Chapter 1

Set Theory

Ex 1.1. Collection of lines with constant slope and changing intercepts (or, vice-versa) does the job. That is,

$$\mathscr{F} = \{L_r | r \in \mathbb{R}\},$$

where, $L_r = \{(x, y) \in \mathbb{R}^2 | x + y = r\}$ are lines with slope -1 and intercept r. Other similar sets can be constructed which can be indexed by $\mathbb{R}$.

Ex 1.2. A few subsethood relations are

$$\begin{aligned}
\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \\
\mathbb{N} \subseteq \mathbb{Q}^+ \subseteq \mathbb{R}^+ \subseteq \mathbb{R} \\
\mathbb{N} \subseteq \mathbb{Q}^* \subseteq \mathbb{R}^* \subseteq \mathbb{R}
\end{aligned}$$

Similar relations can be obtained by comparing elements.

Ex 1.3. The results for $\mathbb{N}$ are given. Other results can be constructed in a similar manner.

$$\begin{aligned}
\mathbb{N} \cup \mathbb{Z} &= \mathbb{Z} \\
\mathbb{N} \cap \mathbb{Z} &= \mathbb{N} \\
\mathbb{N} \cup \mathbb{Q} &= \mathbb{Q} \\
\mathbb{N} \cap \mathbb{Q} &= \mathbb{N} \\
\mathbb{N} \cup \mathbb{Q}^+ &= \mathbb{Q}^+ \\
\mathbb{N} \cap \mathbb{Q}^+ &= \mathbb{N} \\
\mathbb{N} \cup \mathbb{Q}^- &= \{x \in \mathbb{Q} | x < 0 \text{ or } x \in \mathbb{N}\} \\
\mathbb{N} \cap \mathbb{Q}^- &= \emptyset \\
\mathbb{N} \cup \mathbb{Q}^* &= \mathbb{Q}^* \\
\mathbb{N} \cap \mathbb{Q}^* &= \mathbb{N} \\
\mathbb{N} \cup \mathbb{R}^+ &= \mathbb{R}^+ \\
\mathbb{N} \cap \mathbb{R}^+ &= \mathbb{N} \\
\mathbb{N} \cup \mathbb{R}^- &= \{x \in \mathbb{R} | x < 0 \text{ or } x \in \mathbb{N}\} \\
\mathbb{N} \cap \mathbb{R}^- &= \emptyset \\
\mathbb{N} \cup \mathbb{R}^* &= \mathbb{R}^* \\
\mathbb{N} \cap \mathbb{R}^* &= \mathbb{N}
\end{aligned}$$

A few observations that can be made are as follows. The reader can make as many observations as they want to.

1. $A \subset B$ implies $A \cup B = B$ and $A \cap B = A$.

2. Some of the (newly formed) sets are merely a result of union/intersection of known sets.

Ex 1.4. If $\mathscr{F}$ is a pair-wise disjoint family indexed by Λ, then $\bigcap_{\lambda\in\Lambda} A_\lambda \neq \emptyset$ would imply the existence of some element x in all of A_λ's contradicting the pair-wise disjointness. Thus, any pair-wise disjoint family is disjoint.
The family $\{\{1,2\},\{2,3\},\{1,3\}\}$ is disjoint but not pair-wise disjoint, since $\{1,2\}\cap\{2,3\}=\{2\}\neq\emptyset$.

Ex 1.5. Example in Solution to exercise 4 does the job.

Ex 1.6. Prove the two-way subsethood for each of the equalities using the basic principles of logic and properties of conjunction and disjunction operators.

Ex 1.7. Prove the two-way subsethood for each of the equalities using the basic principles of logic and properties of conjunction and disjunction operators.

Ex 1.8. $\mathcal{P}(\emptyset)=\{\emptyset\}$.

