1 PRECALCULUS REVIEW

1.1 Real Numbers, Functions, and Graphs

Preliminary Questions

1. Give an example of numbers a and b such that $a < b$ and $|a| > |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then $a < b$ but $|a| = 3 > 1 = |b|$.

2. Which numbers satisfy $|a| = a$? Which satisfy $|a| = -a$? What about $|-a| = a$?

SOLUTION The numbers $a \geq 0$ satisfy $|a| = a$ and $|-a| = a$. The numbers $a \leq 0$ satisfy $|a| = -a$.

3. Give an example of numbers a and b such that $|a + b| < |a| + |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then

$$|a + b| = |-3 + 1| = |-2| = 2, \qquad \text{but} \qquad |a| + |b| = |-3| + |1| = 3 + 1 = 4$$

Thus, $|a + b| < |a| + |b|$.

4. Are there numbers a and b such that $|a + b| > |a| + |b|$?

SOLUTION No. By the triangle inequality, $|a + b| \leq |a| + |b|$ for all real numbers a and b.

5. What are the coordinates of the point lying at the intersection of the lines $x = 9$ and $y = -4$?

SOLUTION The point $(9, -4)$ lies at the intersection of the lines $x = 9$ and $y = -4$.

6. In which quadrant do the following points lie?

(a) $(1, 4)$ **(b)** $(-3, 2)$ **(c)** $(4, -3)$ **(d)** $(-4, -1)$

SOLUTION

(a) Because both the x- and y-coordinates of the point $(1, 4)$ are positive, the point $(1, 4)$ lies in the first quadrant.

(b) Because the x-coordinate of the point $(-3, 2)$ is negative but the y-coordinate is positive, the point $(-3, 2)$ lies in the second quadrant.

(c) Because the x-coordinate of the point $(4, -3)$ is positive but the y-coordinate is negative, the point $(4, -3)$ lies in the fourth quadrant.

(d) Because both the x- and y-coordinates of the point $(-4, -1)$ are negative, the point $(-4, -1)$ lies in the third quadrant.

7. What is the radius of the circle with equation $(x - 7)^2 + (y - 8)^2 = 9$?

SOLUTION The circle with equation $(x - 7)^2 + (y - 8)^2 = 9$ has radius 3.

8. The equation $f(x) = 5$ has a solution if (choose one):

(a) 5 belongs to the domain of f.

(b) 5 belongs to the range of f.

SOLUTION The correct response is **(b)**: the equation $f(x) = 5$ has a solution if 5 belongs to the range of f.

9. What kind of symmetry does the graph have if $f(-x) = -f(x)$?

SOLUTION If $f(-x) = -f(x)$, then the graph of f is symmetric with respect to the origin.

10. Is there a function that is both even and odd?

SOLUTION Yes. The constant function $f(x) = 0$ for all real numbers x is both even and odd because

$$f(-x) = 0 = f(x)$$

and

$$f(-x) = 0 = -0 = -f(x)$$

for all real numbers x.

Exercises

1. Which of the following equations is incorrect?

(a) $3^2 \cdot 3^5 = 3^7$

(b) $(\sqrt{5})^{4/3} = 5^{2/3}$

(c) $3^2 \cdot 2^3 = 1$

(d) $(2^{-2})^{-2} = 16$

SOLUTION

(a) This equation is correct: $3^2 \cdot 3^5 = 3^{2+5} = 3^7$.

(b) This equation is correct: $(\sqrt{5})^{4/3} = (5^{1/2})^{4/3} = 5^{(1/2)\cdot(4/3)} = 5^{2/3}$.

(c) This equation is incorrect: $3^2 \cdot 2^3 = 9 \cdot 8 = 72 \neq 1$.

(d) This equation is correct: $(2^{-2})^{-2} = 2^{(-2)\cdot(-2)} = 2^4 = 16$.

2. Rewrite as a whole number (without using a calculator):

(a) 7^0

(b) $10^2(2^{-2} + 5^{-2})$

(c) $\frac{(4^3)^5}{(4^5)^3}$

(d) $27^{4/3}$

(e) $8^{-1/3} \cdot 8^{5/3}$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2}$

SOLUTION

(a) $7^0 = 1$

(b) $10^2(2^{-2} + 5^{-2}) = 100(1/4 + 1/25) = 25 + 4 = 29$

(c) $(4^3)^5/(4^5)^3 = 4^{15}/4^{15} = 1$

(d) $(27)^{4/3} = (27^{1/3})^4 = 3^4 = 81$

(e) $8^{-1/3} \cdot 8^{5/3} = (8^{1/3})^5/8^{1/3} = 2^5/2 = 2^4 = 16$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2} = 3 \cdot 2^{1/2} - 3 \cdot 2^2 \cdot 2^{-3/2} = 0$

3. Use the binomial expansion formula to expand $(2 - x)^7$.

SOLUTION Using the binomial expansion formula,

$$\begin{aligned}(2-x)^7 &= \frac{7!}{7!0!}2^7(-x)^0 + \frac{7!}{6!1!}2^6(-x) + \frac{7!}{5!2!}2^5(-x)^2 + \frac{7!}{4!3!}2^4(-x)^3 + \frac{7!}{3!4!}2^3(-x)^4 \\ &\quad + \frac{7!}{2!5!}2^2(-x)^5 + \frac{7!}{1!6!}2(-x)^6 + \frac{7!}{0!7!}2^0(-x)^7 \\ &= 128 - 448x + 672x^2 - 560x^3 + 280x^4 - 84x^5 + 14x^6 - x^7\end{aligned}$$

4. Use the binomial expansion formula to expand $(x + 1)^9$.

SOLUTION Using the binomial expansion formula,

$$\begin{aligned}(x+1)^9 &= \frac{9!}{9!0!}x^9 + \frac{9!}{8!1!}x^8 + \frac{9!}{7!2!}x^7 + \frac{9!}{6!3!}x^6 + \frac{9!}{5!4!}x^5 + \frac{9!}{4!5!}x^4 + \frac{9!}{3!6!}x^3 + \frac{9!}{2!7!}x^2 + \frac{9!}{1!8!}x + \frac{9!}{0!9!} \\ &= x^9 + 9x^8 + 36x^7 + 84x^6 + 126x^5 + 126x^4 + 84x^3 + 36x^2 + 9x + 1\end{aligned}$$

5. Which of (a)–(d) are true for $a = 4$ and $b = -5$?

(a) $-2a < -2b$

(b) $|a| < -|b|$

(c) $ab < 0$

(d) $\frac{1}{a} < \frac{1}{b}$

SOLUTION

(a) True

(b) False; $|a| = 4 > -5 = -|b|$

(c) True

(d) False; $\frac{1}{a} = \frac{1}{4} > -\frac{1}{5} = \frac{1}{b}$

6. Which of (a)–(d) are true for $a = -3$ and $b = 2$?

(a) $a < b$

(b) $|a| < |b|$

(c) $ab > 0$

(d) $3a < 3b$

SOLUTION

(a) True

(b) False; $|a| = 3 > 2 = |b|$

(c) False; $(-3)(2) = -6 < 0$

(d) True

In Exercises 7–12, express the interval in terms of an inequality involving absolute value.

7. $[-2, 2]$

SOLUTION $|x| \le 2$

8. $(-4, 4)$

SOLUTION $|x| < 4$

9. $(0, 4)$

SOLUTION The midpoint of the interval is $c = (0 + 4)/2 = 2$, and the radius is $r = (4 - 0)/2 = 2$; therefore, $(0, 4)$ can be expressed as $|x - 2| < 2$.

10. $[-4, 0]$

SOLUTION The midpoint of the interval is $c = (-4 + 0)/2 = -2$, and the radius is $r = (0 - (-4))/2 = 2$; therefore, the interval $[-4, 0]$ can be expressed as $|x + 2| \le 2$.

11. $[-1, 8]$

SOLUTION The midpoint of the interval is $c = (-1 + 8)/2 = \frac{7}{2}$, and the radius is $r = (8 - (-1))/2 = \frac{9}{2}$; therefore, the interval $[-1, 8]$ can be expressed as $\left|x - \frac{7}{2}\right| \le \frac{9}{2}$.

12. $(-2.4, 1.9)$

SOLUTION The midpoint of the interval is $c = (-2.4 + 1.9)/2 = -0.25$, and the radius is $r = (1.9 - (-2.4))/2 = 2.15$; therefore, the interval $(-2.4, 1.9)$ can be expressed as $|x + 0.25| < 2.15$.

In Exercises 13–16, write the inequality in the form $a < x < b$.

13. $|x| < 8$

SOLUTION $-8 < x < 8$

14. $|x - 12| < 8$

SOLUTION $-8 < x - 12 < 8$ so $4 < x < 20$

15. $|2x + 1| < 5$

SOLUTION $-5 < 2x + 1 < 5$ so $-6 < 2x < 4$ and $-3 < x < 2$

16. $|3x - 4| < 2$

SOLUTION $-2 < 3x - 4 < 2$ so $2 < 3x < 6$ and $\frac{2}{3} < x < 2$

In Exercises 17–22, express the set of numbers x satisfying the given condition as an interval.

17. $|x| < 4$

SOLUTION $(-4, 4)$

18. $|x| \le 9$

SOLUTION $[-9, 9]$

19. $|x - 4| < 2$

SOLUTION The expression $|x - 4| < 2$ is equivalent to $-2 < x - 4 < 2$. Therefore, $2 < x < 6$, which represents the interval $(2, 6)$.

20. $|x + 7| < 2$

SOLUTION The expression $|x + 7| < 2$ is equivalent to $-2 < x + 7 < 2$. Therefore, $-9 < x < -5$, which represents the interval $(-9, -5)$.

21. $|4x - 1| \le 8$

SOLUTION The expression $|4x - 1| \le 8$ is equivalent to $-8 \le 4x - 1 \le 8$ or $-7 \le 4x \le 9$. Therefore, $-\frac{7}{4} \le x \le \frac{9}{4}$, which represents the interval $[-\frac{7}{4}, \frac{9}{4}]$.

