Ninth Edition

Solutions Manual

Automatic Control Systems

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Chapter 2

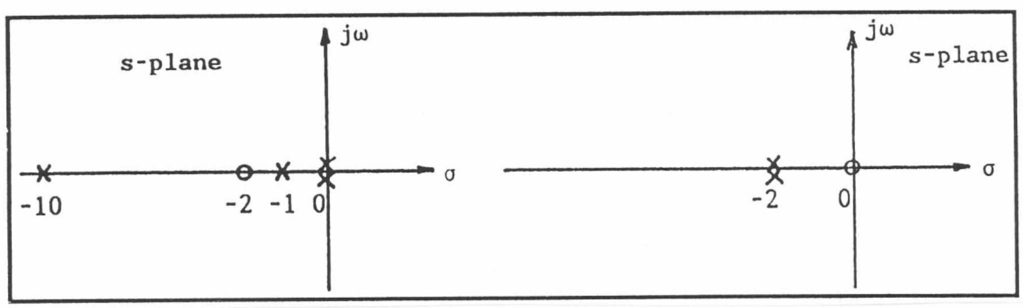
2-1 (a) Poles: $s = 0, 0, -1, -10$;

Zeros: $s = -2, \infty, \infty, \infty$.

(b) Poles: $s = -2, -2$;

Zeros: $s = 0$.

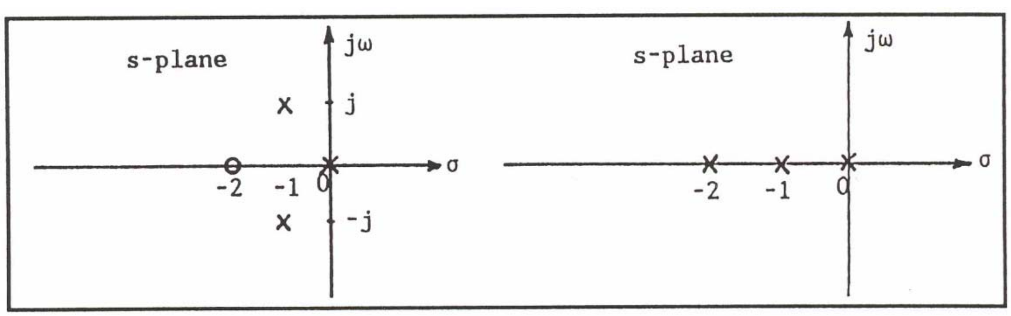
The pole and zero at $s = -1$ cancel each other.



(c) Poles: $s = 0, -1 + j, -1 - j$;

Zeros: $s = -2$.

(d) Poles: $s = 0, -1, -2, \infty$.



2-2) a) $G(s) = \frac{(s+1)}{s(s+2)(s+3)^2}$

b) $G(s) = \frac{s^2}{(s+1)(s+4)}$

c) $G(s) = \frac{s^2-1}{s^2(s+3)(s+1)^2}$

2-3)

MATLAB code:

```
clear all;
s = tf('s')

'Generated transfer function:'
Ga=10*(s+2)/(s^2*(s+1)*(s+10))
'Poles:'
pole(Ga)
'Zeros:'
zero(Ga)

'Generated transfer function:'
Gb=10*s*(s+1)/((s+2)*(s^2+3*s+2))
'Poles:';
pole(Gb)
'Zeros:'
zero(Gb)

'Generated transfer function:'
Gc=10*(s+2)/(s*(s^2+2*s+2))
'Poles:';
pole(Gc)
'Zeros:'
zero(Gc)

'Generated transfer function:'
Gd=pade(exp(-2*s),1)/(10*s*(s+1)*(s+2))
'Poles:';
pole(Gd)
'Zeros:'
zero(Gd)
```

Poles and zeros of the above functions:

(a)

Poles: 0 0 -10 -1

Zeros: -2

(b)

Poles: -2.0000 -2.0000 -1.0000

Zeros: 0 -1

(c)

Poles:

0

-1.0000 + 1.0000i

-1.0000 - 1.0000i

Zeros: -2

Generated transfer function:

(d) using first order Pade approximation for exponential term

Poles:

0

-2.0000

-1.0000 + 0.0000i

-1.0000 - 0.0000i

Zeros:

1

2-4) Mathematical representation:

In all cases substitute $s = j\omega$ and simplify. The use MATLAB to verify.

a)

$$\frac{10(j\omega+2)}{-\omega^2(j\omega+1)(j\omega+10)}$$
$$=\frac{10(j\omega+2)}{-\omega^2(j\omega+1)(j\omega+10)}\times\frac{(-j\omega+1)(-j\omega+10)}{(-j\omega+1)(-j\omega+10)}$$
$$=\frac{10(j\omega+2)(-j\omega+1)(-j\omega+10)}{-\omega^2(\omega^2+1)(\omega^2+100)}$$
$$=R\frac{j\omega+2}{\sqrt{2^2+\omega^2}}\frac{-j\omega+1}{\sqrt{1+\omega^2}}\frac{-j\omega+10}{\sqrt{10^2+\omega^2}}$$
$$=R(e^{j\phi_1}e^{j\phi_2}e^{j\phi_3})$$

$$R=\frac{10\sqrt{2^2+\omega^2}\sqrt{1+\omega^2}\sqrt{10^2+\omega^2}}{-\omega^2(\omega^2+1)(\omega^2+100)};$$
$$\phi_1=\tan^{-1}\frac{\frac{\omega}{\sqrt{2^2+\omega^2}}}{\frac{2}{\sqrt{2^2+\omega^2}}}$$
$$\phi_2=\tan^{-1}\frac{\frac{-\omega}{\sqrt{1+\omega^2}}}{\frac{1}{\sqrt{1+\omega^2}}}$$
$$\phi_3=\tan^{-1}\frac{\frac{-\omega}{\sqrt{10^2+\omega^2}}}{\frac{10}{\sqrt{10^2+\omega^2}}}$$
$$\phi=\phi_1+\phi_2+\phi_3$$

b)

$$\frac{10}{(j\omega+1)^2(j\omega+3)}$$
$$=\frac{10}{(j\omega+1)(j\omega+1)(j\omega+3)}\times\frac{(-j\omega+1)(-j\omega+1)(-j\omega+3)}{(-j\omega+1)(-j\omega+1)(-j\omega+3)}$$
$$=\frac{10(-j\omega+1)(-j\omega+1)(-j\omega+3)}{(\omega^2+1)^2(\omega^2+9)}$$
$$=R\frac{-j\omega+1}{\sqrt{1+\omega^2}}\frac{-j\omega+1}{\sqrt{1+\omega^2}}\frac{-j\omega+3}{\sqrt{9+\omega^2}}$$
$$=R(e^{j\phi_1}e^{j\phi_2}e^{j\phi_3})$$

$$R=\frac{10\sqrt{1+\omega^2}\sqrt{9+\omega^2}}{(\omega^2+1)^2(\omega^2+9)};$$
$$\phi_1=\tan^{-1}\frac{\frac{-\omega}{\sqrt{1+\omega^2}}}{\frac{1}{\sqrt{1+\omega^2}}}$$
$$\phi_2=\tan^{-1}\frac{\frac{-\omega}{\sqrt{1+\omega^2}}}{\frac{1}{\sqrt{1+\omega^2}}}$$
$$\phi_3=\tan^{-1}\frac{\frac{-\omega}{\sqrt{9+\omega^2}}}{\frac{3}{\sqrt{9+\omega^2}}}$$
$$\phi=\phi_1+\phi_2+\phi_3$$

c)

$$\frac{10}{j\omega(j2\omega+2-\omega^2)}$$
$$=\frac{-10j}{\omega(j2\omega+2-\omega^2)}\times\frac{(2-\omega^2-j2\omega)}{(2-\omega^2-j2\omega)}$$
$$=\frac{10(-2\omega-(2-\omega^2)j)}{\omega(4\omega^2+(2-\omega^2)^2)}$$
$$=R\frac{-2\omega-(2-\omega^2)j}{\sqrt{4\omega^2+(2-\omega^2)^2}}$$
$$=R(e^{j\phi})$$

$$R=\frac{10\sqrt{4\omega^2+(2-\omega^2)^2}}{\omega(4\omega^2+(2-\omega^2)^2)}=\frac{10}{\omega\sqrt{4\omega^2+(2-\omega^2)^2}};$$

