11. Suppose f is continuous on $[0, 1]$ and $f(0) = f(1)$.

a) To be proved: $f(a) = f(a + \frac{1}{2})$ for some a in $[0, \frac{1}{2}]$.
Proof: If $f(1/2) = f(0)$ we can take $a = 0$ and be done. If not, let

$$g(x) = f(x + \tfrac{1}{2}) - f(x).$$

Then $g(0) \neq 0$ and

$$g(1/2) = f(1) - f(1/2) = f(0) - f(1/2) = -g(0).$$

Since g is continuous and has opposite signs at $x = 0$ and $x = 1/2$, the Intermediate-Value Theorem assures us that there exists a between 0 and 1/2 such that $g(a) = 0$, that is, $f(a) = f(a + \frac{1}{2})$.

b) To be proved: if $n > 2$ is an integer, then
$f(a) = f(a + \frac{1}{n})$ for some a in $[0, 1 - \frac{1}{n}]$.
Proof: Let $g(x) = f(x + \frac{1}{n}) - f(x)$. Consider the numbers $x = 0$, $x = 1/n$, $x = 2/n$, $\ldots$, $x = (n-1)/n$. If $g(x) = 0$ for any of these numbers, then we can let a be that number. Otherwise, $g(x) \neq 0$ at any of these numbers. Suppose that the values of g at all these numbers has the same sign (say positive). Then we have

$$f(1) > f(\tfrac{n-1}{n}) > \cdots > f(\tfrac{2}{n}) > \tfrac{1}{n} > f(0),$$

which is a contradiction, since $f(0) = f(1)$. Therefore there exists j in the set $\{0, 1, 2, \ldots, n-1\}$ such that $g(j/n)$ and $g((j+1)/n)$ have opposite sign. By the Intermediate-Value Theorem, $g(a) = 0$ for some a between j/n and $(j+1)/n$, which is what we had to prove.