22. $|3x + 5| < 1$

SOLUTION The expression $|3x + 5| < 1$ is equivalent to $-1 < 3x + 5 < 1$ or $-6 < 3x < -4$. Therefore, $-2 < x < -\frac{4}{3}$, which represents the interval $(-2, -\frac{4}{3})$.

In Exercises 23–26, describe the set as a union of finite or infinite intervals.

23. $\{x : |x - 4| > 2\}$

SOLUTION $x - 4 > 2$ or $x - 4 < -2 \Rightarrow x > 6$ or $x < 2 \Rightarrow (-\infty, 2) \cup (6, \infty)$

24. $\{x : |2x + 4| > 3\}$

SOLUTION $2x + 4 > 3$ or $2x + 4 < -3 \Rightarrow 2x > -1$ or $2x < -7 \Rightarrow (-\infty, -\frac{7}{2}) \cup (-\frac{1}{2}, \infty)$

25. $\{x : |x^2 - 1| > 2\}$

SOLUTION $x^2 - 1 > 2$ or $x^2 - 1 < -2 \Rightarrow x^2 > 3$ or $x^2 < -1$ (this will never happen) $\Rightarrow x > \sqrt{3}$ or $x < -\sqrt{3} \Rightarrow (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

26. $\{x : |x^2 + 2x| > 2\}$

SOLUTION $x^2 + 2x > 2$ or $x^2 + 2x < -2 \Rightarrow x^2 + 2x - 2 > 0$ or $x^2 + 2x + 2 < 0$. For the first case, the zeroes are

$$x = -1 \pm \sqrt{3} \Rightarrow (-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty).$$

For the second case, note there are no real zeros. Because the parabola opens upward and its vertex is located above the x-axis, there are no values of x for which $x^2 + 2x + 2 < 0$. Hence, the solution set is $(-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty)$.

27. Match (a)–(f) with (i)–(vi).

(a) $a > 3$

(b) $|a - 5| < \frac{1}{3}$

(c) $\left|a - \frac{1}{3}\right| < 5$

(d) $|a| > 5$

(e) $|a - 4| < 3$

(f) $1 \le a \le 5$

(i) a lies to the right of 3.
(ii) a lies between 1 and 7.
(iii) The distance from a to 5 is less than $\frac{1}{3}$.
(iv) The distance from a to 3 is at most 2.
(v) a is less than 5 units from $\frac{1}{3}$.
(vi) a lies either to the left of -5 or to the right of 5.

SOLUTION

(a) On the number line, numbers greater than 3 appear to the right; hence, $a > 3$ is equivalent to the numbers to the right of 3: **(i)**.

(b) $|a - 5|$ measures the distance from a to 5; hence, $|a - 5| < \frac{1}{3}$ is satisfied by those numbers less than $\frac{1}{3}$ of a unit from 5: **(iii)**.

(c) $|a - \frac{1}{3}|$ measures the distance from a to $\frac{1}{3}$; hence, $|a - \frac{1}{3}| < 5$ is satisfied by those numbers less than 5 units from $\frac{1}{3}$: **(v)**.

(d) The inequality $|a| > 5$ is equivalent to $a > 5$ or $a < -5$; that is, either a lies to the right of 5 or to the left of -5: **(vi)**.

(e) The interval described by the inequality $|a - 4| < 3$ has a center at 4 and a radius of 3; that is, the interval consists of those numbers between 1 and 7: **(ii)**.

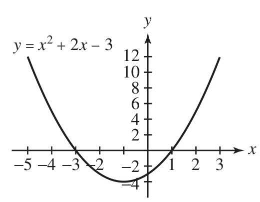
(f) The interval described by the inequality $1 < x < 5$ has a center at 3 and a radius of 2; that is, the interval consists of those numbers less than 2 units from 3: **(iv)**.

28. Describe $\left\{x : \frac{x}{x+1} < 0\right\}$ as an interval. *Hint:* Consider the sign of x and $x + 1$ individually.

SOLUTION Case 1: $x < 0$ and $x + 1 > 0$. This implies that $x < 0$ and $x > -1 \Rightarrow -1 < x < 0$.
Case 2: $x > 0$ and $x < -1$ for which there is no such x. Thus, solution set is therefore $(-1, 0)$.

29. Describe $\{x : x^2 + 2x < 3\}$ as an interval. *Hint:* Plot $y = x^2 + 2x - 3$.

SOLUTION The inequality $x^2 + 2x < 3$ is equivalent to $x^2 + 2x - 3 < 0$. The graph of $y = x^2 + 2x - 3$ is shown here. From this graph, it follows that $x^2 + 2x - 3 < 0$ for $-3 < x < 1$. Thus, the set $\{x : x^2 + 2x < 3\}$ is equivalent to the interval $(-3, 1)$.



30. Describe the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ as a half-infinite interval.

SOLUTION Case 1: If $x \ge 3$, then $|x - 3| = x - 3$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $x - 3 = x - 2 + 1$ or $-3 = -1$. As this is never true, the given equation has no solution for $x \ge 3$.

Case 2: If $2 \le x < 3$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = x - 2 + 1$ or $x = 2$.

Case 3: If $x < 2$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = -(x - 2) = 2 - x$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = 2 - x + 1$ or $1 = 1$. As this is always true, the given equation holds for all $x < 2$.

Combining the results from all three cases, it follows that the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ is equivalent to the half-infinite interval $(-\infty, 2]$.

31. Show that if $a > b$, and $a, b \neq 0$, then $b^{-1} > a^{-1}$, provided that a and b have the same sign. What happens if $a > 0$ and $b < 0$?

SOLUTION Case 1a: If a and b are both positive, then $a > b \Rightarrow 1 > \frac{b}{a} \Rightarrow \frac{1}{b} > \frac{1}{a}$.
Case 1b: If a and b are both negative, then $a > b \Rightarrow 1 < \frac{b}{a}$ (since a is negative) $\Rightarrow \frac{1}{b} > \frac{1}{a}$ (again, since b is negative).
Case 2: If $a > 0$ and $b < 0$, then $\frac{1}{a} > 0$ and $\frac{1}{b} < 0$ so $\frac{1}{b} < \frac{1}{a}$. (See Exercise 6f for an example of this.)

32. Which x satisfies both $|x - 3| < 2$ and $|x - 5| < 1$?

SOLUTION $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow 1 < x < 5$. Also $|x - 5| < 1 \Rightarrow 4 < x < 6$. Since we want an x that satisfies both of these, we need the intersection of the two solution sets, that is, $4 < x < 5$.

33. Show that if $|a - 5| < \frac{1}{2}$ and $|b - 8| < \frac{1}{2}$, then
$|(a + b) - 13| < 1$. *Hint:* Use the triangle inequality ($|a + b| \leq |a| + |b|$).

SOLUTION

$$\begin{aligned} |a + b - 13| &= |(a - 5) + (b - 8)| \\ &\leq |a - 5| + |b - 8| \quad \text{(by the triangle inequality)} \\ &< \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

34. Suppose that $|x - 4| \leq 1$.
(a) What is the maximum possible value of $|x + 4|$?
(b) Show that $|x^2 - 16| \leq 9$.

SOLUTION
(a) $|x - 4| \leq 1$ guarantees $3 \leq x \leq 5$. Thus, $7 \leq x + 4 \leq 9$, so $|x + 4| \leq 9$.
(b) $|x^2 - 16| = |x - 4| \cdot |x + 4| \leq 1 \cdot 9 = 9$

35. Suppose that $|a - 6| \leq 2$ and $|b| \leq 3$.
(a) What is the largest possible value of $|a + b|$?
(b) What is the smallest possible value of $|a + b|$?

SOLUTION $|a - 6| \leq 2$ guarantees $4 \leq a \leq 8$, and $|b| \leq 3$ guarantees $-3 \leq b \leq 3$, so $1 \leq a + b \leq 11$. Based on this information,
(a) the largest possible value of $|a + b|$ is 11; and
(b) the smallest possible value of $|a + b|$ is 1.

36. Prove that $|x| - |y| \leq |x - y|$. *Hint:* Apply the triangle inequality to y and $x - y$.

SOLUTION First note

$$|x| = |x - y + y| \leq |x - y| + |y|$$

by the triangle inequality. Subtracting $|y|$ from both sides of this inequality yields

$$|x| - |y| \leq |x - y|$$

37. Express $r_1 = 0.\overline{27}$ as a fraction. *Hint:* $100r_1 - r_1$ is an integer. Then express $r_2 = 0.2666\ldots$ as a fraction.

SOLUTION Let $r_1 = 0.\overline{27}$. We observe that $100r_1 = 27.\overline{27}$. Therefore, $100r_1 - r_1 = 27.\overline{27} - 0.\overline{27} = 27$ and

$$r_1 = \frac{27}{99} = \frac{3}{11}$$

Now, let $r_2 = 0.2\overline{666}$. Then $10r_2 = 2.\overline{666}$ and $100r_2 = 26.\overline{666}$. Therefore, $100r_2 - 10r_2 = 26.\overline{666} - 2.\overline{666} = 24$ and

$$r_2 = \frac{24}{90} = \frac{4}{15}$$

38. Represent $1/7$ and $4/27$ as repeating decimals.

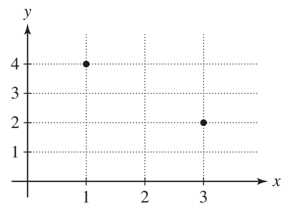
SOLUTION $\frac{1}{7} = 0.\overline{142857}$; $\frac{4}{27} = 0.\overline{148}$

39. Plot each pair of points and compute the distance between them:
(a) $(1, 4)$ and $(3, 2)$
(b) $(2, 1)$ and $(2, 4)$

SOLUTION

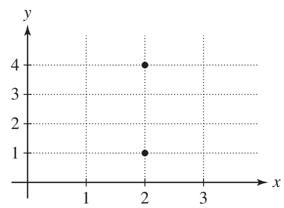
(a) The points $(1, 4)$ and $(3, 2)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(3-1)^2 + (2-4)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$



(b) The points $(2, 1)$ and $(2, 4)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(2-2)^2 + (4-1)^2} = \sqrt{9} = 3$$



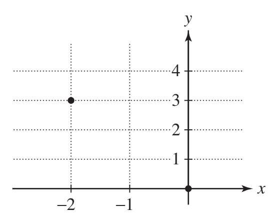
40. Plot each pair of points and compute the distance between them:

(a) $(0, 0)$ and $(-2, 3)$ **(b)** $(-3, -3)$ and $(-2, 3)$

SOLUTION

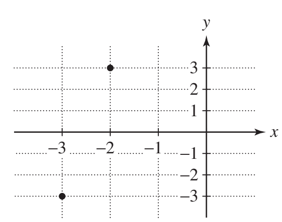
(a) The points $(0, 0)$ and $(-2, 3)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$$



(b) The points $(-3, -3)$ and $(-2, 3)$ are plotted in the figure. The distance between the points is

$$d = \sqrt{(-3-(-2))^2 + (-3-3)^2} = \sqrt{1+36} = \sqrt{37}$$



41. Find the equation of the circle with center $(2, 4)$:

(a) With radius $r = 3$

(b) That passes through $(1, -1)$

SOLUTION (a) The equation of the indicated circle is $(x-2)^2 + (y-4)^2 = 3^2 = 9$.

