Revised Answers Manual to an Introduction to Measure-Theoretic Probability

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Chapter 1

Certain Classes of Sets, Measurability, Pointwise Approximation

1. **(i)** $x \in \underline{\lim}_{n\to\infty} A_n$ if and only if $x \in \cup_{n\geq 1}\cap_{j\geq n} A_j$, so that $x \in \cap_{j\geq n_0} A_j$ for some $n_0 \geq 1$, and then $x \in A_j$ for all $j \geq n_0$, or $x \in \cup_{j\geq n} A_j$ for all $n \geq 1$, so that $x \in \cap_{n\geq 1}\cup_{j\geq 1} A_j\, \overline{\lim}_{n\to\infty} A_n$.

(ii) $\left(\underline{\lim}_{n\to\infty} A_n\right)^c = \left(\cup_{n\geq 1}\cap_{j\geq n} A_j\right)^c = \cap_{n\geq 1}\cup_{j\geq n} A_j^c = \overline{\lim}_{n\to\infty} A_n^c$, $\left(\overline{\lim}_{n\to\infty} A_n\right)^c = \left(\cap_{n\geq 1}\cup_{j\geq n} A_j\right)^c = \cup_{n\geq 1}\cap_{j\geq n} A_j^c = \underline{\lim}_{n\to\infty} A_n^c$.
Let $\lim_{n\to\infty} A_n = A$. Then $\underline{\lim}_{n\to\infty} A_n^c = \left(\overline{\lim}_{n\to\infty} A_n\right)^c = \left(\lim_{n\to\infty} A_n\right)^c = A^c$, and $\overline{\lim}_{n\to\infty} A_n = \left(\underline{\lim}_{n\to\infty} A_n\right)^c = \left(\lim_{n\to\infty} A_n\right)^c = A^c$, so that $\lim_{n\to\infty} A_n^c$ exists and is A^c.

(iii) To show that $\underline{\lim}_{n\to\infty}(A_n \cap B_n) = (\underline{\lim}_{n\to\infty} A_n) \cap (\underline{\lim}_{n\to\infty} B_n)$. Equivalently,

$$\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty}(A_j \cap B_j) = \left(\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} A_j\right) \cap \left(\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} B_j\right).$$

Indeed, let x belong to the left-hand side. Then $x \in \cap_{j=n_0}^{\infty}(A_j \cap B_j)$ for some $n_0 \geq 1$, hence $x \in (A_j \cap B_j)$ for all $j \geq n_0$, and then $x \in A_j$ and $x \in B_j$ for all $j \geq n_0$. Hence $x \in \cap_{j=n_0}^{\infty} A_j$ and $x \in \cap_{j=n_0}^{\infty} B_j$, so that $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty} A_j$ and $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty} B_j$; i.e., x belongs to the right-hand side. Next, let x belong to the right-hand side. Then $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty} A_j$ and $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty} B_j$, so that $x \in \cap_{j=n_1}^{\infty} A_j$ and $x \in \cap_{j=n_2}^{\infty} B_j$ for some $n_1, n_2 \geq 1$. Then $x \in \cap_{j=n_0}^{\infty} A_j$ and $x \in \cap_{j=n_0}^{\infty} B_j$ where $n_0 = \max(n_1, n_2)$, and hence $x \in A_j$ and $x \in B_j$ for all $j \geq n_0$. Thus, $x \in (A_j \cap B_j)$ for all $j \geq n_0$, so that $x \in \cap_{j=n_0}^{\infty}(A_j \cap B_j)$ and hence $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}(A_j \cap B_j)$; i.e., x belongs to the left-hand side.

Next, $\overline{\lim}_{n\to\infty}(A_n \cup B_n) = \overline{\lim}_{n\to\infty}(A_n^c \cap B_n^c)^c = [\underline{\lim}_{n\to\infty}(A_n^c \cap B_n^c)]^c$ (by part (ii)), and this equals to $[(\underline{\lim}_{n\to\infty} A_n^c) \cap (\underline{\lim}_{n\to\infty} B_n^c)]^c$ (by what we just proved), and this equals $[(\overline{\lim}_{n\to\infty} A_n)^c \cap (\overline{\lim}_{n\to\infty} B_n)^c]^c = (\overline{\lim}_{n\to\infty} A_n) \cup (\overline{\lim}_{n\to\infty} B_n)$, as was to be seen.

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(iv) To show that: $\overline{\lim}_{n\to\infty}(A_n \cap B_n) \subseteq (\overline{\lim}_{n\to\infty}A_n) \cap (\overline{\lim}_{n\to\infty}B_n)$ and $\underline{\lim}_{n\to\infty}(A_n \cup B_n) \supseteq (\underline{\lim}_{n\to\infty}A_n) \cup (\underline{\lim}_{n\to\infty}B_n)$.
Suffices to show: $\cap_{n=1}^{\infty}\cup_{j=n}^{\infty}(A_j \cap B_j) \subseteq \left(\cap_{n=1}^{\infty}\cup_{j=n}^{\infty}A_j\right) \cap \left(\cap_{n=1}^{\infty}\cup_{j=n}^{\infty}B_j\right)$.

Indeed, let x belong to the left-hand side. Then $x \in \cup_{j=n}^{\infty}(A_j \cap B_j)$ for all $n \geq 1$, so that $x \in (A_j \cap B_j)$ for some $j \geq n$ and all $n \geq 1$. Then $x \in A_j$ and $x \in B_j$ for some $j \geq n$ and all $n \geq 1$, hence $x \in \cup_{j=n}^{\infty}A_j$ and $x \in \cup_{j=n}^{\infty}B_j$ for all $n \geq 1$, so that $x \in \cap_{n=1}^{\infty}\cup_{j=n}^{\infty}A_j$ and $x \in \cap_{n=1}^{\infty}\cup_{j=n}^{\infty}B_j$, and hence $x \in \left(\cap_{n=1}^{\infty}\cup_{j=n}^{\infty}A_j\right) \cap \left(\cap_{n=1}^{\infty}\cup_{j=n}^{\infty}B_j\right)$; i.e., x belongs to the right-hand side. So, the above inclusion is correct.

Also, to show that : $\left(\cup_{n=1}^{\infty}\cap_{j=n}^{\infty}A_j\right) \cup \left(\cup_{n=1}^{\infty}\cap_{j=n}^{\infty}B_j\right) \subseteq \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}(A_j \cup B_j)$.

Indeed, let x belong to the left-hand side. Then $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}A_j$ or $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}B_j$ or to both. Let $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}A_j$. Then $x \in \cap_{j=n_0}^{\infty}A_j$ for some $n_0 \geq 1$, hence $x \in A_j$ for all $j \geq n_0$, and then $x \in (A_j \cup B_j)$ for all $j \geq n_0$, so that $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}(A_j \cup B_j)$; i.e., x belongs to the right-hand side. Similarly if $x \in \cup_{n=1}^{\infty}\cap_{j=n}^{\infty}B_j$.

An alternative proof of the second part is as follows:

$$\begin{aligned}
\underline{\lim}(A_n \cup B_n) &= \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}(A_k \cup B_k) = \left[\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}(A_k^c \cap B_k^c)\right]^c \\
&= \left[\overline{\lim}(A_k^c \cap B_k^c)\right]^c \supseteq \left[\left(\overline{\lim}A_k^c\right) \cap \left(\overline{\lim}B_k^c\right)\right]^c \\
&\qquad \text{(by the previous part)} \\
&= \left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\right)^c \cup \left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}B_k^c\right)^c \\
&= \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k\right) \cup \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}B_k\right) = (\underline{\lim}A_n) \cup (\underline{\lim}B_n).
\end{aligned}$$

(v) That the inverse inclusions in part (iv) need not hold is demonstrated by the following
Counterexample:
Let $A_{2j-1} = A$, $A_{2j} = A_0$ and $B_{2j-1} = B$, $B_{2j} = B_0$, $j \geq 1$, for some events A, A_0, B and B_0. Then: $\underline{\lim}_{n\to\infty}A_n = A \cap A_0$, $\overline{\lim}_{n\to\infty}A_n = A \cup A_0$, $\underline{\lim}_{n\to\infty}B_n = B \cap B_0$, $\overline{\lim}_{n\to\infty}B_n = B \cup B_0$, $\overline{\lim}_{n\to\infty}(A_n \cap B_n) = (A \cap B) \cup (A_0 \cap B_0)$, $\underline{\lim}_{n\to\infty}(A_n \cup B_n) = (A \cup B) \cap (A_0 \cup B_0)$.
Therefore $(A \cup B) \cap (A_0 \cup B_0)$ need not contain $(A \cup A_0) \cap (B \cup B_0)$, and $(A \cap A_0) \cup (B \cap B_0)$ need not contain $(A \cup B) \cap (A_0 \cup B_0)$.
As a concrete example, take $\Omega = \Re$, $A = (0, 1]$, $A_0 = [2, 3]$, $B = [1, 2]$, $B_0 = [3, 4]$. Then: $(A \cup B) \cap (A_0 \cup B_0) = (0, 2]$, $(A \cup A_0) \cap$

$(B \cup B_0) = ((0,1] \cup [2,3]) \cap ([1,2] \cup [3,4]) = \{1\} \cup \{3\} = \{1,3\} \not\supseteq (0,2]$, and $(A \cap A_0) \cup (B \cap B_0) = \oslash \cup \oslash = \oslash$, $(A \cup B) \cap (A_0 \cup B_0) = (0,2] \cap [2,4] = \{2\}$ not contained in $\oslash$.

(vi) If $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, then by parts (iii) and (iv): $\overline{\lim}_{n\to\infty}(A_n \cap B_n) \subseteq A \cap B$ and $\underline{\lim}_{n\to\infty}(A_n \cap B_n) = A \cap B$. Thus, $A \cap B = \underline{\lim}_{n\to\infty}(A_n \cap B_n) \subseteq \overline{\lim}_{n\to\infty}(A_n \cap B_n) \subseteq A \cap B$, so that $\lim_{n\to\infty}(A_n \cap B_n) = A \cap B$. Likewise: $A \cup B \subseteq \underline{\lim}_{n\to\infty}(A_n \cup B_n) \subseteq \overline{\lim}_{n\to\infty}(A_n \cup B_n) = A \cup B$, so that $\lim_{n\to\infty}(A_n \cup B_n) = A \cup B$.

(vii) Since $A_n \triangle B = (A_n - B) + (B - A_n) = (A_n \cap B^c) + (B \cap A_n^c)$, we have $\lim_{n\to\infty}(A_n \cap B^c) = (\lim_{n\to\infty} A_n) \cap B^c = A \cap B^c$ by part (vi), and $\lim_{n\to\infty}(B \cap A_n^c) = B \cap (\lim_{n\to\infty} A_n^c) = B \cap A^c$ by parts (vi) and (ii). Therefore, by part (vi) again, $\lim_{n\to\infty}(A_n \triangle B) = \lim_{n\to\infty}[(A_n \cap B^c) + (B \cap A_n^c)] = \lim_{n\to\infty}(A_n \cap B^c) + \lim_{n\to\infty}(B \cap A_n^c) = (A \cap B^c) + (B \cap A^c) = A \triangle B$.

