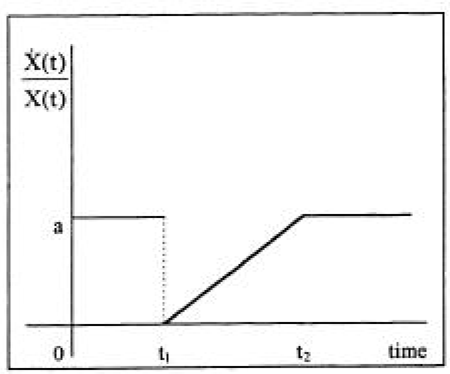
Using the fact that $\ln[X(t)^{\alpha}] = \alpha \ln X(t)$, we have

$$(8)\quad \frac{\dot{Z}(t)}{Z(t)} = \frac{d[\alpha \ln X(t)]}{dt} = \alpha\frac{d\ln X(t)}{dt} = \alpha\frac{\dot{X}(t)}{X(t)},$$

where we have used the fact that α is a constant.

Problem 1.2

(a) Using the information provided in the question, the path of the growth rate of X, $\dot{X}(t)/X(t)$, is depicted in the figure at right.



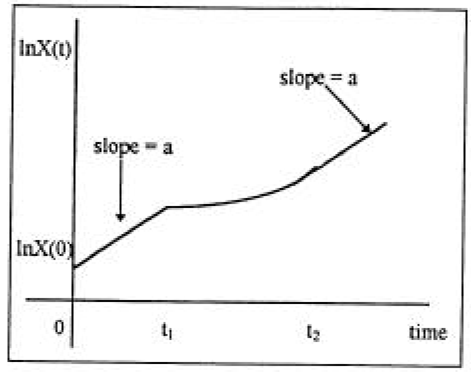
From time 0 to time t_1, the growth rate of X is constant and equal to $a > 0$. At time t_1, the growth rate of X drops to 0. From time t_1 to time t_2, the growth rate of X rises gradually from 0 to a. Note that we have made the assumption that $\dot{X}(t)/X(t)$ rises at a constant rate from t_1 to t_2. Finally, after time t_2, the growth rate of X is constant and equal to a again.

(b) Note that the slope of $\ln X(t)$ plotted against time is equal to the growth rate of $X(t)$. That is, we know

$$\frac{d \ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

(See equation (1.10) in the text.)

From time 0 to time t_1 the slope of $\ln X(t)$ equals $a > 0$. The $\ln X(t)$ locus has an inflection point at t_1, when the growth rate of $X(t)$ changes discontinuously from a to 0. Between t_1 and t_2, the slope of $\ln X(t)$ rises gradually from 0 to a. After time t_2 the slope of $\ln X(t)$ is constant and equal to $a > 0$ again.

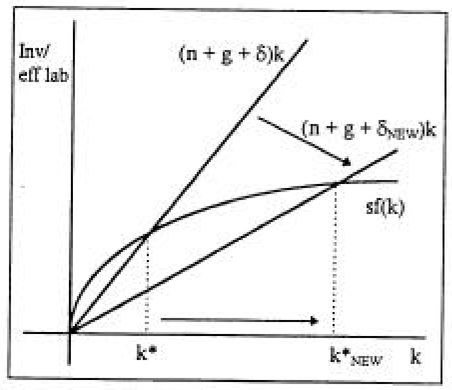


Problem 1.3

(a) The slope of the break-even investment line is given by $(n + g + \delta)$ and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, $sf(k)$ is unaffected.

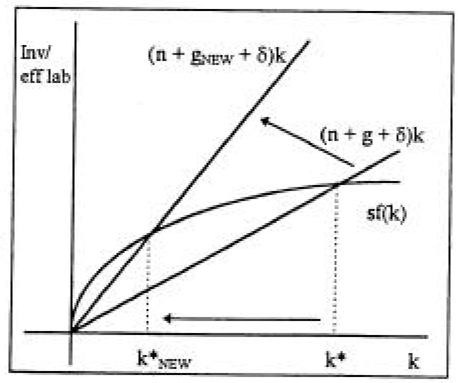
From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k^* to k^*_{NEW}.



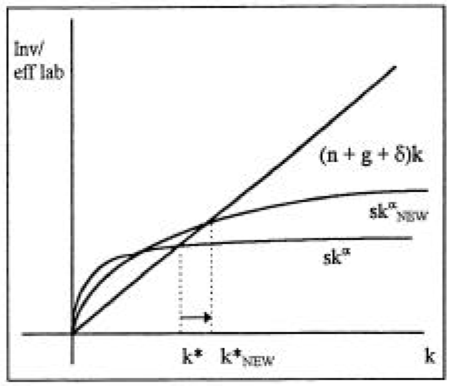
(b) Since the slope of the break-even investment line is given by $(n + g + \delta)$, a rise in the rate of technological progress, g, makes the break-even investment line steeper.

The actual investment curve, $sf(k)$, is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor falls from k^* to k^*_{NEW}.



(c) The break-even investment line, $(n + g + \delta)k$, is unaffected by the rise in capital's share, α.



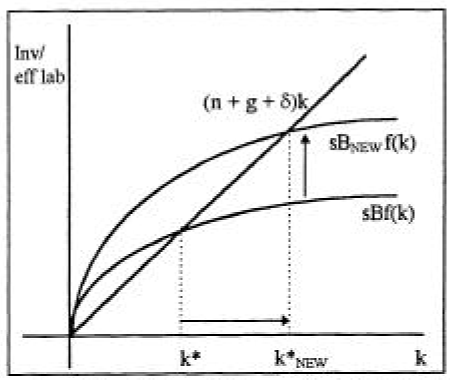
The effect of a change in α on the actual investment curve, sk^{α}, can be determined by examining the derivative $\partial(sk^{\alpha})/\partial\alpha$. It is possible to show that

(1) $$\frac{\partial sk^{\alpha}}{\partial\alpha} = sk^{\alpha}\ln k.$$

For $0 < \alpha < 1$, and for positive values of k, the sign of $\partial(sk^{\alpha})/\partial\alpha$ is determined by the sign of lnk. For $\ln k > 0$, or $k > 1$, $\partial sk^{\alpha}/\partial\alpha > 0$ and so the new actual investment curve lies above the old one. For $\ln k < 0$ or $k < 1$, $\partial sk^{\alpha}/\partial\alpha < 0$ and so the new actual investment curve lies below the old one. At $k = 1$, so that $\ln k = 0$, the new actual investment curve intersects the old one.

In addition, the effect of a rise in α on k* is ambiguous and depends on the relative magnitudes of s and $(n + g + \delta)$. It is possible to show that a rise in capital's share, α, will cause k* to rise if $s > (n + g + \delta)$. This is the case depicted in the figure above.

(d) Suppose we modify the intensive form of the production function to include a non-negative constant, B, so that the actual investment curve is given by $sBf(k)$, $B > 0$.



Then workers exerting more effort, so that output per unit of effective labor is higher than before, can be modeled as an increase in B. This increase in B shifts the actual investment curve up.

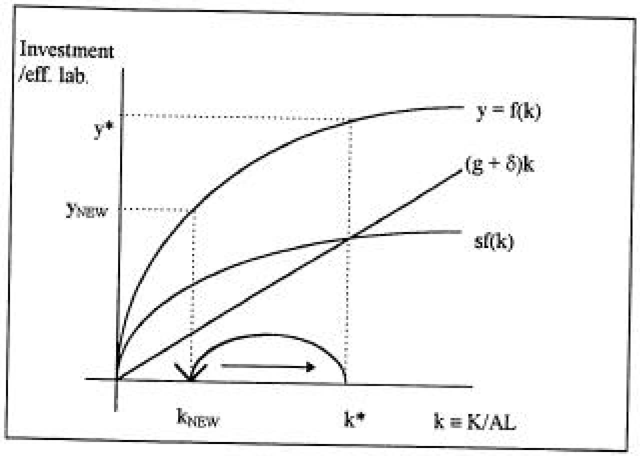
The break-even investment line, $(n + g + \delta)k$, is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k^* to k^*_{NEW}.

Problem 1.4

(a) At some time, call it t_0, there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW}. We can see this by simply looking at the definition, $k = K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y^* to y_{NEW}.

(b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g+\delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y*.



(c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k*. At k*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

Problem 1.5

(a) The equation describing the evolution of the capital stock per unit of effective labor is given by

(1) $\dot{k} = sf(k) - (n+g+\delta)k$.

Substituting in for the intensive form of the Cobb-Douglas, $f(k) = k^{\alpha}$, yields

$$\dot{k} = sk^{\alpha} - (n+g+\delta)k.$$

On the balanced growth path, $\dot{k}$ is zero; investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k*, we have $sk^{*\alpha} = (n+g+\delta)k^*$. Rearranging to solve for k* yields

(2) $k^* = \left[s/(n+g+\delta)\right]^{1/(1-\alpha)}$.

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (2) into the intensive form of the production function, $y = k^{\alpha}$:

(3) $y^* = \left[s/(n+g+\delta)\right]^{\alpha/(1-\alpha)}$.

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1-s)y^*$. Substituting equation (3) into this expression yields

(4) $c^* = (1-s)\left[s/(n+g+\delta)\right]^{\alpha/(1-\alpha)}$.

(b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (2), which expresses the balanced-growth-path level of k, and rearrange it to solve for s:

(5) $s = (n+g+\delta)k^{*1-\alpha}$.

Now substitute equation (5) into equation (4):

$$c^* = \left[1-(n+g+\delta)k^{*1-\alpha}\right]\left[(n+g+\delta)k^{*1-\alpha}/(n+g+\delta)\right]^{\alpha/(1-\alpha)}.$$

After some straightforward algebraic manipulation, this simplifies to

(6) $c^* = k^{*\alpha} - (n + g + \delta)k^*$.
Equation (6) can be easily interpreted. Consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor, which on the balanced growth path is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (6) to maximize c^* with respect to k^*. The first-order condition is given by

$$\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0,$$

or simply

(7) $\alpha k^{*\alpha-1} = (n + g + \delta)$.

Note that equation (7) is just a specific form of $f'(k^*) = (n + g + \delta)$, which is the general condition that implicitly defines the golden-rule level of capital per unit of effective labor. Equation (7) has a graphical interpretation: it defines the level of k at which the slope of the intensive form of the production function is equal to the slope of the break-even investment line.

Solving equation (7) for the golden-rule level of k yields

(8) $k^*_{GR} = [\alpha/(n + g + \delta)]^{1/(1-\alpha)}$.