(b) First, determine the radius as the distance from the center to the indicated point on the circle:

$$r = \sqrt{(2-1)^2 + (4-(-1))^2} = \sqrt{26}$$

Thus, the equation of the circle is $(x-2)^2 + (y-4)^2 = 26$.

42. Find all points in the xy-plane with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from $(2, 3)$.

SOLUTION

- To be located a distance 5 from the origin, the points must lie on the circle $x^2 + y^2 = 25$. This leads to 12 points with integer coordinates:

$$\begin{array}{cccc} (5,0) & (-5,0) & (0,5) & (0,-5) \\ (3,4) & (-3,4) & (3,-4) & (-3,-4) \\ (4,3) & (-4,3) & (4,-3) & (-4,-3) \end{array}$$

- To be located a distance 5 from the point $(2,3)$, the points must lie on the circle $(x-2)^2 + (y-3)^2 = 25$, which implies that we must shift the points listed 2 units to the right and 3 units up. This gives the 12 points

$$\begin{array}{cccc} (7,3) & (-3,3) & (2,8) & (2,-2) \\ (5,7) & (-1,7) & (5,-1) & (-1,-1) \\ (6,6) & (-2,6) & (6,0) & (-2,0) \end{array}$$

43. Determine the domain and range of the function

$$f : \{r, s, t, u\} \to \{A, B, C, D, E\}$$

defined by $f(r) = A$, $f(s) = B$, $f(t) = B$, $f(u) = E$.

SOLUTION The domain is the set $D = \{r, s, t, u\}$; the range is the set $R = \{A, B, E\}$.

44. Give an example of a function whose domain D has three elements and whose range R has two elements. Does a function exist whose domain D has two elements and whose range R has three elements?

SOLUTION Define f by $f : \{a, b, c\} \to \{1, 2\}$, where $f(a) = 1$, $f(b) = 1$, $f(c) = 2$.

There is no function whose domain has two elements and range has three elements. If that happened, one of the domain elements would get assigned to more than one element of the range, which would contradict the definition of a function.

In Exercises 45–52, find the domain and range of the function.

45. $f(x) = -x$

SOLUTION D: all reals; R: all reals

46. $g(t) = t^4$

SOLUTION D: all reals; R: $\{y : y \geq 0\}$

47. $f(x) = x^3$

SOLUTION D: all reals; R: all reals

48. $g(t) = \sqrt{2-t}$

SOLUTION D: $\{t : t \leq 2\}$; R: $\{y : y \geq 0\}$

49. $f(x) = |x|$

SOLUTION D: all reals; R: $\{y : y \geq 0\}$

50. $h(s) = \frac{1}{s}$

SOLUTION D: $\{s : s \neq 0\}$; R: $\{y : y \neq 0\}$

51. $f(x) = \frac{1}{x^2}$

SOLUTION D: $\{x : x \neq 0\}$; R: $\{y : y > 0\}$

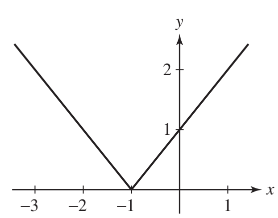
52. $g(t) = \frac{1}{\sqrt{1-t}}$

SOLUTION D: $\{t : t < 1\}$; R: $\{y : y > 0\}$

In Exercises 53–56, determine where f is increasing.

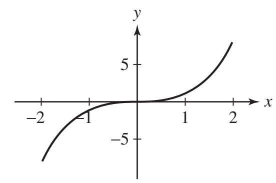
53. $f(x) = |x+1|$

SOLUTION A graph of the function $y = |x+1|$ is shown. From the graph, we see that the function is increasing on the interval $(-1, \infty)$.



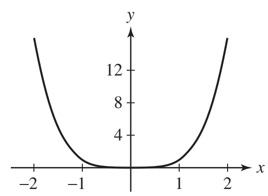
54. $f(x) = x^3$

SOLUTION A graph of the function $y = x^3$ is shown. From the graph, we see that the function is increasing for all real numbers.



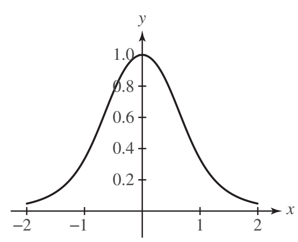
55. $f(x) = x^4$

SOLUTION A graph of the function $y = x^4$ is shown. From the graph, we see that the function is increasing on the interval $(0, \infty)$.



56. $f(x) = \frac{1}{x^4 + x^2 + 1}$

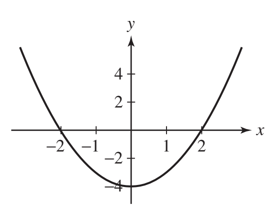
SOLUTION A graph of the function $y = \frac{1}{x^4 + x^2 + 1}$ is shown. From the graph, we see that the function is increasing on the interval $(-\infty, 0)$.



In Exercises 57–62, find the zeros of f and sketch its graph by plotting points. Use symmetry and increase/decrease information where appropriate.

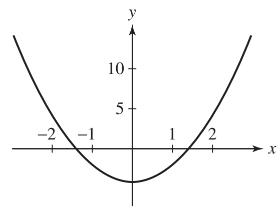
57. $f(x) = x^2 - 4$

SOLUTION Zeros: ± 2
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function); so, y-axis symmetry



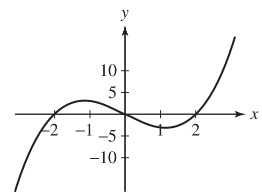
58. $f(x) = 2x^2 - 4$

SOLUTION Zeros: $\pm \sqrt{2}$
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function); so, y-axis symmetry



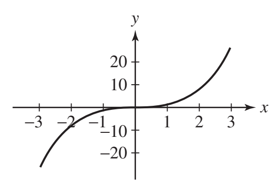
59. $f(x) = x^3 - 4x$

SOLUTION Zeros: 0, ±2; symmetry: $f(-x) = -f(x)$ (odd function); so, origin symmetry



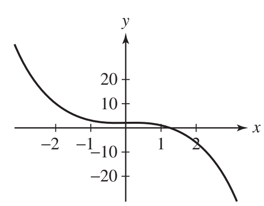
60. $f(x) = x^3$

SOLUTION Zeros: 0; increasing for all x; symmetry: $f(-x) = -f(x)$ (odd function); so, origin symmetry



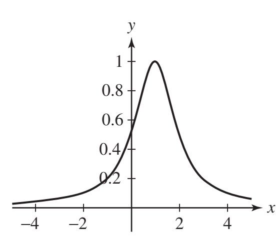
61. $f(x) = 2 - x^3$

SOLUTION This is an x-axis reflection of x^3 translated up 2 units. There is one zero at $x = \sqrt[3]{2}$.



62. $f(x) = \frac{1}{(x-1)^2+1}$

SOLUTION This is the graph of $\frac{1}{x^2+1}$ translated to the right 1 unit. The function has no zeros.



63. Which of the curves in Figure 27 is the graph of a function of x?

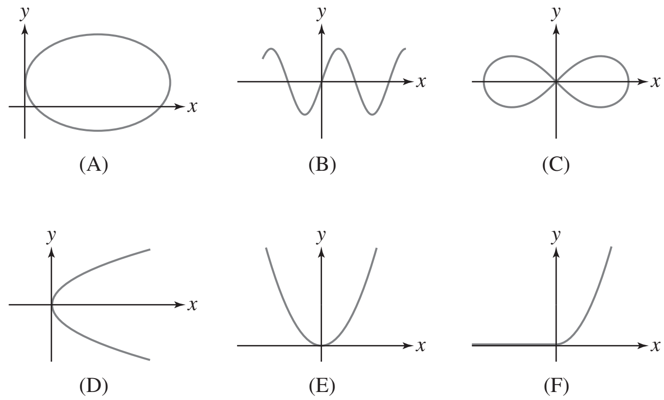


FIGURE 27

SOLUTION (B), (E), and (F) are graphs of functions. (A), (C), and (D) all fail the vertical line test.

64. Of the curves in Figure 27 that are graphs of functions, which is the graph of an odd function? Of an even function?

SOLUTION (B) is the graph of an odd function because the graph is symmetric about the origin; (E) is the graph of an even function because the graph is symmetric about the y-axis.

65. Determine whether the function is even, odd, or neither.

(a) $f(x) = x^5$ **(b)** $g(t) = t^3 - t^2$ **(c)** $F(t) = \frac{1}{t^4 + t^2}$

SOLUTION

(a) Because $f(-x) = (-x)^5 = -x^5 = -f(x)$, $f(x) = x^5$ is an odd function.
(b) Because $g(-t) = (-t)^3 - (-t)^2 = -t^3 - t^2$ equals neither $g(t)$ nor $-g(t)$, $g(t) = t^3 - t^2$ is neither an even function nor an odd function.
(c) Because $F(-t) = \frac{1}{(-t)^4 + (-t)^2} = \frac{1}{t^4 + t^2} = F(t)$, $F(t) = \frac{1}{t^4 + t^2}$ is an even function.