(viii) $A_{2j-1} = B$, $A_{2j} = C$, $j \ge 1$. Then, as in part (v), $\underline{\lim}_{n\to\infty} A_n = B \cap C$ and $\overline{\lim}_{n\to\infty} A_n = B \cup C$. The $\lim_{n\to\infty} A_n$ exists if and only if $B \cap C = B \cup C$, or $B \cup C = (B \cap C^c) + (B^c \cap C) + (B \cap C) = B \cap C$. Then, by the pairwise disjointness of $B \cap C^c$, $B^c \cap C$ and $B \cap C$, we have $B \cap C^c = B^c \cap C = \oslash$. From $B \cap C^c = \oslash$, it follows that $B \subseteq C$, and from $B^c \cap C = \oslash$, it follows that $C \subseteq B$. Therefore $B = C$. Thus, $\lim_{n\to\infty} A_n$ exists if and only if $B = C$. #

2. **(i)** All three sets $\underline{A}$, $\overline{A}$, and A (if it exists) are in $\mathcal{A}$, because they are expressed in terms of A_n, $n \ge 1$, by means of countable operations.

(ii) Let $A_n \uparrow$. Then $\underline{\lim}_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} A_j = \cup_{n=1}^{\infty} A_n$, and $\overline{\lim}_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{j=n}^{\infty} A_j = \cup_{j=n}^{\infty} A_j = \cup_{j=1}^{\infty} A_j = \cup_{n=1}^{\infty} A_n$, so that $\lim_{n\to\infty} A_n = \cup_{n=1}^{\infty} A_n$.

If $A_n \downarrow$, then $A_n^c \uparrow$ and hence $\cap_{n=1}^{\infty} \cup_{j=n}^{\infty} A_j^c = \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} A_j^c = \cup_{n=1}^{\infty} A_n^c$, so that, by taking the complements, $\cup_{n=1}^{\infty} \cap_{j=n}^{\infty} A_j = \cap_{n=1}^{\infty} \cup_{j=n}^{\infty} A_j = \cap_{n=1}^{\infty} A_n$, so that $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$. #

3. **(i)** $\cap_{j\in I} \mathcal{F}_j \ne \oslash$ since, e.g., $\Omega \in \mathcal{F}_j$, $j \in I$. Next, if $A \in \cap_{j\in I} \mathcal{F}_j$ for all $j \in I$, and hence $A^c \in \mathcal{F}_j$ for all $j \in I$, so that $A^c \in \cap_{j\in I} \mathcal{F}_j$. Finally, if $A, B \in \cap_{j\in I} \mathcal{F}_j$, then $A, B \in \mathcal{F}_j$ for all $j \in I$, and hence $A \cup B \in \mathcal{F}_j$ for all $j \in I$, so that $A \cup B \in \cap_{j\in I} \mathcal{F}_j$.

(ii) If $A_i \in \cap_{j\in I} \mathcal{A}_j$, $i = 1, 2, \ldots$, then $A_i \in \mathcal{A}_j$, $i = 1, 2, \ldots$, for all $j \in I$, and hence $\cup_{i=1}^{\infty} A_i \in \mathcal{A}_j$ for all $j \in I$, so that $\cup_{i=1}^{\infty} A_i \in \cap_{j\in I} \mathcal{A}_j$. #

4. Let $\Omega = \Re$, $\mathcal{F} = \{A \subseteq \Re;$ either A or A^c is finite$\}$, and let $A_j = \{1, 2, \ldots, j\}$, $j \ge 1$. Then $\mathcal{F}$ is a field and $A_j \in \mathcal{F}$, $j \ge 1$, but $\cup_{j=1}^{\infty} A_j = \{1, 2, \ldots\} \notin \mathcal{F}$, because neither this set nor its complement is finite.

Also, if $B_j = \{j+1, j+2, \ldots\}$, then $B_j \in \mathcal{F}_j$ since B_j^c is finite, whereas $\cap_{j=1}^{\infty} B_j = \cap_{j=1}^{\infty} A_j^c = \left(\cup_{j=1}^{\infty} A_j\right)^c \notin \mathcal{F}$, as it has been seen already. #

5. Clearly, $\mathcal{C}$ is $\neq \oslash$, every member of $\mathcal{C}$ is a countable union of members of $\mathcal{P}$, and $\mathcal{C}$ is the smallest σ-field containing $\mathcal{P}$, if indeed, is a σ-field. If $B \in \mathcal{C}$, then $B = \cup_{i \in I} A_i$ for some $I \subseteq \mathbb{N} = \{1, 2, \ldots\}$, and then $B^c = \cup_{j \in J} A_j$, where $J = \mathbb{N} - I$, so that $B^c \in \mathcal{C}$. Finally, if $B_j \in \mathcal{C}$, $j = 1, 2, \ldots$, then $B_j = \cup_{i \in I_j} A_{ji}$, where $I_j \subseteq \mathbb{N}$ and $I_i \cap I_j = \oslash$. Then $\cup_{j=1}^{\infty} B_j = \cup_{j=1}^{\infty} \cup_{i \in I_j} A_{ji}$, the union of members of $\mathcal{P}$, so that $\cup_{j=1}^{\infty} B_j$ belongs in $\mathcal{C}$. #

6. Since $\mathcal{C}_j$ and $\mathcal{C}'_j \subseteq \mathcal{C}_0$, $j = 1, \ldots, 8$, it follows that $\sigma(\mathcal{C}_j)$ and $\sigma(\mathcal{C}'_j) \subseteq \sigma(\mathcal{C}_0) = \mathcal{B}$, so that it suffices to show that $\mathcal{B} \subseteq \sigma(\mathcal{C}_j)$ and $\mathcal{B} \subseteq \sigma(\mathcal{C}'_j)$, which are implied, respectively, by $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_j)$ and $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}'_j)$, $j = 1, \ldots, 8$. As an example, consider the classes mentioned in the hint.
So, to show that $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$. In all that follows, all limits are taken as $n \to \infty$. Indeed, for $y_n \downarrow y$, we have $(x, y_n) \in \mathcal{C}_1$ and $\cap_{n=1}^{\infty}(x, y_n) = (x, y] \in \sigma(\mathcal{C}_1)$. Likewise, for $x_n \uparrow x$, we have $(x_n, y) \in \mathcal{C}_1$ and $\cap_{n=1}^{\infty}(x_n, y) = [x, y) \in \sigma(\mathcal{C}_1)$. Next, with x_n and y_n as above, $(x_n, y_n) \in \mathcal{C}_1$ and $\cap_{n=1}^{\infty}(x_n, y_n) = [x, y] \in \sigma(\mathcal{C}_1)$. Also, for $x_n \downarrow -\infty$, we have $(x_n, a) \in \mathcal{C}_1$ and $\cap_{n=1}^{\infty}(x_n, a) = (-\infty, a) \in \sigma(\mathcal{C}_1)$, and likewise $(x_n, a] \in \mathcal{C}_1$ and $\cup_{n=1}^{\infty}(x_n, a] = (-\infty, a] \in \sigma(\mathcal{C}_1)$. Finally, $(b, \infty) = (-\infty, b]^c \in \sigma(\mathcal{C}_1)$, and $[b, \infty) = (-\infty, b)^c \in \sigma(\mathcal{C}_1)$. It follows that $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$.

That $\mathcal{C}_0 \in \sigma(\mathcal{C}'_1)$ is seen as follows. For (x, y), there exist x_n and y_n rationals with $x_n \downarrow x$ and $y_n \uparrow y$, so that $(x, y) = \cup_{n=1}^{\infty} \in \sigma(\mathcal{C}'_j)$. Also, for $y_n \downarrow y$, we have $(x, y_n) \in \sigma(\mathcal{C}'_1)$, as was just proved, and then $\cap_{n=1}^{\infty}(x, y_n) = (x, y] \in \sigma(\mathcal{C}'_1)$. Likewise, with $x_n \uparrow x$, we have $(x_n, y) \in \sigma(\mathcal{C}'_1)$ and then $\cap_{n=1}^{\infty}(x_n, y) = [x, y) \in \sigma(\mathcal{C}'_1)$. Also, with $x_n \uparrow x$ and $y_n \downarrow y$, we have $(x_n, y_n) \in \sigma(\mathcal{C}'_1)$, and $\cap_{n=1}^{\infty}(x_n, y_n) = [x, y] \in \sigma(\mathcal{C}'_1)$. Likewise, with $x_n \downarrow -\infty$, we have $(x_n, a) \in \sigma(\mathcal{C}'_1)$ and $\cup_{n=1}^{\infty}(x_n, a) = (-\infty, a) \in \sigma(\mathcal{C}'_1)$, whereas $(x_n, a] \in \sigma(\mathcal{C}'_1)$, so that $\cup_{n=1}^{\infty}(x_n, a] = (-\infty, a] \in \sigma(\mathcal{C}'_1)$. Finally, $(b, \infty) = (-\infty, b]^c \in \sigma(\mathcal{C}'_1)$ since $(-\infty, b] \in \sigma(\mathcal{C}'_1)$, and $[b, \infty) = (-\infty, b)^c \in \sigma(\mathcal{C}'_1)$ since $(-\infty, b) \in \sigma(\mathcal{C}'_1)$. It follows that $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}'_1)$.

A slightly alternative version of the proof follows. We will show (a) $\sigma(\mathcal{C}_1) = \mathcal{B}$ and (b) $\sigma(\mathcal{C}'_1) = \mathcal{B}$.

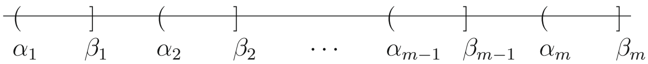
(a) $\sigma(\mathcal{C}_1) = \mathcal{B}$.
That $\sigma(\mathcal{C}_1) \subseteq \mathcal{B}$ is clear; to show $\mathcal{B} \subseteq \sigma(\mathcal{C}_1)$ it suffices to show that $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$. To this end, we show that $(x, y] \in \sigma(\mathcal{C}_1)$. Indeed, $\left(x, y + \frac{1}{n}\right) \in \mathcal{C}_1$, so that $\bigcap_{n=1}^{\infty}\left(x, y + \frac{1}{n}\right) = (x, y] \in \sigma(\mathcal{C}_1)$. Next, $\left(x - \frac{1}{n}, y\right) \in \mathcal{C}_1$, so that $\bigcap_{n=1}^{\infty}\left(x - \frac{1}{n}, y\right) = [x, y) \in \sigma(\mathcal{C}_1)$. Also, $\left(x - \frac{1}{n}, y + \frac{1}{n}\right) \in \mathcal{C}_1$, so that $\bigcap_{n=1}^{\infty}\left(x - \frac{1}{n}, y + \frac{1}{n}\right) = [x, y] \in \sigma(\mathcal{C}_1)$. Next, $(-n, x) \in \mathcal{C}_1$, so that $\bigcup_{n=1}^{\infty}(-n, x) = (-\infty, x) \in \sigma(\mathcal{C}_1)$. Also, $\left(-\infty, x + \frac{1}{n}\right] \in \mathcal{C}_1$, so that $\bigcap_{n=1}^{\infty}\left(-\infty, x + \frac{1}{n}\right] = (-\infty, x] \in \sigma(\mathcal{C}_1)$. Likewise, $(x, n) \in \mathcal{C}_1$, so that $\bigcup_{n=1}^{\infty}(x, n) = (x, \infty) \in \sigma(\mathcal{C}_1)$; and $\left(x - \frac{1}{n}, \infty\right) \in \sigma(\mathcal{C}_1)$, so that $\bigcap_{n=1}^{\infty}\left(x - \frac{1}{n}, \infty\right) = [x, \infty) \in \sigma(\mathcal{C}_1)$. The proof is complete.

(b) $\sigma(\mathcal{C}'_1) = \mathcal{B}$.