(c) To get the saving rate that will yield the golden-rule level of k, substitute equation (8) into (5):

$$s_{GR} = (n + g + \delta)[\alpha/(n + g + \delta)]^{(1-\alpha)/(1-\alpha)},$$

which simplifies to

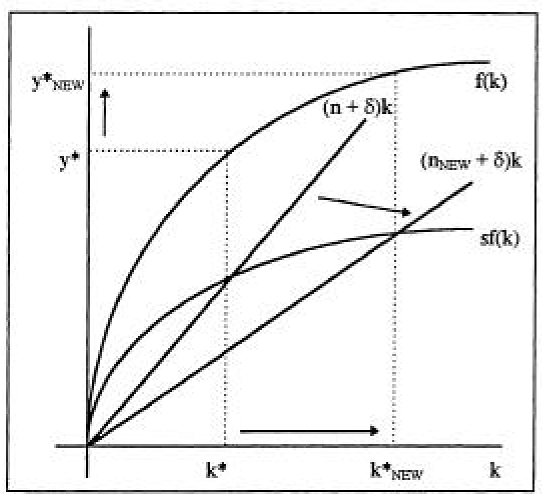
(9) $s_{GR} = \alpha$.

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

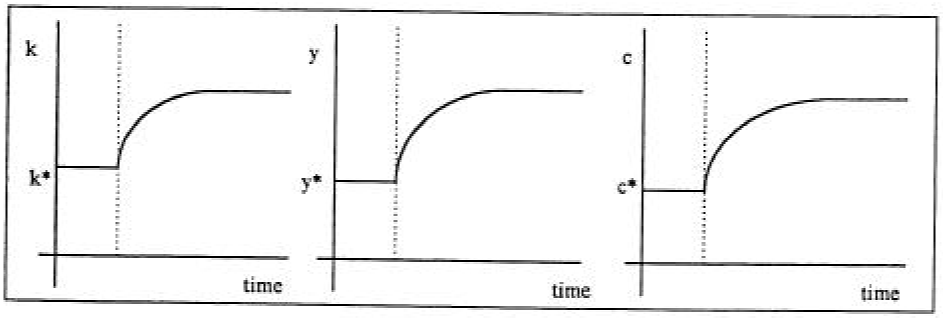
Problem 1.6

(a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus we can define $y = Y/L$ and $k = K/L$.

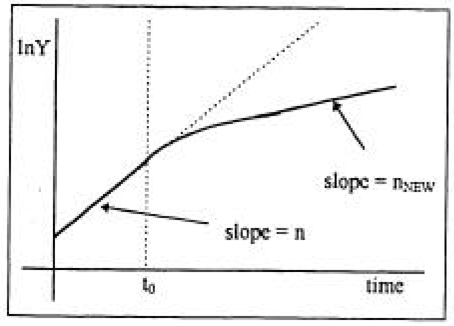
The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k, capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n -- the economy was on a balanced growth path -- the decrease in n causes $\dot{k}$ to become positive. At k^*, actual investment per worker, $sf(k^*)$, now exceeds break-even investment per worker, $(n_{NEW} + \delta)k^*$. Thus k moves to a new higher balanced growth path level. See the figure at right.



As k rises, y -- output per worker -- also rises. Since a constant fraction of output is saved, c -- consumption per worker -- rises as y rises. This is summarized in the figures below.



(b) By definition, output can be written as $Y = Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ -- output per worker is constant -- so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again -- output per worker is constant again -- and so $\dot{Y}/Y = \dot{L}/L = n_{NEW} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Examine the production function $Y = F(K,AL)$. On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and that of AL (which is growing at n_{NEW}). Thus, during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally K, AL, and thus Y are all growing at the new lower n_{NEW}.

Problem 1.7

The derivative of $y^* = f(k^*)$ with respect to n is given by

(1) $\partial y^*/\partial n = f'(k^*)[\partial k^*/\partial n]$.

To find $\partial k^*/\partial n$, use the equation for the evolution of the capital stock per unit of effective labor, $\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, $k = k^*$ and thus $sf(k^*) = (n + g + \delta)k^*$. Taking the derivative of both sides of this expression with respect to n yields

$$sf'(k^*)\frac{\partial k^*}{\partial n} = (n + g + \delta)\frac{\partial k^*}{\partial n} + k^*,$$

and rearranging yields

(2) $$\frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - (n + g + \delta)}.$$

Substituting equation (2) into equation (1) gives us

(3) $\dfrac{\partial y^*}{\partial n} = f'(k^*)\left[\dfrac{k^*}{sf'(k^*) - (n+g+\delta)}\right]$.

Rearranging the condition that implicitly defines k^*, $sf(k^*) = (n + g + \delta)k^*$, and solving for s yields

(4) $s = (n + g + \delta)k^*/f(k^*)$.

Substitute equation (4) into equation (3):

(5) $\dfrac{\partial y^*}{\partial n} = \dfrac{f'(k^*)k^*}{[(n+g+\delta)f'(k^*)k^*/f(k^*)] - (n+g+\delta)}$.

To turn this into the elasticity that we want, multiply both sides of equation (5) by n/y^*:

$$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = \frac{n}{(n+g+\delta)}\frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)] - 1}.$$

Using the definition that $\alpha_K(k^*) = f'(k^*)k^*/f(k^*)$ gives us

(6) $\dfrac{n}{y^*}\dfrac{\partial y^*}{\partial n} = -\dfrac{n}{(n+g+\delta)}\left[\dfrac{\alpha_K(k^*)}{1-\alpha_K(k^*)}\right]$.

Now, with $\alpha_K(k^*) = 1/3$, $g = 2\%$ and $\delta = 3\%$, we need to calculate the effect on y^* of a fall in n from 2% to 1%. Using the midpoint of $n = 0.015$ to calculate the elasticity gives us

$$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{0.015}{(0.015+0.02+0.03)}\left(\frac{1/3}{1-1/3}\right) \cong -0.12.$$

So this 50% drop in the population growth rate, from 2% to 1%, will lead to approximately a 6% increase in the level of output per unit of effective labor, since (-0.50)(-0.12) = 0.06. This calculation illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

Problem 1.8

(a) A permanent increase in the fraction of output that is devoted to investment from 0.15 to 0.18 represents a 20% increase in the saving rate. From equation (1.27) in the text, the elasticity of output with respect to the saving rate is

(1) $\dfrac{s}{y^*}\dfrac{\partial y^*}{\partial s} = \dfrac{\alpha_K(k^*)}{1-\alpha_K(k^*)}$,

where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

Substituting the assumption that $\alpha_K(k^*) = 1/3$ into equation (1) gives us

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)} = \frac{1/3}{1-1/3} = \frac{1}{2}.$$

Thus the elasticity of output with respect to the saving rate is 1/2. So this 20% increase in the saving rate -- from $s = 0.15$ to $s_{NEW} = 0.18$ -- will cause output to rise relative to what it would have been by about 10%. [Note that the analysis has been carried out in terms of output per unit of effective labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10%, output itself is also 10% higher than what it would have been.]

(b) Consumption will rise less than output. Although output winds up 10% higher than what it would have been, the fact that the saving rate is higher means that we are now consuming a smaller fraction of output. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by

(2) $c^* = (1 - s)y^*$.
Taking the derivative with respect to s yields

(3) $$\frac{\partial c^*}{\partial s} = -y^* + (1-s)\frac{\partial y^*}{\partial s}.$$

To turn this into an elasticity, multiply both sides of equation (3) by s/c^*:

$$\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-y^* s}{(1-s)y^*} + (1-s)\frac{\partial y^*}{\partial s}\frac{s}{(1-s)y^*},$$

where we have substituted $c^* = (1 - s)y^*$ on the right-hand side. Simplifying gives us

(4) $$\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-s}{(1-s)} + \frac{\partial y^*}{\partial s}\frac{s}{(1-s)y^*}.$$

From part (a), the second term on the right-hand side of (4), the elasticity of output with respect to the saving rate, equals 1/2. We can use the midpoint between $s = 0.15$ and $s_{NEW} = 0.18$ to calculate the elasticity:

$$\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-0.165}{(1-0.165)} + 0.5 \cong 0.30.$$

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate will cause consumption to be approximately 6% above what it would have been.

(c) The immediate effect of the rise in investment as a fraction of output is that consumption falls. Although y^* does not jump immediately -- it only begins to move toward its new, higher balanced-growth-path level -- we are now saving a greater fraction, and thus consuming a smaller fraction, of this same y^*. At the moment of the rise in s by 3 percentage points -- since $c = (1 - s)y^*$ and y^* is unchanged -- c falls. In fact, the percentage change in c will be the percentage change in $(1 - s)$. Now, $(1 - s)$ falls from 0.85 to 0.82, which is approximately a 3.5% drop. Thus at the moment of the rise in s, consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine the length of time it takes for consumption to return to what it would have been without the increase in the saving rate. After the initial rise in s, s remains constant throughout. Since $c = (1 - s)y$, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y, after a linear approximation, is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta)$ equal to 6% per year and $\alpha_K = 1/3$, this yields a value for λ of about 4%. This means that k and y move about 4% of the remaining distance toward their balanced-growth-path values of k^* and y^* each year. Since c is proportional to y -- $c = (1 - s)y$ -- it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.31) in the text, we could write

(5) $$c(t) - c^* \cong e^{-(1-\alpha_K)(n+g+\delta)t}[c(0) - c^*],$$

or equivalently

(6) $$e^{-\lambda t} = \frac{c(t) - c^*}{c(0) - c^*}.$$

The term on the right-hand side of equation (6) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5% and eventually will be 6% higher than it would have been. Thus it must change by 9.5% on the way to the balanced growth path. It will therefore be equal to what it would have been about 36.8% (3.5%/9.5% ≅ 36.8%) of the way to the new balanced growth path.

Equivalently, this is when the remaining distance to the new balanced growth path is 63.2% of the original distance. In order to determine the length of time this will take, we need to find a t* that solves

(7) $e^{-\lambda t^*} = 0.632$.

Taking logs of both sides of equation (7) yields

$-\lambda t^* = \ln(0.632)$.

Rearranging to solve for t gives us

$t^* = 0.459/0.04$,

and thus

(8) $t^* \cong 11.5$ years.

It will take a fairly long time -- over a decade -- for consumption to return to what it would have been in the absence of the increase in investment as a fraction of output.

Problem 1.9

(a) Define the marginal product of labor as $w \equiv \partial F(K,AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields

(1) $w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$,

as required.

(b) Define the marginal product of capital as $r \equiv [\partial F(K,AL)/\partial K] - \delta$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields

(2) $r \equiv [\partial Y/\partial K] - \delta = ALf'(k)[1/AL] - \delta = f'(k) - \delta$.

Substitute equations (1) and (2) into wL + rK:

$wL + rK = A[f(k) - kf'(k)]L + [f'(k) - \delta]K = ALf(k) - f'(k)[K/AL]AL + f'(k)K - \delta K$.

Simplifying gives us

(3) $wL + rK = ALf(k) - f'(k)K + f'(k)K - \delta K = Alf(k) - \delta K \equiv ALF(K/AL, 1) - \delta K$.

Finally, since F is constant returns to scale, equation (3) can be rewritten as

(4) $wL + rK = F(ALK/AL, AL) - \delta K = F(K, AL) - \delta K$.