66. Determine whether the function is even, odd, or neither.

(a) $f(x) = 2x - x^2$ **(b)** $k(w) = (1 - w)^3 + (1 + w)^3$
(c) $f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}$ **(d)** $g(t) = 2^t - 2^{-t}$

SOLUTION

(a) Because $f(-x) = 2(-x) - (-x)^2 = -2x - x^2$ equals neither $f(x)$ nor $-f(x)$, $f(x) = 2x - x^2$ is neither an even nor an odd function.
(b) Because $k(-w) = (1 - (-w))^3 + (1 + (-w))^3 = (1 + w)^3 + (1 - w)^3 = k(w)$, $k(w) = (1 - w)^3 + (1 + w)^3$ is an even function.
(c) Because

$$f(-t) = \frac{1}{(-t)^4 + (-t) + 1} - \frac{1}{(-t)^4 - (-t) + 1} = \frac{1}{t^4 - t + 1} - \frac{1}{t^4 + t + 1}$$
$$= -\left(\frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}\right) = -f(t)$$

$f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}$ is an odd function.
(d) Because $g(-t) = 2^{-t} - 2^{-(-t)} = 2^{-t} - 2^t = -(2^t - 2^{-t}) = -g(t)$, $g(t) = 2^t - 2^{-t}$ is an odd function.

67. Write $f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4$ as the sum of an even and an odd function.

SOLUTION Let $g(x) = 2x^4 + 12x^2 + 4$ and $h(x) = -5x^3 - 3x$. Then

$$g(-x) = 2(-x)^4 + 12(-x)^2 + 4 = 2x^4 + 12x^2 + 4 = g(x)$$

so that g is an even function,

$$h(-x) = -5(-x)^3 - 3(-x) = 5x^3 + 3x = -h(x)$$

so that h is an odd function, and $f(x) = g(x) + h(x)$.

68. Assume that p is a function that is defined for all x.

(a) Prove that if f is defined by $f(x) = p(x) + p(-x)$ then f is even.
(b) Prove that if g is defined by $g(x) = p(x) - p(-x)$ then g is odd.

SOLUTION
(a) Let $f(x) = p(x) + p(-x)$. Then

$$f(-x) = p(-x) + p(-(-x)) = p(-x) + p(x) = f(x)$$

Because $f(-x) = f(x)$, it follows that f is an even function.
(b) Let $g(x) = p(x) - p(-x)$. Then

$$g(-x) = p(-x) - p(-(-x)) = p(-x) - p(x) = -(p(x) - p(-x)) = -g(x)$$

Because $g(-x) = -g(x)$, it follows that g is an odd function.

69. Assume that p is a function that is defined for $x > 0$ and satisfies $p(a/b) = p(b) - p(a)$. Prove that $f(x) = p\left(\frac{2-x}{2+x}\right)$ is an odd function.

SOLUTION Let $f(x) = p\left(\frac{2-x}{2+x}\right)$. Then

$$f(-x) = p\left(\frac{2-(-x)}{2-x}\right) = p\left(\frac{2+x}{2-x}\right) = p(2+x) - p(2-x) = -(p(2-x) - p(2+x)) = -p\left(\frac{2-x}{2+x}\right) = -f(x)$$

Because $f(-x) = -f(x)$, it follows that f is an odd function.

70. State whether the function is increasing, decreasing, or neither.
(a) Surface area of a sphere as a function of its radius
(b) Temperature at a point on the equator as a function of time
(c) Price of an airline ticket as a function of the price of oil
(d) Pressure of the gas in a piston as a function of volume

SOLUTION
(a) Increasing **(b)** Neither **(c)** Increasing **(d)** Decreasing

In Exercises 71–76, let f be the function shown in Figure 28.

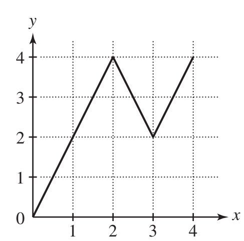


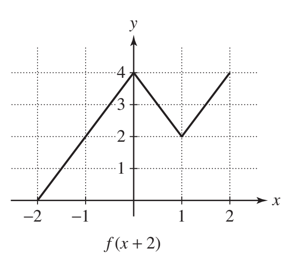
FIGURE 28

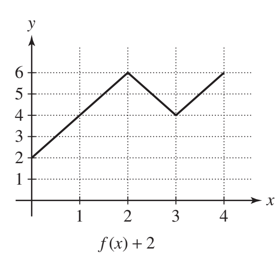
71. Find the domain and range of f.

SOLUTION D: $[0, 4]$; R: $[0, 4]$

72. Sketch the graphs of $y = f(x + 2)$ and $y = f(x) + 2$.

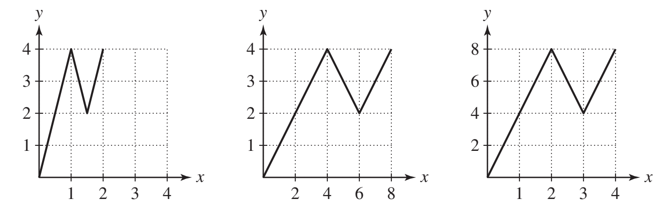
SOLUTION The graph of $y = f(x + 2)$ is obtained by shifting the graph of $y = f(x)$ 2 units to the left (see the graph below on the left). The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ 2 units up (see the graph below on the right).





73. Sketch the graphs of $y = f(2x)$, $y = f(\frac{1}{2}x)$, and $y = 2f(x)$.

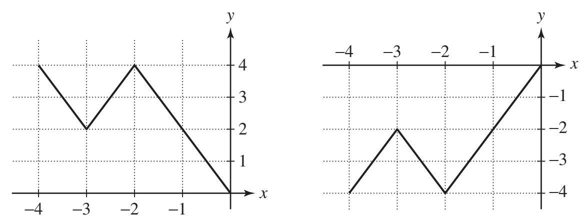
SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below in the middle). The graph of $y = 2f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below on the right).



$f(2x)$ $\qquad$ $f(x/2)$ $\qquad$ $2f(x)$

74. Sketch the graphs of $y = f(-x)$ and $y = -f(-x)$.

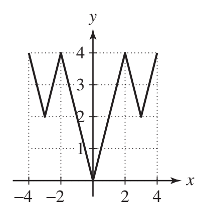
SOLUTION The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across the y-axis (see the graph below on the left). The graph of $y = -f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across both the x- and y-axes, or equivalently, about the origin (see the graph below on the right).



$f(-x)$ $\qquad$ $-f(-x)$

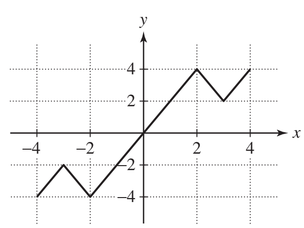
75. Extend the graph of f to $[-4, 4]$ so that it is an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph).



76. Extend the graph of f to $[-4, 4]$ so that it is an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph).



77. Suppose that f has domain $[4, 8]$ and range $[2, 6]$. Find the domain and range of:

(a) $y = f(x) + 3$ $\qquad$ **(b)** $y = f(x + 3)$

(c) $y = f(3x)$ $\qquad$ **(d)** $y = 3f(x)$

SOLUTION

(a) $f(x) + 3$ is obtained by shifting $f(x)$ upward 3 units. Therefore, the domain remains $[4, 8]$, while the range becomes $[5, 9]$.

(b) $f(x + 3)$ is obtained by shifting $f(x)$ left 3 units. Therefore, the domain becomes $[1, 5]$, while the range remains $[2, 6]$.

(c) $f(3x)$ is obtained by compressing $f(x)$ horizontally by a factor of 3. Therefore, the domain becomes $[\frac{4}{3}, \frac{8}{3}]$, while the range remains $[2, 6]$.

(d) $3f(x)$ is obtained by stretching $f(x)$ vertically by a factor of 3. Therefore, the domain remains $[4, 8]$, while the range becomes $[6, 18]$.

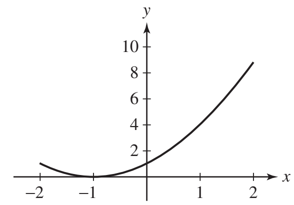
78. Let $f(x) = x^2$. Sketch the graph over $[-2, 2]$ of:

(a) $y = f(x + 1)$ **(b)** $y = f(x) + 1$

(c) $y = f(5x)$ **(d)** $y = 5f(x)$

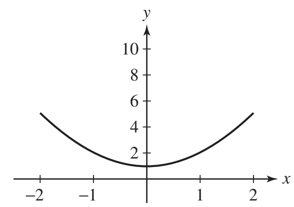
SOLUTION

(a) The graph of $y = f(x + 1)$ is obtained by shifting the graph of $y = f(x)$ 1 unit to the left.



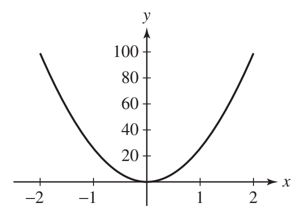
$f(x + 1)$

(b) The graph of $y = f(x) + 1$ is obtained by shifting the graph of $y = f(x)$ 1 unit up.



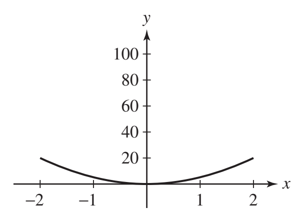
$f(x) + 1$

(c) The graph of $y = f(5x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 5.



$f(5x)$

(d) The graph of $y = 5f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 5.



$5f(x)$

79. Suppose that the graph of $f(x) = x^4 - x^2$ is compressed horizontally by a factor of 2 and then shifted 5 units to the right.

(a) What is the equation for the new graph?

(b) What is the equation if you first shift by 5 and then compress by 2?

(c) GU Verify your answers by plotting your equations.

