CHAPTER 1

DIFFERENCE EQUATIONS

Lecture Suggestions

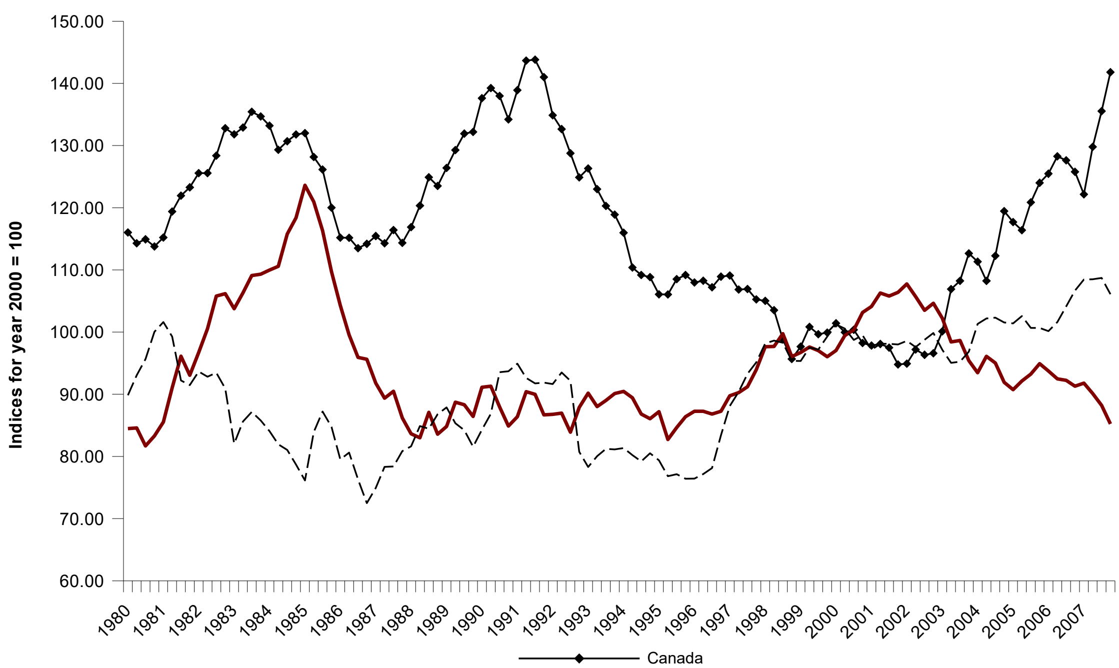
Nearly all students will have some familiarity with the concepts developed in the chapter. A first course in integral calculus makes reference to convergent versus divergent solutions. I draw the analogy between the particular solution to a difference equation and indefinite integrals.

It is essential to convey the fact that difference equations are capable of capturing the types of dynamic models used in economics and political science. Towards this end, I computer-generate a number of simulated series and discuss how their dynamic properties depend on the parameters of the data-generating process. Next, I show the students a number of macroeconomic variables—such as real GDP, real exchange rates, interest rates, and rates of return on stock prices—and ask them to think about the underlying dynamic processes that might be driving each variable. I ask them to think about the economic theory that bears on each of the variables. For example, the figure below shows the three real exchange rate series used in Figure 3.5. Some students see a tendency for the series to revert to a long-run mean value. Nevertheless, the statistical evidence that real exchange rates are actually mean reverting is debatable.

Moreover, there is no compelling theoretical reason to believe that purchasing power parity holds as a long-run phenomenon. The classroom discussion might center on the appropriate way to model the tendency for the levels to meander. At this stage, the precise models are not important. The objective is for students to conceptualize economic data in terms of difference equations.

It is also important to stress the distinction between convergent and divergent solutions. Be sure to emphasize the relationship between characteristic roots and the convergence or divergence of a sequence. Much of the current time-series literature focuses on the issue of unit roots. It is wise to introduce students to the properties of difference equations with unitary characteristic roots at this early stage in the course. Question 5 at the end of this chapter is designed to preview this important issue. The tools to emphasize are the **method of undetermined coefficients** and **lag operators**. Few students will have been exposed to these methods in other classes.

Figure 3.5 Indices of Real Effective Exchange Rates



Answers to Questions

1. Consider the difference equation: $y_t = a_0 + a_1 y_{t-1}$ with the initial condition y_0. Jill solved the difference equation by iterating backwards:

$$
\begin{aligned}
y_t &= a_0 + a_1 y_{t-1} \\
&= a_0 + a_1[a_0 + a_1 y_{t-2}] \\
&= a_0 + a_0 a_1 + a_0(a_1)^2 + ... + a_0(a_1)^{t-1} + (a_1)^t y_0
\end{aligned}
$$

Bill added the homogeneous and particular solutions to obtain: $y_t = a_0/(1 - a_1) + (a_1)^t[y_0 - a_0/(1 - a_1)]$.

a. Show that the two solutions are identical for $|a_1| < 1$.

Answer: The key is to demonstrate:

$$a_0 + a_0 a_1 + a_0(a_1)^2 + ... + a_0(a_1)^{t-1} + (a_1)^t y_0 = a_0/(1 - a_1) + (a_1)^t[y_0 - a_0/(1 - a_1)]$$

First, cancel $(a_1)^t y_0$ from each side and then divide by a_0. The two sides of the equation are identical if:

$$1 + a_1 + (a_1)^2 + ... + (a_1)^{t-1} = 1/(1 - a_1) - (a_1)^t/(1 - a_1)$$

Now, multiply each side by $(1 - a_1)$ to obtain:

$$(1 - a_1)[1 + a_1 + (a_1)^2 + ... + (a_1)^{t-1}] = 1 - (a_1)^t$$

Multiply the two expressions on the left-hand side to obtain:

$$1 - (a_1)^t = 1 - (a_1)^t$$

The two sides of the equation are identical. Hence, Jill and Bob obtained the identical answer.

b. Show that for $a_1 = 1$, Jill's solution is equivalent to: $y_t = a_0 t + y_0$. How would you use Bill's method to arrive at this same conclusion in the case $a_1 = 1$.