$$\phi=\tan^{-1}\frac{-2-\omega^2}{\sqrt{4\omega^2+(2-\omega^2)^2}}\bigg/\frac{-2\omega}{\sqrt{4\omega^2+(2-\omega^2)^2}}$$

d)

$$\frac{e^{-2j\omega}}{10j\omega(j\omega+1)(j\omega+2)}$$
$$=\frac{-j(-j\omega+1)(-j\omega+2)}{10\omega(\omega^2+1)(\omega^2+2)}e^{-2j\omega}$$
$$=R\frac{-j\omega+2}{\sqrt{2^2+\omega^2}}\frac{-j\omega+1}{\sqrt{1+\omega^2}}e^{-2j\omega-j\pi/2}$$
$$=R(e^{j\phi_1}e^{j\phi_2}e^{j\phi_3})$$

$$R=\frac{1}{10\omega\sqrt{2^2+\omega^2}\sqrt{1+\omega^2}};$$

$$\phi_1=\tan^{-1}\frac{-\omega}{\sqrt{2^2+\omega^2}}\bigg/\frac{2}{\sqrt{2^2+\omega^2}}$$

$$\phi_2=\tan^{-1}\frac{-\omega}{\sqrt{1+\omega^2}}\bigg/\frac{1}{\sqrt{1+\omega^2}}$$

$$\phi=\phi_1+\phi_2+\phi_3$$

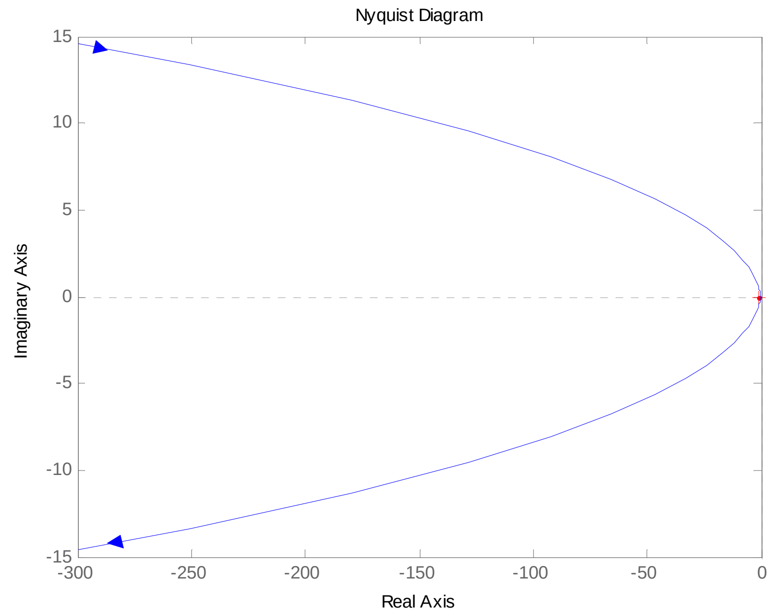
MATLAB code:

```
clear all;

s = tf('s')


'Generated transfer function:'

Ga=10*(s+2)/(s^2*(s+1)*(s+10))

figure(1)
```

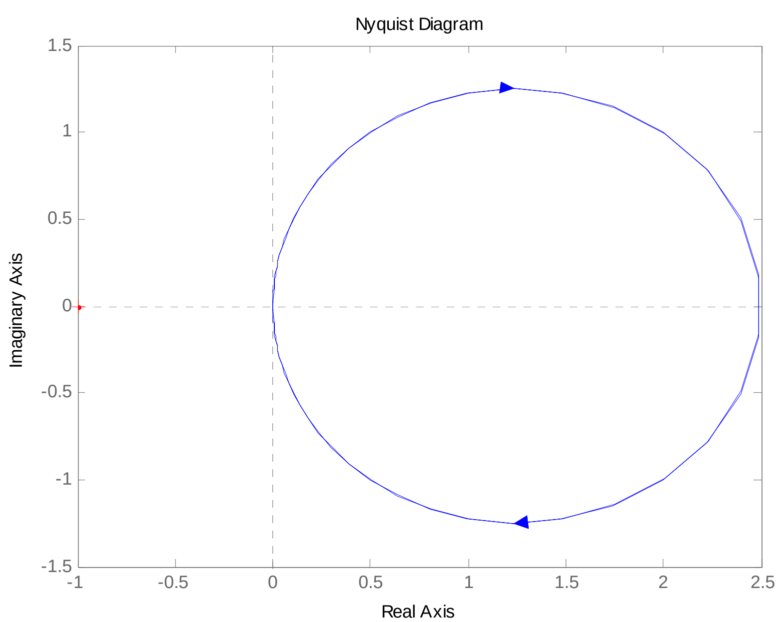
```
Nyquist(Ga)

'Generated transfer function:'
Gb=10*s*(s+1)/((s+2)*(s^2+3*s+2))
figure(2)
Nyquist(Gb)

'Generated transfer function:'
Gc=10*(s+2)/(s*(s^2+2*s+2))
figure(3)
Nyquist(Gc)


'Generated transfer function:'
Gd=pade(exp(-2*s),1)/(10*s*(s+1)*(s+2))
figure(4)
Nyquist(Gd)
```

Nyquist plots (polar plots):

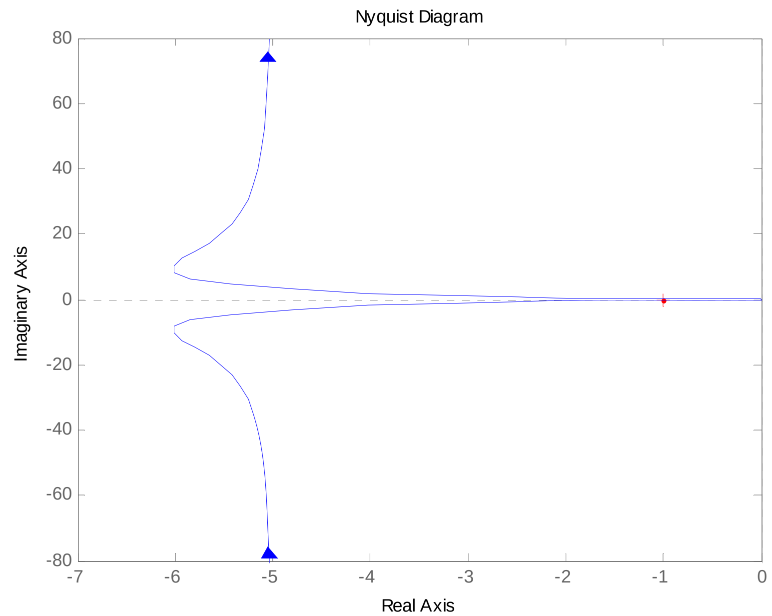
Part(a)


Nyquist Diagram
15
10
5
0
-5
-10
-15
Imaginary Axis
-300
-250
-200
-150
-100
-50
0
Real Axis

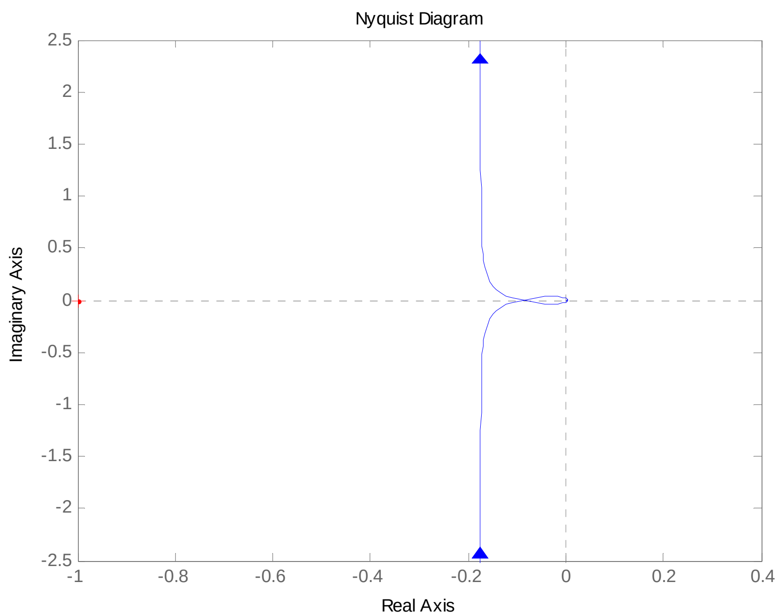
Part(b)


Nyquist Diagram
1.5
1
0.5
0
-0.5
-1
-1.5
Imaginary Axis
-1
-0.5
0
0.5
1
1.5
2
2.5
Real Axis