SOLUTION

(a) Let $f(x) = x^4 - x^2$. After compressing the graph of f horizontally by a factor of 2, we obtain the function $g(x) = f(2x) = (2x)^4 - (2x)^2$. Shifting the graph 5 units to the right then yields

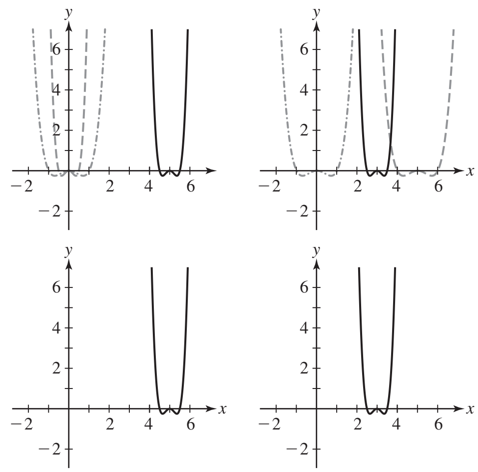
$$h(x) = g(x - 5) = (2(x - 5))^4 - (2(x - 5))^2 = (2x - 10)^4 - (2x - 10)^2$$

(b) Let $f(x) = x^4 - x^2$. After shifting the graph 5 units to the right, we obtain the function $g(x) = f(x - 5) = (x - 5)^4 - (x - 5)^2$. Compressing the graph horizontally by a factor of 2 then yields

$$h(x) = g(2x) = (2x - 5)^4 - (2x - 5)^2$$

(c) The figure below at the top left shows the graphs of $y = x^4 - x^2$ (the dash-dot curve), the graph compressed horizontally by a factor of 2 (the dashed curve), and then shifted right 5 units (the solid curve). Compare this last graph with the graph of $y = (2x - 10)^4 - (2x - 10)^2$ shown at the bottom left.

The figure below at the top right shows the graphs of $y = x^4 - x^2$ (the dash-dot curve), the graph shifted right 5 units (the dashed curve), and then compressed horizontally by a factor of 2 (the solid curve). Compare this last graph with the graph of $y = (2x - 5)^4 - (2x - 5)^2$ shown at the bottom right.



80. Figure 29 shows the graph of $f(x) = |x| + 1$. Match the functions (a)–(e) with their graphs (i)–(v).

(a) $y = f(x - 1)$ **(b)** $y = -f(x)$ **(c)** $y = -f(x) + 2$

(d) $y = f(x - 1) - 2$ **(e)** $y = f(x + 1)$

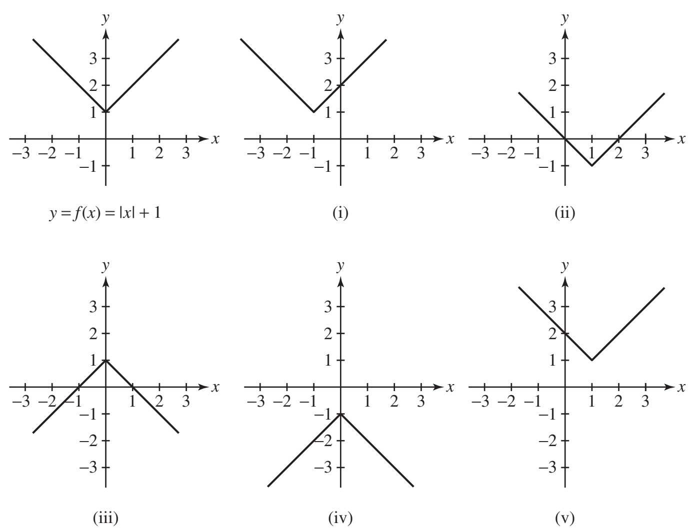


FIGURE 29

SOLUTION

(a) Shift graph to the right 1 unit: (v)

(b) Reflect graph across x-axis: (iv)

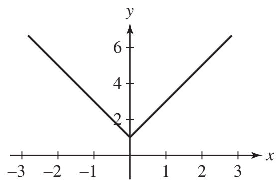
(c) Reflect graph across x-axis and then shift up 2 units: (iii)

(d) Shift graph to the right one unit and down 2 units: (ii)

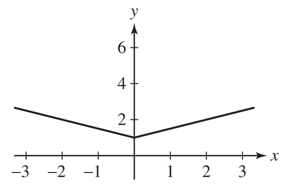
(e) Shift graph to the left 1 unit: (i)

81. Sketch the graph of $y = f(2x)$ and $y = f(\frac{1}{2}x)$, where $f(x) = |x| + 1$ (Figure 29).

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the right).



$f(2x)$



$f(x/2)$

82. Find the function f whose graph is obtained by shifting the parabola $y = x^2$ by 3 units to the right and 4 units down, as in Figure 30.

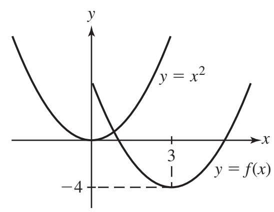
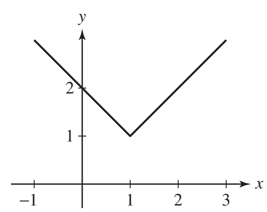


FIGURE 30

SOLUTION The new function is $f(x) = (x - 3)^2 - 4$.

83. Define $f(x)$ to be the larger of x and $2 - x$. Sketch the graph of f. What are its domain and range? Express $f(x)$ in terms of the absolute value function.

SOLUTION



The graph of $y = f(x)$ is shown. Clearly, the domain of f is the set of all real numbers while the range is $\{y \mid y \geq 1\}$. Notice the graph has the standard V shape associated with the absolute value function, but the base of the V has been translated to the point $(1, 1)$. Thus, $f(x) = |x - 1| + 1$.

84. For each curve in Figure 31, state whether it is symmetric with respect to the y-axis, the origin, both, or neither.

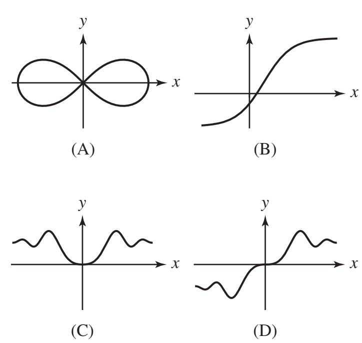


FIGURE 31

SOLUTION

(A) Both

(B) Neither

(C) y-axis

(D) Origin

85. Show that the sum of two even functions is even and the sum of two odd functions is odd.

SOLUTION Even: $(f + g)(-x) = f(-x) + g(-x) \overset{\text{even}}{=} f(x) + g(x) = (f + g)(x)$

Odd: $(f + g)(-x) = f(-x) + g(-x) \overset{\text{odd}}{=} -f(x) + -g(x) = -(f + g)(x)$

86. Suppose that f and g are both odd. Which of the following functions are even? Which are odd?

(a) $y = f(x)g(x)$ **(b)** $y = f(x)^3$

(c) $y = f(x) - g(x)$ **(d)** $y = \frac{f(x)}{g(x)}$

SOLUTION

(a) $f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) \Rightarrow$ even

(b) $f(-x)^3 = [-f(x)]^3 = -f(x)^3 \Rightarrow$ odd

(c) $f(-x) - g(-x) = -f(x) + g(x) = -(f(x) - g(x)) \Rightarrow$ odd

(d) $\frac{f(-x)}{g(-x)} = \frac{-f(x)}{-g(x)} = \frac{f(x)}{g(x)} \Rightarrow$ even

87. Prove that the only function whose graph is symmetric with respect to both the y-axis and the origin is the function $f(x) = 0$.

SOLUTION A circle of radius 1 with its center at the origin is symmetrical with respect to both the y-axis and the origin.

The only function having both symmetries is $f(x) = 0$. For if f is symmetric with respect to the y-axis, then $f(-x) = f(x)$. If f is also symmetric with respect to the origin, then $f(-x) = -f(x)$. Thus, $f(x) = -f(x)$ or $2f(x) = 0$. Finally, $f(x) = 0$.

Further Insights and Challenges

88. Prove the triangle inequality ($|a + b| \le |a| + |b|$) by adding the two inequalities:

$$-|a| \le a \le |a|, \qquad -|b| \le b \le |b|$$

SOLUTION Adding the indicated inequalities gives

$$-(|a| + |b|) \le a + b \le |a| + |b|$$

and this is equivalent to $|a + b| \le |a| + |b|$.

89. Show that a fraction $r = a/b$ in lowest terms has a *finite* decimal expansion if and only if

$$b = 2^n 5^m \quad \text{for some } n, m \ge 0$$

Hint: Observe that r has a finite decimal expansion when $10^N r$ is an integer for some $N \ge 0$ (and hence b divides 10^N).

SOLUTION Suppose r has a finite decimal expansion. Then there exists an integer $N \ge 0$ such that $10^N r$ is an integer, call it k. Thus, $r = k/10^N$. Because the only prime factors of 10 are 2 and 5, it follows that when r is written in lowest terms, its denominator must be of the form $2^n 5^m$ for some integers $n, m \ge 0$.

Conversely, suppose $r = a/b$ is written in lowest terms with $b = 2^n 5^m$ for some integers $n, m \ge 0$. Then $r = \frac{a}{b} = \frac{a}{2^n 5^m}$ or $2^n 5^m r = a$. If $m \ge n$, then $2^m 5^m r = a 2^{m-n}$ or $r = \frac{a 2^{m-n}}{10^m}$ and thus r has a finite decimal expansion (less than or equal to m terms, to be precise). On the other hand, if $n > m$, then $2^n 5^n r = a 5^{n-m}$ or $r = \frac{a 5^{n-m}}{10^n}$ and once again r has a finite decimal expansion.

90. Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$. Show that

$$\frac{p}{10^s - 1} = 0.\overline{p_1 \dots p_s}$$

Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that

$$r = \frac{2}{11} = \frac{18}{10^2 - 1}$$

SOLUTION Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$, and let $\overline{p} = 0.\overline{p_1 \dots p_s}$. Then

$$10^s \overline{p} - \overline{p} = p_1 \dots p_s.\overline{p_1 \dots p_s} - 0.\overline{p_1 \dots p_s} = p_1 \dots p_s = p$$

Thus,

$$\frac{p}{10^s - 1} = \overline{p} = 0.\overline{p_1 \dots p_s}$$

Consider the rational number $r = 2/11$. Because

$$r = \frac{2}{11} = \frac{18}{99} = \frac{18}{10^2 - 1}$$

it follows that the decimal expansion of r is $0.\overline{18}$.