Answer: When $a_1 = 1$, Jill's solution can be written as:

$$
\begin{aligned}
y_t &= a_0(1^0 + 1^1 + 1^2 + ... + 1^{t-1}) + y_0 \\
&= a_0 t + y_0
\end{aligned}
$$

To use Bill's method, find the homogeneous solution from the equation $y_t = y_{t-1}$. Clearly, the homogeneous solution is any arbitrary constant A. The key in finding the particular solution is to realize that the characteristic root is unity. In this instance, the particular solution has the form a_0t. Adding the homogeneous and particular solutions, the general solution is:

$y_t = a_0t + A$

To eliminate the arbitrary constant, impose the initial condition. The general solution must hold for all t including $t = 0$. Hence, at $t = 0$, $y_0 = a_0t + A$ so that $A = y_0$. Hence, Bill's method yields:

$$\boxed{y_t = a_0t + y_0}$$

2. The Cobweb model in section 5 assumed *static* price expectations. Consider an alternative formulation called *adaptive expectations*. Let the expected price in t (denoted by $p_t^{¿}$) be a weighted average of the price in t-1 and the price expectation of the previous period. Formally:

$$p_t^{¿} = \alpha p_{t-1} + (1 - \alpha)p_{t-1}^{¿} \qquad 0 < \alpha \leq 1.$$

Clearly, when $\alpha = 1$, the static and adaptive expectations schemes are equivalent. An interesting feature of this model is that it can be viewed as a difference equation expressing the expected price as a function of its own lagged value and the forcing variable p_{t-1}.

a. Find the homogeneous solution for $p_t^{¿}$

Answer: Form the homogeneous equation $p_t^{¿} - (1 - \alpha)p_{t-1}^{¿} = 0$.
The homogeneous solution is:

$$p_t^{¿} \boxed{= A(1-\alpha)^t}$$

where A is an arbitrary constant and $(1-\alpha)$ is the characteristic root.

b. Use lag operators to find the particular solution. Check your answer by substituting your answer into the original difference equation.

Answer: The particular solution can be written as:

$$[\,1 - (1\text{-}\alpha)L\,]\, p_t^{¿} = \alpha p_{t\text{-}1}$$

or

$p_t^{¿} = \alpha p_{t\text{-}1}/[\,1 - (1\text{-}\alpha)L\,]$ so that:

$$p_t^{¿} \boxed{= \alpha[p_{t\text{-}1} + (1\text{-}\alpha)p_{t\text{-}2} + (1\text{-}\alpha)^2 p_{t\text{-}3} + \ldots\,]}$$

To check the answer, substitute the particular solution into the original difference equation

$$\alpha[p_{t\text{-}1} + (1\text{-}\alpha)p_{t\text{-}2} + (1\text{-}\alpha)^2 p_{t\text{-}3} + \ldots\,] = \alpha p_{t\text{-}1} + (1\text{-}\alpha)\alpha[p_{t\text{-}2} + (1\text{-}\alpha)p_{t\text{-}3} + (1\text{-}\alpha)^2 p_{t\text{-}4} + \ldots\,]$$

It should be clear that the equation holds as an identity.

3. Suppose that the money supply process has the form $m_t = m + \rho m_{t\text{-}1} + \varepsilon_t$ where m is a constant and $0 < \rho < 1$.

a. Show that it is possible to express m_{t+n} in terms of the known value m_t and the sequence $\{\varepsilon_{t+1}, \varepsilon_{t+2}, \ldots, \varepsilon_{t+n})$.

Answer: One method is to use forward iteration. Updating the money supply process one period yields $m_{t+1} = m + \rho m_t + \varepsilon_{t+1}$. Update again to obtain

$$\begin{aligned} m_{t+2} &= m + \rho m_{t+1} + \varepsilon_{t+2} \\ &= m + \rho[m + \rho m_t + \varepsilon_{t+1}] + \varepsilon_{t+2} = m + \rho m + \varepsilon_{t+2} + \rho\varepsilon_{t+1} + \rho^2 m_t \end{aligned}$$

Repeating the process for m_{t+3}

$$\begin{aligned} m_{t+3} &= m + \rho m_{t+2} + \varepsilon_{t+3} \\ &= m + \varepsilon_{t+3} + \rho[m + \rho m + \varepsilon_{t+2} + \rho\varepsilon_{t+1} + \rho^2 m_t] \end{aligned}$$

For any period $t+n$, the solution is

$$\boxed{m_{t+n} = m(1 + \rho + \rho^2 + \rho^3 + ... + \rho^{n-1}) + \varepsilon_{t+n} + \rho\varepsilon_{t+n-1} + ... + \rho^{n-1}\varepsilon_{t+1} + \rho^n m_t}$$

b. Suppose that all values of ε_{t+i} for $i > 0$ have a mean value of zero. Explain how you could use your result in part A to forecast the money supply n-periods into the future.

Answer: The expectation of ε_{t+1} through ε_{t+n} is equal to zero. Hence, the expectation of the money supply n periods into the future is

$$\boxed{m(1 + \rho + \rho^2 + \rho^3 + ... + \rho^{n-1}) + \rho^n m_t}$$

As $n \to \infty$, the forecast approaches $m/(1-\rho)$.

4. Find the particular solutions for each of the following:

a. $y_t = a_1 y_{t-1} + \varepsilon_t + \beta_1 \varepsilon_{t-1}$

Answer: Assuming $|a_1| < 1$, you can use lag operators to write the equation as $(1 - a_1L)y_t = \varepsilon_t + \beta_1\varepsilon_{t-1}$. Hence, $y_t = (\varepsilon_t + \beta_1\varepsilon_{t-1})/(1 - a_1L)$.

Now apply the expression $(1 - a_1L)^{-1}$ to each term in the numerator so that:

$$y_t = \varepsilon_t + a_1\varepsilon_{t-1} + (a_1)^2\varepsilon_{t-2} + (a_1)^3\varepsilon_{t-3} + ... + \beta_1[\varepsilon_{t-1} + a_1\varepsilon_{t-2} + (a_1)^2\varepsilon_{t-3} + ...]$$

$$\boxed{y_t = \varepsilon_t + (a_1+\beta_1)\varepsilon_{t-1} + a_1(a_1+\beta_1)\varepsilon_{t-2} + (a_1)^2(a_1+\beta_1)\varepsilon_{t-3} + (a_1)^3(a_1+\beta_1)\varepsilon_{t-4} + ...}$$

If $a_1 = 1$, the improper form of the particular solution is:

$$y_t = b_0 + \varepsilon_t + (1 + \beta_1)\sum_{i=1}^{\infty} ¿\varepsilon_{t-i}¿$$

where: an initial condition is needed to eliminate the constant b_0 and the non-convergent sequence.

b. $y_t = a_1 y_{t-1} + \varepsilon_{1t} + \beta\varepsilon_{2t}$

Answer: Write the equation as $y_t = \varepsilon_{1t}/(1-a_1L) + \beta\varepsilon_{2t}/(1-a_1L)$. Now, apply $(1 - a_1L)^{-1}$ to each term in the numerator so that:

$$y_t = \varepsilon_{1t} + a_1\varepsilon_{1t-1} + (a_1)^2\varepsilon_{1t-2} + (a_1)^3\varepsilon_{1t-3} + ... + \beta[\varepsilon_{2t} + a_1\varepsilon_{2t-1} + (a_1)^2\varepsilon_{2t-2} + (a_1)^3\varepsilon_{2t-3} + ...]$$