Since, clearly, $\sigma(\mathcal{C}_1') \subseteq \sigma(\mathcal{C}_1)$, it suffices to show that $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_1')$. For $x, y \in \Re$ with $x < y$, there exist $x_n \downarrow x$ and $y_n \uparrow y$ with x_n, y_n rational numbers and $x_n < y_n$ for each n. Since $(x_n, y_n) \in \mathcal{C}_1'$, it follows that $\bigcup_{n=1}^{\infty}(x_n, y_n) = (x, y) \in \sigma(\mathcal{C}_1')$. So $\mathcal{C}_1 \subseteq \sigma(\mathcal{C}_1')$, and hence $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_1')$. The proof is complete. #

7. **(i)** Let $A \in \mathcal{C}$. Then there are the following possible cases:

(a) $A = \sum_{i=1}^{m} I_i$, $I_i = (\alpha_i, \beta_i]$, $i = 1, \ldots, m$.



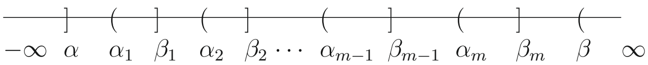
Then $A^c = (-\infty, \alpha_1] + (\beta_1, \alpha_2] + \ldots + (\beta_{m-1}, \alpha_m] + (\beta_m, \infty)$ and this is in $\mathcal{C}$.

(b) A consists only of intervals of the form $(-\infty, \alpha]$. Then there can be only one such interval; i.e., $A = (-\infty, \alpha]$ and hence $A^c = (\alpha, \infty)$ which is in $\mathcal{C}$.

(c) A consists only of intervals of the form (β, ∞). Then there can only be one such interval; i.e., $A = (\beta, \infty)$ so that $A^c = (-\infty, \beta]$ which is in $\mathcal{C}$.

(d) A consists only of intervals of the form $(-\infty, \alpha]$ and (β, ∞). Then A will be as follows: $A = (-\infty, \alpha] + (\beta, \infty)$ $(\alpha < \beta)$, so that $A^c = (\alpha, \infty) \cap (-\infty, \beta] = (\alpha, \beta]$ which is in $\mathcal{C}$.

(e) Finally, let A consist of intervals of all forms. Then A is as below:



Then, clearly,

$$A^c = (\alpha, \alpha_1] + (\beta_1, \alpha_2] + \ldots + (\beta_{m-1}, \alpha_m] + (\beta_m, \beta]$$

which is in $\mathcal{C}$. So, $\mathcal{C}$ is closed under complementation. It is also closed under the union of two sets A and B in $\mathcal{C}$, because, clearly, the union of two such sets is also a member of $\mathcal{C}$. Thus, $\mathcal{C}$ is a field. Next, let $\mathcal{C}_2 = \{(\alpha, \beta];\ \alpha, \beta \in \Re, \alpha < \beta\}$. Then, by Exercise 6, $\sigma(\mathcal{C}_2) = \mathcal{B}$. Also, $\mathcal{C}_2 \subset \mathcal{C}$, so that $\mathcal{B} = \sigma(\mathcal{C}_2) \subseteq \sigma(\mathcal{C})$. Furthermore, $\mathcal{C} \subseteq \sigma(\mathcal{C}_0) = \mathcal{B}$ and hence $\sigma(\mathcal{C}) \subseteq \mathcal{B}$. It follows that $\sigma(\mathcal{C}) = \mathcal{B}$.

(ii) If $A \in \mathcal{C}$, then $A = \sum_{i=1}^{m} I_i$, where I_is are of the forms: (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$, $[\alpha, \beta]$, $(-\infty, \alpha)$, $(-\infty, \alpha]$, (β, ∞), $[\beta, \infty)$. But $(\alpha, \beta)^c = (-\infty, \alpha] + [\beta, \infty)$, $(\alpha, \beta]^c = (-\infty, \alpha] + (\beta, \infty)$, $[\alpha, \beta)^c = (-\infty, \alpha) + (\beta, \infty)$, $[\alpha, \beta]^c = (-\infty, \alpha) + (\beta, \infty)$, $(-\infty, \alpha)^c = [\alpha, \infty)$, $(-\infty, \alpha]^c = (\alpha, \infty)$, $(\beta, \infty)^c = (-\infty, \beta]$, and $[\beta, \infty)^c = (-\infty, \beta)$. Then, considering all possibilities as in part (i), we conclude that $A^c \in \mathcal{C}$ in all cases. Next, for A as above and $B = \sum_{j=1}^{n} J_j$ with J_j being from among the above intervals, it follows that $A \cup B$ is a finite sum of intervals as above, and hence $A \cup B \in \mathcal{C}$. Thus, $\mathcal{C}$ is a field. Finally, from $\mathcal{C}_0 \subset \mathcal{C} \subset \mathcal{B}$, it follows that $\mathcal{B} = \sigma(\mathcal{C}_0) \subseteq \sigma(\mathcal{C}) \subseteq \mathcal{B}$, so that $\sigma(\mathcal{C}) = \mathcal{B}$. #

8. Clearly, $\mathcal{F}_A$ is $\neq \oslash$ since, for example, $A = A \cap \Omega$ and hence $A \in \mathcal{F}_A$. Next, for $B \in \mathcal{F}_A$, it follows that $B = A \cap C$, $C \in \mathcal{F}$, and B_A^c(=complement of B with respect to A)$=A \cap C^c \in \mathcal{F}_A$ since $C^c \in \mathcal{F}$. Finally, for $B_1, B_2 \in \mathcal{F}_A$, it follows that $B_i = A_i \cap C_i$, $C_i \in \mathcal{F}$, $i = 1, 2$, and then $B_1 \cup B_2 = A \cap (C_1 \cup C_2) \in \mathcal{F}_A$, since $C_1 \cup C_2 \in \mathcal{F}$. #

9. That $\mathcal{A}_A \neq \oslash$ and that it is closed under complementation is as in Exercise 8. For $B_i \in \mathcal{A}_A$, $i = 1, 2, \ldots$, it follows that $B_i = A \cap C_i$ for some $C_i \in \mathcal{A}$, $i \geq 1$, and $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty (A \cap C_i) = A \cap \left(\cup_{i=1}^\infty C_i\right) \in \mathcal{A}_A$ since $\cup_{i=1}^\infty C_i \in \mathcal{A}$. Thus, $\mathcal{A}_A$ is a σ-field. Since $\mathcal{F} \subseteq \mathcal{A}$, it follows that $\mathcal{F}_A \subseteq \mathcal{A}_A$ and hence $\sigma(\mathcal{F}_A) \subseteq \mathcal{A}_A$. Since for every $\mathcal{F} \subseteq \mathcal{A}_i$, $i \in I$, it follows $\mathcal{F}_A \subseteq \mathcal{A}_{i,A}$, $i \in I$, then $\sigma(\mathcal{F}_A) \subseteq \cap_{i \in I} \mathcal{A}_{i,A}$. Also, $\sigma(\mathcal{F}_A) = \cap_{j \in J} \mathcal{A}_j^*$ for all σ-fields of subsets of A with $\mathcal{A}_j^* \supseteq \mathcal{F}_A$. In order to show that $\sigma(\mathcal{F}_A) = \mathcal{A}_A$, it must be shown that for every σ-field $\mathcal{A}^*$ of subsets of A with $\mathcal{A}^* \supseteq \mathcal{F}_A$, we have $\mathcal{A}^* \supseteq \mathcal{A}_A$. That this is, indeed, the case is seen as follows. Define the class $\mathcal{M}$ by : $\mathcal{M} = \{C \in \mathcal{A};\ A \cap C \in \mathcal{A}^*\}$. Then, clearly, $\mathcal{F} \subseteq \mathcal{M} \subseteq \mathcal{A}$ and $\mathcal{M}_A (= \mathcal{M} \cap A) \subseteq \mathcal{A}^*$. This is so because, for $C \in \mathcal{F}$, it follows that $C \cap A \in \mathcal{F}_A$ and hence $C \cap A \in \mathcal{A}^*$ ($\supseteq \mathcal{F}_A$). Also, with $\mathcal{M}_A = \{C \subseteq A; C = M \cap A, M \in \mathcal{M}\}$, it follows that $\mathcal{M}_A \subseteq \mathcal{A}^*$ from the definition of $\mathcal{M}$. We assert that $\mathcal{M}$ is a monotone class. Indeed, let $C_n \in \mathcal{M}$ with $C_n \uparrow$ or $C_n \downarrow$. Then, for the case that $C_n \uparrow$, $A \cap (\lim_{n\to\infty} C_n) = A \cap \left(\cup_{n=1}^\infty C_n\right) = \cup_{n=1}^\infty \left(A \cap C_n\right) \in \mathcal{A}^*$ since $A \cap C_n \in \mathcal{A}^*$, $n \geq 1$, so that $\lim_{n\to\infty} C_n \in \mathcal{M}$. Likewise, for $C_n \downarrow$, $A \cap (\lim_{n\to\infty} C_n) = A \cap (\cap_{n=1}^\infty C_n) = \cap_{n=1}^\infty (A \cap C_n) \in \mathcal{A}^*$ since $A \cap C_n \in \mathcal{A}^*$, $n \geq 1$, so that $\lim_{n\to\infty} C_n \in \mathcal{M}$. So $\mathcal{M}$ is a monotone class $\supseteq \mathcal{F}$, and hence $\mathcal{M} \supseteq$ minimal monotone class $\mathcal{M}_0$, say, $\supseteq \mathcal{F}$. Since $\mathcal{F}$ is a field, it follows that $\mathcal{M}_0$ is a σ-field and indeed $\mathcal{M}_0 = \mathcal{A}$ (by Theorem 6). Finally, $\mathcal{A} = \mathcal{M}_0 \subseteq \mathcal{M}$ implies $\mathcal{A}_A = \mathcal{M}_{0,A} \subseteq \mathcal{M}_A \subseteq \mathcal{A}^*$, as was to be seen. #

10. Set $\mathcal{F} = \cup_{n=1}^\infty \mathcal{A}_n$, and let $A \in \mathcal{F}$. Then $A \in \mathcal{A}_n$ for some n, so that $A^c \in \mathcal{A}_n$ and hence $A \in \mathcal{F}$. Next, let $A, B \in \mathcal{F}$. Then $A \in \mathcal{A}_{n_1}$, $B \in \mathcal{A}_{n_2}$ for some n_1 and n_2, and let $n_0 = \max(n_1, n_2)$. Then $A, B \in \mathcal{A}_{n_0}$, so that $A \cup B \in \mathcal{A}_{n_0}$ and $A \cup B \in \mathcal{F}$. Then, $A^c \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$, so that $\mathcal{F}$ is a field.
It need not be a σ-field.
Counterexample: Let $\Omega = \Re$ and let $\mathcal{A}_n = \{A \subseteq [-n, n];\ \text{either } A \text{ or } A^c \text{ is countable}\}$, $n \geq 1$. Then $\mathcal{A}_n$ is a σ-field (by Example 8) and $\mathcal{A}_n \uparrow$. However, $\mathcal{F}$ is not a σ-field because, if $A_n = \{\text{rationals in } [-n, n]\}$, $n \geq 1$, and if we set $A = \cup_{n=1}^\infty A_n$, then $A \notin \mathcal{F}$, because otherwise $A \in \mathcal{A}_n$ for some n, which cannot happen. #

11. Set $\mathcal{M} \cap_{j \in I} \mathcal{M}_j$ and let $A_n \in \mathcal{M}$, $n \geq 1$, where the A_ns form a monotone sequence. Then $A_n \in \mathcal{M}_j$ for each $j \in I$ and all $n \geq 1$, so that $\lim_{n\to\infty} A_n$ is also in $\mathcal{M}_j$. Since this is true for all $j \in I$, it follows that $\lim_{n\to\infty} A_n$ is in $\mathcal{M}$, and $\mathcal{M}$ is a monotone class. #

12. Let $\Omega = \{1, 2, \ldots\}$, $\mathcal{M} = \{\oslash, \{1, \ldots, n\}, \{n, n+1, \ldots\}, n \geq 1, \Omega\}$. Then $\mathcal{M}$ is a monotone class, but not a field, because, e.g., if $A = \{1, \ldots, n\}$ and $B = \{n-2, n-1, \ldots\}$ ($n \geq 3$), then $A, B \in \mathcal{M}$, but $A \cap B = \{n-2, n-1, n\} \notin \mathcal{M}$.