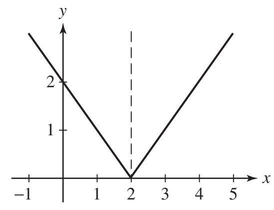
91. A function f is symmetric with respect to the vertical line $x = a$ if $f(a - x) = f(a + x)$.

(a) Draw the graph of a function that is symmetric with respect to $x = 2$.

(b) Show that if f is symmetric with respect to $x = a$, then $g(x) = f(x + a)$ is even.

SOLUTION

(a) There are many possibilities, one of which is



$y = |x - 2|$

(b) Let $g(x) = f(x + a)$. Then

$$\begin{aligned} g(-x) = f(-x + a) &= f(a - x) \\ &= f(a + x) \qquad \text{(symmetry with respect to } x = a\text{)} \\ &= g(x) \end{aligned}$$

Thus, $g(x)$ is even.

92. Formulate a condition for f to be symmetric with respect to the point $(a, 0)$ on the x-axis.

SOLUTION In order for $f(x)$ to be symmetric with respect to the point $(a, 0)$, the value of f at a distance x units to the right of a must be opposite the value of f at a distance x units to the left of a. In other words, $f(x)$ is symmetrical with respect to $(a, 0)$ if $f(a + x) = -f(a - x)$.

1.2 Linear and Quadratic Functions

Preliminary Questions

1. What is the slope of the line $y = -4x - 9$?

SOLUTION The slope of the line $y = -4x - 9$ is -4, given by the coefficient of x.

2. Are the lines $y = 2x + 1$ and $y = -2x - 4$ perpendicular?

SOLUTION The slopes of perpendicular lines are negative reciprocals of one another. Because the slope of $y = 2x + 1$ is 2 and the slope of $y = -2x - 4$ is -2, these two lines are *not* perpendicular.

3. When is the line $ax + by = c$ parallel to the y-axis? To the x-axis?

SOLUTION The line $ax + by = c$ will be parallel to the y-axis when $b = 0$ and parallel to the x-axis when $a = 0$.

4. Suppose $y = 3x + 2$. What is Δy if x increases by 3?

SOLUTION Because $y = 3x + 2$ is a linear function with slope 3, increasing x by 3 will lead to $\Delta y = 3(3) = 9$.

5. What is the minimum of $f(x) = (x + 3)^2 - 4$?

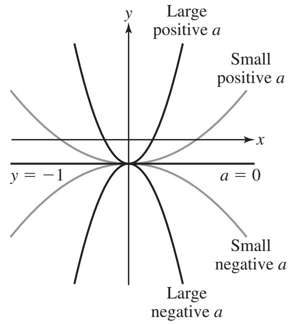
SOLUTION Because $(x + 3)^2 \geq 0$, it follows that $(x + 3)^2 - 4 \geq -4$. Thus, the minimum value of $(x + 3)^2 - 4$ is -4.

6. What is the result of completing the square for $f(x) = x^2 + 1$?

SOLUTION Because there is no x term in $x^2 + 1$, completing the square on this expression leads to $(x - 0)^2 + 1$.

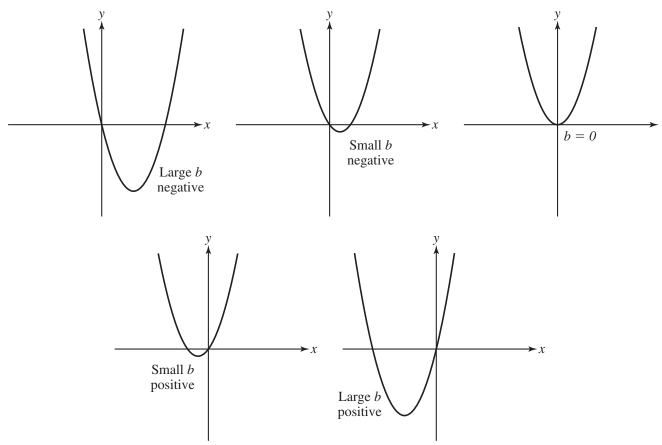
7. Describe how the parabolas $y = ax^2 - 1$ change as a changes from $-\infty$ to ∞.

SOLUTION First, note that the graph of $y = ax^2 - 1$ passes through the point $(0, -1)$ for all values of a. When a is large negative, the parabola $y = ax^2 - 1$ opens downward and is narrow. As a increases toward zero, the parabola becomes wider but remains opening downward. When $a = 0$, the parabola has "opened out" to become the horizontal line $y = -1$. For positive a, the parabola opens upward, wide for small values of a but then becoming more and more narrow as a increases. See the figure.



8. Describe how the parabolas $y = x^2 + bx$ change as b changes from $-\infty$ to ∞.

SOLUTION The parabolas open upward and have x-intercepts at $(0, 0)$ and $(-b, 0)$. Moreover, the vertex is located at $(-b/2, -b^2/4)$. For b negative and increasing toward zero, the nonzero x-intercept is positive and moves toward the origin, while the vertex moves up and to the left toward the origin. When $b = 0$, there is a single x-intercept, which is also the vertex, at the origin. For b positive and increasing, the nonzero x-intercept is negative and moves away from the origin, while the vertex moves down and to the left. See the figures.



Exercises

In Exercises 1–4, find the slope, the y-intercept, and the x-intercept of the line with the given equation.

1. $y = 3x + 12$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term; that is, $m = 3$ and the y-intercept is 12. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 3x + 12$ or $x = -4$.

2. $y = 4 - x$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term; that is, $m = -1$ and the y-intercept is 4. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 4 - x$ or $x = 4$.

3. $4x + 9y = 3$

SOLUTION To determine the slope and y-intercept, we first solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{4}{9}x + \frac{1}{3}$. From here, we see that the slope is $m = -\frac{4}{9}$ and the y-intercept is $\frac{1}{3}$. To determine the x-intercept, substitute $y = 0$ and solve for x: $4x = 3$ or $x = \frac{3}{4}$.

4. $y - 3 = \frac{1}{2}(x - 6)$

SOLUTION The equation is in point-slope form, so we see that $m = \frac{1}{2}$. Substituting $x = 0$ yields $y - 3 = -3$ or $y = 0$. Thus, the x- and y-intercepts are both 0.

In Exercises 5–8, find the slope of the line.

5. $y = 3x + 2$

SOLUTION $m = 3$

6. $y = 3(x - 9) + 2$

SOLUTION $m = 3$

7. $3x + 4y = 12$

SOLUTION First, solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x + 3$. The slope of the line is therefore $m = -\frac{3}{4}$.

8. $3x + 4y = -8$

SOLUTION First, solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x - 2$. The slope of the line is therefore $m = -\frac{3}{4}$.

In Exercises 9–20, find the equation of the line with the given description.

9. Slope 3, y-intercept 8

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = 3x + 8$.

10. Slope -2, y-intercept 3

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = -2x + 3$.

11. Slope 3, passes through $(7, 9)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 9 = 3(x - 7)$ or $y = 3x - 12$.

12. Slope -5, passes through $(0, 0)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 0 = -5(x - 0)$ or $y = -5x$.

13. Horizontal, passes through $(0, -2)$

SOLUTION A horizontal line has a slope of 0. Using the point-slope form for the equation of a line, we have $y - (-2) = 0(x - 0)$ or $y = -2$.

14. Passes through $(-1, 4)$ and $(2, 7)$

SOLUTION The slope of the line that passes through $(-1, 4)$ and $(2, 7)$ is

$$m = \frac{7 - 4}{2 - (-1)} = 1$$

Using the point-slope form for the equation of a line, we have $y - 7 = 1(x - 2)$ or $y = x + 5$.

15. Parallel to $y = 3x - 4$, passes through $(1, 1)$

SOLUTION Because the equation $y = 3x - 4$ is in slope-intercept form, we can readily identify that it has a slope of 3. Parallel lines have the same slope, so the slope of the requested line is also 3. Using the point-slope form for the equation of a line, we have $y - 1 = 3(x - 1)$ or $y = 3x - 2$.

16. Passes through $(1, 4)$ and $(12, -3)$

SOLUTION The slope of the line that passes through $(1, 4)$ and $(12, -3)$ is

$$m = \frac{-3 - 4}{12 - 1} = \frac{-7}{11}$$

Using the point-slope form for the equation of a line, we have $y - 4 = -\frac{7}{11}(x - 1)$ or $y = -\frac{7}{11}x + \frac{51}{11}$.

17. Perpendicular to $3x + 5y = 9$, passes through $(2, 3)$

SOLUTION We start by solving the equation $3x + 5y = 9$ for y to obtain the slope-intercept form for the equation of a line. This yields

$$y = -\frac{3}{5}x + \frac{9}{5}$$

from which we identify the slope as $-\frac{3}{5}$. Perpendicular lines have slopes that are negative reciprocals of one another, so the slope of the desired line is $m_\perp = \frac{5}{3}$. Using the point-slope form for the equation of a line, we have $y - 3 = \frac{5}{3}(x - 2)$ or $y = \frac{5}{3}x - \frac{1}{3}$.

18. Vertical, passes through $(-4, 9)$

SOLUTION A vertical line has the equation $x = c$ for some constant c. Because the line needs to pass through the point $(-4, 9)$, we must have $c = -4$. The equation of the desired line is then $x = -4$.

19. Horizontal, passes through $(8, 4)$

SOLUTION A horizontal line has slope 0. Using the point-slope form for the equation of a line, we have $y - 4 = 0(x - 8)$ or $y = 4$.

20. Slope 3, x-intercept 6

SOLUTION If the x-intercept is 6, then the line passes through the point $(6, 0)$. Using the point-slope form for the equation of a line, we have $y - 0 = 3(x - 6)$ or $y = 3x - 18$.

21. Find the equation of the perpendicular bisector of the segment joining $(1, 2)$ and $(5, 4)$ (Figure 12). *Hint:* The midpoint Q of the segment joining (a, b) and (c, d) is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

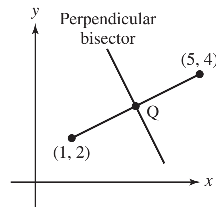


FIGURE 12

SOLUTION The slope of the segment joining $(1, 2)$ and $(5, 4)$ is

$$m = \frac{4-2}{5-1} = \frac{1}{2}$$

and the midpoint of the segment (Figure 12) is

$$\text{midpoint} = \left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3, 3)$$

The perpendicular bisector has slope $-1/m = -2$ and passes through $(3, 3)$, so its equation is $y - 3 = -2(x - 3)$ or $y = -2x + 9$.