Alternatively, you can use the Method of Undetermined Coefficients and write the challenge solution in the form:

$$\boxed{\begin{array}{l} y_t = \Sigma c_i\varepsilon_{1t-i} + \Sigma d_i\varepsilon_{2t-i} \\ \textit{where}\text{ the coefficients satisfy: } c_i = (a_1)^i \text{ and } d_i = \beta(a_1)^i. \end{array}}$$

5. The *Unit Root Problem* in time-series econometrics is concerned with characteristic roots that are equal to unity. In order to preview the issue:

a. Find the homogeneous solution to each of the following.

i) $y_t = a_0 + 1.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - 1.5y_{t-1} + .5y_{t-2} = 0$. The homogeneous solution will take the form $y_t = \alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - 1.5A\alpha^{t-1} + 0.5A\alpha^{t-2} = 0$$

Next, divide by $A\alpha^{t-2}$ to obtain the characteristic equation

$$\alpha^2 - 1.5\alpha + 0.5 = 0$$

The two characteristic roots are $\alpha_1 = 1$, $\alpha_2 = 0.5$. The linear combination of the two homogeneous solutions is also a solution. Hence, letting A_1 and A_2 be two arbitrary constants, the complete homogeneous solution is:

$$\boxed{A_1 + A_2(0.5)^t}$$

ii) $y_t = a_0 + y_{t-2} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - y_{t-2} = 0$. The homogeneous solution will take the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - A\alpha^{t-2} = 0$$

Next, divide by $A\alpha^{t-2}$ to obtain the characteristic equation $\alpha^2 - 1 = 0$. The two characteristic roots are $\alpha_1 = 1$, $\alpha_2 = -1$. The linear combination of the two homogeneous solutions is also a solution. Hence, letting A_1 and A_2 be two arbitrary constants, the complete homogeneous solution is:

$$\boxed{A_1 + A_2(-1)^t}$$

iii) $y_t = a_0 + 2y_{t-1} - y_{t-2} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - 2y_{t-1} + y_{t-2} = 0$. The homogeneous solution always takes the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$$A\alpha^t - 2A\alpha^{t-1} + A\alpha^{t-2} = 0$$

Next, divide by $A\alpha^{t-2}$ to obtain the characteristic equation

$$\alpha^2 - 2\alpha + 1 = 0$$

The two characteristic roots are $\alpha_1 = 1$, and $\alpha_2 = 1$; hence there is a repeated root.

The linear combination of the two homogeneous solutions is also a solution. Letting A_1 and A_2 be two arbitrary constants, the complete homogeneous solution is:

$$\boxed{A_1 + A_2 t}$$

iv) $y_t = a_0 + y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t$

Answer: The homogeneous equation is $y_t - y_{t-1} - 0.25y_{t-2} + 0.25y_{t-3} = 0$. The homogeneous solution always takes the form $y_t = A\alpha^t$. To form the characteristic equation, first substitute this *challenge solution* into the homogeneous equation to obtain

$A\alpha^t - A\alpha^{t-1} - 0.25A\alpha^{t-2} + 0.25A\alpha^{t-3} = 0$

Next, divide by $A\alpha^{t-3}$ to obtain the characteristic equation

$\alpha^3 - \alpha^2 - 0.25\alpha + 0.25 = 0$

The three characteristic roots are $\alpha_1 = 1$, $\alpha_2 = 0.5$, and $\alpha_3 = -0.5$. The linear combination of the three homogeneous solutions is also a solution. Hence, letting A_1, A_2 and A_3 be three arbitrary constants, the complete homogeneous solution is:

$$\boxed{A_1 + A_2(0.5)^t + A_3(-0.5)^t}$$

b. Show that each of the backward-looking particular solutions is not convergent.

i) $y_t = a_0 + 1.5y_{t-1} - .5y_{t-2} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - 1.5L + 0.5L^2)y_t = a_0 + \varepsilon_t$. Factoring the polynomial yields $(1 - L)(1 - 0.5L)y_t = a_0 + \varepsilon_t$. Although the expression $(a_0 + \varepsilon_t)/(1 - 0.5L)$ is convergent, $(a_0 + \varepsilon_t)/(1 - L)$ does not converge.

ii) $y_t = a_0 + y_{t-2} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - L)(1 + L)y_t = a_0 + \varepsilon_t$. It is clear that neither $(a_0 + \varepsilon_t)/(1 - L)$ nor $(a_0 + \varepsilon_t)/(1 + L)$ converges.

iii) $y_t = a_0 + 2y_{t-1} - y_{t-2} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - L)(1 - L)y_t = a_0 + \varepsilon_t$. Here there are two characteristic roots that equal unity. Dividing $(a_0 + \varepsilon_t)$ by either of the $(1 - L)$ expressions does not lead to a convergent result.

iv) $y_t = a_0 + y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t$

Answer: Using lag operators, write the equation as $(1 - L)(1 - 0.5L)(1 + 0.5L)y_t = a_0 + \varepsilon_t$. The expressions $(a_0 + \varepsilon_t)/(1 + 0.5L)$ and $(a_0 + \varepsilon_t)/(1 - 0.5L)$ are convergent, but the expression $(a_0 + \varepsilon_t)/(1 - L)$ is not convergent.

c. Show that equation (i) can be written entirely in first-differences; i.e., $\Delta y_t = a_0 + .5\Delta y_{t-1} + \varepsilon_t$. Find the particular solution for Δy_t.

Answer: Subtract y_{t-1} from each side of $y_t = a_0 + 1.5y_{t-1} - .5y_{t-2} + \varepsilon_t$ to obtain

$y_t - y_{t-1} = a_0 + 0.5y_{t-1} - .5y_{t-2} + \varepsilon_t$ so that
$\Delta y_t = a_0 + 0.5y_{t-1} - 0.5y_{t-2} + \varepsilon_t$
$\quad = a_0 + 0.5\Delta y_{t-1} + \varepsilon_t$

The particular solution for $y_t^{¿} = a_0 + 0.5\, y_{t-1}^{¿} + \varepsilon_t$ is given by
$y_t^{¿} = (a_0 + \varepsilon_t)/(1 - 0.5L)$ so that:

$$y_t^{¿}\boxed{= 2a_0 + \varepsilon_t + 0.5\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.125\varepsilon_{t-3} + \ldots.}$$

d. Similarly transform the other equations into their first-difference form. Find the backward-looking particular solution, if it exists, for the transformed equations.

