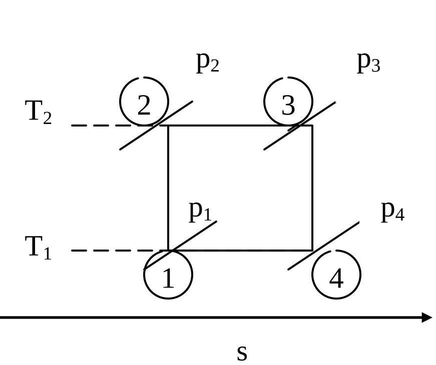
Chapter 1 Solutions

Problem 1.1

The Carnot cycle sets the limit on thermal efficiency of a heat engine operating between two temperature limits. Show that ideal Carnot efficiency is:

$$\eta_{th} = 1 - \frac{T_1}{T_2}$$

What is the thermal efficiency if T_1=288 K and T_2=2000 K?



Solution:

Following conservation of energy, the amount of work done by the system per unit mass is:

$$w = \oint dq$$

For a reversible heat engine operating between two reservoirs at temperatures T_H and T_L,

$$\oint dq = \oint T ds = (T_H - T_L)\Delta s$$

Therefore, net work, per unit mass is w

$$w = (T_H - T_L)\Delta s$$

The heat input in the cycle takes place between stations 2 and 3, i.e.,

$$q_{in} = \int_2^3 T ds = T_H \Delta s$$

Therefore thermal efficiency of the cycle is:

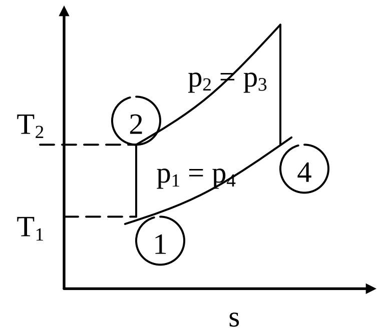
$$\eta_{th} = \frac{w}{q_{in}} = \frac{(T_H - T_L)\Delta s}{T_H \Delta s} = 1 - \frac{T_L}{T_H}$$

For the cycle defined in the diagram, the thermal efficiency is

$$\eta_{th} = 1 - \frac{T_1}{T_2}$$

Thermal efficiency: $\eta_t = 1 - \frac{T_1}{T_2} = 1 - \frac{288\ K}{2000\ K} \Rightarrow \eta_t = 0.856\ or\ 85.6\%$

Problem 1.2 The ideal Brayton cycle operates between two pressure limits as shown. It is the model of an airbreathing jet engine, such as a turbojet or ramjet engine.



Show that ideal Brayton cycle efficiency is:

$$\eta_{th} = 1 - \frac{T_1}{T_2}$$

What is the thermal efficiency of the Brayton That has T_1=288 K and T_2=864 K? Note that maximum cycle temperature T_3 has no effect on cycle thermal efficiency.

Solution: Net cycle heat exchange is:

$$\oint \delta q = \oint T ds$$

Gibbs equation is:

Tds = dh - vdp

Therefore for a constant pressure process, Tds = dh,

$$\int_2^3 T ds = \int_2^3 dh = h_3 - h_2 = c_p(T_3 - T_2)$$

In a cycle, the net work output is equal to the net heat input (according to the 1st law of thermo)

$$\oint \delta w = \oint \delta q = c_p(T_3 - T_2) - c_p(T_4 - T_1)$$

By definition, cycle thermal efficiency is:

$$\eta_B = \frac{W_{net}}{q_{in}} = \frac{q_{in} - q_{out}}{q_{in}}$$

Thus,

$$\eta_B = 1 - \frac{q_{4-1}}{q_{2-3}} = 1 - \frac{c_p(T_4 - T_1)}{c_p(T_3 - T_2)} = 1 - \frac{T_1\left(\frac{T_4}{T_1} - 1\right)}{T_2\left(\frac{T_3}{T_2} - 1\right)}$$

Since processes 1-2 and 3-4 are isentropic and $p_3 = p_2$ and $p_4 = p_1$, we can write:

$$\frac{T_4}{T_1} = \frac{T_4}{T_3}\frac{T_3}{T_2}\frac{T_2}{T_1} = \left(\frac{p_4}{p_3}\right)^{\frac{\gamma-1}{\gamma}} \frac{T_3}{T_2}\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{T_3}{T_2}$$

Therefore, the ideal Brayton cycle efficiency is simplified to:

$$\eta_B = 1 - \frac{T_1}{T_2}$$

Thermal efficiency: $\eta_t = 1 - \frac{T_1}{T_2} = 1 - \frac{288\ K}{864\ K} \Rightarrow \eta_t \approx 0.667\ \ or\ 66.7\%$

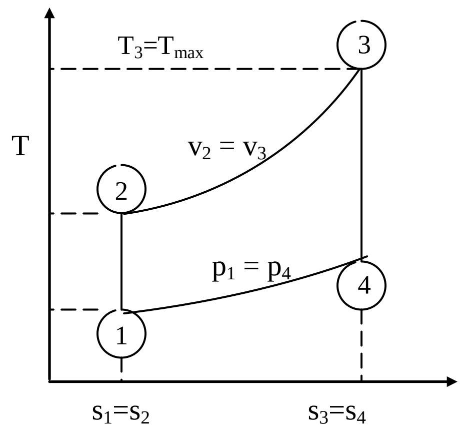
The Brayton cycle operates between two isobars (constant pressure lines), therefore, it is the pressure ratio that sets the thermal efficiency of the ideal Brayton cycle. The maximum cycle temperature changes the amount of heat input and the work output in the same proportion such that the ratio remains constant.