22. Intercept–Intercept Form Show that if $a, b \neq 0$, then the line with x-intercept $x = a$ and y-intercept $y = b$ has equation (Figure 13)

$$\frac{x}{a} + \frac{y}{b} = 1$$

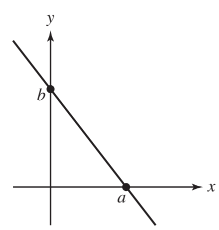


FIGURE 13

SOLUTION The line passes through the points $(a, 0)$ and $(0, b)$. Thus, $m = -\frac{b}{a}$. Using the point-slope form for the equation of a line yields $y - 0 = -\frac{b}{a}(x - a) \Rightarrow y = -\frac{b}{a}x + b \Rightarrow \frac{b}{a}x + y = b \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

23. Find an equation of the line with x-intercept $x = 4$ and y-intercept $y = 3$.

SOLUTION From Exercise 22, $\frac{x}{4} + \frac{y}{3} = 1$ or $3x + 4y = 12$.

24. Find y such that $(3, y)$ lies on the line of slope $m = 2$ through $(1, 4)$.

SOLUTION In order for the point $(3, y)$ to lie on the line through $(1, 4)$ of slope 2, the slope of the segment connecting $(1, 4)$ and $(3, y)$ must have slope 2. Therefore,

$$m = \frac{y-4}{3-1} = \frac{y-4}{2} = 2 \Rightarrow y - 4 = 4 \Rightarrow y = 8$$

25. Determine whether there exists a constant c such that the line $x + cy = 1$:

(a) Has slope 4

(b) Passes through $(3, 1)$

(c) Is horizontal

(d) Is vertical

SOLUTION

(a) Rewriting the equation of the line in slope-intercept form gives $y = -\frac{x}{c} + \frac{1}{c}$. To have slope 4 requires $-\frac{1}{c} = 4$ or $c = -\frac{1}{4}$.

(b) Substituting $x = 3$ and $y = 1$ into the equation of the line gives $3 + c = 1$ or $c = -2$.

(c) From (a), we know the slope of the line is $-\frac{1}{c}$. There is no value for c that will make this slope equal to 0.

(d) With $c = 0$, the equation becomes $x = 1$. This is the equation of a vertical line.

26. Determine whether there exists a constant c such that the line $cx - 2y = 4$:

(a) Has slope 4
(b) Passes through $(1, -4)$
(c) Is horizontal
(d) Is vertical

SOLUTION

(a) Rewriting the equation of the line in slope-intercept form gives $y = \frac{c}{2}x - 2$. To have slope 4 requires $\frac{c}{2} = 4$ or $c = 8$.
(b) Substituting $x = 1$ and $y = -4$ into the equation of the line gives $c + 8 = 4$ or $c = -4$.
(c) A horizontal line has slope zero. From (a), we know the slope of the line is $\frac{c}{2}$, so to have slope zero requires $\frac{c}{2} = 0$ or $c = 0$.
(d) A vertical line has equation $x = a$ for some constant a. There is no value for the constant c that will make the equation $cx - 2y = 4$ have the correct form.

27. Suppose that the number of Bob's Bits computers that can be sold when its price is P (in dollars) is given by a linear function $N(P)$, where $N(1000) = 10{,}000$ and $N(1500) = 7500$.

(a) Determine $N(P)$.
(b) What is the slope of the graph of $N(P)$, including units? Describe what the slope represents.
(c) What is the change ΔN in the number of computers sold if the price is increased by $\Delta P = \$100$?

SOLUTION

(a) We first determine the slope of the line:

$$m = \frac{10000 - 7500}{1000 - 1500} = \frac{2500}{-500} = -5$$

Knowing that $N(1000) = 10000$, it follows that

$$N - 10000 = -5(P - 1000) \quad \text{or} \quad N(P) = -5P + 15000$$

(b) The slope of the graph of $N(P)$ is -5 computers/dollar. The slope represents the rate of change in the number of computers sold with respect to the price of the computer. In particular, five fewer computers are sold for every \$1 increase in price of the computer.
(c) $\Delta N = -5\Delta P = -5(100) = -500$. If the price of the computer is increased by \$100, 500 fewer computers will be sold.

28. Suppose that the demand for Colin's kidney pies is linear in the price P. Further, assume that he can sell 100 pies when the price is \$5.00 and 40 pies when the price is \$10.00.

(a) Determine the demand N (number of pies sold) as a function of the price P (in dollars).
(b) What is the slope of the graph of $N(P)$, including units? Describe what the slope represents.
(c) Determine the revenue $R = N \times P$ for prices $P = 5, 6, 7, 8, 9, 10$ and then choose a price to maximize the revenue.

SOLUTION

(a) We first determine the slope of the line:

$$m = \frac{100 - 40}{5 - 10} = \frac{60}{-5} = -12$$

Knowing that $N(5) = 100$, it follows that

$$N - 100 = -12(P - 5) \quad \text{or} \quad N(P) = -12P + 160$$

(b) The slope of the graph of $N(P)$ is -12 pies/dollar. The slope represents the rate of change in the number of pies sold with respect to the price of the pie. In particular, 12 fewer pies are sold for every \$1 increase in price of the pie.
(c) The table displays the revenue for prices $P = 5, 6, 7, 8, 9, 10$.

Price (P) (in dollars)	Demand (N) (number of pies sold)	Revenue ($N \times P$) (in dollars)
5	100	500
6	88	528
7	76	532
8	64	512
9	52	468
10	40	400

To determine the price that will maximize revenue, note that the revenue function, $R(P)$, is given by

$$R(P) = N(P) \times P = (160 - 12P)P = -12P^2 + 160P$$

Completing the square on the revenue function yields

$$R(P) = -12\left(P^2 - \frac{40}{3}P + \frac{400}{9}\right) + \frac{1600}{9} = -12\left(P - \frac{20}{3}\right)^2 + \frac{1600}{6}$$

From here, it follows that revenue is maximized when $P = \frac{20}{3}$, or roughly \$6.67.

29. In each case, identify the slope and give its meaning with the appropriate units.

(a) The function $N = -70t + 5000$ models the enrollment at Maple Grove College during the fall of 2018, where N represents the number of students and t represents the time in weeks since the start of the semester.

(b) The function $C = 3.5n + 700$ represents the cost (in dollars) to rent the Shakedown Street Dance Hall for an evening if n people attend the dance.

SOLUTION

(a) The slope is -70 students/week. During the fall of 2018, enrollment dropped by 70 students per week.

(b) The slope is 3.5 dollars/person. The rental cost for the Shakedown Street Dance Hall increases \$3.50 for each additional person in attendance.

30. In each case, identify the slope and give its meaning with the appropriate units.

(a) The function $N = 3.9T - 178.8$ models the the number of times, N, that a cricket chirps in a minute when the temperature is $T°$ Celsius.

(b) The function $V = 47,500d$ gives the volume (V, in gallons) of molasses in the storage tank in relation to the depth (d, in feet) of the molasses.

SOLUTION

(a) The slope is 3.9 chirps per minute/°C. The cricket chirp rate increases by 3.9 chirps per minute for every 1°C increase in temperature.

(b) The slope is 47,500 gal/ft. The volume of molasses in the storage tank increases by 47,500 gal for every 1-ft increase in depth of the molasses in the tank.

31. Materials expand when heated. Consider a metal rod of length L_0 at temperature T_0. If the temperature is changed by an amount ΔT, then the rod's length approximately changes by $\Delta L = \alpha L_0 \Delta T$, where α is the thermal expansion coefficient and ΔT is not an extreme temperature change. For steel, $\alpha = 1.24 \times 10^{-5}\,°\mathrm{C}^{-1}$.

(a) A steel rod has length $L_0 = 40$ cm at $T_0 = 40°$C. Find its length at $T = 90°$C.

(b) Find its length at $T = 50°$C if its length at $T_0 = 100°$C is 65 cm.

(c) Express length L as a function of T if $L_0 = 65$ cm at $T_0 = 100°$C.

SOLUTION

(a) With $T = 90°$C and $T_0 = 40°$C, $\Delta T = 50°$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(40)(50) = 0.0248 \qquad \text{and} \qquad L = L_0 + \Delta L = 40.0248 \text{ cm}$$

(b) With $T = 50°$C and $T_0 = 100°$C, $\Delta T = -50°$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(65)(-50) = -0.0403 \qquad \text{and} \qquad L = L_0 + \Delta L = 64.9597 \text{ cm}$$

(c) $L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T) = 65(1 + \alpha(T - 100))$

32. Do the points $(0.5, 1)$, $(1, 1.2)$, $(2, 2)$ lie on a line?

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.2 - 1}{1 - 0.5} = \frac{0.2}{0.5} = 0.4$$

while the second pair of data points yields a slope of

$$\frac{2 - 1.2}{2 - 1} = \frac{0.8}{1} = 0.8$$

Because the slopes are not equal, the three points do not lie on a line.

33. Find b such that $(2, -1)$, $(3, 2)$, and $(b, 5)$ lie on a line.

SOLUTION The slope of the line determined by the points $(2, -1)$ and $(3, 2)$ is

$$\frac{2 - (-1)}{3 - 2} = 3$$

To lie on the same line, the slope between $(3, 2)$ and $(b, 5)$ must also be 3. Thus, we require

$$\frac{5 - 2}{b - 3} = \frac{3}{b - 3} = 3$$

or $b = 4$.

34. Find an expression for the velocity v as a linear function of t that matches the following data:

t (s)	0	2	4	6
v (m/s)	39.2	58.6	78	97.4

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{58.6 - 39.2}{2 - 0} = 9.7$$

while the second pair of data points yields a slope of

$$\frac{78 - 58.6}{4 - 2} = 9.7$$

and the last pair of data points yields a slope of

$$\frac{97.4 - 78}{6 - 4} = 9.7$$

Thus, the data suggest a linear function with slope 9.7. Finally,

$$v - 39.2 = 9.7(t - 0) \Rightarrow v = 9.7t + 39.2$$

35. The period T of a pendulum is measured for pendulums of several different lengths L. Based on the following data, does T appear to be a linear function of L?