i) $y_t = a_0 + y_{t-2} + \varepsilon_t$,

Answer: Subtract y_{t-1} from each side to form $y_t - y_{t-1} = a_0 - y_{t-1} + y_{t-2} + \varepsilon_t$ or
$\Delta y_t = a_0 - \Delta y_{t-1} + \varepsilon_t$ so that

$$y_t^{¿} = a_0 - y_{t-1}^{¿} + \varepsilon_t$$

Note that the first difference Δy_t has characteristic root that is equal to -1. The proper form of the backward-looking solution does not exist for this equation. If you attempt the challenge solution $y_t^{¿} = b_0 + \Sigma\alpha_i\varepsilon_{t-i}$, you find:

$$b_0 + \alpha_0\varepsilon_t + \alpha_1\varepsilon_{t-1} + \alpha_2\varepsilon_{t-2} + \alpha_3\varepsilon_{t-3} + \ldots = a_0 - b_0 - \alpha_0\varepsilon_{t-1} - \alpha_1\varepsilon_{t-2} - \alpha_2\varepsilon_{t-3} - \ldots + \varepsilon_t$$

Matching coefficients on like terms yields

$b_0 = a_0 - b_0 \qquad \Rightarrow b_0 = a_0/2$
$\alpha_0 = 1$
$\alpha_1 = -\alpha_0 \qquad \Rightarrow \alpha_1 = -1$

and
$\alpha_i = (-1)^i$

Note that in part f, students are asked to solve an equation of this form with a given initial condition.

ii) $y_t = a_0 + 2y_{t-1} - y_{t-2} + \varepsilon_t$

Answer: Subtract y_{t-1} from each side to obtain $y_t - y_{t-1} = a_0 + y_{t-1} - y_{t-2} + \varepsilon_t$ so that:

$\Delta y_t = a_0 + \Delta y_{t-1} + \varepsilon_t$

Using the definition of $y^{¿}_t$ it follows that $y^{¿}_t = a_0 + y^{¿}_{t-1} + \varepsilon_t$. Again, a proper form for the particular solution does not exist. The improper form is:

$y^{¿}_t = a_0 t + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots$

Notice that the second difference $\Delta^2 y_t$ does have a convergent solution since

$\Delta y^{¿}_t = a_0 + \varepsilon_t$

iii) $y_t = a_0 + y_{t-1} + 0.25y_{t-2} - 0.25y_{t-3} + \varepsilon_t$

Answer: Subtract y_{t-1} from each side and note that $0.25y_{t-2} - 0.25y_{t-3} = 0.25\Delta y_{t-2}$ so that:

$\Delta y_t = a_0 + 0.25\Delta y_{t-2} + \varepsilon_t$ or
$y^{¿}_t = a_0 + 0.25\, y^{¿}_{t-2} + \varepsilon_t$

Write the equation as $(1 - 0.25L^2)\, y^{¿}_t = a_0 + \varepsilon_t$. Since $(1 - 0.25L^2) = (1 - 0.5L)(1 + 0.5L)$, it follows that

$$y^{¿}_t \boxed{= (a_0 + \varepsilon_t)/[(1 - 0.5L)(1 + 0.5L)]}$$

e. Write equations *i* through *iv* using lag operators:

Answer

i. $(1 - 1.5L + 0.5L^2)y_t = \varepsilon_t$ ii. $(1 - L^2)y_t = \varepsilon_t$

iii. $(1 - 2L + L^2)y_t = \varepsilon_t$ iv. $(1 - L - 0.25L^2 + 0.25L^3)y_t = \varepsilon_t$

Note that each of the polynomials has the expression $(1 - L)$ as a factor.

f. Given the initial condition y_0, find the solution for: $y_t = a_0 - y_{t-1} + \varepsilon_t$.

Answer: You can use iteration or the Method of Undetermined Coefficients to verify that the solution is:

$$y_t = \sum_{i=1}^{t} \text{¿}(-1)^{i+t}\varepsilon_i + (-1)^t y_0 + \frac{a_0}{2}[1 - (-1)^t]\text{¿}$$

Using the iterative method, $y_1 = a_0 + \varepsilon_1 - y_0$ and $y_2 = a_0 + \varepsilon_2 - y_1$ so that:

$y_2 = a_0 + \varepsilon_2 - a_0 - \varepsilon_1 + y_0 = \varepsilon_2 - \varepsilon_1 + y_0$

Since $y_3 = a_0 + \varepsilon_3 - y_2$, it follows that $y_3 = a_0 + \varepsilon_3 - \varepsilon_2 + \varepsilon_1 - y_0$. Continuing in this fashion yields:

$y_4 = a_0 + \varepsilon_4 - y_3 = a_0 + \varepsilon_4 - a_0 - \varepsilon_3 + \varepsilon_2 - \varepsilon_1 + y_0 = \varepsilon_4 - \varepsilon_3 + \varepsilon_2 - \varepsilon_1 + y_0$

To confirm the solution for y_t note that $(-1)^{i+t}$ is positive for even values of $(i+t)$ and negative for odd values of $(i+t)$, $(-1)^t$ is positive for even values of t, and $(a_0/2)[1 - (-1)^t]$ equals zero when t is even and a_0 when t is odd.

6. For each of the following, calculate the characteristic roots and the discriminant d in order to describe the adjustment process.

i. $y_t = 0.75y_{t-1} - 0.125y_{t-2}$

Answer:
The discriminant is $(-0.75)^2 - 4*(0.125) = 0.25$ and the characteristic roots are $r_1 = 0.25$ and $r_2 = 0.5$. The roots are real and distinct. Since both roots are positive and less than unity, convergence is direct.

ii. $y_t = 1.5y_{t-1} - 0.75y_{t-2}$

Answer:
The discriminant is $(-1.5)^2 - 4*(0.75) = 0.866i$ (so that the roots are imaginary). Note that $r_1 = 0.075 - 0.433i$ and $r_2 = 0.075 - 0.433i$. Since $| a_2 | < 1$, convergence is oscillatory.