Part(c)


Nyquist Diagram
Imaginary Axis
Real Axis
80
60
40
20
0
-20
-40
-60
-80
-7
-6
-5
-4
-3
-2
-1
0

Part(d)


Nyquist Diagram
Imaginary Axis
Real Axis
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4

2-5) In all cases find the real and imaginary axis intersections.

$$G(j\omega)=\frac{10}{(j\omega-2)}=\frac{10(-j\omega+2)}{(\omega^2+4)}=\frac{10}{\sqrt{(\omega^2+4)}}\frac{2-j\omega}{\sqrt{(\omega^2+4)}};$$

$$\mathrm{Re}\{G(j\omega)\}=\cos\phi=\frac{2}{\sqrt{(\omega^2+4)}},$$

a) $$\mathrm{Im}\{G(j\omega)\}=\sin\phi=\frac{-\omega}{\sqrt{(\omega^2+4)}},$$

$$\phi=\tan^{-1}\frac{2}{\sqrt{(\omega^2+4)}}\Big/\frac{-\omega}{\sqrt{(\omega^2+4)}}$$

$$R=\frac{10}{\sqrt{(\omega^2+4)}}$$

$$\lim_{\omega\to 0}G(j\omega)=5;\phi=\tan^{-1}1/_{-0}=-90^\circ$$

$$\lim_{\omega\to\infty}G(j\omega)=0;\phi=\tan^{-1}0/_{-1}=-180^\circ$$

Real axis intersection @ $j\omega=0$

Imaginary axis intersection does not exist.

b&c) $lim_{\omega\to 0}\,G(j\omega)=1\angle 0^\circ$

$lim_{\omega\to\infty}\,G(j\omega)=0\angle -180^\circ$

$$G(j\omega)=\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+2\xi\left(j\frac{\omega}{\omega_n}\right)}=\frac{\left(1-\left(\frac{\omega}{\omega_n}\right)^2-2\xi\left(j\frac{\omega}{\omega_n}\right)\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\left(\frac{\omega}{\omega_n}\right)^2}$$

Therefore:

$$\mathrm{Re}\{\,G(j\omega)\,\}=\frac{1-\left(\frac{\omega}{\omega_n}\right)^2}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\left(\frac{\omega}{\omega_n}\right)^2}$$

$$\mathrm{Im}\,\{G(j\omega)\}=-\frac{2\xi\left(j\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\left(\frac{\omega}{\omega_n}\right)^2}$$

If $\mathrm{Re}\{G(j\omega)\} = 0$ ⇨ $\omega = \omega_n$

If $\mathrm{Im}\{G(j\omega)\} = 0$ ⇨ $\begin{cases} \omega = 0 \\ \omega \to 0 \\ \omega \to \infty \end{cases}$

If $\omega = \omega_n$ ⇨ $\begin{cases} G(j\omega_n) = \dfrac{1}{j2\xi} \\ \angle G(j\omega_n) = -90^o \end{cases}$

If $\omega = \omega_n$ and $\xi = 1$ ⇨ $G(j\omega_n) = \dfrac{1}{2j}$

If $\omega = \omega_n$ and $\xi \to 0$ ⇨ $G(j\omega_n) \to \infty$

If $\omega = \omega_n$ and $\xi \to \infty$ ⇨ $G(j\omega_n) \to 0$

d) $G(j\omega) = \dfrac{T\omega - j}{\omega(1+\omega^2 T^2)}$

$\lim_{\omega\to 0} G(j\omega) = \infty \angle -90^o$

$\lim_{\omega\to\infty} G(j\omega) = 0 \angle -180^o$

e) $|G(j\omega)| = \left|\dfrac{e^{-j\omega L}}{1+j\omega T}\right| = \dfrac{1}{\sqrt{1+\omega^2 T^2}}$

$\angle G(j\omega) = \angle\dfrac{1}{1+j\omega T} + \angle e^{-j\omega L} = \tan^{-1}(\omega T) - \omega L$

2-6

MATLAB code:

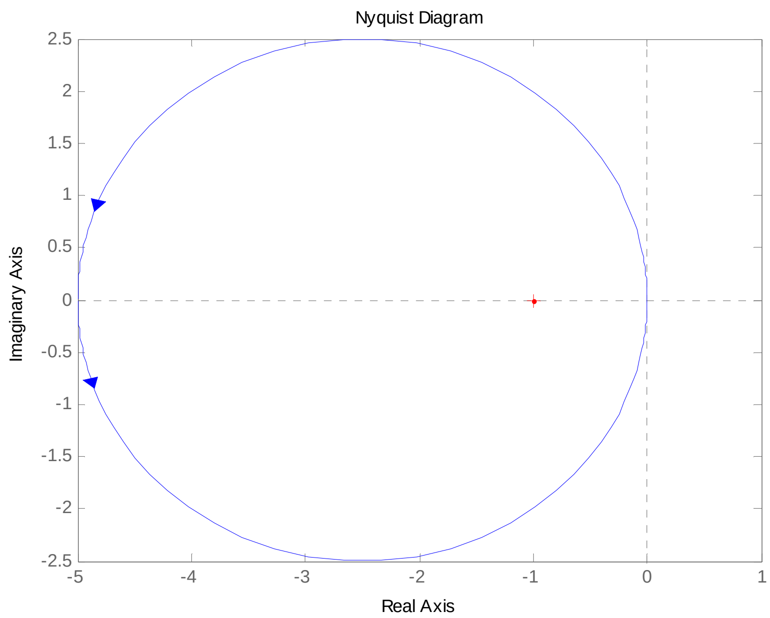
```
clear all;
s = tf('s')

%Part(a)
Ga=10/(s-2)
figure(1)
nyquist(Ga)

%Part(b)
zeta=0.5;  %asuuming a value for zeta <1
wn=2*pi*10 %asuuming a value for wn
Gb=1/(1+2*zeta*s/wn+s^2/wn^2)
figure(2)
nyquist(Gb)

%Part(c)
zeta=1.5;  %asuuming a value for zeta >1
wn=2*pi*10
Gc=1/(1+2*zeta*s/wn+s^2/wn^2)
figure(3)
nyquist(Gc)

%Part(d)
T=3.5 %assuming value for parameter T
Gd=1/(s*(s*T+1))
figure(4)
nyquist(Gd)
```

```
%Part(e)

T=3.5

L=0.5

Ge=pade(exp(-1*s*L),2)/(s*T+1)

figure(5)

hold on;

nyquist(Ge)
```

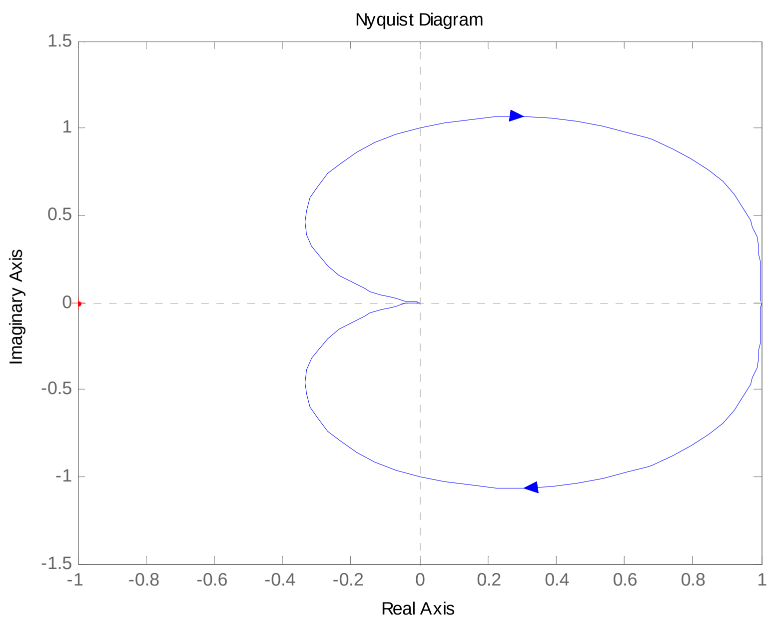
notes: In order to use Matlab Nyquist command, parameters needs to be assigned with values, and Pade approximation needs to be used for exponential term in part (e).