(c) As shown above, $r = f'(k) - \delta$. Since δ is a constant and since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r. In other words, on a balanced growth path, $\dot{r}/r = 0$. Thus the Solow model does exhibit the property that the return to capital is constant over time.

Since capital is paid its marginal product, the share of output going to capital is rK/Y. On a balanced growth path,

(5) $$\frac{\dot{\left(rK/Y\right)}}{\left(rK/Y\right)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n+g) - (n+g) = 0.$$

Thus, on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to one, this implies that the share of output going to labor is also constant on the balanced growth path.

We need to determine the growth rate of the marginal product of labor, w, on a balanced growth path. As shown above, $w = A[f(k) - kf'(k)]$. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

(6) $$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{\dot{[f(k) - kf'(k)]}}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - \dot{k}f'(k) - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

On a balanced growth path $\dot{k} = 0$ and so $\dot{w}/w = g$. That is, on a balanced growth path, the marginal product of labor rises at the rate of growth of the effectiveness of labor.

(d) As shown in part (c), the growth rate of the marginal product of labor is

$$(6)\quad \frac{\dot{w}}{w} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)}.$$

If $k < k^*$, then as k moves toward k^*, $\dot{w}/w > g$. This is true because the denominator of the second term on the right-hand side of equation (6) is positive because f(k) is a concave function. The numerator of that same term is positive because k and $\dot{k}$ are positive and $f''(k)$ is negative. Thus, as k rises toward k^*, the marginal product of labor grows faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path. As we move from k to k^*, however, the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r, is

$$(7)\quad \frac{\dot{r}}{r} = \frac{\dot{[f'(k)]}}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}.$$

As k rises toward k^*, this growth rate is negative since $f'(k) > 0$, $f''(k) < 0$ and $\dot{k} > 0$. Thus, as the economy moves from k to k^*, the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.

Problem 1.10

(a) By definition a balanced growth path occurs when all the variables of the model are growing at constant rates. Despite the differences between this model and the usual Solow model, it turns out that we can again show that the economy will converge to a balanced growth path by examining the behavior of k = K/AL.

Taking the time derivative of both sides of the definition of k = K/AL gives us

$$(1)\quad \dot{k} = \left(\frac{\dot{K}}{AL}\right) = \frac{\dot{K}(AL) - K\left[\dot{L}A - \dot{A}L\right]}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL}\left[\frac{\dot{L}A + \dot{A}L}{AL}\right] = \frac{\dot{K}}{AL} - k\left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A}\right).$$

Substituting the capital-accumulation equation, $\dot{K} = \left[\partial F(K, AL)/\partial K\right]K - \delta K$, and the constant growth rates of the labor force and technology, $\dot{L}/L = n$ and $\dot{A}/A = g$, into equation (1) yields

$$(2)\quad \dot{k} = \frac{\left[\partial F(K, AL)/\partial K\right]K - \delta K}{AL} - (n + g)k = \frac{\partial F(K, AL)}{\partial K}k - \delta k - (n + g)k.$$

Substituting $\partial F(K,AL)/\partial K = f'(k)$ into equation (2) gives us $\dot{k} = f'(k)k - \delta k - (n+g)k$ or simply

$$(3)\quad \dot{k} = \left[f'(k) - (n + g + \delta)\right]k.$$

Capital per unit of effective labor will be constant when $\dot{k} = 0$, i.e. when $[f'(k) - (n + g + \delta)]\, k = 0$. This condition holds if k = 0 (a case we will ignore) or $f'(k) - (n + g + \delta) = 0$. Thus the balanced-growth-path level of the capital stock per unit of effective labor is implicitly defined by $f'(k^*) = (n + g + \delta)$. Since capital per unit of effective labor, k = K/AL, is constant on the balanced growth path, K must grow at the same rate as AL, which grows at rate n + g. Since the production function has constant returns to capital and effective labor, which both grow at rate n + g on the balanced growth path, output must also grow at rate n + g on the balanced growth path. Thus we have found a balanced growth path where all the variables of the model grow at constant rates.

The next step is to show that the economy actually converges to this balanced growth path. At $k = k^*$, $f'(k) = (n + g + \delta)$. If $k > k^*$, $f'(k) < (n + g + \delta)$. This follows from the assumption that $f''(k) < 0$ which means that $f'(k)$ falls as k rises. Thus if $k > k^*$, we have $\dot{k} < 0$ so that k will fall toward its balanced-growth-path value. If $k < k^*$, $f'(k) > (n + g + \delta)$. Again, this follows from the assumption that $f''(k) < 0$ which means that $f'(k)$ rises as k falls. Thus if $k < k^*$, we have $\dot{k} > 0$ so that k will rise toward its balanced-growth-path value. Thus, regardless of the initial value of k (as long as it is not zero), the economy will converge to a balanced growth path at k^*, where all the variables in the model are growing at constant rates.

(b) The golden-rule level of k -- the level of k that maximizes consumption per unit of effective labor -- is defined implicitly by $f'(k^{GR}) = (n + g + \delta)$. Graphically, this occurs when the slope of the production function equals the slope of the break-even investment line. Note that this is exactly the level of k that the economy converges to in this model where all capital income is saved and all labor income is consumed.

In this model, we are saving capital's contribution to output, which is the marginal product of capital times the amount of capital. If that contribution exceeds break-even investment, $(n + g + \delta)k$, then k rises. If it is less than break-even investment, k falls. Thus k settles down to a point where saving, the marginal product of capital times k, equals break-even investment, $(n + g + \delta)k$. That is, the economy settles down to a point where $f'(k)k = (n + g + \delta)k$ or equivalently $f'(k) = (n + g + \delta)$.

Problem 1.11

(a) The production function with capital-augmenting technological progress is given by

(1) $Y(t) = [A(t)K(t)]^{\alpha} L(t)^{1-\alpha}$.

Dividing both sides of equation (1) by $A(t)^{\alpha/(1-\alpha)}L(t)$ yields

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}\left[\frac{L(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{1-\alpha},$$

and simplifying:

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}K(t)}{L(t)}\right]^{\alpha} A(t)^{-\alpha} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}A(t)^{-1}K(t)}{L(t)}\right]^{\alpha},$$

and thus finally

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi = \alpha/(1 - \alpha)$, $k(t) = K(t)/A(t)^{\phi}L(t)$ and $y(t) = Y(t)/A(t)^{\phi}L(t)$ yields

(2) $y(t) = k(t)^{\alpha}$.

In order to analyze the dynamics of k(t), take the time derivative of both sides of $k(t) = K(t)/A(t)^{\phi}L(t)$:

$$\dot{k}(t) = \frac{\dot{K}(t)\left[A(t)^{\phi}L(t)\right] - K(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^{2}},$$

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)^{\phi}L(t)} - \frac{K(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $k(t) \equiv K(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(3) $\dot{k}(t) = \dot{K}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t)$.

The evolution of the total capital stock is given by the usual

(4) $\dot{K}(t) = sY(t) - \delta K(t)$.

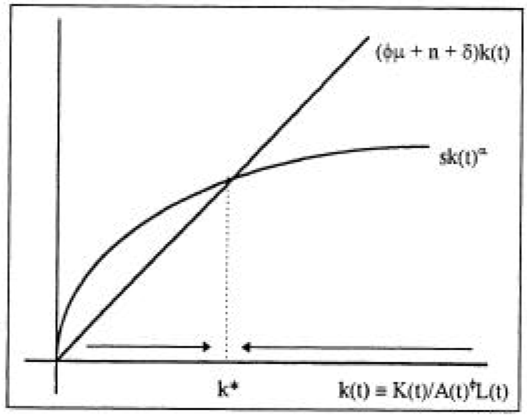
Substituting equation (4) into (3) gives us

$$\dot{k}(t) = sY(t)/A(t)^{\phi}L(t) - \delta K(t)/A(t)^{\phi}L(t) - (\phi\mu + n)k(t) = sy(t) - (\phi\mu + n + \delta)k(t).$$

Finally, using equation (2), $y(t) = k(t)^{\alpha}$, we have

(5) $\dot{k}(t) = sk(t)^{\alpha} - (\phi\mu + n + \delta)k(t)$.

Equation (5) is very similar to the basic equation governing the dynamics of the Solow model with labor-augmenting technological progress. Here, however, we are measuring in units of $A(t)^{\phi}L(t)$ rather than in units of effective labor, $A(t)L(t)$. Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{k}(t)$. See the figure at right.



When actual investment per unit of $A(t)^{\phi}L(t)$, $sk(t)^{\alpha}$, exceeds break-even investment per unit of $A(t)^{\phi}L(t)$, given by $(\phi\mu + n + \delta)k(t)$, k will rise toward k*. When actual investment per unit of $A(t)^{\phi}L(t)$ falls short of break-even investment per unit of $A(t)^{\phi}L(t)$, k will fall toward k*. Ignoring the case in which the initial level of k is zero, the economy will converge to a situation in which k is constant at k*. Since $y = k^{\alpha}$, y will also be constant when the economy converges to k*.

The total capital stock, K, can be written as $A^{\phi}Lk$. Thus when k is constant, K will be growing at the constant rate of $\phi\mu + n$. Similarly, total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$ as well. Since L and A grow at constant rates by assumption, we have found a balanced growth path where all the variables of the model grow at constant rates.

(b) The production function is now given by

(6) $Y(t) = J(t)^{\alpha}L(t)^{1-\alpha}$.

Define $\bar{J}(t) \equiv J(t)/A(t)$. The production function can then be written as

(7) $Y(t) = \left[A(t)\bar{J}(t)\right]^{\alpha}L(t)^{1-\alpha}$.

Proceed as in part (a). Divide both sides of equation (7) by $A(t)^{\alpha/(1-\alpha)}L(t)$ and simplify to obtain

(8) $$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{\bar{J}(t)}{A(t)^{\alpha/(1-\alpha)}L(t)}\right]^{\alpha}.$$

Now, defining $\phi = \alpha/(1-\alpha)$, $\bar{j}(t) \equiv \bar{J}(t)/A(t)^{\phi}L(t)$ and $y(t) \equiv Y(t)/A(t)^{\phi}L(t)$ yields

(9) $y(t) = \bar{j}(t)^{\alpha}$.

In order to analyze the dynamics of $\bar{j}(t)$, take the time derivative of both sides of $\bar{j}(t) = \bar{J}(t)/A(t)^{\phi}L(t)$:

$$\dot{\bar{j}} = \frac{\dot{\bar{J}}(t)\left[A(t)^{\phi}L(t)\right] - \bar{J}(t)\left[\phi A(t)^{\phi-1}\dot{A}(t)L(t) + \dot{L}(t)A(t)^{\phi}\right]}{\left[A(t)^{\phi}L(t)\right]^{2}},$$

$$\dot{\bar{j}}(t) = \frac{\dot{\bar{J}}(t)}{A(t)^{\phi}L(t)} - \frac{\bar{J}(t)}{A(t)^{\phi}L(t)}\left[\phi\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right],$$

and then using $\bar{j}(t) = \bar{J}(t)/A(t)^{\phi}L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields

(10) $\dot{\bar{j}}(t) = \dot{\bar{J}}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)\bar{j}(t)$.