iii. $y_t = 1.8y_{t-1} - 0.81y_{t-2}$

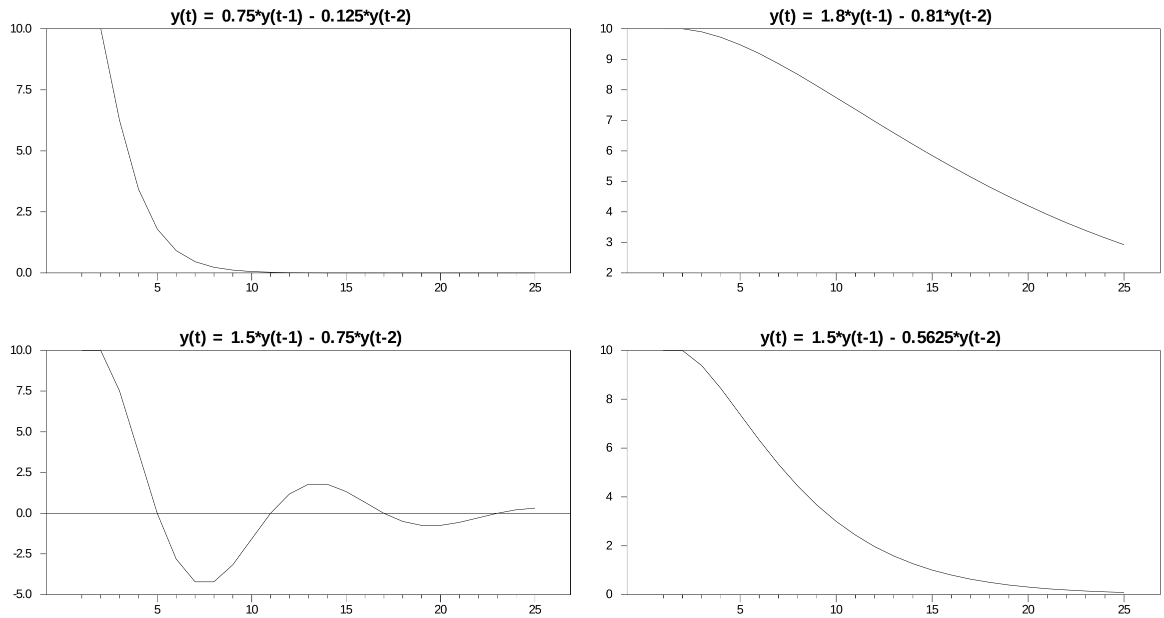
Answer:
The discriminant is $(-1.8)^2 - 4*(0.81) = 0$ so that the roots are repeated. Since $| a_1 | < 2$, there is convergence. Note that $r_1 = r_2 = 0.9$.

iv. $y_t = 1.5y_{t-1} - 0.5625y_{t-2}$

Answer:
The discriminant is $(-1.5)^2 - 4*(0.5625) = 0$ so that the roots are repeated. Since $| a_1 | < 2$, there is convergence. Note that $r_1 = r_2 = 0.75$.

b. Suppose $y_1 = y_2 = 10$. Use a spreadsheet program to calculate and plot the next 25 realizations of the series above.

The Four Impulse Responses



The shapes in *iii* and *iv* are due, in part, to the homogeneous solution $t(a_1/2)^t$.

7. A researcher estimated the following relationship for the inflation rate (π_t):

$$\pi_t = -.05 + 0.7\pi_{t-1} + 0.6\pi_{t-2} + \varepsilon_t$$

a. Suppose that in periods 0 and 1, the inflation rate was 10% and 11%, respectively. Find the homogeneous, particular, and general solutions for the inflation rate.

Answer: The homogeneous equation is $\pi_t - 0.7\pi_{t-1} - 0.6\pi_{t-2} = 0$. Try the challenge solution $\pi_t = A\alpha^t$, so that the characteristic equation is:

$$A\alpha^t - 0.7A\alpha^{t-1} - 0.6A\alpha^{t-2} = 0 \text{ or}$$
$$\alpha^2 - 0.7\alpha - 0.6 = 0$$

The characteristic roots are: $\alpha_1 = 1.2$, $\alpha_2 = -0.5$. Thus, the homogeneous solution is:

$$\boxed{\pi_t = A_1(1.2)^t + A_2(-0.5)^t}$$

The backward-looking particular solution is explosive. Try the challenge solution: $\pi_t = b + \Sigma b_i \varepsilon_{t-i}$. For this to be a solution, it must satisfy

$$b + b_0\varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + b_3\varepsilon_{t-3} + \ldots = -0.05 + 0.7(b + b_0\varepsilon_{t-1} + b_1\varepsilon_{t-2} + b_2\varepsilon_{t-3} + b_3\varepsilon_{t-4} + \ldots) + 0.6(b + b_0\varepsilon_{t-2} + b_1\varepsilon_{t-3} + b_2\varepsilon_{t-4} + b_3\varepsilon_{t-5} + \ldots) + \varepsilon_t$$

Matching coefficients on like terms yields:

$b = -0.05 + 0.7b + 0.6b \qquad \Rightarrow b = 1/6$
$b_0 = 1$
$b_1 = 0.7b_0 \qquad \Rightarrow b_1 = 0.7$
$b_2 = 0.7b_1 + 0.6b_0 \qquad \Rightarrow b_2 = 0.49 + 0.6 = 1.09$

All successive values for b_i satisfy the explosive difference equation

$$\boxed{b_i = 0.7b_{i-1} + 0.6b_{i-2}}$$

If you continue in this fashion, the successive values of the b_i are:
$b_3 = 1.183$; $b_4 = 1.4821$; $b_5 = 1.74727$; $b_6 = 2.11235$; $b_7 = 2.527007$...

Note that the forward-looking solution is not satisfactory here unless you are willing to assume perfect foresight. However, this is inconsistent with the presence of the error term. (After all, the regression would not have to be estimated if everyone had perfect foresight.) The point is that the forward-looking solution expresses the current inflation rate in terms of the future values of the $\{\varepsilon_t\}$ sequence. If $\{\varepsilon_t\}$ is assumed to be a white-noise process, it does not make economic sense to posit that the current inflation rate is determined by the future realizations of ε_{t+i}.

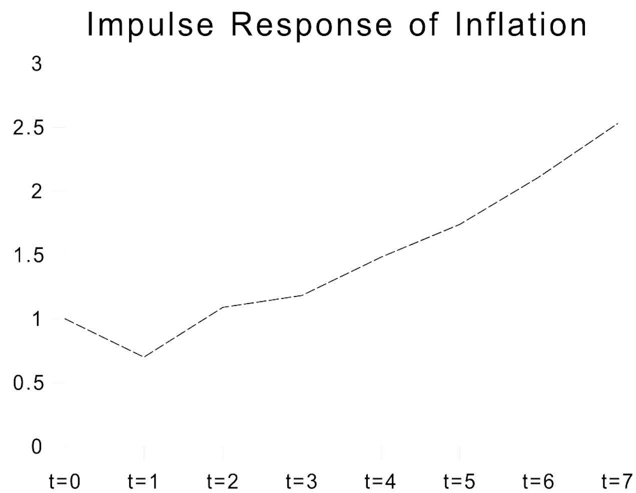
Although the backward-looking particular solution is not convergent, imposing the initial conditions on the particular solution yields finite values for all π_t (as long as t is finite). Consider the general solution

$$\pi_t = 1/6 + \varepsilon_t + 0.7\varepsilon_{t-1} + b_2\varepsilon_{t-2} + \ldots + b_{t-2}\varepsilon_2 + b_{t-1}\varepsilon_1 + b_t\varepsilon_0 + b_{t+1}\varepsilon_{-1} + \ldots + A_1(1.2)^t + A_2(-0.5)^t$$

For $t = 0$ and $t = 1$:

$0.10 = 1/6 + \varepsilon_0 + 0.7\varepsilon_{-1} + b_2\varepsilon_{-2} + ... + A_1 + A_2$
$0.11 = 1/6 + \varepsilon_1 + 0.7\varepsilon_0 + b_2\varepsilon_{-1} + ... + A_1(1.2) + A_2(-0.5)$

These last two equations define A_1 and A_2. Inserting the solutions for A_1 and A_2 into the general solution for π_t eliminates the arbitrary constants.



b. Discuss the shape of the impulse response function. Given that the U.S. is not headed for runaway inflation, why do you believe that the researcher's equation is poorly estimated?