Nyquist diagrams are as follows:

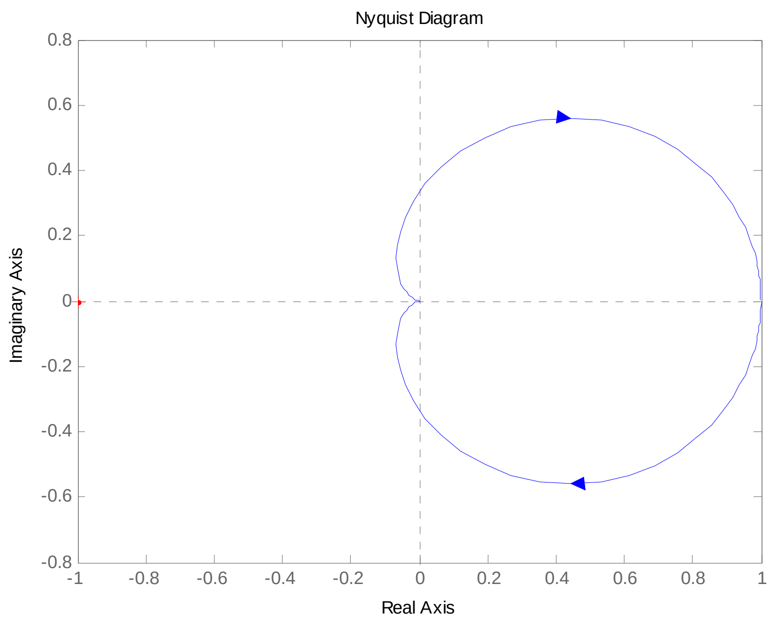
Part(a)


Nyquist Diagram
Imaginary Axis
Real Axis
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-5
-4
-3
-2
-1
0
1

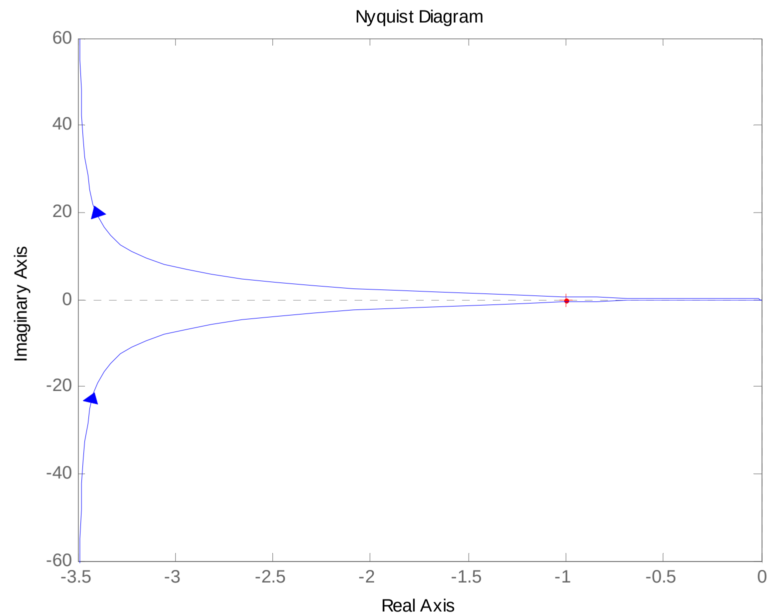
Part(b)


Nyquist Diagram
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

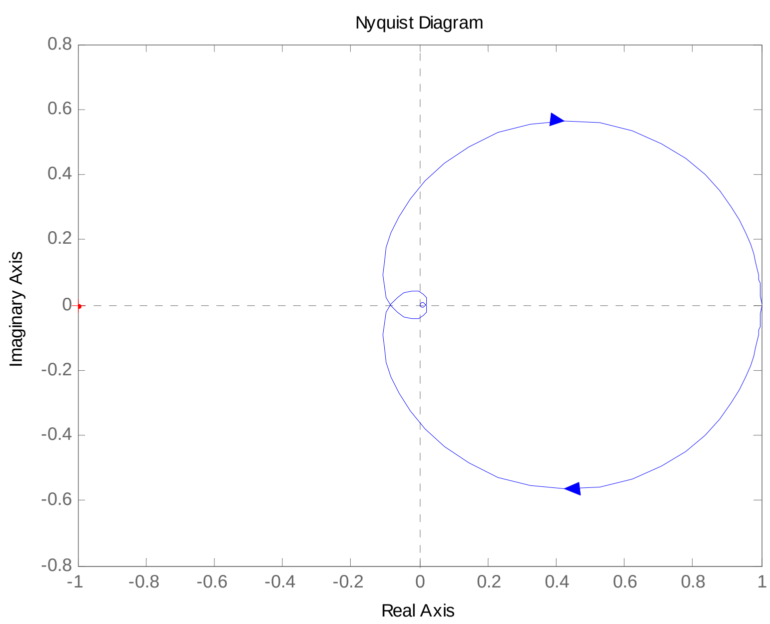
Part(c)


Nyquist Diagram
Imaginary Axis
Real Axis
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Part(d)


Nyquist Diagram
Imaginary Axis
Real Axis
60
40
20
0
-20
-40
-60
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0

Part(e)


Nyquist Diagram
Imaginary Axis
Real Axis
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

2-7) a) $G(j\omega) = \frac{2(j2\omega+1)}{j\omega(0.1j\omega+1)(0.02j\omega+1)}$

Steps for plotting |G|:

(1) For ω < 0.1, asymptote is $\frac{2}{j\omega}$

Break point: ω = 0.5
Slope = -1 or -20 dB/decade

(2) For 0.5 < ω < 10
Break point: ω = 10
Slope = -1+1 = 0 dB/decade

(3) For 10 < ω < 50:
Break point: ω = 50
Slope = -1 or -20 dB/decade

(4) For ω > 50
Slope = -2 or -40 dB/decade

Steps for plotting ∠ G

(1) $\angle \frac{2}{j\omega} = -90^o$

(2) $\angle \frac{1}{2j\omega+1} = \begin{cases} \omega \to 0: \ \angle \frac{1}{2j\omega+1} \to -90^o \\ \omega \to \infty \ : \ \angle \frac{1}{2j\omega+1} \to 0^o \end{cases}$

(3) $\angle \frac{1}{0.1j\omega+1} = \begin{cases} \omega \to 0: \ \angle \frac{1}{0.1j\omega+1} \to 0^o \\ \omega \to \infty: \ \angle \frac{1}{0.1j\omega+1} \to -90^o \end{cases}$

(4) $\angle \frac{1}{0.02j\omega+1} = \begin{cases} \omega \to 0: \ \angle \frac{1}{0.02j\omega+1} \to -90^o \\ \omega \to \infty \ : \ \angle \frac{1}{0.02j\omega+1} \to 0 \end{cases}$

b) Let's convert the transfer function to the following form:

$G(j\omega) = \frac{25}{10j\omega\left(-\frac{2.5}{10}\omega^2+j\frac{\omega}{10}\right)+1} \Rightarrow G(s) = \frac{5}{2}\frac{1}{s\left(\frac{s^2}{4}+0.1s+1\right)}$

Steps for plotting |G|:

(1) Asymptote: ω < 1 $|G(j\omega)| \cong 2.5/\omega$

Slope: -1 or -20 dB/decade
$|G(j\omega)|_{\omega=1} = 2.5$

(2) $\omega_n = 2$ and $\xi = 0.1$ for second-order pole
break point: $\omega = 2$
slope: -3 or -60 dB/decade
$|G(j\omega)|_{\omega=2} = \frac{1}{2\xi} = 5$

Steps for plotting $\angle G(j\omega)$:

(1) for term 1/s the phase starts at -90° and at $\omega = 2$ the phase will be -180°
(2) for higher frequencies the phase approaches -270°

c) Convert the transfer function to the following form:

$$G(j\omega) = \frac{0.01j\omega - \omega^2 + 1}{-\omega^2\left(0.01j\omega - \frac{\omega^2}{9} + 1\right)}$$

for term $\frac{1}{\omega^2}$, slope is -2 (-40 dB/decade) and passes through $|G(j\omega)|_{\omega=1} = 1$

(1) the breakpoint: $\omega = 1$ and slope is zero
(2) the breakpoint: $\omega = 2$ and slope is -2 or -40 dB/decade
$|G(j\omega)|_{\omega=1} = 2\xi = 0.01$ below the asymptote
$|G(j\omega)|_{\omega=1} = \frac{1}{2\xi} = \frac{1}{0.02} = 50$ above the asymptote

Steps for plotting $\angle G$:

(1) phase starts from -180° due to $\frac{1}{s^2}$

(2) $\angle G(j\omega)|_{\omega=1} = 0$
(3) $\angle G(j\omega)|_{\omega=2} = -180^\circ$

d) $G(j\omega) = \frac{1}{1+2\xi\left(j\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2}$

Steps for plotting the $|G|$:

(1) Asymptote for $\frac{\omega}{\omega_n} < 1$ is zero

(2) Breakpoint: $\frac{\omega}{\omega_n} = 1$, slope = -1 or -10 dB/decade

(3) As ξ is a damping ratio, then the magnitude must be obtained for various ξ when $0 \le \xi \le 1$

The high frequency slope is twice that of the asymptote for the single-pole case

Steps for plotting ∠ G:

(1) The phase starts at 0^o and falls -1 or -20 dB/decade at $\frac{\omega}{\omega_n}$ = 0.2 and approaches -180^o at $\frac{\omega}{\omega_n}$ = 5. For $\frac{\omega}{\omega_n}$ > 5, the phase remains at -180^o.

