

33. To be proved: if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then $\lim_{x \rightarrow a} f(x)g(x) = LM$.

Proof: Let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon/(2(1 + |M|))$ if $0 < |x - a| < \delta_1$. Since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_2 > 0$ such that $|g(x) - M| < \epsilon/(2(1 + |L|))$ if $0 < |x - a| < \delta_2$. By Exercise 32, there exists $\delta_3 > 0$ such that $|g(x)| < 1 + |M|$ if $0 < |x - a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $|x - a| < \delta$, then

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\leq |(f(x) - L)g(x)| + |L(g(x) - M)| \\ &= |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\epsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\epsilon}{2(1 + |L|)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\lim_{x \rightarrow a} f(x)g(x) = LM$.