Answer: The impulse response function is given by the $\{b_i\}$ sequence. The impact of an ε_t shock on the rate of inflation is 1. Only 70% of this initial effect remains for one period. After this one-time decay, the effect of the ε_t shock on π_{t+2}, π_{t+3}, ... explodes. You can see the impulse response function in the accompanying chart. The impulse responses imply that the inflation rate will grow exponentially. Given that there will not be runaway inflation, we would want to disregard the estimated model.

8. Consider the stochastic process: $y_t = a_0 + a_2y_{t-2} + \varepsilon_t$.

a. Find the homogeneous solution and determine the stability condition.

Answer: The homogeneous solution has the form $y_t = A\alpha^t$. Form the characteristic equation by substitution of the challenge solution into the original equation, so that:

$A\alpha^t - a_2A\alpha^{t-2} = 0$ so that $\alpha^2 = a_2$.

The two characteristic roots are $\alpha_1 = \sqrt{a_2}$ and $\alpha_2 = -\sqrt{a_2}$. The stability condition is for a_2 to be less than unity in absolute value.

b. Find the particular solution using the Method of Undetermined Coefficients.

Answer: Try the challenge solution $y_t = b + \Sigma b_i \varepsilon_{t-i}$. For this to be a solution, it must satisfy

$$b + b_0\varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2} + b_3\varepsilon_{t-3} + \ldots = a_0 + a_2(b + b_0\varepsilon_{t-2} + b_1\varepsilon_{t-3} + b_2\varepsilon_{t-4} + b_3\varepsilon_{t-5} + \ldots) + \varepsilon_t$$

Matching coefficients on like terms

$b = a_0 + a_2 b \quad \Rightarrow b = a_0/(1-a_2)$
$b_0 = 1$
$b_1 = 0$
$b_2 = a_2 b_0 \quad \Rightarrow b_2 = a_2$
$b_3 = a_2 b_1 \quad \Rightarrow b_3 = 0$ (since $b_1 = 0$)

Continuing in this fashion, it follows that:

$$\boxed{b_i = (a_2)^{i/2} \text{ if } i \text{ is even and } 0 \text{ if } i \text{ is odd.}}$$

9. For each of the following, verify that the posited solution satisfies the difference equation. The symbols c, c_0, and a_0 denote constants.

Equation	Solution
a. $y_t - y_{t-1} = 0$	$y_t = c$
b. $y_t - y_{t-1} = a_0$	$y_t = c + a_0 t$
c. $y_t - y_{t-2} = 0$	$y_t = c + c_0(-1)^t$
d. $y_t - y_{t-2} = \varepsilon_t$	$y_t = c + c_0(-1)^t + \varepsilon_t + \varepsilon_{t-2} + \varepsilon_{t-4} + \ldots$

Answer: Substitute each posited solution into the original difference.

a. Since $y_t = c$ and $y_{t-1} = c$, it immediately follows that $c - c = 0$.
b. Since $y_{t-1} = c + a_0(t-1)$, it follows that $c + a_0 t - c - a_0(t-1) = a_0$.

c. The issue is whether $c + c_0(-1)^t - c - c_0(-1)^{t-2} = 0$? Since $(-1)^t = (-1)^{t-2}$, the posited solution is correct.

d. Does $c + c_0(-1)^t + \varepsilon_t + \varepsilon_{t-2} + \varepsilon_{t-4} + \ldots - c - c_0(-1)^{t-2} - \varepsilon_{t-2} - \varepsilon_{t-4} - \varepsilon_{t-6} - \ldots = \varepsilon_t$? Since $c_0(-1)^t = c_0(-1)^{t-2}$, the posited solution is correct.

10. Part 1: For each of the following, determine whether $\{y_t\}$ represents a stable process. Determine whether the characteristic roots are real or imaginary and whether the real parts are positive or negative.

a. $y_t - 1.2y_{t-1} + .2y_{t-2}$

b. $y_t - 1.2y_{t-1} + .4y_{t-2}$

c. $y_t - 1.2y_{t-1} - 1.2y_{t-2}$

d. $y_t + 1.2y_{t-1}$

e. $y_t - 0.7y_{t-1} - 0.25y_{t-2} + 0.175y_{t-3} = 0$ [Hint: $(x - 0.5)(x + 0.5)(x - 0.7) = x^3 - 0.7x^2 - 0.25x + 0.175$]

Answers:

a. The characteristic equation $\alpha^2 - 1.2\alpha + 0.2 = 0$ has roots $\alpha_1 = 1$ and $\alpha_2 = 0.2$. The unit root means that the $\{y_t\}$ sequence is not convergent.

b. The characteristic equation $\alpha^2 - 1.2\alpha + 0.4 = 0$ has roots $\alpha_1, \alpha_2 = 0.6 \pm 0.2i$. The roots are imaginary. The $\{y_t\}$ sequence exhibits damped wave-like oscillations.

c. The characteristic equation $\alpha^2 - 1.2\alpha - 1.2 = 0$ has roots $\alpha_1 = 1.85$ and $\alpha_2 = -0.65$. One of the roots is outside the unit circle so that the $\{y_t\}$ sequence is explosive.

d. The characteristic equation $\alpha + 1.2 = 0$ has the root $\alpha = -1.2$. The $\{y_t\}$ sequence has explosive oscillations.

e. The characteristic equation $\alpha^3 - 0.7\alpha^2 - 0.25\alpha + 0.175 = 0$ has roots $\alpha_1 = 0.7$, $\alpha_2 = 0.5$ and $\alpha_3 = -0.5$. Although all roots are real, there are damped oscillations due to the presence of the term $(-0.5)^t$.

Part 2: Write each of the above equations using lag operators. Determine the characteristic roots of the inverse characteristic equation.