(2) As ξ is a damping ratio, the phase angles must be obtained for various ξ when $0 \le \xi \le 1$

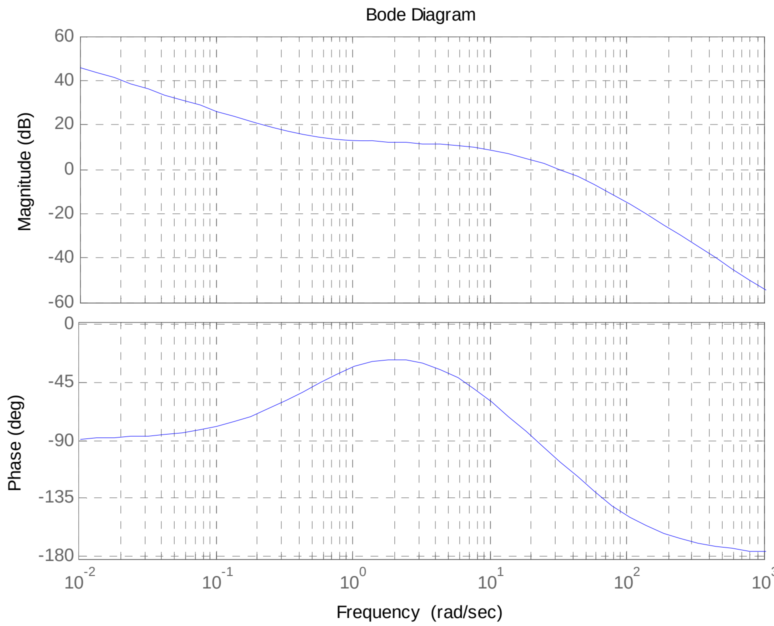
2-8) Use this part to confirm the results from the previous part.

MATLAB code:

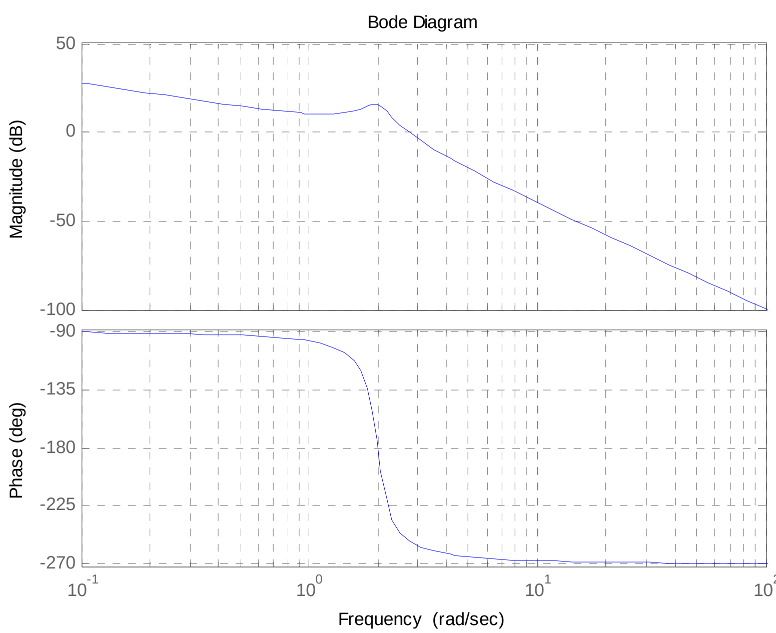
```
s = tf('s')


'Generated transfer function:'

Ga=2000*(s+0.5)/(s*(s+10)*(s+50))

figure(1)

bode(Ga)

grid on;


'Generated transfer function:'

Gb=25/(s*(s+2.5*s^2+10))

figure(2)

bode(Gb)

grid on;


'Generated transfer function:'

Gc=(s+100*s^2+100)/(s^2*(s+25*s^2+100))

figure(3)

bode(Gc)

grid on;
```

```
'Generated transfer function:'
zeta = 0.2
wn=8
Gd=1/(1+2*zeta*s/wn+(s/wn)^2)
figure(4)
bode(Gd)
grid on;

'Generated transfer function:'
t=0.3
'from pade approzimation:'
exp_term=pade(exp(-s*t),1)
Ge=0.03*(exp_term+1)^2/((exp_term-1)*(3*exp_term+1)*(exp_term+0.5))
figure(5)
bode(Ge)
grid on;
```

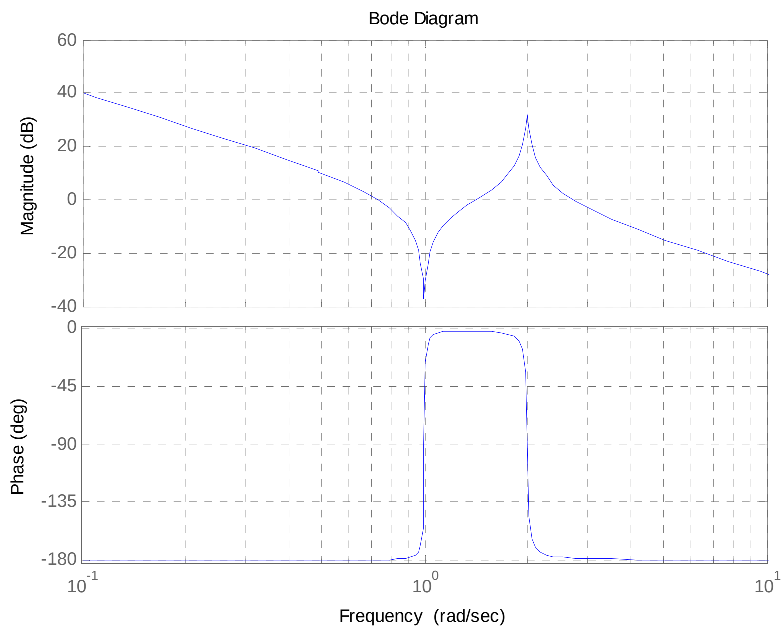
Part(a)


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

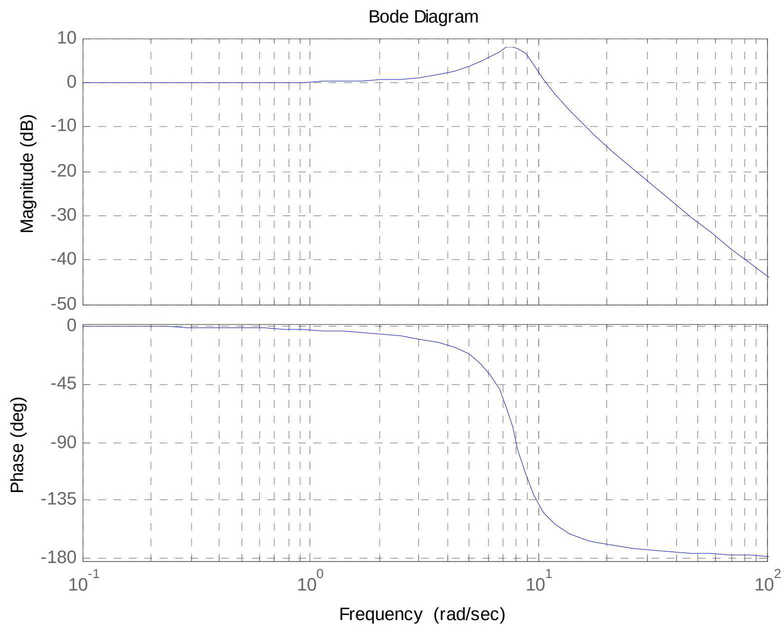
Part(b)