Ex 1.9. The three sets cannot be equal since the elements in all three sets are different. $\mathbb{R}^3$ contains 3-tuples, while $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ and $\mathbb{R}\times(\mathbb{R}\times\mathbb{R})$ contains 2-tuple. Also, the first entry in a 2-tuple of $(\mathbb{R}\times\mathbb{R})\times\mathbb{R}$ is again a 2-tuple, while for elements in $\mathbb{R}\times(\mathbb{R}\times\mathbb{R})$, the second entry is a 2-tuple.

Ex 1.10. $\prod_{\lambda\in\emptyset} A_\lambda=\emptyset$, because if not so, the choice function $f:\emptyset\to\bigcup_{\lambda\in\emptyset} A_\lambda$ will be such that $f(\lambda)\in A_\lambda$. This is possible only when we have such λ's. Many also argue that since $\emptyset$ does not have any elements, this is vacously true. However, even in such an argument, the empty product is not defined. If there is a difficulty to the reader in understanding this, the reader may take it as a convention.

Ex 1.11. $(x_1,x_2,\cdots,x_n)\,L\,(y_1,y_2,\cdots,y_n)$ if and only if

$$\begin{aligned}&(x_1\neq y_1\Rightarrow x_1R_1y_1)\ \text{ and }\ (x_1=y_1\text{ and }x_2\neq y_2\Rightarrow x_2R_2y_2)\ \text{ and }\ \cdots\ \text{ and}\\&(x_1=y_1,x_2=y_2,\cdots,x_{n-1}=y_{n-1}\Rightarrow x_nR_ny_n)\end{aligned}$$

Ex 1.12.

$$\begin{aligned}\leq&:\text{Refexive, Anti-symmetric, Transitive}\\\subseteq&:\text{Refexive, Anti-symmetric, Transitive}\\=&:\text{Refexive, Symmetric, Anti-symmetric, Transitive}\\|&:\text{Refexive, Anti-symmetric, Transitive}\\\equiv&:\text{Refexive, Symmetric, Transitive}\end{aligned}$$

Ex 1.13. Generalising modulo relation on $\mathbb{Z}$ for $n\in\mathbb{N}$, we have $m_1\equiv m_2 \pmod n$ if and only if $n|m_1-m_2$.
The equivalence classes for this relation are $\{[0],[1],[2],\cdots,[n-1]\}$, and an integer m is in an equivalence class j if and only if the remainder after dividing m by n is r.

Ex 1.14. The equivalence classes for $=$ relation on X for each $x\in X$ are $[x]=\{x\}$.
The equivalence classes for $\equiv$ (modulo relation) are as in solution to exercise 13.

Ex 1.15. In the set $X=\{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\}\}$ equipped with $\subseteq$ relation, there is no least element.

Ex 1.16. In the example of solution to exercise 15, the sets $\{1\},\{2\},\{3\}$ are minimal elements and the sets $\{1,2\},\{2,3\},\{1,3\}$ are maximal elements..

Ex 1.17. In general, it is not true. For $X=\{1,2,3\}$, the POSET $(\mathcal{P}(X),\subseteq)$ has a least element, namely $\emptyset$, but the subset $\{\{1\},\{2\},\{3\}\}$ of $\mathcal{P}(X)$ has no least element.

Ex 1.18. $(x, y) \in ((0,0), (1,1))$ if and only if $(0,0) \leq (x, y)$, $(x, y) \leq (1,1)$, $(x, y) \neq (0,0)$ and $(x, y) \neq (1,1)$.
$(0,0) \leq (x, y)$ if and only if $0 \leq x^2 + y^2$. Also, $(x, y) \neq (0,0)$ implies $x^2 + y^2 > 0$. This covers all of $\mathbb{R}^2$ except for $(0,0)$.
$(x, y) \leq (1,1)$ if and only if $x^2 = y^2 \leq 2$ and if $x^2 + y^2 = 2$, then $\tan^{-1}\left(\frac{y}{x}\right) \leq \tan^{-1} 1 = \frac{\pi}{4}$. This covers everything (strictly) inside the circle centered at $(0,0)$ of radius $\sqrt{2}$ and the semi-circle *below* the line $y = x$.
The required interval will be the intersection of the two regions obtained above.