The next step is to get an expression for $\dot{\bar{J}}(t)$. Take the time derivative of both sides of $\bar{J}(t) = J(t)/A(t)$:

$$\dot{\bar{J}}(t) = \frac{\dot{J}(t)A(t) - J(t)\dot{A}(t)}{A(t)^{2}} = \frac{\dot{J}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)}\frac{J(t)}{A(t)}.$$

Now use $\bar{J}(t) = J(t)/A(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$ to obtain

$$\dot{\bar{J}}(t) = \frac{sA(t)Y(t)}{A(t)} - \frac{\delta J(t)}{A(t)} - \mu\bar{J}(t),$$

or simply

(11) $\dot{\bar{J}}(t) = sY(t) - (\mu + \delta)\bar{J}(t)$.

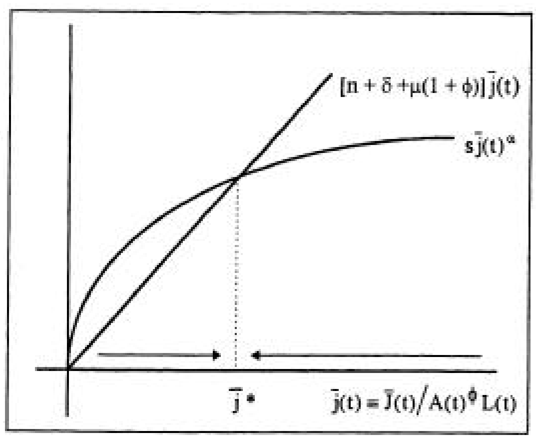
Substitute equation (11) into equation (10):

$$\dot{\bar{j}}(t) = sY(t)/A(t)^{\phi}L(t) - (\mu+\delta)\bar{J}(t)/A(t)^{\phi}L(t) - (\phi\mu + n)\bar{j}(t) = sy(t) - [n + \delta + \mu(1+\phi)]\bar{j}(t).$$

Finally, using equation (9), $y(t) = \bar{j}(t)^{\alpha}$, we have

(12) $\dot{\bar{j}}(t) = s\bar{j}(t)^{\alpha} - [n + \delta + \mu(1+\phi)]\bar{j}(t)$.

Using the same graphical technique as in the basic Solow model, we can graph both components of $\dot{\bar{j}}(t)$.



See the figure at right. Ignoring the possibility that the initial value of $\bar{j}$ is zero, the economy will converge to a situation where $\bar{j}$ is constant at $\bar{j}^*$. Since $y = \bar{j}^{\alpha}$, y will also be constant when the economy converges to $\bar{j}^*$.

The level of total output, Y, can be written as $A^{\phi}Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$.

By definition, $\bar{J} = A^{\phi}L\bar{j}$. Once the economy converges to the situation where $\bar{j}$ is constant, $\bar{J}$ grows at the constant rate of $\phi\mu + n$. Since $J = \bar{J}A$, the effective capital stock, J, grows at rate $\phi\mu + n + \mu$ or $n + \mu(1 + \phi)$. Thus the economy does converge to a balanced growth path where all the variables of the model are growing at constant rates.

(c) On the balanced growth path, $\dot{\bar{j}}(t) = 0$ and thus from equation (12):

$$s\bar{j}^{\alpha} = [n + \delta + \mu(1+\phi)]\bar{j} \quad \Rightarrow \quad \bar{j}^{1-\alpha} = s/[n + \delta + \mu(1+\phi)],$$

and thus

(13) $\bar{j}^{*} = [s/(n + \delta + \mu(1+\phi))]^{1/(1-\alpha)}$.

Substitute equation (13) into equation (9) to get an expression for output per unit of $A(t)^{\phi}L(t)$ on the balanced growth path:

(14) $y^{*} = [s/(n + \delta + \mu(1+\phi))]^{\alpha/(1-\alpha)}$.

Take the derivative of y^* with respect to s:

$$\frac{\partial y^*}{\partial s} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right].$$

In order to turn this into an elasticity, multiply both sides by s/y^* using the expression for y^* from equation (14) on the right-hand side:

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{\alpha/(1-\alpha)-1}\left[\frac{1}{n+\delta+\mu(1+\phi)}\right]s\left[\frac{s}{n+\delta+\mu(1+\phi)}\right]^{-\alpha/(1-\alpha)}.$$

Simplifying yields

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \left[\frac{\alpha}{1-\alpha}\right]\left[\frac{n+\delta+\mu(1+\phi)}{s}\right]\left[\frac{s}{n+\delta+\mu(1+\phi)}\right],$$

and thus finally

(15) $\dfrac{\partial y^*}{\partial s}\dfrac{s}{y^*} = \dfrac{\alpha}{1-\alpha}$.

(d) A first-order Taylor approximation of $\dot{y}$ around the balanced-growth-path value of $y = y^*$ will be of the form

(16) $\dot{y} \cong \partial\dot{y}/\partial y\big|_{y=y^*}[y - y^*]$.

Taking the time derivative of both sides of equation (9) yields

(17) $\dot{y} = \alpha\bar{j}^{\alpha-1}\dot{\bar{j}}$.

Substitute equation (12) into equation (17):

$$\dot{y} = \alpha\bar{j}^{\alpha-1}\left[s\bar{j}^{\alpha} - (n+\delta+\mu(1+\phi))\bar{j}\right],$$

or

(18) $\dot{y} = s\alpha\bar{j}^{2\alpha-1} - \alpha\bar{j}^{\alpha}[n+\delta+\mu(1+\phi)]$.

Equation (18) expresses $\dot{y}$ in terms of $\bar{j}$. We can express $\bar{j}$ in terms of y: since $y = \bar{j}^{\alpha}$, we can write $\bar{j} = y^{1/\alpha}$. Thus $\partial\dot{y}/\partial y$ evaluated at $y = y^*$ is given by

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = \left[\left.\frac{\partial\dot{y}}{\partial\bar{j}}\right|_{y=y^*}\right]\left[\left.\frac{\partial\bar{j}}{\partial y}\right|_{y=y^*}\right] = \left[s\alpha(2\alpha-1)\bar{j}^{2(\alpha-1)} - \alpha^2\bar{j}^{\alpha-1}(n+\delta+\mu(1+\phi))\right]\left[\frac{1}{\alpha}y^{(1-\alpha)/\alpha}\right].$$

Now, $y^{(1-\alpha)/\alpha}$ is simply $\bar{j}^{1-\alpha}$ since $y = \bar{j}^{\alpha}$ and thus

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{y=y^*} = s(2\alpha-1)\bar{j}^{2(\alpha-1)+(1-\alpha)} - \alpha\bar{j}^{\alpha-1+(1-\alpha)}[n+\delta+\mu(1+\phi)] = s(2\alpha-1)\bar{j}^{\alpha-1} - \alpha[n+\delta+\mu(1+\phi)].$$

Finally, substitute out for s by rearranging equation (13) to obtain $s = \bar{j}^{1-\alpha}[n + \delta + \mu(1+\phi)]$ and thus

$$\left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = \bar{j}^{1-\alpha}[n+\delta+\mu(1+\phi)](2\alpha-1)\bar{j}^{\alpha-1} - \alpha[n+\delta+\mu(1+\phi)],$$

or simply

$$(19)\quad \left.\frac{\partial \dot{y}}{\partial y}\right|_{y=y^*} = -(1-\alpha)[n+\delta+\mu(1+\phi)].$$

Substituting equation (19) into equation (16) gives the first-order Taylor expansion:

$$(20)\quad \dot{y} \cong -(1-\alpha)[n+\delta+\mu(1+\phi)]\,[y-y^*].$$

Solving this differential equation (as in the text) yields

$$(21)\quad y(t) - y^* = e^{-(1-\alpha)[n+\delta+\mu(1+\phi)]}\,[y(0)-y^*].$$

This means that the economy moves fraction $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$ of the remaining distance toward y^* each year.

(e) The elasticity of output with respect to s is the same in this model as in the basic Solow model. The speed of convergence is faster in this model. In the basic Solow model, the rate of convergence is given by $(1 - \alpha)[n + \delta + \mu]$, which is less than the rate of convergence in this model, $(1 - \alpha)[n + \delta + \mu(1 + \phi)]$, since $\phi \equiv \alpha/(1 - \alpha)$ is positive.

Problem 1.12

(a) The growth-accounting technique of Section 1.7 yields the following expression for the growth rate of output per person:

$$(1)\quad \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t),$$

where $\alpha_K(t)$ is the elasticity of output with respect to capital at time t and R(t) is the Solow residual. Now imagine applying this growth-accounting equation to a Solow economy that is on its balanced growth path. On the balanced growth path, the growth rates of output per worker and capital per worker are both equal to g, the growth rate of A. Thus equation (1) implies that growth accounting would attribute a fraction α_K of growth in output per worker to growth in capital per worker. It would attribute the rest -- fraction $1 - \alpha_K$ -- to technological progress, as this is what would be left in the Solow residual. So with our usual estimate of $\alpha_K = 1/3$, growth accounting would attribute about 67% of the growth in output per worker to technological progress and about 33% of the growth in output per worker to growth in capital per worker.

(b) In an accounting sense, the result in part (a) would be true, but in a deeper sense it would not: the reason that the capital-labor ratio grows at rate g on the balanced growth path is because the effectiveness of labor is growing at rate g. That is, the growth in the effectiveness of labor -- the growth in A -- raises output per worker through two channels. First, by directly raising output but also by (for a given saving rate) increasing the resources devoted to capital accumulation and thereby raising the capital-labor ratio. Growth accounting attributes the rise in output per worker through the second channel to growth in the capital-labor ratio, and not to its underlying source. Thus, although growth accounting is often instructive, it is not appropriate to interpret it as shedding light on the underlying determinants of growth.

Problem 1.13

(a) Ordinary least squares (OLS) yields a biased estimate of the slope coefficient of a regression if the explanatory variable is correlated with the error term. We are given that

(1) $\ln\left[\left(Y/N\right)_{1979}\right] - \ln\left[\left(Y/N\right)_{1870}\right]^{*} = a + b\ln\left[\left(Y/N\right)_{1870}\right]^{*} + \varepsilon$, and

(2) $\ln\left[\left(Y/N\right)_{1870}\right] = \ln\left[\left(Y/N\right)_{1870}\right]^{*} + u$,

where ε and u are assumed to be uncorrelated with each other and with the true unobservable 1870 income per person variable, $\ln[(Y/N)_{1870}]^{*}$.