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

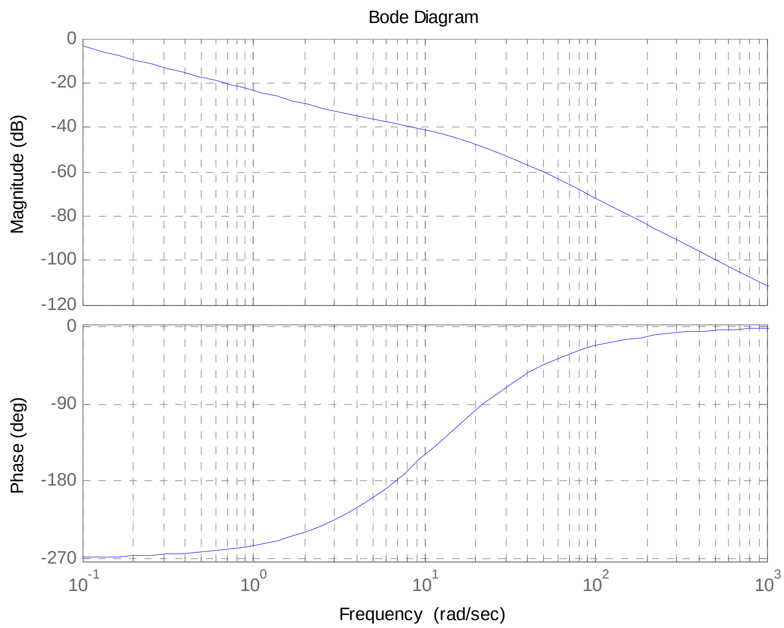
Part(c)


Bode Diagram
Magnitude (dB)
60
40
20
0
-20
-40
Phase (deg)
0
-45
-90
-135
-180
10^-1
10^0
10^1
Frequency (rad/sec)

Part(d)


Bode Diagram
Magnitude (dB)
10
0
-10
-20
-30
-40
-50
Phase (deg)
0
-45
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Part(e)


Bode Diagram
Magnitude (dB)
0
-20
-40
-60
-80
-100
-120
Phase (deg)
0
-90
-180
-270
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

2-9)

a)

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 3 \\ -1 & -3 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

b)

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -1 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

2-10) We know that:

$$\begin{cases} G(s) = \int_0^\infty g(t)e^{-st}dt & (1) \\ g(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} g(t)e^{-st}dt & (2) \end{cases}$$

Partial integration of equation (1) gives:

$$G(s) = \left[-\frac{g(t)e^{-st}}{s} \right]_0^\infty + \frac{1}{s}\int_0^\infty g'(t)e^{-st}dt$$

$$\Rightarrow sG(s) = g(0) + \mathcal{L}\{g'(t)\}$$

$$\Rightarrow \mathcal{L}\{g'(t)\} \leftrightarrow sG(s) - g(0)$$

Differentiation of both sides of equation (1) with respect to s gives:

$$\frac{dG(s)}{ds} = \int_{-\infty}^{\infty} -(t)g(t)e^{-st}dt = \int_{-\infty}^{\infty} (-tg(t))e^{-st}dt$$

Comparing with equation (1), we conclude that:

$$\mathcal{L}^{-1}\left\{\frac{dG(s)}{ds}\right\} \leftrightarrow -tg(t)$$

2-11) Let g(t) = $\int_{-\infty}^{t} x(\tau)d\tau$ then $x(t) = \frac{dg(t)}{dt}$

Using Laplace transform and differentiation property, we have X(s) = sG(s)

Therefore G(s) = $\frac{1}{s}X(s)$, which means:

$$\mathcal{L}\left\{\int_{-\infty}^{\infty} x(\tau)d\tau\right\} \quad \leftrightarrow \quad \frac{1}{s}X(s)$$

2-12) By Laplace transform definition:

$$\mathcal{L}\{g(t-T)u(t-T)\} = \int_{T}^{\infty} g(t-T)e^{-st}dt$$

Now, consider τ = t - T, then:

$$\mathcal{L}\{g(t-T)\} = \int_{0}^{\infty} g(\tau)e^{-s(\tau+T)}d\tau = e^{-sT}\int_{0}^{\infty} g(\tau)e^{-s\tau}d\tau$$

Which means: $\mathcal{L}\{g(t-T)\} \leftrightarrow e^{-sT}G(s)$

2-13) Consider:

$$f(t) = g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau$$

By Laplace transform definition:

$$F(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau\right]e^{-st}dt$$

$$= \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} g_2(t-\tau)e^{-st}dt\right]d\tau$$

By using time shifting theorem, we have:

$$F(s) = \int_{-\infty}^{\infty} g_1(\tau)[e^{-s\,\tau}G_2(s)]d\tau$$

$$= \left[\int_{-\infty}^{\infty} g_1(\tau)e^{-s\tau}d\tau\right]G_2(s) = G_1(s)\cdot G_2(s)$$

Let's consider g(t) = g_1(t) ·g_2(t)

$$G(s) = \int_0^{\infty} g_1(t)g_2(t)e^{-st}dt$$

By inverse Laplace Transform definition, we have

$$g_1(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} G_1(p)e^{pt}dp$$

Then

$$G(s) = \int_{c-j\infty}^{c+j\infty} G_1(p)dp\ \int_0^{\infty} f_2(t)e^{-(s-p)t}dt$$

Where

$$G_2(s-p) = \int_0^{\infty} f_2(t)e^{-(s-p)t}dt$$

therefore:

$$G(s) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} G_1(p)G_2(p-s)dp = \mathrm{G_1(s)} * \mathrm{G_2(s)}$$

2-14) a) We know that

$$\mathcal{L}\left\{\frac{dg(t)}{dt}\right\} = \int_0^{\infty}\frac{dg(t)}{dt}e^{-st}dt = \mathrm{sG(s) + g(0)}$$

When s → ∞ , it can be written as:

$$\lim_{s\to\infty}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = \lim_{s\to\infty}[sG(s) - g(0)]$$

As

$$lim_{x\to\infty}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = 0$$

Therefore: $lim_{s\to\infty}\, sG(s) = g(0)$

b) By Laplace transform differentiation property:

$$\lim_{s\to 0}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt = \lim_{s\to 0}[sG(s) - g(0)]$$

As

$$\lim_{s\to 0}\int_0^\infty \frac{dg(t)}{dt}e^{-st}dt \;= \int_0^\infty \frac{dg(t)}{dt}dt = g(\infty) - g(0)$$

Therefore

$$\lim_{s\to 0}[sG(s)] - g(0) = g(\infty) - g(0)$$

which means:

$$\lim_{s\to 0} sG(s) = g(\infty)$$

2-15)

MATLAB code:

```
clear all;

syms t

s=tf('s')


f1 = (sin(2*t))^2

L1=laplace(f1)
```

```
% f2 = (cos(2*t))^2 = 1-(sin(2*t))^2   ===> L(f2)=1/s-L(f1) ===>
L2= 1/s - 8/s/(s^2+16)


f3 = (cos(2*t))^2
L3=laplace(f3)


'verified as L2 equals L3'
```

MATLAB solution for $L\{\sin^2 2t\}$ is:

8/s/(s^2+16)

Calculating $L\{\cos^2 2t\}$ based on $L\{\sin^2 2t\}$

$L\{\cos^2 2t\}$ = (s^3 + 8 s)/(s^4 + 16 s^2)

verifying $L\{\cos^2 2t\}$:

(8+s^2)/s/(s^2+16)

2-16) (a)

$$G(s) = \frac{5}{(s+5)^2}$$

(b)

$$G(s) = \frac{4s}{(s^2+4)} + \frac{1}{s+2}$$

(c)

$$G(s) = \frac{4}{s^2+4s+8}$$

(d)

$$G(s) = \frac{1}{s^2+4}$$

(e)

$$G(s) = \sum_{k=0}^{\infty} e^{kT(s+5)} = \frac{1}{1-e^{-T(s+5)}}$$

2-17) Note: %section (e) requires assignment of T and a numerical loop calculation

MATLAB code:

```
clear all;

syms t u


f1 = 5*t*exp(-5*t)

L1=laplace(f1)


f2 = t*sin(2*t)+exp(-2*t)

L2=laplace(f2)


f3 = 2*exp(-2*t)*sin(2*t)

L3=laplace(f3)


f4 = sin(2*t)*cos(2*t)

L4=laplace(f4)


%section (e) requires assignment of T and a numerical loop calculation
```