Ex 1.19. $f(0)$ is not defined, and therefore f is not a function.

Ex 1.20.
1. $f(x) = e^x$.
2. $f(x) = (x-1)(x-2)$.
3. $f(x) = x^2$.
4. $f(x) = x$.

Ex 1.21. This is not possible since if the domain is non-empty, there is atleast one image making the codomain non-empty.

Ex 1.22. $f : \{1,2\} \to \{1,2,3\}$ and $g : \{1,2,3\} \to \{1,2\}$ defined as $f(1) = 1$, $f(2) = 2$, $g(1) = 1$, $g(2) = g(3) = 2$. These two functions are as required in all three parts.

Ex 1.23. For $X \neq Y$, if we have functions $f : X \to Y$ and $g : Y \to X$, then the two functions $f \circ g : Y \to Y$ and $g \circ f : X \to X$ are well-defined but cannot be equal since the domains are not equal.

Ex 1.24. $f : \{1,2,3\} \to \{1,2\}$ defined as $f(1) = 1$, $f(2) = f(3) = 2$.
$f : \{1,2\} \to \{1,2,3\}$ defined as $f(1) = 1$, $f(2) = 2$.

Ex 1.25. Since id_X and id_Y are bijective, both $g \circ f$ and $f \circ g$ are bijective. This implies both f and g are bijective.
We also have $f^{-1} = f^{-1} \circ id_Y = f^{-1} \circ g(f \circ g) = (f^{-1} \circ f) \circ g = id_X \circ g = g$.
Finally, $(f^{-1})^{-1} \circ f^{-1} = id_Y$ and $f^{-1} \circ (f^{-1})^{-1} = id_X$. Therefore, $f = (f^{-1})^{-1}$.

Ex 1.26. Functions in solution 24 can be used in appropriate order to obtain required results.

Ex 1.27. If $a = b$ and $c = d$, both the intervals are singleton sets so that there is a bijection between them.
If $a = b$ and $c \neq d$, then $[a, b]$ is a singleton set while $[c, d]$ contains at least two points (c and d), so that any function from $[a, b]$ to $[c, d]$ is not onto and therefore not a bijection.
If $a \neq b$ and $c = d$, using similar arguments, any function is not one-to-one and therefore not a bijection.

Ex 1.28.

$$f(x) = \left(\frac{d-c}{b-a}\right) x + c$$

using proper composition of scaling and translation of intervals.

Ex 1.29. The required bijection is a composition of bijections between the intervals $(a, b), (0,1)$; $(0,1), [0,1]$; and $[0,1] \to [c, d]$ as mentioned in the text and exercises.
If $a = b$, then $(a, b) = \emptyset$ so that no bijection is possible.
If $a \neq b$ and $c = d$, then $[c, d]$ is a singleton set while (a, b) contains at least two distinct points, namely $\frac{a+b}{2}$ and $\frac{a+b}{4}$ so that again, no bijection is possible.

Ex 1.30. For $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$, let $A_1 = [0,2]$ and $A_2 = [-2,2]$ so that $A_1 \subset A_2$ but $f(A_1) = f(A_2) = [0,4]$.

Ex 1.31. If for every two sets A_1 and A_2 of the domain, we have $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, we will prove that f is injective. Let $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. Let $A_1 = \{x_1\}$ and $A_2 = A_1^c$ so that $x_2 \in A_2$. Now, $f(A_1) \cap f(A_2) = \emptyset$, by hypothesis so that $f(x_2) \notin f(A_1)$, which is a contradiction. Therefore, f must be one-to-one.
Conversely, if f is one-to-one, let $y \in f(A_1) \cap f(A_2)$. Then, $\exists x_1 \in A_1$ and $x_2 \in A_2$ such that $f(x_1) = y = f(x_2)$. Since f is injective, $x_1 = x_2$ and therefore, $y \in f(A_1 \cap A_2)$, thereby providing the required equality.