Substituting equation (2) into (1) and rearranging yields

(3) $\ln\left[\left(Y/N\right)_{1979}\right] - \ln\left[\left(Y/N\right)_{1870}\right] = a + b\ln\left[\left(Y/N\right)_{1870}\right] + \left[\varepsilon - (1+b)u\right]$.

Running an OLS regression on model (3) will yield a biased estimate of b if $\ln[(Y/N)_{1870}]$ is correlated with the error term, $[\varepsilon - (1 + b)u]$. In general, of course, this will be the case since u is the measurement error that helps to determine the value of $\ln[(Y/N)_{1870}]$ that we get to observe. However, in the special case in which the true value of b = -1, the error term in model (3) is simply ε. Thus OLS will be unbiased since the explanatory variable will no longer be correlated with the error term.

(b) Measurement error in the dependent variable will not cause a problem for OLS estimation and is, in fact, one of the justifications for the disturbance term in a regression model. Intuitively, if the measurement error is in 1870 income per capita, the explanatory variable, there will be a bias toward finding convergence. If 1870 income per capita is overstated, growth is understated. This looks like convergence: a "high" initial income country growing slowly. Similarly, if 1870 income per capita is understated, growth is overstated. This also looks like convergence: a "low" initial income country growing quickly.

Suppose instead that it is only 1979 income per capita that is subject to random, mean-zero measurement error. When 1979 income is overstated, so is growth for a given level of 1870 income. When 1979 income is understated, so is growth for a given 1870 income. Either case is equally likely: overstating 1979 income for any given 1870 income is just as likely as understating it (or more precisely, measurement error is on average equal to zero). Thus there is no reason for this to systematically cause us to see more or less convergence than there really is in the data.

Problem 1.14

What is needed for a balanced growth path is that K and Y are each growing at a constant rate. The equation of motion for capital, $\dot{K}(t) = sY(t) - \delta K(t)$, implies the growth rate of K is

(1) $\dfrac{\dot{K}(t)}{K(t)} = s\dfrac{Y(t)}{K(t)} - \delta$.

As in the model in the text, Y/K must be constant in order for the growth rate of K to be constant. That is, the growth rates of Y and K must be equal.

Taking logs of both sides of the production function, $Y(t) = K(t)^{\alpha}R(t)^{\beta}T(t)^{\gamma}\left[A(t)L(t)\right]^{1-\alpha-\beta-\gamma}$, yields

(2) $\ln Y(t) = \alpha\ln K(t) + \beta\ln R(t) + \gamma\ln T(t) + (1 - \alpha - \beta - \gamma)[\ln A(t) + \ln L(t)]$.

Differentiating both sides of (2) with respect to time gives us

(3) $g_Y(t) = \alpha g_K(t) + \beta g_R(t) + \gamma g_T(t) + (1 - \alpha - \beta - \gamma)\left[g_A(t) + g_L(t)\right]$.

Substituting in the facts that the growth rates of R, T, and L are all equal to n and the growth rate of A is equal to g gives us

$$g_Y(t) = \alpha g_K(t) + \beta n + \gamma n + (1 - \alpha - \beta - \gamma)(n + g).$$

Simplifying gives us

$$(4)\quad \begin{aligned} g_Y(t) &= \alpha g_K(t) + (\beta + \gamma)n + (1 - \alpha)n - (\beta + \gamma)n + (1 - \alpha - \beta - \gamma)g \\ &= \alpha g_K(t) + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g \end{aligned}$$

Using the fact that g_Y and g_K must be equal on a balanced growth path leaves us with

$$g_Y = \alpha g_Y + (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g,$$

$$(1 - \alpha)g_Y = (1 - \alpha)n + (1 - \alpha - \beta - \gamma)g,$$

and thus the growth rate of output on the balanced growth path is given by

$$(5)\quad \tilde{g}_Y^{bgp} = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$$

The growth rate of output per worker on the balanced growth path is

$$\tilde{g}_{Y/L}^{bgp} = \tilde{g}_Y^{bgp} - \tilde{g}_L^{bgp}.$$

Using equation (5) and the fact that L grows at rate n, we can write

$$\tilde{g}_{Y/L}^{bgp} = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g}{1 - \alpha} - n = \frac{(1 - \alpha)n + (1 - \alpha - \beta - \gamma)g - (1 - \alpha)n}{1 - \alpha}.$$

And thus finally

$$(6)\quad \tilde{g}_{Y/L}^{bgp} = \frac{(1 - \alpha - \beta - \gamma)g}{1 - \alpha}.$$

Equation (6) is identical to equation (1.50) in the text.

SOLUTIONS TO CHAPTER 2

Problem 2.1

(a) The firm's problem is to choose the quantities of capital, K, and effective labor, AL, in order to minimize costs, wAL + rK, subject to the production function, Y = ALf(k). Set up the Lagrangian:

$$\mathcal{L} = wAL + rK + \lambda\left[Y - ALf(K/AL)\right].$$

The first-order conditions are given by

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda\left[ALf'(K/AL)(1/AL)\right] = 0 \quad \Rightarrow \quad r = \lambda f'(k), \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial (AL)} = w - \lambda\left[f(K/AL) + ALf'(K/AL)(-K)/(AL)^2\right] = 0 \quad \Rightarrow \quad w = \lambda\left[f(k) - kf'(k)\right]. \quad (2)$$

Dividing equation (1) by equation (2) gives us

$$(3) \quad \frac{r}{w} = \frac{f'(k)}{f(k) - kf'(k)}.$$

Equation (3) implicitly defines the cost-minimizing choice of k. Clearly this choice does not depend upon the level of output, Y. Note that equation (3) is the standard cost-minimizing condition: the ratio of the marginal cost of the two inputs, capital and effective labor, must equal the ratio of the marginal products of the two inputs.

(b) Since, as shown in part (a), each firm chooses the same value of k and since we are told that each firm has the same value of A, we can write the total amount produced by the N cost-minimizing firms as

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} AL_i f(k) = Af(k)\sum_{i=1}^{n} L_i = A\overline{L}f(k),$$

where $\overline{L}$ is the total amount of labor employed.

The single firm also has the same value of A and would choose the same value of k; the choice of k does not depend on Y. Thus if it used all of the labor employed by the N cost-minimizing firms, $\overline{L}$, the single firm would produce $Y = A\overline{L}f(k)$. This is exactly the same amount of output produced in total by the N cost-minimizing firms.

Problem 2.2

(a) The individual's problem is to maximize lifetime utility given by

$$(1) \quad U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta},$$

subject to the lifetime budget constraint given by

(2) $P_1C_1 + P_2C_2 = W$,

where W represents lifetime income.

Rearrange the budget constraint to solve for C_2:

(3) $C_2 = W/P_2 - C_1P_1/P_2$.

Substitute equation (3) into equation (1):

$$(4) \quad U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{\left[W/P_2 - C_1P_1/P_2\right]^{1-\theta}}{1-\theta}.$$

Now we can solve the unconstrained problem of maximizing utility, as given by equation (4), with respect to first period consumption, C_1. The first-order condition is given by

$$\partial U/\partial C_1 = C_1^{-\theta} + (1/1+\rho)C_2^{-\theta}(-P_1/P_2) = 0 \quad \Rightarrow \quad C_1^{-\theta} = (1/1+\rho)(P_1/P_2)C_2^{-\theta},$$

or simply

(5) $C_1 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2$.

In order to solve for C_2, substitute equation (5) into equation (3):

$$C_2 = W/P_2 - (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}C_2(P_1/P_2) \quad \Rightarrow \quad C_2\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right] = W/P_2,$$

or simply

(6) $C_2 = \dfrac{W/P_2}{\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}$.

Finally, to get the optimal choice of C_1, substitute equation (6) into equation (5):

(7) $C_1 = \dfrac{(1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}(W/P_2)}{\left[1+(1+\rho)^{1/\theta}(P_2/P_1)^{(1-\theta)/\theta}\right]}$.

(b) From equation (5), the optimal ratio of first-period to second-period consumption is

(8) $C_1/C_2 = (1+\rho)^{1/\theta}(P_2/P_1)^{1/\theta}$.

Taking the log of both sides of equation (8) yields

(9) $\ln(C_1/C_2) = (1/\theta)\ln(1+\rho) + (1/\theta)\ln(P_2/P_1)$.

The elasticity of substitution between C_1 and C_2, defined in such a way that it is positive, is given by

$$\frac{\partial(C_1/C_2)}{\partial(P_2/P_1)}\frac{(P_2/P_1)}{(C_1/C_2)} = \frac{\partial\left[\ln(C_1/C_2)\right]}{\partial\left[\ln(P_2/P_1)\right]} = \frac{1}{\theta},$$

where we have used equation (9) to find the derivative. Thus higher values of θ imply that the individual is less willing to substitute consumption between periods.

Problem 2.3

(a) We can use analysis similar to the intuitive derivation of the Euler equation in Section 2.2 of the text. Think of the household's consumption at two moments of time. Specifically, consider a short (formally infinitesimal) period of time Δt from $(t_0 - \varepsilon)$ to $(t_0 + \varepsilon)$.

Imagine the household reducing consumption per unit of effective labor, c, at $(t_0 - \varepsilon)$ -- an instant before the confiscation of wealth -- by a small (again, infinitesimal) amount Δc. It then invests this additional saving and consumes the proceeds at $(t_0 + \varepsilon)$. If the household is optimizing, the marginal impact of this change on lifetime utility must be zero.

This experiment would have a utility cost of $u'(c_{before})\Delta c$. Ordinarily, since the instantaneous rate of return is r(t), c at time $(t_0 + \varepsilon)$ could be increased by $e^{[r(t)-n-g]\Delta t}\Delta c$. But here, half of that increase will be confiscated. Thus the utility benefit would be $[1/2]u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c$. Thus for the path of consumption to be utility-maximizing, it must satisfy

(1) $u'(c_{before})\Delta c = \dfrac{1}{2}u'(c_{after})e^{[r(t)-n-g]\Delta t}\Delta c$.

Rather informally, we can cancel the Δc's and allow $\Delta t \to 0$, leaving us with

(2) $u'(c_{before}) = \frac{1}{2}u'(c_{after})$.

Thus there will be a discontinuous jump in consumption at the time of the confiscation of wealth. Specifically, consumption will jump down. Intuitively, the household's consumption will be high before t_0 because it will have an incentive not to save so as to avoid the wealth confiscation.

(b) In this case, from the viewpoint of an individual household, its actions will not affect the amount of wealth that is confiscated. For an individual household, essentially a predetermined amount of wealth will be confiscated at time t_0 and thus the household's optimization and its choice of consumption path would take this into account. The household would still prefer to smooth consumption over time and there will not be a discontinuous jump in consumption at time t_0.

Problem 2.4

We need to solve the household's problem assuming log utility and in per capita terms rather than in units of effective labor. The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt$,

subject to

(2) $\int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt$.