Problem 1.3

Humphrey cycle operates a constant-volume combustor instead of a constant-pressure cycle like Brayton. Show that:

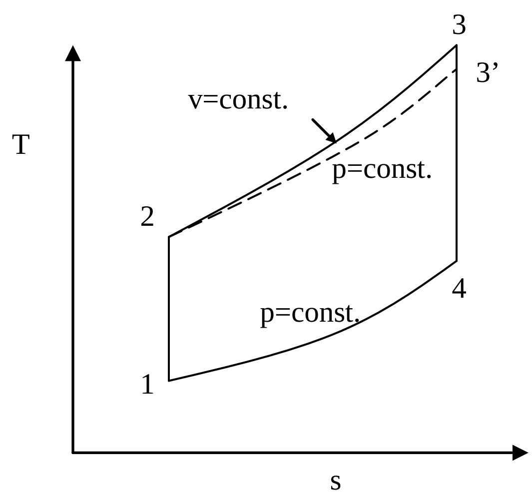
$$\eta_{th} = 1 - \gamma \frac{T_1}{T_2}\left[\left(\frac{T_3}{T_2}\right)^{\frac{1}{\gamma}} - 1\right] / \left[\frac{T_3}{T_2} - 1\right]$$

is the thermal efficiency of an ideal Humphrey cycle.



Let us use the same T_1 as in Problems 1.1 and 1.2, i.e., T_1=288 K. Let use the same temperature T_2 as in Problem 1.2, i.e., T_2=864 K.

Finally, let us use the same maximum cycle temperature as in Carnot (Problem 1.1), i.e., T_{max}=2000 K. With the ratio of specific heats γ=1.4, calculate the thermal efficiency of Humphrey cycle. Compare the answer with Brayton cycle efficiency.



Net cycle heat exchange is:

$$\oint \delta q = \oint T ds = \int_2^3 T ds + \int_4^1 T ds$$

Since Gibbs equation is

Tds = de + pdv

And the process from 2 to 3 is constant volume heating, Tds = de for a constant volume process,

$$\int_2^3 Tds = \int_2^3 de = e_3 - e_2 = c_v(T_3 - T_2)$$

Another form of Gibbs equation is
Tds = dh - vdp

Therefore for a constant pressure process, Tds = dh, therefore

$$\int_4^1 Tds = \int_4^1 dh = h_1 - h_4 = c_p(T_1 - T_4)$$

In a cycle, the net work output is equal to the net heat input (according to the 1st law of thermo)

$$\oint \delta w = \oint \delta q = c_v(T_3 - T_2) - c_p(T_4 - T_1)$$

Thermal investment in the cycle is the integral of δq from 2 to 3.

$$\int_2^3 \delta q = \int_2^3 Tds = \int_2^3 de = e_3 - e_2 = c_v(T_3 - T_2)$$

Therefore thermal efficiency of the ideal Humphrey cycle is:

$$\eta_{th} = \frac{c_v(T_3 - T_2) - c_p(T_4 - T_1)}{c_v(T_3 - T_2)} = 1 - \gamma\frac{T_4 - T_1}{T_3 - T_2} = 1 - \gamma\left(\frac{T_1}{T_2}\right)\frac{\frac{T_4}{T_1} - 1}{\frac{T_3}{T_2} - 1}$$

Now, we show that

$$\frac{T_4}{T_1} = \left(\frac{T_3}{T_2}\right)^{1/\gamma}$$

Using chain rule, we may write:

$$\frac{T_4}{T_1} = \frac{T_4}{T_{3'}}\frac{T_{3'}}{T_2}\frac{T_2}{T_1} = \left(\frac{p_4}{p_{3'}}\right)^{(\gamma-1)/\gamma} . \frac{T_{3'}}{T_2} . \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma} = \frac{T_{3'}}{T_2}$$

Note that $p_{3'}=p_2$ and $p_4=p_1$.

$$\frac{T_{3'}}{T_2} = \frac{T_{3'}}{T_3}\frac{T_3}{T_2} = \left(\frac{p_2}{p_3}\right)^{\frac{\gamma-1}{\gamma}} . \frac{T_3}{T_2} = \left(\frac{T_2}{v_2}\frac{v_3}{T_3}\right)^{\frac{\gamma-1}{\gamma}} . \frac{T_3}{T_2} = \left(\frac{T_3}{T_2}\right)^{1-\frac{\gamma-1}{\gamma}} = \left(\frac{T_3}{T_2}\right)^{1/\gamma}$$ note that $v_3=v_2$

Therefore, we show that the thermal efficiency of the ideal Humphrey cycle is:

$$\eta_{th} = 1 - \gamma\frac{T_1}{T_2}\left[\left(\frac{T_3}{T_2}\right)^{\frac{1}{\gamma}} - 1\right] / \left[\frac{T_3}{T_2} - 1\right] \qquad \text{QED}$$