Ex 1.32. The proof goes same as that given in text for union and intersection of two sets.

Ex 1.33. If $B_2 = \emptyset$, then clearly, such a case cannot arise (hypothesis is not possible). Therefore, consider $B_2 \neq \emptyset$. Suppose that $B_1 \subset B_2$ and $f^{-1}(B_1) = f^{-1}(B_2)$. Let $y \in B_2$ such that $y \notin B_1$. Let x be a pre-image of y so that $x \in f^{-1}(B_2) = f^{-1}(B_1)$ so that $f(x) = y \in B_1$ is a contradiction. Therefore, this case cannot ever arise.

Ex 1.34. The proof follows similar lines as for union, given in text.

Ex 1.35. The proof follows similar lines as for union and intersection of two sets.

Ex 1.36. If $m \neq n$, then either there is no injection from I_m to I_n or there is no such surjection. Therefore, if there is a bijection (both injection and surjection), then $m = n$ must hold.

Ex 1.37. Define $g : A \to I_n$ as $g(a) = \min(f^{-1}(a))$. Since f is surjection, g is well-defined. Also, min is unique so that g is one-to-one. Hence, $|A| \leq n$.

Ex 1.38. Let $f : A \to I_m$ and $g : B \to I_n$ be bijections. Then, define $h : A \cup B \to I_{m+n}$ as

$$h(x) = \begin{cases} f(x), & x \in A \\ m + g(x), & x \in B \end{cases}$$

This is the required bijection so that $|A \cup B| = |A| + |B|$.

Ex 1.39. The function $f : (0,1) \to \mathbb{R}$ defined as $f(x) = \tan\left(\frac{\pi}{2}(2x-1)\right)$ is the required bijection. There may be other bijections, the reader would want to construct.

Ex 1.40. Define $f : \mathbb{Z} \to \mathbb{N}$ as

$$f(x) = \begin{cases} 1, & x = 0 \\ 2x, & x > 0 \\ -2x+1, & x < 0 \end{cases}$$

Ex 1.41. Define $f : \mathcal{P}(\mathbb{N}) \to [0,1)$ as

$$f(S) = \begin{cases} 0, & S = \emptyset \\ 0.a_1a_2a_3\cdots, & S \neq \emptyset \end{cases}$$

where $a_n = \begin{cases} 1, & n \in S \\ 0, & n \notin S \end{cases}$. This is one of the one-to-one function.

Ex 1.42. Use minimum of the set of pre-images as used in the text.

Ex 1.43.
1. $g \circ f : A \to \mathbb{N}$ is an injection, where $g : \mathbb{B} \to \mathbb{N}$ is a bijection. Thus, A is countable.
2. $f \circ g : \mathbb{N} \to B$ is a surjection, where $g : \mathbb{N} \to A$ is a bijectio. Then, B is countable.
3. If $B \subseteq A$, then $f : B \to A$ defined as $f(x) = x$ is an injection. From part (1), we have B to be countable.

Ex 1.44. If $\mathbb{Q}^c$ were countable, then $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would have been countable.

Problem Set

1. (a) $x + \dfrac{1}{x} > 2$ is equivalent to $(x-1)^2 > 0$. Therefore, the set is $\mathbb{R} \setminus \{1\}$.
   (b) Set of non-negative real numbers.
   (c) $\mathbb{N}$.
   (d) $\mathbb{N}$.
   (e) $\left\{x \in \mathbb{R} \middle| x = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2a}\right\}$, whenever $a^2 - 4b \geq 0$, otherwise, it is $\emptyset$.
   (f) $\emptyset$.
   (g) $\left\{\dfrac{a+b-c}{2}, \dfrac{a+b+c}{2}\right\}$.
   (h) If $|a-b| = c$, then the set is $\mathbb{R}$, otherwise $\emptyset$.
   (i) Divide $\mathbb{R}^2$ using $y = \pm x$ and then decide what regions lie in the given set.
   (j) $\emptyset$.

