7. TRUE or FALSE

a) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.
TRUE, because if $\lim_{x\to a}\big(f(x)+g(x)\big)$ were to exist then

$$\begin{aligned}\lim_{x\to a} g(x) &= \lim_{x\to a}\big(f(x)+g(x)-f(x)\big)\\ &= \lim_{x\to a}\big(f(x)+g(x)\big) - \lim_{x\to a} f(x)\end{aligned}$$

would also exist.

b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.
FALSE. Neither $\lim_{x\to 0} 1/x$ nor $\lim_{x\to 0}(-1/x)$ exist, but $\lim_{x\to 0}\big((1/x)+(-1/x)\big) = \lim_{x\to 0} 0 = 0$ exists.

c) If f is continuous at a, then so is $|f|$.
TRUE. For any two real numbers u and v we have

$$\big||u|-|v|\big| \le |u-v|.$$

This follows from

$$\begin{aligned}&|u| = |u-v+v| \le |u-v|+|v|, \quad \text{and}\\ &|v| = |v-u+u| \le |v-u|+|u| = |u-v|+|u|.\end{aligned}$$

Now we have

$$\big||f(x)|-|f(a)|\big| \le |f(x)-f(a)|$$

so the left side approaches zero whenever the right side does. This happens when $x\to a$ by the continuity of f at a.

d) If $|f|$ is continuous at a, then so is f.
FALSE. The function $f(x) = \begin{cases} -1 & \text{if } x<0\\ 1 & \text{if } x\ge 0\end{cases}$ is discontinuous at $x=0$, but $|f(x)|=1$ everywhere, and so is continuous at $x=0$.

e) If $f(x)<g(x)$ in an interval around a and if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then $L<M$.
FALSE. Let $g(x) = \begin{cases} x^2 & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$ and let $f(x) = -g(x)$. Then $f(x)<g(x)$ for all x, but $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. (Note: under the given conditions, it is TRUE that $L\le M$, but not necessarily true that $L<M$.)