Answers: Rewrite each using lag operators in order to obtain the inverse characteristic equation.

a. $(1 - 1.2L + 0.2L^2)y_t$ has the inverse characteristic equation $1 - 1.2L + 0.2L^2 = 0$. Solving this quadratic equation for the two values of L (called L_1 and L_2) yield the characteristic roots of the *inverse characteristic equation*. Here, $L_1 = 1.0$ and $L_2 = 5.0$. Since one root lies on the unit circle, the $\{y_t\}$ sequence is not convergent. Note that these roots are the reciprocals of the roots found in Part 1.

b. $(1 - 1.2L + 0.4L^2)y_t$ has the inverse characteristic equation $1 - 1.2L + 0.4L^2 = 0$. The roots are $L_1, L_2 = 1.5 \pm 0.5i$. The roots of the inverse characteristic equation are *outside* the unit circle so that the $\{y_t\}$ sequence exhibits convergent wave-like oscillations.

c. $(1 - 1.2L - 1.2L^2)y_t$ has the inverse characteristic equation $1 - 1.2L - 1.2L^2 = 0$. The roots are -1.54 and 0.54. One of the *inverse* characteristic roots is *inside* the unit circle so that the $\{y_t\}$ sequence is explosive.

d. The inverse characteristic equation $(1 + 1.2L)y_t$ has the inverse characteristic root: $L = -1/1.2 = -0.8333$. Since this *inverse* characteristic root is negative and lies *inside* the unit circle, the $\{y_t\}$ sequence has explosive oscillations.

e. $(1 - 0.7L - 0.25L^2 + 0.175L^3)y_t$ has the inverse characteristic equation $1 - 0.7L - 0.25L^2 + 0.175L^3 = 0$. Factoring yields the equivalent representation $(1 - 0.5L)(1 + 0.5L)(1 - 0.7L) = 0$. The inverse characteristic roots are 2.0, -2.0, and 1.0/0.7 = 1.429. All the inverse characteristic roots lie outside of the unit circle.

11. Consider the stochastic difference equation: $y_t = 0.8y_{t-1} + \varepsilon_t - 0.5\varepsilon_{t-1}$.

a. Suppose that the initial conditions are such that: $y_0 = 0$ and $\varepsilon_0 = \varepsilon_{-1} = 0$. Now suppose that $\varepsilon_1 = 1$. Determine the values y_1 through y_5 by forward iteration.

Answer: If we assume that all future values of $\{\varepsilon_t\} = 0$ we can find the solution. In essence, this is the method used to obtain the impulse response function.

$$\boxed{y_1 = 1, y_2 = 0.3, y_3 = 0.24, y_4 = 0.192, y_5 = 0.1536}$$

b. Find the homogeneous and particular solutions.

Answer: The solution to the homogeneous equation $y_t - 0.8y_{t-1} = 0$ is $y_t = A(0.8)^t$.
Using lag operators, the particular solution is $y_t = (\varepsilon_t - 0.5\varepsilon_{t-1})/(1 - 0.8L)$. If we apply

1/(1-0.8L) to ε_t and -0.5ε_{t-1}, we obtain

$$y_t = \varepsilon_t + 0.8\varepsilon_{t-1} + (0.8)^2\varepsilon_{t-2} + (0.8)^3\varepsilon_{t-3} + \ldots -0.5[\varepsilon_{t-1} + 0.8\varepsilon_{t-2} + (0.8)^2\varepsilon_{t-3} + \ldots]$$
$$= \varepsilon_t + (0.8 - 0.5)\varepsilon_{t-1} + 0.8(0.8 - 0.5)\varepsilon_{t-2} + 0.8^2(0.8 - 0.5)\varepsilon_{t-3} + \ldots$$

$$\boxed{y_t = \varepsilon_t + 0.3\varepsilon_{t-1} + 0.8(0.3)\varepsilon_{t-2} + 0.8^2(0.3)\varepsilon_{t-3} + \ldots}$$

c. Impose the initial conditions in order to obtain the general solution.

Answer: Combining the homogeneous and particular solutions yields the general solution

$$y_t = \varepsilon_t + 0.3\varepsilon_{t-1} + 0.8(0.3)\varepsilon_{t-2} + 0.8^2(0.3)\varepsilon_{t-3} + \ldots + A(0.8)^t.$$

Now impose the initial condition $y_0 = 0$ and $\varepsilon_0 = \varepsilon_{-1} = 0$ to obtain

$0 = \varepsilon_0 + 0.3\varepsilon_{-1} + 0.8(0.3)\varepsilon_{-2} + 0.8^2(0.3)\varepsilon_{-3} + \ldots + A$. Hence,
$A = -\varepsilon_0 - 0.3\varepsilon_{-1} - 0.8(0.3)\varepsilon_{-2} - 0.8^2(0.3)\varepsilon_{-3} + \ldots$

Hence, $A = 0$ if the system began in initial equilibrium. Now substitute for A to obtain

$$y_t = \varepsilon_t + 0.3\sum_{i=0}^{t-2} ¿(0.8)^i\varepsilon_{t-i-1}¿$$

d. Trace out the time path of an ε_t shock on the entire time path of the $\{y_t\}$ sequence.

Answer: $\partial y_t/\partial\varepsilon_t = 1$; $\partial y_{t+1}/\partial\varepsilon_t = \partial y_t/\partial\varepsilon_{t-1} = 0.3$; $\partial y_{t+2}/\partial\varepsilon_t = \partial y_t/\partial\varepsilon_{t-2} = 0.3(0.8)$; $\partial y_{t+3}/\partial\varepsilon_t = \partial y_t/\partial\varepsilon_{t-3} = 0.3(0.8)^2$; and for $i \geq 1$:

$$\boxed{\partial y_{t+i}/\partial\varepsilon_t = \partial y_t/\partial\varepsilon_{t-i} = 0.3(0.8)^{i-1}}$$

12. Use Equation (1.5) to determine the restrictions on α and β necessary to ensure that the $\{y_t\}$ process is stable.

Answer: To determine stability, it is only necessary to examine the homogeneous portion of (1.5); i.e., $y_t - \alpha(1+\beta)y_{t-1} + \alpha\beta y_{t-2} = 0$ where $0 < \alpha < 1$ and $\beta > 0$.

In terms of the notation used in Figure 1.6, $a_1 = \alpha(1+\beta)$ and $a_2 = -\alpha\beta$. Given that α and β are positive, $a_1 > 0$ and $a_2 < 0$. Thus, the point labeled α_2 could correspond to $\alpha(1+\beta)$ units along the a_1 axis and $-\alpha\beta$ units along the a_2 axis. The stability conditions for a second-order difference equation are:

$a_1 + a_2 < 1$
$a_2 < 1 + a_1$
$-a_2 < 1$ (since $a_2 < 0$).

Note that $a_1 + a_2 = \alpha(1+\beta) - \alpha\beta = \alpha$. Since $0 < \alpha < 1$, the first stability condition is always satisfied. To satisfy the second condition (i.e., $a_2 < 1 + a_1$), it is necessary to restrict the coefficients such that $-\alpha\beta < 1 + \alpha(1+\beta)$; simple manipulation yields: $0 < 1 + \alpha + 2\alpha\beta$. Since α and β are positive, the second stability condition necessarily holds. The third condition (i.e., $-a_2 < 1$) is equivalent to $\alpha\beta < 1$ or $\beta < 1/\alpha$. Hence, to ensure stability, it is necessary to restrict β to be less than $1/\alpha$.