2. $A \cap B$ is the set of all integers divisible by the lcm of m and n.

3. Use rules of logic to prove the assertions directly.

4. (a) $A \setminus A = \emptyset$.
   (b) $A \setminus B = A$.
   (c) $A \setminus B = \emptyset$.

5. All symmetric matrices except for the zero matrix.

6. Use the rules of logic to prove the assertions directly.

7. Use the rules of logic to prove the assertions directly.

8. $\mathscr{C} = \left\{C_{((x,y),r)} \middle| ((x,y), r) \in \mathbb{R}^2 \times (0, \infty)\right\}$, where $C_{((x,y),r)} = \left\{(x', y') \in \mathbb{R}^2 \middle| (x'-x)^2 + (y'-y)^2 = r\right\}$. This family is not disjoint, since the two circles centered at $(0,0)$ and $(1,0)$ of radius 1 is not disjoint.

9. Use induction to prove the assertion.

10. The set cannot be written as a Cartesian product of two sets, because if so, all other points which do not satisfy the given relation will also be included.

11. Use rules of logic to prove/disprove the assertions directly.

12. Use rules of logic to prove/disprove the assertions directly.

13. Yes.

14. Use the vertical line test for graphs of functions to deduce the answers.

    (a) No.
    (b) Yes.
    (c) No.

15. It is injective, since odd numbers are mapped to odd numbers and even numbers are mapped to even numbers and the individual mappings are injective.

16. No. $\det \mathbf{0} = \det\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right) = 0$ but $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \neq \mathbf{0}$.

17. (a) Not surjective. 2 has no pre-image.
    (b) Not surjective. $f(x)$ has a minimum at $x = -\frac{1}{2}$ and the value of minimum is $\frac{1}{4}$.
    (c) Not surjective. $e^x > 0$.
    (d) Surjective.
    (e) Not surjective. $|\sin x| \leq 1$.

18. $h : \mathbb{N} \to A \cup B$ is defined as
$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \setminus A \end{cases}$$
is a surjective function.

19. (a) $f^{-1}(x) = x - 1$.
    (b) $f^{-1}(x) = \frac{1-x}{1+x}$.
    (c) $f^{-1}(x) = \sqrt{\frac{y}{1-y}}$, where the square root under consideration is non-negative.

20. Use the associativity of composition and uniqueness of inverse to prove the assertion.

21. f is surjective but it need not be injective.

22. $f^{-1}([0,1]) = \bigcup\limits_{n\in\mathbb{Z}} [2n\pi, (2n+1)\pi]$.

23. It is the set of all nilpotent matrices.

24. R is an equivalence relation with equivalence classes consisting of $\{(0,0)\}$, union of first and third quadrant and union of second and fourth quadrant.

25. R is an equivalence relation. The equivalence classes are circles centered at $(0,0)$, and the set $\{(0,0)\}$.

26. No. Because if so X and $\mathcal{P}(X)$ would be in bijection, contradicting Cantor's theorem. This is due to the fact that there is an injective map $f : X \to \mathcal{P}(X)$ defined as $f(x) = \{x\}$.

27. Use the methods used in text to prove the assertions.

28. No. Infact, a bijection can be produced from $\mathbb{N}^{\mathbb{N}}$ to $[0,1)$ using the decimal expansion and we know that $[0,1)$ is uncountable.

29. No. If so, then $X = A \cup (X \setminus A)$ would be countable.

30. The maximal elements are H_p, where p is a prime.

31. A finite totally ordered set can be listed as $a_1 < a_2 < \cdots < a_n$ so that a_1 is the minimum and a_n is maximum.

32. Singleton sets are examples where minimal elements are same as maximal elements.