Now let $W = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt$.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt + \lambda \left[W - \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \right].$$

The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-1} \frac{L(t)}{H} - \lambda e^{-R(t)} \frac{L(t)}{H} = 0.$$

Canceling the $L(t)/H$ yields

(3) $e^{-\rho t} C(t)^{-1} = \lambda e^{-R(t)}$,

which implies

(4) $C(t) = e^{-\rho t} \lambda^{-1} e^{R(t)}$.

Substituting this into the budget constraint, equation (2), gives us

(5) $\int_{t=0}^{\infty} e^{-R(t)} \left[e^{-\rho t} \lambda^{-1} e^{R(t)} \right] \frac{L(t)}{H} dt = W$.

Since $L(t) = e^{nt} L(0)$, this implies

(6) $\lambda^{-1} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-(\rho - n)t} dt = W$.

As long as $\rho - n > 0$ (which it must be), the integral is equal to $1/(\rho - n)$ and thus λ^{-1} is given by

(7) $\lambda^{-1} = \dfrac{W}{L(0)/H}(\rho - n)$.

Substituting equation (7) into equation (4) yields

(8) $C(t) = e^{R(t)-\rho t}\left[\dfrac{W}{L(0)/H}(\rho - n)\right]$.

Initial consumption is therefore

(9) $C(0) = \dfrac{W}{L(0)/H}(\rho - n)$.

Note that C(0) is consumption per person, W is wealth per household and L(0)/H is the number of people per household. Thus W/[L(0)/H] is wealth per person. This equation says that initial consumption per person is a constant fraction of initial wealth per person, and (ρ - n) can be interpreted as the marginal propensity to consume out of wealth. With logarithmic utility, this propensity to consume is independent of the path of the real interest rate. Also note that the bigger is the household's discount rate ρ -- the more the household discounts the future -- the bigger is the fraction of wealth that it initially consumes.

Problem 2.5

The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize

(1) $U = \int_{t=0}^{\infty} e^{-\rho t}\dfrac{C(t)^{1-\theta}}{1-\theta}\dfrac{L(t)}{H}dt$,

subject to

(2) $\int_{t=0}^{\infty} e^{-rt}C(t)\dfrac{L(t)}{H}dt = W$,

where W denotes the household's initial wealth plus the present value of its lifetime labor income, i.e. the right-hand side of equation (2.6) in the text. Note that the real interest rate, r, is assumed to be constant.

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt + \lambda\left[W - \int_{t=0}^{\infty} e^{-rt}C(t)\frac{L(t)}{H}dt\right].$$

The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t}C(t)^{-\theta}\frac{L(t)}{H} - \lambda e^{-rt}\frac{L(t)}{H} = 0.$$

Canceling the L(t)/H yields

(3) $e^{-\rho t}C(t)^{-\theta} = \lambda e^{-rt}$.

Differentiate both sides of equation (3) with respect to time:

$$e^{-\rho t}\left[-\theta C(t)^{-\theta-1}\dot{C}(t)\right] - \rho e^{-\rho t}C(t)^{-\theta} + r\lambda e^{-rt} = 0.$$

This can be rearranged to obtain

(4) $-\theta\dfrac{\dot{C}(t)}{C(t)}e^{-\rho t}C(t)^{-\theta} - \rho e^{-\rho t}C(t)^{-\theta} + r\lambda e^{-rt} = 0$.

Now substitute the first-order condition, equation (3), into equation (4):

$$-\theta\frac{\dot{C}(t)}{C(t)}\lambda e^{-rt} - \rho\lambda e^{-rt} + r\lambda e^{-rt} = 0.$$

Canceling the λe^{-rt} and solving for the growth rate of consumption, $\dot{C}(t)/C(t)$, yields

(5) $$\frac{\dot{C}(t)}{C(t)} = \frac{r-\rho}{\theta}.$$

Thus with a constant real interest rate, the growth rate of consumption is a constant. If $r > \rho$ -- that is, if the rate that the market pays to defer consumption exceeds the household's discount rate -- consumption will be rising over time. The value of θ determines the magnitude of consumption growth if r exceeds ρ. A lower value of θ -- and thus a higher value of the elasticity of substitution, $1/\theta$ -- means that consumption growth will be higher for any given difference between r and ρ.

We now need to solve for the path of $C(t)$. First, note that equation (5) can be rewritten as

(6) $$\frac{\partial \ln C(t)}{\partial t} = \frac{r-\rho}{\theta}.$$

Integrate equation (6) forward from time $\tau = 0$ to time $\tau = t$:

$$\ln C(t) - \ln C(0) = \left[(r-\rho)/\theta\right]\tau\Big|_{\tau=0}^{\tau=t},$$

which simplifies to

(7) $$\ln\left[C(t)/C(0)\right] = \left[(r-\rho)/\theta\right]t.$$

Taking the exponential function of both sides of equation (7) yields

$$C(t)/C(0) = e^{\left[(r-\rho)/\theta\right]t},$$

and thus

(8) $$C(t) = C(0)e^{\left[(r-\rho)/\theta\right]t}.$$

We can now solve for initial consumption, $C(0)$, by using the fact that it must be chosen to satisfy the household's budget constraint. Substitute equation (8) into equation (2):

$$\int_{t=0}^{\infty} e^{-rt}C(0)e^{\left[(r-\rho)/\theta\right]t}\frac{L(t)}{H}dt = W.$$

Using the fact that $L(t) = L(0)e^{nt}$ yields

(9) $$\frac{C(0)L(0)}{H}\int_{t=0}^{\infty} e^{-\left[\rho - r + \theta(r-n)\right]t/\theta}dt = W.$$

As long as $[\rho - r + \theta(r - n)]/\theta > 0$, we can solve the integral:

(10) $$\int_{t=0}^{\infty} e^{-\left[\rho - r + \theta(r-n)\right]t/\theta}dt = \frac{\theta}{\rho - r + \theta(r-n)}.$$

Substitute equation (10) into equation (9) and solve for $C(0)$:

(11) $$C(0) = \frac{W}{L(0)/H}\left[\frac{(\rho - r)}{\theta} + (r-n)\right].$$

Finally, to get an expression for consumption at each instant in time, substitute equation (11) into equation (8):

(12) $$C(t) = e^{\left[(r-\rho)/\theta\right]t}\frac{W}{L(0)/H}\left[\frac{(\rho - r)}{\theta} + (r-n)\right].$$

Problem 2.6

(a) The equation describing the dynamics of the capital stock per unit of effective labor is

(1) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$.

For a given k, the level of c that implies $\dot{k} = 0$ is given by c = f(k) - (n + g)k. Thus a fall in g makes the level of c consistent with $\dot{k} = 0$ higher for a given k. That is, the $\dot{k} = 0$ curve shifts up. Intuitively, a lower g makes break-even investment lower at any given k and thus allows for more resources to be devoted to consumption and still maintain a given k. Since (n + g)k falls proportionately more at higher levels of k, the $\dot{k} = 0$ curve shifts up more at higher levels of k. See the figure.

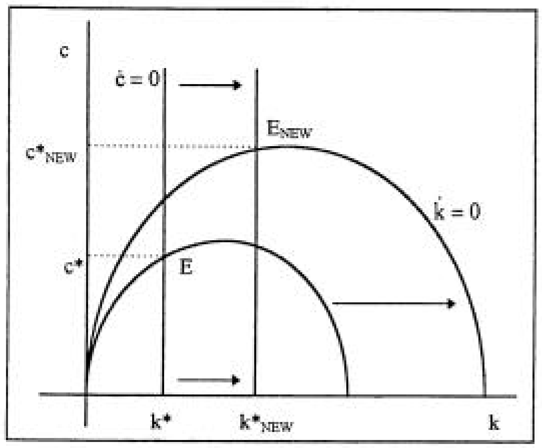
(b) The equation describing the dynamics of consumption per unit of effective labor is given by

(2) $$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}.$$

Thus the condition required for $\dot{c} = 0$ is given by f ' (k) = ρ + θg. After the fall in g, f ' (k) must be lower in order for $\dot{c} = 0$. Since f '' (k) is negative this means that the k needed for $\dot{c} = 0$ therefore rises. Thus the $\dot{c} = 0$ curve shifts to the right.

(c) At the time of the change in g, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.

In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time of the shock. In order for the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.



However, we cannot tell whether the new saddle path passes above or below the original point E. Thus we cannot tell whether c jumps up or down and in fact, if the new saddle path passes right through point E, c might even remain the same at the instant that g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

(d) On a balanced growth path, the fraction of output that is saved and invested is given by [f(k*) - c*]/f(k*). Since k is constant, or $\dot{k} = 0$ on a balanced growth path then, from equation (1), we can write f(k*) - c* = (n + g)k*. Using this, we can rewrite the fraction of output that is saved on a balanced growth path as

(3) s = [(n + g)k*]/f(k*).

Differentiating both sides of equation (3) with respect to g yields

(4) $$\frac{\partial s}{\partial g} = \frac{f(k^*)\left[(n+g)(\partial k^*/\partial g) + k^*\right] - (n+g)k^* f'(k^*)(\partial k^*/\partial g)}{\left[f(k^*)\right]^2},$$

which simplifies to

(5) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[f(k^*) - k^* f'(k^*)](\partial k^*/\partial g) + f(k^*)k^*}{[f(k^*)]^2}$.

Since k* is defined by $f'(k^*) = \rho + \theta g$, differentiating both sides of this expression gives us $f''(k^*)(\partial k^*/\partial g) = \theta$. Solving for $\partial k^*/\partial g$ gives us

(6) $\partial k^*/\partial g = \theta / f''(k^*) < 0$.

Substituting equation (6) into equation (5) yields

(7) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[f(k^*) - k^* f'(k^*)]\theta + f(k^*)k^* f''(k^*)}{[f(k^*)]^2 f''(k^*)}$.

The first term in the numerator is positive, whereas the second is negative and so the sign of $\partial s/\partial g$ is ambiguous. Thus we cannot tell whether the fall in g raises or lowers the saving rate on the new balanced growth path.

(e) When the production function is Cobb-Douglas, $f(k) = k^{\alpha}$, $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha - 1)k^{\alpha-2}$. Substituting these facts into equation (7) yields

(8) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)[k^{*\alpha} - k^* \alpha k^{*\alpha-1}]\theta + k^{*\alpha} k^* \alpha(\alpha-1)k^{*\alpha-2}}{k^{*\alpha} k^{*\alpha} \alpha(\alpha-1)k^{*\alpha-2}}$,

which simplifies to

(9) $\dfrac{\partial s}{\partial g} = \dfrac{(n+g)k^{*\alpha}(1-\alpha)\theta - (1-\alpha)k^{*\alpha} \alpha k^{*\alpha-1}}{[-(1-\alpha)k^{*\alpha}(\alpha k^{*\alpha-1})(\alpha k^{*\alpha-1})/\alpha]}$,

which implies

(10) $\dfrac{\partial s}{\partial g} = -\alpha \dfrac{[(n+g)\theta - (\rho + \theta g)]}{(\rho + \theta g)^2}$.