(a) $g(t) = 5te^{-5t}u_s(t)$

Answer: 5/(s+5)^2

(b) $g(t) = (t\sin 2t + e^{-2t})u_s(t)$

Answer: 4*s/(s^2+4)^2+1/(s+2)

(c) $g(t) = 2e^{-2t}\sin 2t\, u_s(t)$

Answer: 4/(s^2+4*s+8)

(d) $g(t) = \sin 2t \cos 2t\, u_s(t)$

Answer: 2/(s^2+16)

(e) $g(t) = \sum_{k=0}^{\infty} e^{-5kT}\, \delta(t - kT)$ where $\delta(t)$ = unit-impulse function

```
%section (e) requires assignment of T and a numerical loop calculation
```

2-18 (a)

$$g(t) = u_s(t) - 2u_s(t-1) + 2u_s(t-2) - 2u_s(t-3) + \cdots$$

$$G(s) = \frac{1}{s}\left(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + \cdots\right) = \frac{1-e^{-s}}{s\left(1+e^{-s}\right)}$$

$$g_T(t) = u_s(t) - 2u_s(t-1) + u_s(t-2) \qquad 0 \le t \le 2$$

$$G_T(s) = \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{1}{s}\left(1-e^{-s}\right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-2k)u_s(t-2k) \qquad G(s) = \sum_{k=0}^{\infty}\frac{1}{s}(1-e^{-s})^2 e^{-2ks} = \frac{1-e^{-s}}{s(1+e^{-s})}$$

(b)

$$g(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 4(t-1)u_s(t-1) - 4(t-1.5)u_s(t-1.5) + \cdots$$

$$G(s) = \frac{2}{s^2}\left(1 - 2e^{-0.5s} + 2e^{-s} - 2e^{-1.5s} + \cdots\right) = \frac{2\left(1-e^{-0.5s}\right)}{s^2\left(1+e^{-0.5s}\right)}$$

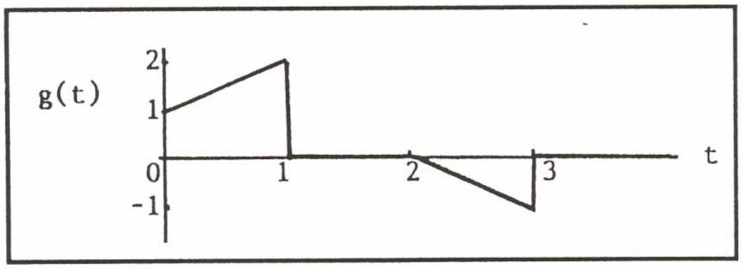
$$g_T(t) = 2tu_s(t) - 4(t-0.5)u_s(t-0.5) + 2(t-1)u_s(t-1) \qquad 0 \le t \le 1$$

$$G_T(s) = \frac{2}{s^2}\left(1 - 2e^{-0.5s} + e^{-s}\right) = \frac{2}{s^2}\left(1-e^{-0.5s}\right)^2$$

$$g(t) = \sum_{k=0}^{\infty} g_T(t-k)u_s(t-k) \qquad G(s) = \sum_{k=0}^{\infty}\frac{2}{s^2}\left(1-e^{-0.5s}\right)^2 e^{-ks} = \frac{2\left(1-e^{-0.5s}\right)}{s^2\left(1+e^{-0.5s}\right)}$$

2-19)

$$g(t) = (t+1)u_s(t) - (t-1)u_s(t-1) - 2u_s(t-1) - (t-2)u_s(t-2) + (t-3)u_s(t-3) + u_s(t-3)$$



$$G(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} + e^{-3s}\right) + \frac{1}{s}\left(1 - 2e^{-s} + e^{-3s}\right)$$

2-20)

$$\mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt = \int_0^{\frac{T}{2}} e^{-st}dt + \int_{\frac{T}{2}}^{T} (-1)e^{-st}dt$$

$$= \frac{1 - e^{-\frac{Ts}{2}}}{s} + \frac{e^{-Ts} - e^{-\frac{Ts}{2}}}{s} = \frac{1}{s}\left[1 - e^{-\frac{Ts}{2}}\right]^2$$

2-21)

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^L \frac{e^{-st}}{L^2}dt - \int_L^{\frac{L}{2}} \frac{e^{-st}}{L^2}dt$$

$$= \left[-\frac{e^{-st}}{sL^2}\right]_0^L + \left[\frac{e^{-st}}{sL^2}\right]_L^{2L} = \frac{1 - e^{-Lt}}{sL^2} + \frac{e^{-2Lt} - e^{-Lt}}{sL^2}$$

$$= \frac{1}{sL^2}(1 - e^{-Lt})^2$$

2-22) $\mathcal{L}\left\{\frac{d^3y(t)}{dt^2}\right\} = s^3Y(s) - s^2y''(0) - sy'(0) - y(0)$

$\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s) - sy'(0) - y(0)$

$\mathcal{L}\left\{\frac{dy(t)}{dt^2}\right\} = sY(s) - y(0)$

$\mathcal{L}\{-e^{-t}u_s(t)\} = -\frac{1}{s+1}$

⇨ $s^3Y(s) + s - s + 2s^2Y(s) + 2s - 2 - sY(s) + 2Y(s) = -\frac{1}{s+1}$

⇨ $(s^3 + 2s^2 - s + 2)Y(s) + 2s - 2 = -\frac{1}{s+1}$

⇨ $Y(s) = \frac{2s^2-3}{(s+2)(s^2+1)(s+1)}$

2-23

MATLAB code:

```
clear all;

syms t u s x1 x2 Fs


f1 = exp(-2*t)

L1=laplace(f1)/(s^2+5*s+4);


Eq2=solve('s*x1=1+x2','s*x2=-2*x1-3*x2+1','x1','x2')

f2_x1=Eq2.x1

f2_x2=Eq2.x2


f3=solve('(s^3-s+2*s^2+s+2)*Fs=-1+2-(1/(1+s))','Fs')
```

Here is the solution provided by MATLAB:

Part (a): F(s)=1/(s+2)/(s^2+5*s+4)

Part (b): X1(s)= (4+s)/(2+3*s+s^2)

X2(s)= (s-2)/(2+3*s+s^2)

Part (c): F(s) = s/(1+s)/(s^3+2*s^2+2)

2-24)

MATLAB code:

```
clear all;
syms s Fs
f3=solve('s^2*Fs-Fs=1/(s-1)','Fs')
Answer from MATLAB: Y(s)=1/(s-1)/(s^2-1)
```

2-25)

MATLAB code:

```
clear all;
syms s CA1 CA2 CA3
v1=1000;
v2=1500;
v3=100;
k1=0.1
k2=0.2
k3=0.4

f1='s*CA1=1/v1*(1000+100*CA2-1100*CA1-k1*v1*CA1)'
f2='s*CA2=1/v2*(1100*CA1-1100*CA2-k2*v2*CA2)'
f3='s*CA3=1/v3*(1000*CA2-1000*CA3-k3*v3*CA3)'
Sol=solve(f1,f2,f3,'CA1','CA2','CA3')
CA1=Sol.CA1
CA3=Sol.CA2
CA4=Sol.CA3
```

Solution from MATLAB:

CA1(s) =
1000*(s*v2+1100+k2*v2)/(1100000+s^2*v1*v2+1100*s*v1+s*v1*k2*v2+1100*s*v2+1100*k2*v2+k1*v1*s*v2+1100*k1*v1+k1*v1*k2*v2)

CA3(s) =

1100000/(1100000+s^2*v1*v2+1100*s*v1+s*v1*k2*v2+1100*s*v2+1100*k2*v2+k1*v1*s*v2+1100*k1*v1+k1*v1*k2*v2)

CA4 (s)=

1100000000/(1100000000+1100000*s*v3+1000*s*v1*k2*v2+1100000*s*v1+1000*k1*v1*s*v2+1000*k1*v1*k2*v2+1100*s*v1*k3*v3+1100*s*v2*k3*v3+1100*k2*v2*s*v3+1100*k2*v2*k3*v3+1100*k1*v1*s*v3+1100*k1*v1*k3*v3+1100000*k1*v1+1000*s^2*v1*v2+1100000*s*v2+1100000*k2*v2+1100000*k3*v3+s^3*v1*v2*v3+1100*s^2*v1*v3+1100*s^2*v2*v3+s^2*v1*v2*k3*v3+s^2*v1*k2*v2*v3+s*v1*k2*v2*k3*v3+k1*v1*s^2*v2*v3+k1*v1*s*v2*k3*v3+k1*v1*k2*v2*s*v3+k1*v1*k2*v2*k3*v3)