As another example, let $\Omega = (0, 1)$ and $\mathcal{M} = \{(0, 1 - \frac{1}{n}], n \geq 1, \Omega\}$. Then $\mathcal{M}$ is a monotone class and $(0, \frac{1}{2}] \in \mathcal{M}$, but $(0, \frac{1}{2}]^c = (\frac{1}{2}, 1) \notin \mathcal{M}$.

Still as a third example, let $\Omega = \Re$ and let $\mathcal{M} = \{\oslash, (0, n), (-n, 0), n \geq 1, (0, \infty), (-\infty, 0)\}$. Then $\mathcal{M}$ is a monotone class, but not a field since, for $A = (-1, 0)$ and $B = (0, 1)$, we have $A, B, \in \mathcal{M}$, but $A \cup B = (-1, 1) \notin \mathcal{M}$. #

13. **(i)** For $\omega = (\omega_1, \omega_2) \in E^c$, we have $\omega \notin E = A \times B$, so that either $\omega_1 \notin A$ or $\omega_2 \notin B$ or both. Let $\omega_1 \notin A$. Then $\omega_1 \in A^c$ and $(\omega_1, \omega_2) \in A^c \times \Omega_2$, whether or not $\omega_2 \in B$. Hence $E^c \subseteq (A \times B^c) + (A^c \times \Omega_2)$. If $\omega_1 \in A$, then $\omega_2 \notin B$, so that $(\omega_1, \omega_2) \in A \times B^c$ and $E^c \subseteq (A \times B^c) + (A^c \times \Omega_2)$. Next, if $(\omega_1, \omega_2) \in A \times B^c$, then $\omega_1 \in A$ and $\omega_2 \notin B$, so that $(\omega_1, \omega_2) \notin E$ and hence $(\omega_1, \omega_2) \in E^c$. If $(\omega_1, \omega_2) \in A^c \times \Omega_2$, then $\omega_1 \notin A$ and hence $(\omega_1, \omega_2) \notin A \times B = E$ whether or not $\omega_2 \in B$. Thus $(\omega_1, \omega_2) \in E^c$. In both cases, $(A \times B^c) + (A^c \times \Omega_2) \supseteq E^c$ and equality follows. The second equality is entirely symmetric.

(ii) Let $(\omega_1, \omega_2) \in E_1 \cap E_2$, so that $(\omega_1, \omega_2) \in E_1$ and $(\omega_1, \omega_2) \in E_2$ and hence $\omega_1 \in A_1, \omega_2 \in B_1$, and $\omega_1 \in A_2$, $\omega_2 \in B_2$. It follows that $\omega_1 \in A_1 \cap A_2$, $\omega_2 \in B_1 \cap B_2$ and hence $(\omega_1, \omega_2) \in (A_1 \cap A_2) \times (B_1 \cap B_2)$. Next, $(\omega_1, \omega_2) \in (A_1 \cap A_2) \times (B_1 \cap B_2)$, so that $\omega_1 \in A_1 \cap A_2$ and $\omega_2 \in B_1 \cap B_2$. Thus, $\omega_1 \in A_1, \omega_1 \in A_2$ and $\omega_2 \in B_1, \omega_2 \in B_2$, so that $(\omega_1, \omega_2) \in A_1 \cap B_1$ and $(\omega_1, \omega_2) \in A_2 \cap B_2$, or $(\omega_1, \omega_2) \in E_1 \cap E_2$, so that equality occurs. The second conclusion is immediate.

(iii) Indeed, $E_1 \cap F_1 = (A_1 \cap A_1') \times (B_1 \cap B_1')$ and $E_2 \cap F_2 = (A_2 \cap A_2') \times (B_2 \cap B_2')$, by part (ii), and the first equality follows. Next, again by part (ii), and replacing E_1 by $(A_1 \cap A_1') \times (B_1 \cap B_1')$ and E_2 by $(A_2 \cap A_2') \times (B_2 \cap B_2')$, we obtain the second equality. The third equality is immediate. Finally, the last conclusion is immediate. #

14. **(i)** Either by the inclusion process or as follows:

$$
\begin{aligned}
&(A_1 \times B_1) - (A_2 \times B_2)\\
&\quad = (A_1 \times B_1) \cap (A_2 \times B_2)^c\\
&\quad = (A_1 \times B_1) \cap [(A_2 \times B_2^c) + (A_2^c \times \Omega_2)] \text{ (by Lemma 2)}\\
&\quad = (A_1 \times B_1) \cap (A_2 \times B_2^c) + (A_1 \times B_1) \cap (A_2^c \times \Omega_2)\\
&\quad = (A_1 \cap A_2) \times (B_1 \cap B_2^c) + (A_1 \cap A_2^c) \times (B_1 \cap \Omega_2) \text{ (clearly)}\\
&\quad = (A_1 \cap A_2) \times (B_1 - B_2) + (A_1 - A_2) \times B_1.
\end{aligned}
$$

(ii) Let $A \times B = \oslash$. Then $(x, y) \in A \times B$, so that $x \in A$ and $y \in B$. Also, $(x, y) \in \oslash$ and this can happen only if at least one of A or B is $= \oslash$. On the other hand, if at least one of A or B is $= \oslash$, then, clearly, $A \times B = \oslash$.

(iii) Let $A_1 \times B_1 \subseteq A_2 \times B_2$. Then $(x, y) \in A_1 \times B_1$, so that $x \in A_1$ and $y \in B_1$. Also, $(x, y) \in A_2 \times B_2$ implies $x \in A_2$ and $y \in B_2$. Thus, $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Next, let $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Then $A_1 \times B_1 \subseteq A_2 \times B_2$ since $(x, y) \in A_1 \times B_1$ if and only if $x \in A_1$ and $y \in B_1$. Hence, $x \in A_2$ and $y \in B_2$ or $(x, y) \in A_2 \times B_2$.

(iv) $A_1 \times B_1 \neq \oslash$ and $A_2 \times B_2 \neq \oslash$. Then $A_1 \times B_1 = A_2 \times B_2$ or $A_1 \times B_1 \subseteq A_2 \times B_2$ and then (by (iii)), $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Also, $A_2 \times B_2 = A_1 \times B_1$ or $A_2 \times B_2 \subseteq A_1 \times B_1$, and then (by (iii) again), $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$.

So, both $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$, and therefore $A_1 = A_2$. Likewise, $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$ so that $B_1 = B_2$.

(v)

$$A \times B = (A_1 \times B_1) + (A_2 \times B_2) \qquad (*)$$

From $\oslash = (A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$ and part (ii), we have that at least one of $A_1 \cap A_2$, $B_1 \cap B_2$ is $\oslash$. Let $A_1 \cap A_2 = \oslash$. Then the claim is that $A = A_1 + A_2$. In fact, $(x, y) \in A \times B$ implies $x \in A$ (and $y \in B$). Also, (x, y) belonging to the right-hand side of (*) implies $(x, y) \in A_1 \times B_1$ or $(x, y) \in A_2 \times B_2$. Let $(x, y) \in A_1 \times B_1$. Then $x \in A_1$ (and $y \in B_1$), so that $A \subseteq A_2$. On the other hand, $(x, y) \in A_2 \times B_2$ implies $x \in A_2$ (and $y \in B_2$), so that $A \subseteq A_2$. Thus, $A \subseteq A_1 + A_2$. Next, let again (x, y) belong to the right-hand side of (*). Then $(x, y) \in A_1 \times B_1$ or $(x, y) \in A_2 \times B_2$. Now $(x, y) \in A_1 \times B_1$ implies that $x \in A_1$ (and $y \in B_1$). Also, (x, y) belonging to the left-hand side of (*) implies $(x, y) \in A \times B$, so that $x \in A$ (and $y \in B$). Hence $A_1 \subseteq A$. Likewise, $(x, y) \in A_2 \times B_2$ implies $A_2 \subseteq A$, so that $A_1 + A_2 \subseteq A$, and hence $A = A_1 + A_2$. Next, let $A = A_1 + A_2$. Then $A \times B = (A_1 + A_2) \times B = (A_1 \times B) + (A_2 \times B)$. Also, $A \times B = (A_1 \times B_1) + (A_2 \times B_2)$. Thus, $(A_1 \times B) + (A_2 \times B) = (A_1 \times B_1) + (A_2 \times B_2)$. (x, y) belonging to the left-hand side of (*) implies $(x, y) \in A_1 \times B$ or $(x, y) \in A_2 \times B$. $(x, y) \in A_1 \times B$ yields $y \in B$ (and $x \in A_1$). Same if $(x, y) \in A_2 \times B$. Also, (x, y) belonging to the right-hand side of (*) implies $(x, y) \in A_1 \times B_1$ or $(x, y) \in A_2 \times B_2$. For $(x, y) \in A_1 \times B_1$, we have $y \in B_1$ (and $x \in A_1$), so that $B \subseteq B_1$. For $(x, y) \in A_2 \times B_2$, we have $B \subseteq B_2$ likewise. Next, let again (x, y) belong to the right-hand side of (*). Then $(x, y) \in A_1 \times B_1$ or $(x, y) \in A_2 \times B_2$. For $(x, y) \in A_1 \times B_1$, we have $y \in B_1$ (and $x \in A_1$). Thus $B_1 \subseteq B$. For $(x, y) \in A_2 \times B_2$, we have $B_2 \subseteq B$. It follows that $B = B_1 = B_2$.

To summarize: $A_1 \cap A_2 = \oslash$ implies $A = A_1 + A_2$ and $B = B_1 = B_2$. Likewise, $B_1 \cap B_2 = \oslash$ implies $B = B_1 + B_2$ and $A = A_1 = A_2$. Furthermore, $A_1 \cap A_2 = \oslash$ and $B_1 \cap B_2 = \oslash$ cannot happen simultaneously. Indeed, $A_1 \cap A_2 = \oslash$ implies $A = A_1 + A_2$, and $B_1 \cap B_2 = \oslash$ implies $B = B_1 + B_2$. Then $A \times B = (A_1 + A_2) \times (B_1 + B_2) = (A_1 \times B_1) + (A_2 \times B_2) + (A_1 \times B_2) + (A_2 \times B_1)$. Also, $A \times B = (A_1 \times B_1) + (A_2 \times B_2)$, so that: $(A_1 \times B_1) + (A_2 \times B_2) + (A_1 \times B_2) + (A_2 \times B_1) = (A_1 \times B_1) + (A_2 \times B_2)$. Then $(A_1 \times B_2) + (A_2 \times B_1) = \oslash$ implies $(A_1 \times B_2) = (A_2 \times B_1) = \oslash$, so that at least one of $A_1, A_2, B_1, B_2 = \oslash$ (by part (ii)). However, this is not possible by the fact that $A_1 \times B_1 \neq \oslash$, $A_2 \times B_2 \neq \oslash$. #

15. **(i)** If either A or $B = \oslash$, then, clearly, $A \times B = \oslash$. Next, if $A \times B = \oslash$, and $A \neq \oslash$ and $B \neq \oslash$, then there exist $\omega_1 \in A$ and $\omega_2 \in B$, so that $(\omega_1, \omega_2) \in A \times B$, a contradiction.

