

38. To be proved: if $f(x) \leq g(x) \leq h(x)$ in an open interval containing $x = a$ (say, for $a - \delta_1 < x < a + \delta_1$, where $\delta_1 > 0$), and if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then also $\lim_{x \rightarrow a} g(x) = L$.

Proof: Let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|f(x) - L| < \epsilon/3$. Since $\lim_{x \rightarrow a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then $|h(x) - L| < \epsilon/3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x - a| < \delta$, then

$$\begin{aligned} |g(x) - L| &= |g(x) - f(x) + f(x) - L| \\ &\leq |g(x) - f(x)| + |f(x) - L| \\ &\leq |h(x) - f(x)| + |f(x) - L| \\ &= |h(x) - L + L - f(x)| + |f(x) - L| \\ &\leq |h(x) - L| + |f(x) - L| + |f(x) - L| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus $\lim_{x \rightarrow a} g(x) = L$.