Thus, finally, we have

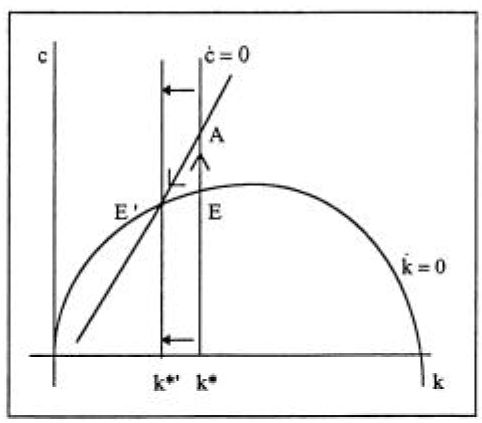
(11) $\dfrac{\partial s}{\partial g} = -\alpha \dfrac{(n\theta - \rho)}{(\rho + \theta g)^2} = \alpha \dfrac{(\rho - n\theta)}{(\rho + \theta g)^2}$.

Problem 2.7

The two equations of motion are

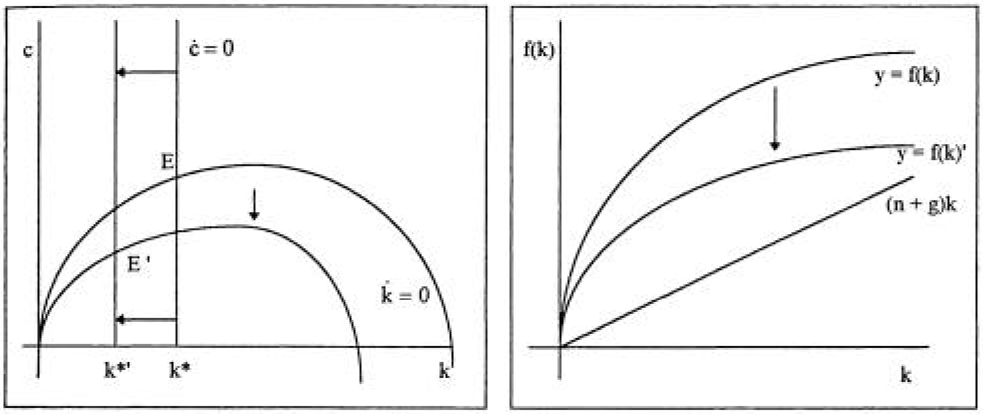
(1) $\dfrac{\dot{c}(t)}{c(t)} = \dfrac{f'(k(t)) - \rho - \theta g}{\theta}$, and (2) $\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$.

(a) A rise in θ or a fall in the elasticity of substitution, 1/θ, means that households become less willing to substitute consumption between periods. It also means that the marginal utility of consumption falls off more rapidly as consumption rises. If the economy is growing, this tends to make households value present consumption more than future consumption.



The capital-accumulation equation is unaffected. The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. Since $f''(k) < 0$, the $f'(k)$ that makes $\dot{c} = 0$ is now higher. Thus the value of k that satisfies $\dot{c} = 0$ is lower. The $\dot{c} = 0$ locus shifts to the left. The economy moves up to point A on the new saddle path; people consume more now. Movement is then down along the new saddle path until the economy reaches point E '. At that point, c^* and k^* are lower than their original values.

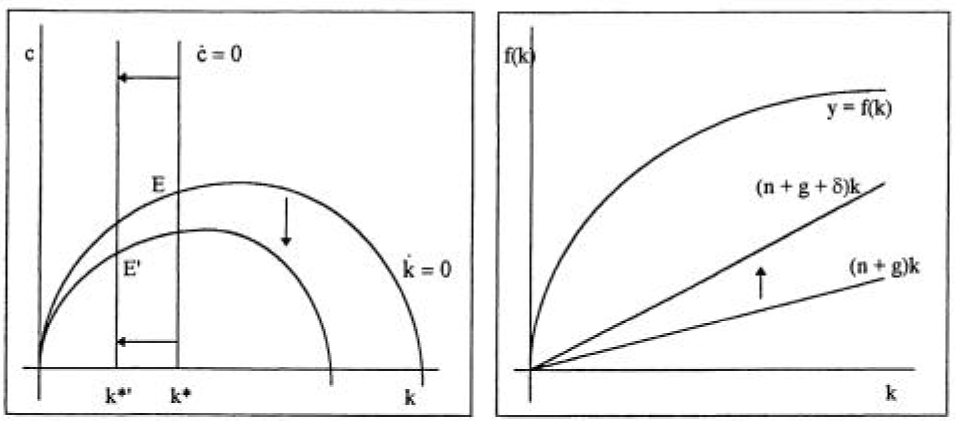
(b) We can assume that a downward shift of the production function means that for any given k, both f(k) and f ' (k) are lower than before.



The condition required for $\dot{k} = 0$ is given by $c = f(k) - (n+g)k$. We can see from the figure on the right that the $\dot{k} = 0$ locus will shift down more at higher levels of k. Also, since for a given k, f ' (k) is lower now, the golden-rule k will be lower than before. Thus the $\dot{k} = 0$ locus shifts as depicted in the figure.

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. For a given k, f ' (k) is now lower. Thus we need a lower k to keep f ' (k) the same and satisfy the $\dot{c} = 0$ equation. Thus the $\dot{c} = 0$ locus shifts left. The economy will eventually reach point E ' with lower c^* and lower k^*. Whether c initially jumps up or down depends upon whether the new saddle path passes above or below point E.

(c) With a positive rate of depreciation, $\delta > 0$, the new capital-accumulation equation is
(3) $\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$.



The level of saving and investment required just to keep any given k constant is now higher -- and thus the amount of consumption possible is now lower -- than in the case with no depreciation. The level of extra investment required is also higher at higher levels of k. Thus the $\dot{k} = 0$ locus shifts down more at higher levels of k.

In addition, the real return on capital is now $f'(k(t)) - \delta$ and so the household's maximization will yield

$$(4)\quad \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta}.$$

The condition required for $\dot{c} = 0$ is $f'(k) = \delta + \rho + \theta g$. Compared to the case with no depreciation, $f'(k)$ must be higher and k lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left. The economy will eventually wind up at point E ' with lower levels of c* and k*. Again, whether c jumps up or down initially depends upon whether the new saddle path passes above or below the original equilibrium point of E.

Problem 2.7

With a positive depreciation rate, $\delta > 0$, the Euler equation and the capital-accumulation equation are given by

$$(1)\quad \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta}, \qquad \text{and} \qquad (2)\quad \dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t).$$

We begin by taking first-order Taylor approximations to (1) and (2) around k = k* and c = c*. That is, we can write

$$(3)\quad \dot{c} \cong \frac{\partial \dot{c}}{\partial k}[k - k^*] + \frac{\partial \dot{c}}{\partial c}[c - c^*], \qquad \text{and} \qquad (4)\quad \dot{k} \cong \frac{\partial \dot{k}}{\partial k}[k - k^*] + \frac{\partial \dot{k}}{\partial c}[c - c^*],$$

where $\partial\dot{c}/\partial k$, $\partial\dot{c}/\partial c$, $\partial\dot{k}/\partial k$ and $\partial\dot{k}/\partial c$ are all evaluated at k = k* and c = c*.

Define $\tilde{c} \equiv c - c^*$ and $\tilde{k} \equiv k - k^*$. Since c* and k* are constants, $\dot{c}$ and $\dot{k}$ are equivalent to $\dot{\tilde{c}}$ and $\dot{\tilde{k}}$ respectively. We can therefore rewrite (3) and (4) as

$$(5)\quad \dot{\tilde{c}} \cong \frac{\partial \dot{c}}{\partial k}\tilde{k} + \frac{\partial \dot{c}}{\partial c}\tilde{c}, \qquad \text{and} \qquad (6)\quad \dot{\tilde{k}} \cong \frac{\partial \dot{k}}{\partial k}\tilde{k} + \frac{\partial \dot{k}}{\partial c}\tilde{c}.$$

Using equations (1) and (2) to compute these derivatives yields

$$(7)\quad \left.\frac{\partial \dot{c}}{\partial k}\right|_{bgp} = \frac{f''(k^*)c^*}{\theta}, \qquad (8)\quad \left.\frac{\partial \dot{c}}{\partial c}\right|_{bgp} = \frac{f'(k^*) - \delta - \rho - \theta g}{\theta} = 0,$$

$$(9)\quad \left.\frac{\partial \dot{k}}{\partial k}\right|_{bgp} = f'(k^*) - (n + g + \delta), \qquad (10)\quad \left.\frac{\partial \dot{k}}{\partial c}\right|_{bgp} = -1.$$

Substituting equations (7) and (8) into (5) and equations (9) and (10) into (6) yields

$$(11)\quad \dot{\tilde{c}} \cong \frac{f''(k^*)c^*}{\theta}\tilde{k}, \text{ and}$$

$$\begin{aligned}(12)\quad \dot{\tilde{k}} &\cong [f'(k^*) - (n + g + \delta)]\,\tilde{k} - \tilde{c} \\ &\cong [(\delta + \rho + \theta g) - (n + g + \delta)]\,\tilde{k} - \tilde{c} \\ &\cong \beta\tilde{k} - \tilde{c}.\end{aligned}$$

The second line of equation (12) uses the fact that (1) implies that $f'(k^*) = \delta + \rho + \theta g$. The third line uses the definition of $\beta = \rho - n - (1 - \theta)g$.

Dividing equation (11) by $\tilde{c}$ and dividing equation (12) by $\tilde{k}$ yields

(13) $\frac{\dot{\tilde{c}}}{\tilde{c}} \cong \frac{f''(k^*)c^*}{\theta}\frac{\tilde{k}}{\tilde{c}}$, and (14) $\frac{\dot{\tilde{k}}}{\tilde{k}} \cong \beta - \frac{\tilde{c}}{\tilde{k}}$.

Note that these are exactly the same as equations (2.31) and (2.32) in the text; adding a positive depreciation rate does not alter the expressions for the growth rates of $\tilde{c}$ and $\tilde{k}$. Thus equation (2.36), the expression for μ, the constant growth rate of both $\tilde{c}$ and $\tilde{k}$ as the economy moves toward the balanced growth path, is still valid. Thus

(15) $\mu_1 = \frac{\beta - \sqrt{\beta^2 - 4f''(k^*)c^*/\theta}}{2}$,

where we have chosen the negative growth rate so that c and k are moving toward c^* and k^*, not away from them.

Now consider the Cobb-Douglas production function, $f(k) = k^\alpha$. Thus

(16) $f'(k^*) = \alpha k^{*\alpha-1} = r^* + \delta$, and (17) $f''(k^*) = \alpha(\alpha-1)k^{*\alpha-2}$.