(ii) Both directions of the first assertion are immediate. Without the assumption E_1 and $E_2 \neq \oslash$, the result need not be true. Indeed, let $\Omega_1 = \Omega_2$, $A_1 \neq \oslash$, $B_1 = A_2 = B_2 = \oslash$. Then $E_1 = E_2 = \oslash$, but $A_1 \not\subseteq A_2$. #

16. **(i)** If at least one of $A_1, \ldots, A_n$ is $= \oslash$, then, clearly, $A_1 \times \ldots \times A_n = \oslash$. Next, let $E = \oslash$ and suppose that $A_i \neq \oslash, i = 1, \ldots, n$. Then there exists $\omega_i \in A_i, i = 1, \ldots, n$, so that $(\omega_1, \ldots, \omega_n) \in E$, a contradiction.

(ii) Let $\omega = (\omega_1, \ldots, \omega_n) \in E \cap F$, or $(\omega_1, \ldots, \omega_n) \in (A_1 \times \ldots \times A_n) \cap (B_1 \times \ldots \times B_n)$. Then $(\omega_1, \ldots, \omega_n) \in A_1 \times \ldots \times A_n$ and $(\omega_1, \ldots, \omega_n) \in B_1 \times \ldots \times B_n$. It follows that $\omega_i \in A_i$ and $\omega_i \in B_i, i = 1, \ldots, n$, so that $\omega_i \in A_i \cap B_i, i = 1, \ldots, n$, and hence $(\omega_1, \ldots, \omega_n) \in (A_1 \cap B_1) \times \ldots \times (A_n \cap B_n)$. Next, let $(\omega_1, \ldots, \omega_n) \in (A_1 \cap B_1) \times \ldots \times (A_n \cap B_n)$. Then $\omega_i \in A_i \cap B_i, i = 1, \ldots, n$, so that $\omega_i \in A_i$ and $\omega_i \in B_i, i = 1, \ldots, n$. It follows that $(\omega_1, \ldots, \omega_n) \in A_1 \times \ldots \times A_n$ and $(\omega_1, \ldots, \omega_n) \in B_1 \times \ldots \times B_n$, so that $(\omega_1, \ldots, \omega_n) \in (A_1 \times \ldots \times A_n) \cap (B_1 \times \ldots \times B_n)$. #

17. We have $E = F + G$ and E, F, G are all $\neq \oslash$. This implies that A_i, B_i, and $C_i, i = 1, \ldots, n$ are all $\neq \oslash$; this is so by Exercise 16(i). Furthermore, by Exercise 16(ii):

$$F \cap G = (B_1 \times \ldots \times B_n) \cap (C_1 \times \ldots \times C_n) = (B_1 \cap C_1) \times \ldots \times (B_n \cap C_n),$$

whereas $F \cap G = \oslash$. It follows that $B_j \cap C_j = \oslash$ for at least one $j, 1 \leq j \leq n$. Without loss of generality, suppose that $B_1 \cap C_1 = \oslash$. Then we shall show that $A_1 = B_1 + C_1$ and $A_i = B_i = C_i, i = 2, \ldots, n$. To this end, let $\omega_j \in A_j$, $j = 1, \ldots, n$. Then $(\omega_1, \ldots, \omega_n) \in A_1 \times \ldots \times A_n$ or $(\omega_1, \ldots, \omega_n) \in E$ or $(\omega_1, \ldots, \omega_n) \in (F + G)$. Hence $(\omega_1, \ldots, \omega_n) \in F$ or $(\omega_1, \ldots, \omega_n) \in G$. Let $(\omega_1, \ldots, \omega_n) \in F$. Then $(\omega_1, \ldots, \omega_n) \in B_1 \times \ldots \times B_n$ and hence $\omega_1 \in B_1$ or $\omega_1 \in (B_1 \cup C_1)$, so that $A_1 \subseteq B_1 \cup C_1$. Likewise if $(\omega_1, \ldots, \omega_n) \in G$. Next, let $\omega_j \in B_j, j = 1, \ldots, n$. Then $(\omega_1, \ldots, \omega_n) \in B_1 \times \ldots \times B_n$ or $(\omega_1, \ldots, \omega_n) \in F$ or $(\omega_1, \ldots, \omega_n) \in E$ or $(\omega_1, \ldots, \omega_n) \in (A_1 \times \ldots \times A_n)$, hence $\omega_1 \in A_1$, which implies that $B_1 \subseteq A_1$. By taking $\omega_j \in C_j, j = 1, \ldots, n$ and arguing as before, we conclude that $C_1 \subseteq A_1$. From $B_1 \subseteq A_1$ and $C_1 \subseteq A_1$, we obtain $B_1 \cup C_1 \subseteq A_1$. Since also $A_1 \subseteq B_1 \cup C_1$, we get $A_1 = B_1 \cup C_1$. Since $B_1 \cap C_1 = \oslash$, we have then $A_1 = B_1 + C_1$.

It remains for us to show that $A_i = B_i = C_i, i = 2, \ldots, n$. Without loss of generality, it suffices to show that $A_2 = B_2 = C_2$, the remaining cases being treated symmetrically. As before, let $\omega_j \in A_j, j = 1, \ldots, n$. Then $(\omega_1, \ldots, \omega_n) \in (A_1 \times \ldots \times A_n)$ or $(\omega_1, \ldots, \omega_n) \in E$ or $(\omega_1, \ldots, \omega_n) \in (F+G)$. Hence either $(\omega_1, \ldots, \omega_n) \in F$ or $(\omega_1, \ldots, \omega_n) \in G$. Let $(\omega_1, \ldots, \omega_n) \in F$. Then $(\omega_1, \ldots, \omega_n) \in B_1 \times \ldots \times B_n$ and hence $\omega_2 \in B_2$, so that $A_2 \subseteq B_2$.

Likewise $A_2 \subseteq C_2$ if $(\omega_1, \ldots, \omega_n) \in G$. Next, let $(\omega_1, \ldots, \omega_n) \in B_1 \times \ldots \times B_n$ or $(\omega_1, \ldots, \omega_n) \in F$ or $(\omega_1, \ldots, \omega_n) \in (F + G)$ or $(\omega_1, \ldots, \omega_n) \in E$ or $(\omega_1, \ldots, \omega_n) \in (A_1 \times \ldots \times A_n)$ and hence $\omega_2 \in A_2$, so that $B_2 \subseteq A_2$. It follows that $A_2 = B_2$. We arrive at the same conclusion $A_2 = B_2$ if we take $(\omega_1, \ldots, \omega_n) \in G$. So, to sum it up, $A_1 = B_1 + C_1$, and $A_2 = B_2 = C_2$, and by symmetry, $A_i = B_i = C_i, i = 3, \ldots, n$.

A variation to the above proof is as follows.

Let $E = F + G$ or $A_1 \times \ldots \times A_n = (B_1 \times \ldots \times B_n) + (C_1 \times \ldots \times C_n)$, and let $(\omega_1, \ldots, \omega_n) \in E$. Then $(\omega_1, \ldots, \omega_n) \in A_1 \times \ldots \times A_n$, so that $\omega_i \in A_i, i = 1, \ldots, n$. Then $\omega_i \in B_i, i = 1, \ldots, n$ or $\omega_i \in C_i, i = 1, \ldots, n$ (but not both). So, $A_i = B_i \cup C_i, i = 1, \ldots, n$ and $A_j = B_j + C_j$ for at least one j. Consider the case $n = 2$, and without loss of generality suppose that $A_1 = B_1 + C_1$, $A_2 = B_2 \cup C_2$. Then, clearly:

$$\begin{aligned} A_1 \times A_2 &= (B_1 + C_1) \times (B_2 \cup C_2) \\ &= (B_1 \times B_2) \cup (C_1 \times C_2) \cup (B_1 \times C_2) \cup (C_1 \times B_2). \end{aligned}$$

However, $A_1 \times A_2 = (B_1 \times B_2) + (C_1 \times C_2)$, and this implies that $B_1 \times C_2 \subseteq B_1 \times B_2$ and $C_1 \times B_2 \subseteq B_1 \times C_2$, hence $C_2 \subseteq B_2$ and $B_2 \subseteq C_2$, so that $B_2 = C_2 (= A_2)$. Next, assume the assertion to be true for n and consider:

$$A_1 \times \ldots \times A_n \times A_{n+1} = (B_1 \times \ldots \times B_n \times B_{n+1}) + (C_1 \times \ldots \times C_n \times C_{n+1}),$$

or $A^n \times A_{n+1} = (B^n \times B_{n+1}) = (C^n \times C_{n+1})$, where $A^n = A_1 \times \ldots \times A_n$, $B^n = B_1 \times \ldots \times B_n$ and $C^n = C_1 \times \ldots \times C_n$. Apply the reasoning used in the case $n = 2$ by replacing A_1 by A^n and A_2 by A_{n+1} (so that B_1, B_2 and C_1, C_2 are replaced, respectively, by B^n, B_{n+1} and C^n, C_{n+1}) to get that:

$$A^n = B^n + C^n, \;\; A_{n+1} = B_{n+1} \cup C_{n+1}.$$

The first union is a "+" by the induction hypothesis. The second union may or may not be a "+" as of now. Then:

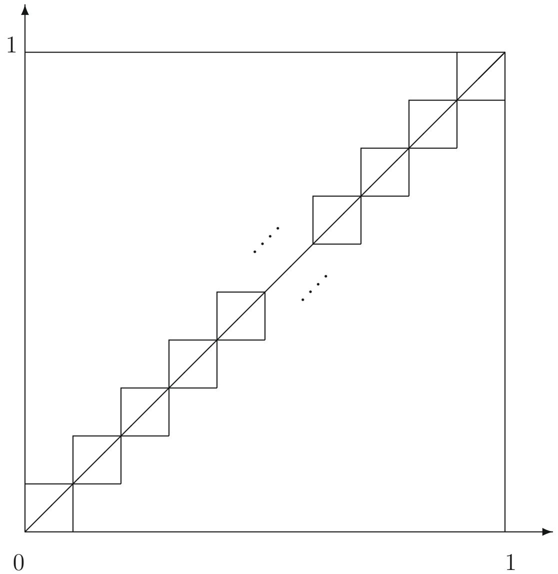
$$\begin{aligned} A^n \times A_{n+1} &= (B^n \cup C^n) \times (B_{n+1} \cup C_{n+1}) \\ &= (B^n \times B_{n+1}) \cup (C^n \times C_{n+1}) \cup (B^n \times C_{n+1}) \cup (C^n \times B_{n+1}). \end{aligned}$$

However, $A^n \times A_{n+1} = (B^n \times B_{n+1}) + (C^n \times C_{n+1})$. Therefore $B^n \times C_{n+1} \subseteq B^n \times B_{n+1}$ and $C^n \times B_{n+1} \subseteq C^n \times C_{n+1}$, so that $C_{n+1} \subseteq B_{n+1}$ and $B_{n+1} \subseteq C_{n+1}$, and hence $B_{n+1} = C_{n+1}$. The proof is completed. #

18. The only properties of the σ-fields $\mathcal{A}_1$ and $\mathcal{A}_2$ used in the proof of Theorem 7 is that $\mathcal{A}_i, i = 1, 2$ are closed under the intersection of two sets in them and also closed under complementations. Since these properties hold also for the case that $\mathcal{A}_i, i = 1, 2$ are fields, $\mathcal{F}_i, i = 1, 2$, the proof is completed. #

19. $\mathcal{C}$ as defined here need not be a σ-field. Here is a *Counterexample*: $\Omega_1 = \Omega_2 = [0, 1]$. For $n \geq 2$, let $I_{n1} = [0, \frac{1}{n}]$, $I_{nj} = (\frac{j-1}{n}, \frac{j}{n}]$, $j = 2, \ldots, n$, and set $E_{nj} = I_{nj} \times I_{nj}$, $j = 1, \ldots, n$. Also, let

$Q_n = \sum_{j=1}^{n} E_{nj}, n \geq 2$. Then Q_n belongs to the field of all finite sums of rectangles. Furthermore, it is clear that $\cap_{n=2}^{\infty} Q_n = D$, where D is the main diagonal determined by the origin and the point (1,1). (See picture below.) However, D is not in the class of all countable sums of rectangles, since it cannot be written as such. D is written as $D = \cup_{x \in [0,1]}(x, x)$, an uncountable union.