13. Consider the following two stochastic difference equations

i. $y_t = 3 + 0.75y_{t-1} - 0.125y_{t-2} + \varepsilon_t$ ii. $y_t = 3 + 0.25y_{t-1} + 0.375y_{t-2} + \varepsilon_t$

a. Use the method of undetermined coefficients to find the particular solution for each equation.

i. Let $y_t = c + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$. The task is to equate the coefficients of:

$$c + c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + \ldots = 3 + 0.75[c + c_0\varepsilon_{t-1} + c_1\varepsilon_{t-2} + c_2\varepsilon_{t-3} + \ldots]$$
$$- 0.125[c + c_0\varepsilon_{t-2} + c_1\varepsilon_{t-3} + c_2\varepsilon_{t-4} + \ldots] + \varepsilon_t$$

Grouping the constant terms: $c = 3 + 0.75c - 0.125c$ so that $c = 3/(1 - 0.75 + 0.125) = 8$

Grouping terms with ε_t: $c_0 = 1$

Grouping terms with ε_{t-1}: $c_1 = 0.75c_0$ so that $c_1 = 0.75$

Grouping terms with ε_{t-2}: $c_2 = 0.75c_1 - 0.125c_0$ so that $c_2 = 0.438$

Note that for $i \geq 2$, all c_i satisfy $c_i = 0.75c_{i-1} - 0.125c_{i-2}$. This can be viewed as a 2nd-order difference equation in the c_i with initial conditions $c_0 = 1$ and $c_1 = 0.75$. The two roots of c_i

$= 0.75c_{i-1} - 0.125c_{i-2}$ are $r_1 = 0.25$ and $r_2 = 0.5$. Hence, the c_i satisfy:

$c_i = A_0(0.25)^i + A_1(0.5)^i$ where A_0 and A_1 are arbitrary constants. Imposing the initial conditions:

$1 = A_0 + A_1$ and $0.75 = A_0(0.25) + A_1(0.5)$ yields $A_0 = -1$ and $A_1 = 2$. Hence, the coefficients are the values of $c_i = -(0.25)^i + 2(0.5)^i$. You can verify that $c_2 = 0.438$, $c_3 = 0.234$, and $c_4 = 0.121$.

ii. Again, let $y_t = c + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$. Now, the task is to equate the coefficients of:

$$c + c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + \ldots = 3 + 0.25[c + c_0\varepsilon_{t-1} + c_1\varepsilon_{t-2} + c_2\varepsilon_{t-3} + \ldots]$$
$$+ 0.375[c + c_0\varepsilon_{t-2} + c_1\varepsilon_{t-3} + c_2\varepsilon_{t-4} + \ldots] + \varepsilon_t$$

Grouping the constant terms: $c = 3 + 0.25c + 0.375c$ so that $c = 3/(1 - 0.25 - 0.375) = 8$

Grouping terms with ε_t: $c_0 = 1$

Grouping terms with ε_{t-1}: $c_1 = 0.25c_0$ so that $c_1 = 0.25$.

Note that for $i \geq 2$, all c_i satisfy $c_i = 0.25c_{i-1} + 0.375c_{i-2}$. This can be viewed as a 2nd-order difference equation in the c_i with initial conditions $c_0 = 1$ and $c_1 = 0.25$. The two roots of $c_i = 0.25c_{i-1} + 0.375c_{i-2}$ are $r_1 = 0.75$ and $r_2 = -0.5$. Hence, the c_i satisfy:

$c_i = A_0(0.75)^i + A_1(-0.5)^i$ where A_0 and A_1 are arbitrary constants. Imposing the initial conditions:

$1 = A_0 + A_1$ and $0.25 = 0.75A_0 - 0.5A_1$yields $A_0 = 0.6$ and $A_1 = .04$. Hence, the coefficients are the values of $c_i = 0.6(0.75)^i + 0.4(-0.5)^i$. For example $c_5 = 0.6(0.75)^5 + 0.4(-0.5)^5 = 0.13$.

b. Find the homogeneous solutions for each equation.

i. For $y_t = 3 + 0.75y_{t-1} - 0.125y_{t-2} + \varepsilon_t$, the homogeneous equation is:

$y_t - 0.75y_{t-1} + 0.125y_{t-2} = 0$

Let $y_t^h = Ar^t$ where A is an arbitrary constant and r is the characteristic root. The value of r must satisfy $r^2 - 0.75r + 0.125 = 0$ so that $r_1 = 0.25$ and $r_2 = 0.5$. As such $y_t^h = A_0(0.25)^t + A_1(0.5)^t$.

ii. For $y_t = 3 + 0.25y_{t-1} + 0.375y_{t-2} + \varepsilon_t$, the homogeneous equation is:

$y_t - 0.25y_{t-1} - 0.375y_{t-2} = 0.$

Let $y_t^h = Ar^t$ where A is an arbitrary constant and r is the characteristic root. The value of r must satisfy $r^2 - 0.725r - 0.375 = 0$ so that $r_1 = 0.75$ and $r_2 = -0.5$. As such $y_t^h = A_0(0.75)^t + A_1(-0.5)^t$.

c. For each process, suppose that $y_0 = y_1 = 8$ and that all values of ε_t for $t = 1, 0, -1, -2, \ldots = 0$. Use the method illustrated by equations (1.75) and (1.76) to find that values of the constants A_1 and A_2.

i. We can combine the homogeneous and particular solutions to obtain $y_t = 8 + \sum c_i \varepsilon_{t-i} + A_0(0.25)^t + A_1(0.5)^t$ where A_0 and A_1 are arbitrary constants and the c_i are given by $c_i = -(0.25)^i + 2(0.5)^i$. Given that all values of $\varepsilon_t = 0$, we have

$y_t = 8 + A_0(0.25)^t + A_1(0.5)^t$

Since $y_0 = y_1 = 8$, it must be the case that $0 = A_0 + A_1$ and $0 = 0.25A_0 + 0.5A_1$. Hence $A_0 = A_1 = 0$.

ii. We can combine the homogeneous and particular solutions to obtain

$y_t = 8 + \sum c_i \varepsilon_{t-i} + A_0(0.75)^t + A_1(-0.5)^t$ where A_0 and A_1 are arbitrary constants and the c_i are given by $c_i = 0.6(0.75)^i + 0.4(-0.5)^i$. Given that all values of $\varepsilon_t = 0$, we have

$y_t = 8 + A_0(0.75)^t + A_1(-0.5)^t$

Since $y_0 = y_1 = 8$, it must be the case that $0 = A_0 + A_1$ and $0 = 0.25A_0 + 0.5A_1$. Hence $A_0 = A_1 = 0$.