Squaring both sides of equation (16) gives us

(18) $(r^* + \delta)^2 = \alpha^2 k^{*2\alpha-2}$,

and so equation (17) can be rewritten as

(19) $f''(k^*) = \frac{(r^*+\delta)^2(\alpha-1)}{\alpha k^{*\alpha}} = \frac{\alpha-1}{\alpha}\frac{(r^*+\delta)^2}{f(k^*)}$.

In addition, defining s^* to be the saving rate on the balanced growth path, we can write the balanced-growth-path level of consumption as

(20) $c^* = (1 - s^*)f(k^*)$.

Substituting equations (19) and (20) into (15) yields

$$\mu_1 = \frac{\beta - \sqrt{\beta^2 - 4\left(\frac{\alpha-1}{\alpha}\right)\frac{(r^*+\delta)^2}{f(k^*)\theta}(1-s^*)f(k^*)}}{2}.$$

Canceling the $f(k^*)$ and multiplying through by the minus sign yields

(21) $\mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(r^*+\delta)^2(1-s^*)}}{2}$.

On the balanced growth path, the condition required for $\dot{c} = 0$ is given by $r^* = \rho + \theta g$ and thus

(22) $r^* + \delta = \rho + \theta g + \delta$.

In addition, actual saving, $s^*f(k^*)$, equals break-even investment, $(n + g + \delta)k^*$, and thus

(23) $s^* = \frac{(n+g+\delta)k^*}{f(k^*)} = \frac{(n+g+\delta)}{k^{*\alpha-1}} = \frac{\alpha(n+g+\delta)}{(r^*+\delta)}$,

where we have used equation (16), $r^* + \delta = \alpha k^{*\alpha-1}$. From equation (23), we can write

(24) $(1-s^*) = \frac{(r^*+\delta) - \alpha(n+g+\delta)}{(r^*+\delta)}$.

Substituting equations (22) and (24) into equation (21) yields

$$(25)\quad \mu_1 = \frac{\beta - \sqrt{\beta^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(\rho+\theta g+\delta)\left[\rho+\theta g+\delta-\alpha(n+g+\delta)\right]}}{2}.$$

Equation (25) is analogous to equation (2.38) in the text. It expresses the rate of adjustment in terms of the underlying parameters of the model. Keeping the values in the text -- $\alpha = 1/3$, $\rho = 4\%$, $n = 2\%$, $g = 1\%$ and $\theta = 1$ -- and using $\delta = 3\%$ yields a value for μ_1 of approximately - 8.8%. This is faster convergence than the -5.4% obtained with no depreciation.

Problem 2.9

(a) The real after-tax rate of return on capital is now given by $(1 - \tau)f'(k(t))$. Thus the household's maximization would now yield the following expression describing the dynamics of consumption per unit of effective labor:

$$(1)\quad \frac{\dot{c}(t)}{c(t)} = \frac{\left[(1-\tau)f'(k(t)) - \rho - \theta g\right]}{\theta}.$$

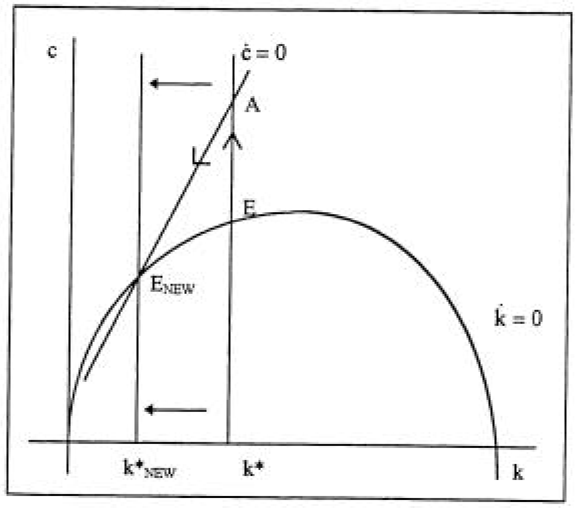
The condition required for $\dot{c} = 0$ is given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return must equal $\rho + \theta g$. Compared to the case without a tax on capital, $f'(k)$, the pre-tax rate of return on capital, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left.

The equation describing the dynamics of the capital stock per unit of effective labor is still given by

$$(2)\quad \dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

For a given k, the level of c that implies $\dot{k} = 0$ is given by $c(t) = f(k) - (n + g)k$. Since the tax is rebated to households in the form of lump-sum transfers, this $\dot{k} = 0$ locus is unaffected.

(b) At time 0, when the tax is put in place, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k* on the old balanced growth path.



In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time that the tax is introduced. This jump in c is not inconsistent with the consumption-smoothing behavior implied by the household's optimization problem since the tax was unexpected and could not be prepared for.

In order for the economy to reach the new balanced growth path, it should be clear what must occur. At time 0, c jumps up so that the economy is on the new saddle path. In the figure, the economy jumps from point E to a point such as A. Since the return to saving and accumulating capital is now lower than before, people switch away from saving and into consumption.

After time 0, the economy will gradually move down the new saddle path until it eventually reaches the new balanced growth path at E_{NEW}.

(c) On the new balanced growth path at E_{NEW}, the distortionary tax on investment income has caused the economy to have a lower level of capital per unit of effective labor as well as a lower level of consumption per unit of effective labor.

(d) (i) From the analysis above, we know that the higher is the tax rate on investment income, τ, the lower will be the balanced-growth-path level of k^*, all else equal. In terms of the above story, the higher is τ the more that the $\dot{c} = 0$ locus shifts to the left and hence the more that k^* falls. Thus $\partial k^*/\partial\tau < 0$.

On a balanced growth path, the fraction of output that is saved and invested, the saving rate, is given by $[f(k^*) - c^*]/f(k^*)$. Since k is constant, or $\dot{k} = 0$, on a balanced growth path then from $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$ we can write $f(k^*) - c^* = (n + g)k^*$. Using this we can rewrite the fraction of output that is saved on a balanced growth path as

(3) $s = [(n + g)k^*]/f(k^*)$.

Use equation (3) to take the derivative of the saving rate with respect to the tax rate, τ:

(4) $$\frac{\partial s}{\partial\tau} = \frac{(n+g)(\partial k^*/\partial\tau)f(k^*) - (n+g)k^* f'(k^*)(\partial k^*/\partial\tau)}{f(k^*)^2}.$$

Simplifying yields

$$\frac{\partial s}{\partial\tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau} - \frac{(n+g)}{f(k^*)}\frac{k^* f'(k^*)}{f(k^*)}\frac{\partial k^*}{\partial\tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau}\left[1 - \frac{k^* f'(k^*)}{f(k^*)}\right].$$

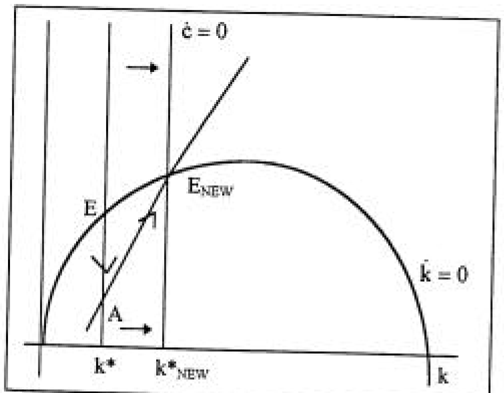
Recall that $k^*f'(k^*)/f(k^*) = \alpha_K(k^*)$ is capital's (pre-tax) share in income, which must be less than one. Since $\partial k^*/\partial\tau < 0$ we can write

(5) $$\frac{\partial s}{\partial\tau} = \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial\tau}\left[1 - \alpha_K(k^*)\right] < 0$$

Thus the saving rate on the balanced growth path is decreasing in τ.

(d) (ii) Citizens in low-τ, high-k^*, high-saving countries do not have the incentive to invest in low-saving countries. From part (a), the condition required for $\dot{c} = 0$ is $(1 - \tau)f'(k) = \rho + \theta g$.. That is, the after-tax rate of return must equal $\rho + \theta g$. Assuming preferences and technology are the same across countries so that ρ, θ and g are the same across countries, the after-tax rate of return will be the same across countries. Since the after-tax rate of return is thus the same in low-saving countries as it is in high-saving countries, there is no incentive to shift saving from a high-saving to a low-saving country.

(e) Should the government subsidize investment instead and fund this with a lump-sum tax? This would lead to the opposite result from above and the economy would have higher c and k on the new balanced growth path.



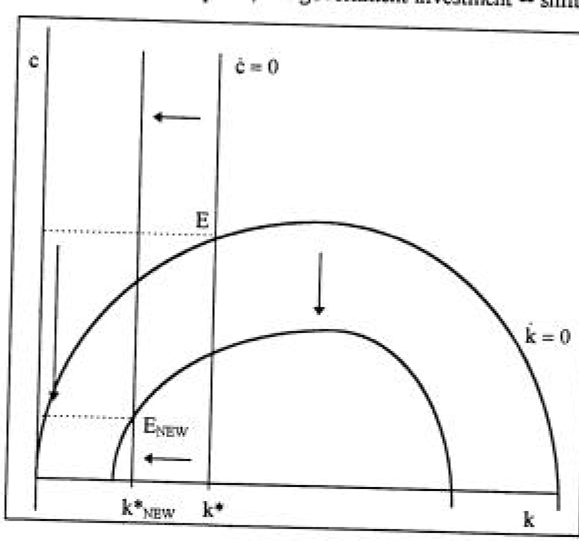
The answer is no. The original market outcome is already the one that would be chosen by a central planner attempting to maximize the lifetime utility of a representative household subject to the capital-accumulation equation. It therefore gives the household the highest possible lifetime utility.

Starting at point E, the implementation of the subsidy would lead to a short-term drop in consumption at point A, but would eventually result in permanently higher consumption at point E_{NEW}. It would turn out that the utility lost from the short-term sacrifice would outweigh the utility gained in the long-term (all in present value terms, appropriately discounted). This is the same type of argument used to explain the reason that households do not choose to consume at the golden-rule level. See Section 2.4 for a more complete description of the welfare implications of this model.

(f) Suppose the government does not rebate the tax revenue to households but instead uses it to make government purchases. Let G(t) represent government purchases per unit of effective labor. The equation describing the dynamics of the capital stock per unit of effective labor is now given by

(2 ') $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$.

The fact that the government is making purchases that do not add to the capital stock -- it is assumed to be government consumption, not government investment -- shifts down the $\dot{k} = 0$ locus.



After the imposition of the tax, the $\dot{c} = 0$ locus shifts to the left, just as it did in the case in which the government rebated the tax to households. In the end, k* falls to k^*_{NEW} just as in the case where the government rebated the tax. Consumption per unit of effective labor on the new balanced growth path at E_{NEW} is lower than in the case where the tax is rebated by the amount of the government purchases, which is $\tau f'(k)k$.

Finally, whether the level of c jumps up or down initially depends upon whether the new saddle path passes above or below the original balanced-growth-path point of E.