Note: In the picture, the first rectangle $E_{n1} = [0, \frac{1}{n}] \times [0, \frac{1}{n}]$, and the subsequent rectangles E_{nj} are: $E_{nj} = (\frac{j-1}{n}, \frac{j}{n}], j = 2, 3, \ldots, n$. #

20. That $\mathcal{C} \neq \oslash$ is obvious. For $A \in \mathcal{C}$, there exists $A' \in \mathcal{A}'$ such that $A = X^{-1}(A')$. Then $A^c = [X^{-1}(A')]^c = X^{-1}[(A')^c]$ with $(A')^c \in \mathcal{A}'$. Thus $A^c \in \mathcal{C}$. Finally, if $A_j \in \mathcal{C}, j = 1, 2, \ldots,$ then $A_j = X^{-1}(A'_j)$ with $A'_j \in \mathcal{A}'$, and hence $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} X^{-1}(A'_j) = X^{-1}\left(\cup_{j=1}^{\infty} A'_j\right)$ with $\cup_{j=1}^{\infty} A'_j \in \mathcal{A}'$, so that $\cup_{j=1}^{\infty} A_j \in \mathcal{C}$, and $\mathcal{C}$ is a σ-field. #

21. That $\mathcal{C}' \neq \oslash$ is obvious. For $A' \in \mathcal{C}'$, there exists $A \in \mathcal{A}$ such that $A = X^{-1}(A')$. Then $X^{-1}[(A')^c] = [X^{-1}(A')]^c = A^c \in \mathcal{A}$, so that $(A')^c \in \mathcal{C}'$. Finally, for $A'_j \in \mathcal{C}', j = 1, 2, \ldots,$ there exists $A_j \in \mathcal{A}$ such that $A_j = X^{-1}(A'_j)$ and $X^{-1}\left(\cup_{j=1}^{\infty} A'_j\right) = \cup_{j=1}^{\infty} X^{-1}(A'_j) = \cup_{j=1}^{\infty} A_j \in \mathcal{A}$, so that $\cup_{j=1}^{\infty} A'_j \in \mathcal{C}'$. It follows that $\mathcal{C}'$ is a σ-field. #

22. A simple example is the following. Let $\Omega = \{a, b, c, d\}$, $\mathcal{A} = \{\oslash, \{a\}, \{b, c, d\}, \Omega\}$, $X(a) = X(b) = 1$, $X(c) = 2$, $X(d) = 3$. Then $\Omega' = \{1, 2, 3\}$ and $X(\{a\}) = \{1\}$, $X(\{b, c, d\}) = \{1, 2, 3\}$, so that $\mathcal{C}' = \{\oslash, \{1\}, \{1, 2, 3\}\}$ which is not a σ-field. #

23. Let $X = \sum_{i=1}^{n} \alpha_i I_{A_i}$ and suppose that $A_i \in \mathcal{A}, i = 1, \ldots, n$. Then for any $B \in \mathcal{B}$, $X^{-1}(B) = \cup A_i$ where the union is taken over those is for which $\alpha_i \in B$.

Since this union is in $\mathcal{A}$, it follows that X is a r.v. Next, let X be a r.v. Then, by assuming without loss of generality that $\alpha_i \neq \alpha_j, i \neq j$, we have $X^{-1}(\{\alpha_i\}) = A_i \in \mathcal{A}$ since $\{\alpha_i\} \in \mathcal{B}, i = 1, \ldots, n$. Clearly, the same reasoning applies when $X = \sum_{i=1}^{\infty} \alpha_i I_{A_i}$. #

24. Let ω belong to the right-hand side. Then $X(\omega) < r$ and $Y(\omega) < x - r$ for some $r \in Q$, so that $X(\omega) + Y(\omega) < x$ and hence ω belongs to the left-hand side. Next, let ω belong to the left-hand side, so that $X(\omega) + Y(\omega) < x$ or $X(\omega) < x - Y(\omega)$. But then there exists $r \in Q$ such that $X(\omega) < r < x - Y(\omega)$ or $X(\omega) < r$ and $r < x - Y(\omega)$ or $X(\omega) < r$ and $Y(\omega) < x - r$, so that ω belongs to the right-hand side. #

25. If X is a r.v., then so is $|X|$, because for all $x \geq 0$, we have $|X|^{-1}((-\infty, x)) = (|X| < x) = (-x < X < x) \in \mathcal{A}$, since X is a r.v. That the converse is not necessarily true is seen by the following simple example. Take $\Omega = \{a, b, c, d\}$, $\mathcal{A} = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}$, and define X by: $X(a) = -1, X(b) = 1, X(c) = -2, X(d) = 2$. Then $\Omega' = \{-2, -1, 1, 2\}$, and let $\mathcal{A}' = \mathcal{P}(\Omega')$. We have $|X|^{-1}(\{1\}) = \{a, b\}, |X|^{-1}(\{2\}) = \{c, d\}, |X|^{-1}(\{-2\}) = |X|^{-1}(\{-1\}) = \emptyset$, and all these sets are in $\mathcal{A}$, so that $|X|$ is measurable. However, $X^{-1}(\{-1\}) = \{a\}$ and $X^{-1}(\{-2\}) = \{c\}$, none of which belongs in $\mathcal{A}$, so that X is not measurable.

As another example, let B be a non-Borel set in $\Re$, and define X by: $X(\omega) = 1, \omega \in B$, and $X(\omega) = -1, \omega \in B^c$. Then X is not $\mathcal{B}$-measurable as $X^{-1}(\{1\}) = B \notin \mathcal{B}$, but $|X|^{-1}(\{1\}) = \Re \in \mathcal{B}$. #

26. $X + Y$ is measurable by Exercise 24. Next, $(-Y \leq y) = (Y \geq -y) \in \mathcal{A}$, so that $-Y$ is measurable. Then $X + (-Y) = X - Y$ is measurable. Now, if Z is measurable, then so is Z^2 because, for $z \geq 0$, $(Z^2 \leq z) = (-\sqrt{z} \leq Z \leq \sqrt{z}) \in \mathcal{A}$. Thus, if X, Y are measurable, then so are $(X+Y)^2$ and $(X-Y)^2$, and therefore so is: $(X+Y)^2 - (X-Y)^2$. But $(X+Y)^2 - (X-Y)^2 = 4XY$. Thus, $4XY$ is measurable, and then so is, clearly, XY.
Finally, if $P(Y \neq 0) = 1$, then, for $y \neq 0$, $(\frac{1}{Y} \leq y) = (Y \geq \frac{1}{y}) \in \mathcal{A}$, so that $\frac{1}{Y}$ is measurable. Thus, X and Y are measurable, and $P(Y \neq 0) = 1$, so that X and $\frac{1}{Y}$ are measurable. Then $X \times \frac{1}{Y} = \frac{X}{Y}$ is measurable. #

27. Since $\sigma(\mathcal{T}_m) = \mathcal{B}^m$, it suffices to show (by Theorem 2) that $f^{-1}(\mathcal{T}_m) \subseteq \mathcal{B}^m$ for f to be measurable. By continuity of f, $f^{-1}(\mathcal{T}_m) \subseteq \mathcal{T}_n \subseteq \mathcal{B}^n$, since $\sigma(\mathcal{T}_n) = \mathcal{B}^n$. Thus, f is measurable. Then, for $B \in \mathcal{B}^m$, $[f(X)]^{-1} = X^{-1}[f^{-1}(B)] \in \mathcal{A}$, since $f^{-1}(B) \in \mathcal{B}^n$ and X is measurable. #

28. For any r.v. Z, it holds: $Z = Z^+ - Z^-$ and $|Z| = Z^+ + Z^-$. Hence $Z^+ = \frac{1}{2}(|Z| + Z)$, $Z^- = \frac{1}{2}(|Z| - Z)$.
Applying this to X, Y and $X + Y$, we get:

$$X^+ = \frac{1}{2}(|X| + X), \; Y^+ = \frac{1}{2}(|Y| + Y), \; (X+Y)^+ = \frac{1}{2}[|X+Y| + (X+Y)].$$

Hence

$$X^+ + Y^+ = \frac{1}{2}[(|X| + |Y|) + (X+Y)] \geq \frac{1}{2}[|X+Y| + (X+Y)] = (X+Y)^+.$$

Likewise,

$$X^- = \frac{1}{2}(|X| - X),\ Y^- = \frac{1}{2}(|Y| - Y),\ (X+Y)^- = \frac{1}{2}[|X+Y| - (X+Y)]$$

and hence

$$X^- + Y^- = \frac{1}{2}[(|X|+|Y|) - (X+Y)] \geq \frac{1}{2}[|X+Y| - (X+Y)] = (X+Y)^-.$$

Alternative proof:
Let $X + Y \leq 0$. Then $(X+Y)^+ = 0 = 0 + 0 \leq X^+ + Y^+$. Let $X + Y > 0$. Then $(X+Y)^+ = X + Y \leq X^+ + Y^+$, because $X = X^+ - X^- \leq X^+$ and $Y = Y^+ - Y^- \leq Y^+$. Thus, $(X+Y)^+ \leq X^+ + Y^+$. Again, let $X + Y < 0$. Then $(X+Y)^- = -(X+Y) = -X - Y \leq X^- + Y^-$, because $X = X^+ - X^-$ or $-X = X^- - X^+ \leq X^-$ and $Y = Y^+ - Y^-$ or $-Y = Y^- - Y^+ \leq Y^-$. Next, let $X + Y \geq 0$. Then $(X+Y)^- = 0 = 0 + 0 \leq X^- + Y^-$, so that $(X+Y)^- \leq X^- + Y^-$. So, again: $(X+Y)^+ \leq X^+ + Y^+$ and $(X+Y)^- \leq X^- + Y^-$. #

29. **(i)** From the definition of B_m, we have: $B_1 = A_1$, and for $m \geq 2$, $B_m = A_1^c \cap \ldots \cap A_{m-1}^c \cap A_m$.

(ii) For $i \neq j$ (e.g., $i < j$), B_i is either A_1 (for $i = 1$) or $B_i = A_1^c \cap \ldots \cap A_{i-1}^c \cap A_i$, whereas $B_j = A_1^c \cap \ldots \cap A_{j-1}^c \cap A_j$, and $B_i \cap B_j = \oslash$, because B_i contains A_i and B_j contains A_i^c (since $i \leq j - 1$).