2-26) (a)

$$G(s)=\frac{1}{3s}-\frac{1}{2(s+2)}+\frac{1}{3(s+3)} \qquad g(t)=\frac{1}{3}-\frac{1}{2}e^{-2t}+\frac{1}{3}e^{-3t} \qquad t\geq 0$$

(b)

$$G(s)=\frac{-2.5}{s+1}+\frac{5}{(s+1)^2}+\frac{2.5}{s+3} \qquad g(t)=-2.5e^{-t}+5te^{-t}+2.5e^{-3t} \qquad t\geq 0$$

(c)

$$G(s)=\left(\frac{50}{s}-\frac{20}{s+1}-\frac{30s+20}{s^2+4}\right)e^{-s} \qquad g(t)=\left[50-20e^{-(t-1)}-30\cos 2(t-1)-5\sin 2(t-1)\right]u_s(t-1)$$

(d)

$$G(s)=\frac{1}{s}-\frac{s-1}{s^2+s+2}=\frac{1}{s}+\frac{1}{s^2+s+2}-\frac{s}{s^2+s+2}$$ Taking the inverse Laplace transform,

$$g(t)=1+1.069e^{-0.5t}\left[\sin 1.323t+\sin\left(1.323t-69.3^{\circ}\right)\right]=1+e^{-0.5t}\left(1.447\sin 1.323t-\cos 1.323t\right) \qquad t\geq 0$$

(e) $g(t)=0.5t^2e^{-t} \qquad t\geq 0$

(f)Try using MATLAB

```
>> b=num*2
```

```
b =

    2     2     2

>>num =

    1     1     1

>> denom1=[1 1]

denom1 =

    1     1

>> denom2=[1 5 5]

denom2 =

    1     5     5

>> num*2

ans =

    2     2     2

>> denom=conv([1 0],conv(denom1,denom2))

denom =

    1     6    10     5     0

>> b=num*2

b =

    2     2     2

>> a=denom

a =

    1     6    10     5     0

>> [r, p, k] = residue(b,a)

r =

  -0.9889
```

2.5889

-2.0000

0.4000

p =

-3.6180

-1.3820

-1.0000

0

k = [ ]

If there are no multiple roots, then

The number of poles n is

$$\frac{b}{a} = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \ldots + \frac{r_n}{s+p_n} + k$$

In this case, p_1 and k are zero. Hence,

$$G(s) = \frac{0.4}{s} - \frac{0.9889}{s+3.6180} + \frac{2.5889}{s+1.3820} - \frac{2}{s+1}$$

$$g(t) = 0.4 - 0.9889e^{-3.618t} + 1.3820e^{-2.5889t} - 2e^{-t}$$

(g) $G(s) = \frac{2}{(s+1)(s+2)} + \frac{2e^{-s}}{s+1}$

$$= \frac{2}{s+1} - \frac{2}{s+2} + \frac{2e^{-s}}{s+1}$$

⇨ $\mathcal{L}^{-1}\{\mathrm{G(s)}\} = 2\mathrm{e}^{-\mathrm{t}} - 2\mathrm{e}^{-2\mathrm{t}} + 2\mathrm{e}^{-(\mathrm{t}-1)}u(t-1)$

(h) $G(s) = \frac{2s+1}{(s+1)(s+2)(s+3)} = -\frac{\frac{1}{2}}{s+1} + \frac{3}{s+2} - \frac{5}{2(s+3)}$

⇨ $\mathcal{L}^{-1}\{G(s)\} = -\frac{1}{2}e^{-t} + 3e^{-2t} - \frac{5}{2}e^{-3t}$

(i) $G(s) = \frac{3s^3+10s^2+8s+5}{s^{34}+5s^3+7s^2+5s+6} = \frac{1}{s+2} + \frac{1}{s+3} + \frac{s}{s^2+1}$

⇨ $\mathcal{L}^{-1}\{G(s)\} = e^{-2t} + e^{-3t} + cost$

2-27

MATLAB code:

```
clear all;
syms s

f1=1/(s*(s+2)*(s+3))
F1=ilaplace(f1)

f2=10/((s+1)^2*(s+3))
F2=ilaplace(f2)

f3=10*(s+2)/(s*(s^2+4)*(s+1))*exp(-s)
F3=ilaplace(f3)

f4=2*(s+1)/(s*(s^2+s+2))
F4=ilaplace(f4)

f5=1/(s+1)^3
F5=ilaplace(f5)

f6=2*(s^2+s+1)/(s*(s+1.5)*(s^2+5*s+5))
F6=ilaplace(f6)

s=tf('s')
f7=(2+2*s*pade(exp(-1*s),1)+4*pade(exp(-2*s),1))/(s^2+3*s+2) %using Pade command
for exponential term
```

```
[num,den]=tfdata(f7,'v') %extracting the polynomial values

syms s

f7n=(-2*s^3+6*s+12)/(s^4+6*s^3+13*s^2+12*s+4) %generating sumbolic function for ilaplace

F7=ilaplace(f7n)


f8=(2*s+1)/(s^3+6*s^2+11*s+6)

F8=ilaplace(f8)


f9=(3*s^3+10^s^2+8*s+5)/(s^4+5*s^3+7*s^2+5*s+6)

F9=ilaplace(f9)
```

Solution from MATLAB for the Inverse Laplace transforms:

Part (a): $$G(s) = \frac{1}{s(s+2)(s+3)}$$

G(t)=-1/2*exp(-2*t)+1/3*exp(-3*t)+1/6

To simplify:

syms t

digits(3)

vpa(-1/2*exp(-2*t)+1/3*exp(-3*t)+1/6)

ans =-.500*exp(-2.*t)+.333*exp(-3.*t)+.167

Part (b): $$G(s) = \frac{10}{(s+1)^2(s+3)}$$

G(t)= 5/2*exp(-3*t)+5/2*exp(-t)*(-1+2*t)

Part (c): $G(s)=\dfrac{100(s+2)}{s(s^2+4)(s+1)}e^{-s}$

G(t)=Step(t-1)*(-4*cos(t-1)^2+2*sin(t-1)*cos(t-1)+4*exp(-1/2*t+1/2)*cosh(1/2*t-1/2)-4*exp(-t+1)-cos(2*t-2)-2*sin(2*t-2)+5)

Part (d): $G(s)=\dfrac{2(s+1)}{s(s^2+s+2)}$

G(t)= 1+1/7*exp(-1/2*t)*(-7*cos(1/2*7^(1/2)*t)+3*7^(1/2)*sin(1/2*7^(1/2)*t))

To simplify:

syms t

digits(3)

vpa(1+1/7*exp(-1/2*t)*(-7*cos(1/2*7^(1/2)*t)+3*7^(1/2)*sin(1/2*7^(1/2)*t)))

ans = 1.+.143*exp(-.500*t)*(-7.*cos(1.32*t)+7.95*sin(1.32*t))

Part (e): $G(s)=\dfrac{1}{(s+1)^3}$

G(t)= 1/2*t^2*exp(-t)

Part (f): $G(s)=\dfrac{2(s^2+s+1)}{s(s+1.5)(s^2+5s+5)}$

G(t)= 4/15+28/3*exp(-3/2*t)-16/5*exp(-5/2*t)*(3*cosh(1/2*t*5^(1/2))+5^(1/2)*sinh(1/2*t*5^(1/2)))

Part (g): $G(s)=\dfrac{2+2se^{-s}+4e^{-2s}}{s^2+3s+2}$

G(t)= 2*exp(-2*t)*(7+8*t)+8*exp(-t)*(-2+t)

Part (h): $$G(s)=\frac{2s+1}{s^3+6s^2+11s+6}$$

G(t)= -1/2*exp(-t)+3*exp(-2*t)-5/2*exp(-3*t)

Part (i): $$G(s)=\frac{3s^3+10s^2+8s+5}{s^4+5s^3+7s^2+5s+6}$$

G(t)= -7*exp(-2*t)+10*exp(-3*t)-

1/10*ilaplace(10^(2*s)/(s^2+1)*s,s,t)+1/10*ilaplace(10^(2*s)/(s^2+1),s,t)+1/10*sin(t)*(10+dirac(t)*(-exp(-

3*t)+2*exp(-2*t)))