(iii) Let $\omega = \sum_{m=1}^{\infty} B_m$. Then either $\omega \in B_1 = A_1$, and hence $\omega \in \cup_{n=1}^{\infty} A_n$, or $\omega \notin A_i, i = 1, \ldots, n-1$ and $\omega \in A_n$, so that $\omega \in \cup_{n=1}^{\infty} A_n$. Thus, $\sum_{m=1}^{\infty} B_m \subseteq \cup_{n=1}^{\infty} A_n$. Next, let $\omega \in \cup_{n=1}^{\infty} A_n$. Then either $\omega \in A_1 = B_1$, so that $\omega \in \sum_{m=1}^{\infty} B_m$, or $\omega \notin A_i, i = 1, \ldots, n-1$ and $\omega \in A_n$. Then $\omega \in B_n$, so that $\omega \in \sum_{m=1}^{\infty} B_m$. #

30. **(i)** We have $\underline{\lim}_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$, so that $\omega \in (\underline{\lim}_{n\to\infty} A_n)$ or $\omega \in \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$, therefore $\omega \in \cap_{k=n_0}^{\infty} A_k$ for some n_0, and hence $\omega \in A_k$ for all $k \geq n_0$. Next, let $\omega \in A_n$ for all but finitely many ns; i.e., $\omega \in A_n$ for all $n \geq n_0$. Then $\omega \in \cap_{k=n_0}^{\infty} A_k$ and hence $\omega \in \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$, which completes the proof.

(ii) Here $\overline{\lim}_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$, and hence $\omega \in (\overline{\lim}_{n\to\infty} A_n)$ or $\omega \in \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ implies that $\omega \in \cup_{k=n}^{\infty} A_k$ for $n \geq 1$. From $\omega \in \cup_{k=1}^{\infty} A_k$, let k_1 be the first k for which $\omega \in A_{k_1}$. Next, consider $\cup_{k=k_1+1}^{\infty} A_k$, and from $\omega \in \cup_{k=k_1+1}^{\infty} A_k$, let k_2 be the first k ($\geq k_1 + 1$) for which $\omega \in A_{k_2}$. Continuing like this, we get that ω belongs to infinitely many A_ns. In the other way around, if ω belongs to infinitely many A_ns, that means that there exist $1 < k_1 < k_2 < \ldots$ such that $\omega \in A_{k_j}, j = 1, 2, \ldots$ Then $\omega \in \cup_{k=k_j}^{\infty} A_k, j \geq 1$, and hence $\omega \in \cup_{k=n}^{\infty} A_k$ for $1 \leq n \leq k_1$ and $k_j < n < k_{j+1}, j \geq 1$. Thus, $\omega \in \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ and the result follows. #

31. From $A_k \subseteq B_k, k \geq 1$, we have $\cup_{k=n}^{\infty} A_k \subseteq \cup_{k=n}^{\infty} B_k, n \geq 1$, and hence $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k \subseteq \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} B_k$ or $\overline{\lim}_{n\to\infty} A_n \subseteq \overline{\lim}_{n\to\infty} B_n$ or (A_n i.o.) $\subseteq$ (B_n i.o.) (by Exercise 2). #

32. We have $\underline{\lim}_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$ and $\cap_{k=n}^{\infty} A_k = \cap_{k=n}^{\infty}\{r \in (1 - \frac{1}{k+1}, 1 + \frac{1}{k});\ r \in Q\} = \{1\}$ for all n, so that $\cup_{n=1}^{\infty}\cap_{k=n}^{\infty} A_k = \{1\}$; i.e., $\underline{\lim}_{n\to\infty} A_n = \{1\}$. Next, $\overline{\lim}_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ and $\cup_{k=n}^{\infty} A_k = \cup_{k=n}^{\infty}\{r \in (1 - \frac{1}{k+1}, 1 + \frac{1}{k});\ r \in Q\} = \{r \in (1 - \frac{1}{n+1}, 1 + \frac{1}{n});\ r \in Q\}$, so that $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \cap_{n=1}^{\infty}\{r \in (1 - \frac{1}{n+1}, 1 + \frac{1}{n});\ r \in Q\} = \{1\}$. Thus, $\underline{\lim}_{n\to\infty} A_n = \overline{\lim}_{n\to\infty} A_n = \{1\} = \lim_{n\to\infty} A_n$. #

33. Here $\underline{\lim}_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$, and consider the $\cap_{k=n}^{\infty} A_k$ for n odd or even. Then

$$\bigcap_{k=2n-1}^{\infty} A_k = \Big(\bigcap_{\substack{k \text{ odd} \\ \geq 2n-1}} A_k\Big) \cap \Big(\bigcap_{\substack{k \text{ even} \\ \geq 2n}} A_k\Big),$$

and

$A_{2n-1} \cap A_{2n+1} \cap \ldots = [-1, \frac{1}{2n-1}] \cap [-1, \frac{1}{2n+1}] \cap \ldots = [-1, 0]$, $A_{2n} \cap A_{2n+2} \cap \ldots = [0, \frac{1}{2n}) \cap [0, \frac{1}{2n+2}) \cap \ldots = \{0\}$, so that $\cap_{k=2n-1}^{\infty} A_k = [-1, 0] \cap \{0\} = \{0\}$. Next,

$$\bigcap_{k=2n}^{\infty} A_k = \Big(\bigcap_{\substack{k \text{ even} \\ \geq 2n}} A_k\Big) \cap \Big(\bigcap_{\substack{k \text{ odd} \\ \geq 2n+1}} A_k\Big),$$

and

$A_{2n} \cap A_{2n+2} \cap \ldots = [0, \frac{1}{2n}) \cap [0, \frac{1}{2n+2}) \cap \ldots = \{0\}$, $A_{2n+1} \cap A_{2n+3} \cap \ldots = [-1, \frac{1}{2n+1}] \cap [-1, \frac{1}{2n+3}] \cap \ldots = [-1, 0]$, so that $\cap_{k=2n}^{\infty} A_k = \{0\} \cap [-1, 0] = \{0\}$. It follows that $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_n = \{0\} = \underline{\lim}_{n\to\infty} A_n$.
Next, $\overline{\lim}_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$, and consider the $\cup_{k=n}^{\infty} A_k$ for odd and even values of n. We have

$$\bigcup_{k=2n-1}^{\infty} A_k = \Big(\bigcup_{\substack{k \text{ odd} \\ \geq 2n-1}} A_k\Big) \cup \Big(\bigcup_{\substack{k \text{ even} \\ \geq 2n}} A_k\Big),$$

and

$A_{2n-1} \cup A_{2n+1} \cup \ldots = [-1, \frac{1}{2n-1}] \cup [-1, \frac{1}{2n+1}] \cup \ldots = [-1, \frac{1}{2n-1}]$, $A_{2n} \cup A_{2n+2} \cup \ldots = [0, \frac{1}{2n}) \cup [0, \frac{1}{2n+2}) \cup \ldots = [0, \frac{1}{2n})$, so that $\cup_{k=2n-1}^{\infty} A_k = [-1, \frac{1}{2n-1}] \cup [0, \frac{1}{2n}) = [-1, \frac{1}{2n-1}]$. Next,

$$\bigcup_{k=2n}^{\infty} A_k = \Big(\bigcup_{\substack{k \text{ even} \\ \geq 2n}} A_k\Big) \cup \Big(\bigcup_{\substack{k \text{ odd} \\ \geq 2n+1}} A_k\Big),$$

and

$A_{2n} \cup A_{2n+2} \cup \ldots = [0, \frac{1}{2n}) \cup [0, \frac{1}{2n+2}) \cup \ldots = [0, \frac{1}{2n})$, $A_{2n+1} \cup A_{2n+3} \cup \ldots = [-1, \frac{1}{2n+1}] \cup [-1, \frac{1}{2n+3}] \cup \ldots = [-1, \frac{1}{2n+1}]$, so that $\cup_{k=2n}^{\infty} A_k = [0, \frac{1}{2n}) \cup$

$[-1, \frac{1}{2n+1}] = [-1, \frac{1}{2n})$. It follows that

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = [-1, 1] \cap [-1, \tfrac{1}{2}) \cap [-1, \tfrac{1}{3}] \cap [-1, \tfrac{1}{4}) \cap \ldots$$
$$= [-1, 0] = \overline{\lim_{n\to\infty}} A_n.$$

So, $\underline{\lim}_{n\to\infty} A_n = \{0\}$ and $\overline{\lim}_{n\to\infty} A_n = [-1, 0]$, so that the $\lim_{n\to\infty} A_n$ does not exist. #

34. **(i)** We have:

$$\{[0, 1), [1, 2), \ldots, [n-1, n)\} \subset \{[0, 1), [1, 2), \ldots, [n-1, n), [n, n+1)\}$$

and hence $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$. That $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ follows by the fact that, e.g., $[n, n+1)$ cannot belong in $\mathcal{A}_n$ since all members of $\mathcal{A}_n$ are $\subseteq [0, n)$.

(ii) Let $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$ but not in $\mathcal{A}_1, \ldots, A_n \in \mathcal{A}_n$ but not in $\mathcal{A}_{n-1}, \ldots,$ and set $A = \cup_{i=1}^{\infty} A_i$. Then $A \notin \cup_{n=1}^{\infty} \mathcal{A}_n$, because otherwise, $A \in \mathcal{A}_n$ for some n. However, this is not possible since $\cup_{i=n+1}^{\infty} A_i \notin \mathcal{A}_n$.

(iii) $\mathcal{A}_1 = \{\oslash, [0, 1), [0, 1)^c = (-\infty, 0) \cup [1, \infty), \Re\}$, $\mathcal{A}_2 = \{\oslash, [0, 1), [1, 2), (-\infty, 0) \cup [1, \infty), (-\infty, 1) \cup [2, \infty), [0, 2), (-\infty, 0) \cup [2, \infty), \Re\}$. #

35. **(i)** First, observe that all intersections $A_1' \cap \ldots \cap A_n'$ are pairwise disjoint, so that their unions are, actually, sums. Next, if A and B are in $\mathcal{C}$, it is clear that $A \cup B$ is a sum of intersections $A_1' \cap \ldots \cap A_n'$ (the sum of those intersections in A and those intersections in B), so that $A \cup B$ is in $\mathcal{C}$. Now, if $A \in \mathcal{C}$, then A^c is the sum of all those intersections $A_1' \cap \ldots \cap A_n'$ which are not part of A. Hence A^c is also in $\mathcal{C}$, and $\mathcal{C}$ is a field.

(ii) In forming $A_1' \cap \ldots \cap A_n'$, we have 2 choices at each one of the n steps. Thus, there are 2^n sets of the form $A_1' \cap \ldots \cap A_n'$. Next, in forming their sums, we select k of those members at a time, where $k = 0, 1, \ldots, 2^n$. Therefore the total number of sums is: $\binom{2^n}{0} + \binom{2^n}{1} + \ldots + \binom{2^n}{2^n} = 2^{2^n}$. #

36. **(i)** If $\omega \in A$, then $f(\omega) \in f(A)$ and $\omega \in f^{-1}[f(A)]$. For a concrete example, take $f : \Re \to [0, 1)$ where $f(x) = x^2$, and let $A = [0, 1)$. Then $f(A) = f([0, 1]) = [0, 1)$, and $f^{-1}([0, 1)) = (-1, 1)$. It follows that $f^{-1}[f(A)] = f^{-1}([0, 1)) = (-1, 1) \supset [0, 1) = A$.

(ii) Let $\omega' \in f[f^{-1}(B)]$ which implies that there exists $\omega \in f^{-1}(B)$ such that $f(\omega) = \omega'$. Also, $\omega \in f^{-1}(B)$ implies that $f(\omega) \in B$. Since also $f(\omega) = \omega'$, it follows that $\omega' \in B$. Thus $f[f^{-1}(B)] \subseteq B$.

For a concrete example, let $f : \Re \to \Re$ with $f(x) = c$. Take $B = (c-1, c+1)$, so that $f^{-1}[(c-1, c+1)] = \Re$ and $f(\Re) = \{c\} \subset (c-1, c+1)$. That is, $f[f^{-1}(B)] = \{c\} \subset (c-1, c+1) = B$. #

37. **(i)** Since $X^{-1}(\{-1\}) = A_1$, $X^{-1}(\{1\}) = A_1^c \cap A_2$, and $X^{-1}(\{0\}) = A_1^c \cap A_2^c$, and $A_1, A_1^c \cap A_2, A_1^c \cap A_2^c$ are in $\mathcal{A}$, X is a r.v.

(ii) We have $X^{-1}(\{-1\}) = \{a, b\}$, $X^{-1}(\{1\}) = \{c\}$, $X^{-1}(\{2\}) = \{d\}$, and neither $\{c\}$ nor $\{d\}$ are in $\mathcal{A}$. Then X is not $\mathcal{A}$-measurable.

(iii) We have $X^{-1}(\{-2\}) = \{-2\}$, $X^{-1}(\{-1\}) = \{-1\}$, $X^{-1}(\{0\}) = \{0\}$, $X^{-1}(\{1\}) = \{1\}$, $X^{-1}(\{2\}) = \{2\}$, so that $X^{-1}(\mathcal{B})$ is the field induced in Ω by the partition: $\{\{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}\}$.
The values taken on by X^2 are 0, 1, 4, and $(X^2)^{-1}(\{0\}) = \{0\}$, $(X^2)^{-1}(\{1\}) = \{-1, 1\}$, $(X^2)^{-1}(\{4\}) = \{-2, 2\}$, so that the field induced by X^2 is the one generated by the sets $\{0\}$, $\{-1, 1\}$, $\{-2, 2\}$, and it is, clearly, strictly contained in the one induced by X. #

38. For a fixed k, let $\mathcal{A}_{k,n} = (X_k, \ldots, X_{k+n-1})^{-1}(\mathcal{B})$. Then the σ-fields $\mathcal{A}_{k,n}$, $n \geq 1$, form a nondecreasing sequence and therefore $\mathcal{F}_k = \cup_{n=1}^{\infty} \mathcal{A}_{k,n}$ is a field (but it may fail to be a σ-field; see Exercise 10 in this chapter) and $\mathcal{B}_k = \sigma(\mathcal{F}_k)$. Likewise, $\mathcal{B}_l = \sigma(\mathcal{F}_l)$ where $\mathcal{F}_l = \cup_{n=1}^{\infty} \mathcal{A}_{l,n}$.

However, $\cup_{n=k}^{\infty} \mathcal{A}_n \supseteq \cup_{n=l}^{\infty} \mathcal{A}_n$, so that $\mathcal{B}_k = \sigma(\cup_{n=k}^{\infty} \mathcal{A}_n) \supseteq \sigma(\cup_{n=l}^{\infty} \mathcal{A}_n) = \mathcal{B}_l$. This is so by the way the σ-fields $\mathcal{B}_k$ and $\mathcal{B}_l$ are generated (see Theorem 2(ii) in this chapter). #

39. Since S_k is a function of the X_js, $j = 1, \ldots, k$, $k = 1, \ldots, n$ it follows that $\sigma(S_k) \subseteq \sigma(X_1, \ldots, X_n)$, $k = 1, \ldots, n$. Hence $\cup_{k=1}^{n} \sigma(S_k) \subseteq \sigma(X_1, \ldots, X_n)$ and then $\sigma(\cup_{k=1}^{n} \sigma(S_k)) \subseteq \sigma(X_1, \ldots, X_n)$ or $\sigma(S_1, \ldots, S_n) \subseteq \sigma(X_1, \ldots, X_n)$. Next, $X_k = S_k - S_{k-1}$, $k = 1, \ldots, n$ $(S_0 = 0)$, so that X_k is a function of the S_js, $k = 1, \ldots, n$. Then, as above, $\sigma(X_1, \ldots, X_n) \subseteq \sigma(S_1, \ldots, S_n)$, and equality follows. #

40. Consider the function $f : \Re \to \Re$ defined by $y = f(x) = x + c$. Then, clearly, $f(B) = B_c$. The existing inverse of f, f^{-1}, is given by: $x = f^{-1}(y) = x - c$, and it is clear that $(f^{-1})(B_c) = B$. By setting $g = f^{-1}$, so that $g^{-1} = f$, we have that $g^{-1}(B)(= f(B)) = B_c$. So, g^{-1} is continuous and hence measurable, and $g^{-1}(B) = B_c$. Since B is measurable then so is B_c. #

41. **(i)** Clearly, $\mathcal{F} \neq \oslash$. Next, to show that $\mathcal{F}$ is closed under complementation. Indeed, if $A \in \mathcal{F}$, then

$$\begin{aligned} A &= \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m_i} A_i^j \\ &= (A_1^1 \cap \ldots \cap A_1^{m_1}) \cup \ldots \cup (A_n^1 \cap \ldots \cap A_n^{m_n}) \end{aligned}$$

with all $A_1^1, \ldots, A_1^{m_1}, \ldots, A_n^1, \ldots, A_n^{m_n}$ in $\mathcal{F}_1$, so that

$$\begin{aligned} A^c &= [A_1^1 \cap \ldots \cap A_1^{m_1}) \cup \ldots \cup (A_n^1 \cap \ldots \cap A_n^{m_n})]^c \\ &= [(A_1^1)^c \cup \ldots \cup (A_1^{m_1})^c] \cap \ldots \cap [(A_n^1)^c \cup \ldots \cup (A_n^{m_n})^c] \\ &= \bigcup_{i_1=1}^{m_1} \cdots \bigcup_{i_n=1}^{m_n} [(A_1^{i_1})^c \cap \ldots \cap (A_n^{i_n})^c]. \end{aligned}$$

The fact that $A_1^{i_1}, \ldots, A_n^{i_n}$ are in $\mathcal{F}_1$ implies that $(A_1^{i_1})^c, \ldots, (A_n^{i_n})^c$ are also in $\mathcal{F}_1$, as follows from the definition of $\mathcal{F}_1$. So, A^c is a finite union of a finite intersection of members of $\mathcal{F}_1$, and hence $A^c \in \mathcal{F}_3 (= \mathcal{F})$,

by the definition of $\mathcal{F}_3$. Next, let $A, B \in \mathcal{F}$. To show that $A \cup B \in \mathcal{F}$. Indeed, $A, B \in \mathcal{F}$ implies that $A = A_1 \cup \ldots \cup A_m = (A_1^1 \cap \ldots \cap A_1^{k_1}) \cup \ldots \cup (A_m^1 \cap \ldots \cap A_m^{k_m})$ with $A_i^1, \ldots, A_i^{k_i}$ in $\mathcal{F}_1, i = 1, \ldots, m$, $B = B_1 \cup \ldots \cup B_n = (B_1^1 \cap \ldots \cap B_1^{l_1}) \cup \ldots \cup (B_n^1 \cap \ldots \cap B_n^{l_n})$ with $B_j^1, \ldots, B_j^{l_j}$ in $\mathcal{F}_1$, $j = 1, \ldots, n$,
so that

$$\begin{aligned} A \cup B &= [(A_1^1 \cap \ldots \cap A_1^{k_1}) \cup \ldots \cup (A_m^1 \cap \ldots \cap A_m^{k_m})] \cup \\ &\quad [(B_1^1 \cap \ldots \cap B_1^{l_1}) \cup \ldots \cup (B_n^1 \cap \ldots \cap B_n^{l_n})] \\ &= (A_1^1 \cap \ldots \cap A_1^{k_1}) \cup \ldots \cup (A_m^1 \cap \ldots \cap A_m^{k_m}) \cup \\ &\quad (B_1^1 \cap \ldots \cap B_1^{l_1}) \cup \ldots \cup (B_n^1 \cap \ldots \cap B_n^{l_n}), \end{aligned}$$

which is a finite union of finite intersections of members of $\mathcal{F}_1$. It follows that $A \cup B$ is in $\mathcal{F}_3 (= \mathcal{F})$, so that $\mathcal{F}$ is a field.

(ii) Trivially, $\mathcal{C} \subseteq \mathcal{F}$, so that $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{F}$. To show that $\mathcal{F} \subseteq \mathcal{F}(\mathcal{C})$. Let $A \in \mathcal{F}$. Then, by part (i), $A = (A_1^1 \cap \ldots \cap A_1^{m_1}) \cup \ldots \cup (A_n^1 \cap \ldots \cap A_n^{m_n})$ with all $A_1^1, \ldots, A_1^{m_1}, \ldots, A_n^1, \ldots, A_n^{m_n}$ in $\mathcal{F}_1$.
Clearly, $\mathcal{F}_1 \subseteq \mathcal{F}(\mathcal{C})$ by the definition of $\mathcal{F}_1$. Thus, $A_i^1, \ldots, A_i^{m_i}$ are in $\mathcal{F}(\mathcal{C})$, for $i = 1, \ldots, n$, and then the intersections $A_i^1 \cap \ldots \cap A_i^{m_i}, i = 1, \ldots, n$ are in $\mathcal{F}(\mathcal{C})$, and therefore so is their union $(A_1^1 \cap \ldots \cap A_1^{m_1}) \cup \ldots \cup (A_n^1 \cap \ldots \cap A_n^{m_n})$. Since this union is A, it follows that $A \in \mathcal{F}(\mathcal{C})$. Thus, $\mathcal{F} \subseteq \mathcal{F}(\mathcal{C})$, and the proof is completed. #
Remark: In Exercise 41, in the proof that $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$, the following property was used (in a slightly different notation for simplification); namely, $(C_1^1 \cup \ldots \cup C_1^{m_1}) \cap \ldots \cap (C_n^1 \cup \ldots \cup C_n^{m_n}) = \cup_{i_1=1}^{m_1} \ldots \cup_{i_n=1}^{m_n} (C_1^{i_1} \cap \ldots \cap C_n^{i_n})$.
This is justified as follows: Let ω belong to the right-hand side. Then ω belongs to at leats one of the $m_1 \times \ldots \times m_n$ members of the union, for example, $\omega \in (C_1^{i_1'} \cap \ldots \cap C_n^{i_n'})$ for some $1 \leq i_1' \leq m_1, \ldots, 1 \leq i_n' \leq m_n$. But then $\omega \in (C_1^1 \cup \ldots \cup C_1^{m_1}), \ldots, \omega \in (C_n^1 \cup \ldots \cup C_n^{m_n})$, and therefore $\omega \in [C_1^1 \cup \ldots \cup C_1^{m_1}) \cap \ldots \cap (C_n^1 \cup \ldots \cup C_n^{m_n})]$, or ω belongs to the left-hand side. Next, let ω belong to the left-hand side. Then $\omega \in (C_1^1 \cup \ldots \cup C_1^{m_1}), \ldots, \omega \in (C_n^1 \cup \ldots \cup C_n^{m_n})$, so that $\omega \in C_1^{i_1'}, \ldots, \omega \in C_n^{i_n'}$ for some $1 \leq i_1' \leq m_1, \ldots, 1 \leq i_n' \leq m_n$. But then $\omega \in (C_1^{i_1'} \cap \ldots \cap C_n^{i_n'})$, and $C_1^{i_1'} \cap \ldots \cap C_n^{i_n'}$ is one of the $m_1 \times \ldots \times m_n$ members of the union on the right-hand side. It follows that ω belongs to the right-hand side, and the justification is completed. #

42. Let $A \in \mathcal{A}$. Then $A = \cup_{i=1}^{\infty} A_i, A_i = A_i^1 \cap A_i^2 \cap \ldots$ with $A_i^1, A_i^2, \ldots$ in $\mathcal{A}_1, i \geq 1$. Then

$$A^c = (\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c = \bigcap_{i=1}^{\infty} (A_i^1 \cap A_i^2 \cap \ldots)^c$$