L (cm)	20	30	40	50
T (s)	0.9	1.1	1.27	1.42

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.1 - 0.9}{30 - 20} = 0.02$$

while the second pair of data points yields a slope of

$$\frac{1.27 - 1.1}{40 - 30} = 0.017$$

and the last pair of data points yields a slope of

$$\frac{1.42 - 1.27}{50 - 40} = 0.015$$

Because the three slopes are not equal, T does not appear to be a linear function of L.

36. Show that f is linear of slope m if and only if

$$f(x + h) - f(x) = mh \quad \text{(for all } x \text{ and } h\text{)}$$

That is to say, prove the following two statements:

(a) f is linear of slope m implies that $f(x + h) - f(x) = mh$ (for all x and h).

(b) $f(x + h) - f(x) = mh$ (for all x and h) implies that f is linear of slope m.

SOLUTION

(a) First, suppose $f(x)$ is linear. Then the slope between $(x, f(x))$ and $(x + h, f(x + h))$ is

$$m = \frac{f(x + h) - f(x)}{h} \Rightarrow mh = f(x + h) - f(x)$$

(b) Conversely, suppose $f(x + h) - f(x) = mh$ for all x and for all h. Then

$$m = \frac{f(x + h) - f(x)}{h} = \frac{f(x + h) - f(x)}{x + h - x}$$

which is the slope between $(x, f(x))$ and $(x + h, f(x + h))$. Since this is true for all x and h, f must be linear (it has constant slope).

37. Find the roots of the quadratic polynomials:

(a) $f(x) = 4x^2 - 3x - 1$

(b) $f(x) = x^2 - 2x - 1$

SOLUTION

(a) $x = \frac{3 \pm \sqrt{9-4(4)(-1)}}{2(4)} = \frac{3 \pm \sqrt{25}}{8} = 1$ or $-\frac{1}{4}$

(b) $x = \frac{2 \pm \sqrt{4-(4)(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

In Exercises 38–45, complete the square and find the minimum or maximum value of the quadratic function.

38. $y = x^2 + 2x + 5$

SOLUTION $y = x^2 + 2x + 1 - 1 + 5 = (x+1)^2 + 4$; therefore, the minimum value of the quadratic polynomial is 4, and this occurs at $x = -1$.

39. $y = x^2 - 6x + 9$

SOLUTION $y = (x-3)^2$; therefore, the minimum value of the quadratic polynomial is 0, and this occurs at $x = 3$.

40. $y = -9x^2 + x$

SOLUTION $y = -9(x^2 - x/9) = -9(x^2 - \frac{x}{9} + \frac{1}{324}) + \frac{9}{324} = -9(x - \frac{1}{18})^2 + \frac{1}{36}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{36}$, and this occurs at $x = \frac{1}{18}$.

41. $y = x^2 + 6x + 2$

SOLUTION $y = x^2 + 6x + 9 - 9 + 2 = (x+3)^2 - 7$; therefore, the minimum value of the quadratic polynomial is -7, and this occurs at $x = -3$.

42. $y = 2x^2 - 4x - 7$

SOLUTION $y = 2(x^2 - 2x + 1 - 1) - 7 = 2(x^2 - 2x + 1) - 7 - 2 = 2(x-1)^2 - 9$; therefore, the minimum value of the quadratic polynomial is -9, and this occurs at $x = 1$.

43. $y = -4x^2 + 3x + 8$

SOLUTION $y = -4x^2 + 3x + 8 = -4(x^2 - \frac{3}{4}x + \frac{9}{64}) + 8 + \frac{9}{16} = -4(x - \frac{3}{8})^2 + \frac{137}{16}$; therefore, the maximum value of the quadratic polynomial is $\frac{137}{16}$, and this occurs at $x = \frac{3}{8}$.

44. $y = 3x^2 + 12x - 5$

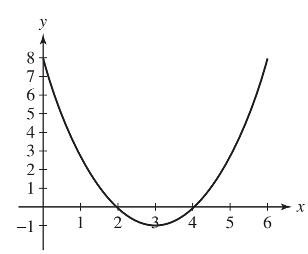
SOLUTION $y = 3(x^2 + 4x + 4) - 5 - 12 = 3(x+2)^2 - 17$; therefore, the minimum value of the quadratic polynomial is -17, and this occurs at $x = -2$.

45. $y = 4x - 12x^2$

SOLUTION $y = -12(x^2 - \frac{x}{3}) = -12(x^2 - \frac{x}{3} + \frac{1}{36}) + \frac{1}{3} = -12(x - \frac{1}{6})^2 + \frac{1}{3}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{3}$, and this occurs at $x = \frac{1}{6}$.

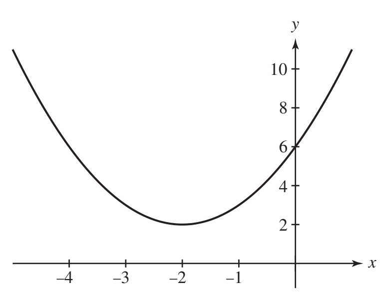
46. Sketch the graph of $y = x^2 - 6x + 8$ by plotting the roots and the minimum point.

SOLUTION $y = x^2 - 6x + 9 - 9 + 8 = (x-3)^2 - 1$ so the vertex is located at $(3, -1)$ and the roots are $x = 2$ and $x = 4$. This is the graph of x^2 moved right 3 units and down 1 unit.



47. Sketch the graph of $y = x^2 + 4x + 6$ by plotting the minimum point, the y-intercept, and one other point.

SOLUTION $y = x^2 + 4x + 4 - 4 + 6 = (x+2)^2 + 2$ so the minimum occurs at $(-2, 2)$. If $x = 0$, then $y = 6$ and if $x = -4$, $y = 6$. This is the graph of x^2 moved left 2 units and up 2 units.



48. If the alleles A and B of the cystic fibrosis gene occur in a population with frequencies p and $1 - p$ (where p is between 0 and 1), then the frequency of heterozygous carriers (carriers with both alleles) is $2p(1 - p)$. Which value of p gives the largest frequency of heterozygous carriers?

SOLUTION Let

$$f = 2p - 2p^2 = -2\left(p^2 - p + \frac{1}{4}\right) + \frac{1}{2} = -2\left(p - \frac{1}{2}\right)^2 + \frac{1}{2}$$

Then $p = \frac{1}{2}$ yields a maximum.

49. For which values of c does $f(x) = x^2 + cx + 1$ have a double root? No real roots?

SOLUTION A double root occurs when $c^2 - 4(1)(1) = 0$ or $c^2 = 4$. Thus, $c = \pm 2$.
There are no real roots when $c^2 - 4(1)(1) < 0$ or $c^2 < 4$. Thus, $-2 < c < 2$.

50. Let $f(x) = x^2 + x - 1$.

(a) Show that the lines $y = x + 3$, $y = x - 1$, and $y = x - 3$ intersect the graph of f in two, one, and zero points, respectively.

(b) Sketch the graph of f and the three lines from (a).

(c) Describe the relationship between the graph of f and the lines $y = x + c$ as c changes from $-\infty$ to ∞.

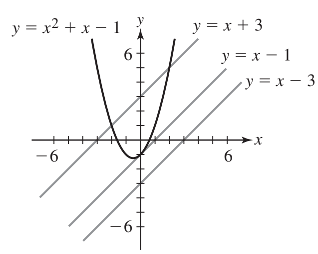
SOLUTION

(a) The x-coordinates of any points of intersection between the line $y = x + 3$ and the graph of f must satisfy the equation $x^2 + x - 1 = x + 3$, or equivalently, $x^2 - 4 = (x + 2)(x - 2) = 0$. This equation has two solutions, $x = -2$ and $x = 2$, so the line $y = x + 3$ intersects the graph of f in two points: $(-2, 1)$ and $(2, 5)$.

The x-coordinates of any points of intersection between the line $y = x - 1$ and the graph of f must satisfy the equation $x^2 + x - 1 = x - 1$, or equivalently, $x^2 = 0$. This equation has one solution, $x = 0$, so the line $y = x - 1$ intersects the graph of f in one point: $(0, -1)$.

Finally, The x-coordinates of any points of intersection between the line $y = x - 3$ and the graph of f must satisfy the equation $x^2 + x - 1 = x - 3$, or equivalently, $x^2 = -2$. This equation has no real solutions, so the line $y = x - 3$ does not intersect the graph of f.

(b) The figure shows the graph of f and the three lines from (a).



(c) The lines $y = x + c$ have slope 1 and y-intercept $(0, c)$. For large negative c, the line $y = x + c$ lies below the graph of f. As c increases, the line moves vertically upward and when $c = -1$ makes contact with the graph of f at the single intersection point $(0, -1)$. As c increases beyond -1, the line moves farther upward, intersecting the graph of f in two points for all such c.

51. Let $f(x) = x^2 + 2x - 21$.

(a) Show that the lines $y = 3x - 25$, $y = 6x - 25$, and $y = 9x - 25$ intersect the graph of f in zero, one, and two points, respectively.

(b) Sketch the graph of f and the three lines from (a).

(c) Describe the relationship between the graph of f and the lines $y = cx - 25$ as c changes from 0 to ∞.

SOLUTION

(a) The x-coordinates of any points of intersection between the line $y = 3x - 25$ and the graph of f must satisfy the equation $x^2 + 2x - 21 = 3x - 25$, or equivalently, $x^2 - x + 4 = 0$. For the latter equation, the quadratic formula yields

$$x = \frac{1 \pm \sqrt{-15}}{2}$$

Because there are no real solutions, the line $y = 3x - 25$ does not intersect the graph of f.

The x-coordinates of any points of intersection between the line $y = 6x - 25$ and the graph of f must satisfy the equation $x^2 + 2x - 21 = 6x - 25$, or equivalently, $x^2 - 4x + 4 = (x - 2)^2 = 0$. This equation has one solution, $x = 2$, so the line $y = 6x - 25$ intersects the graph of f in one point: $(2, -13)$.

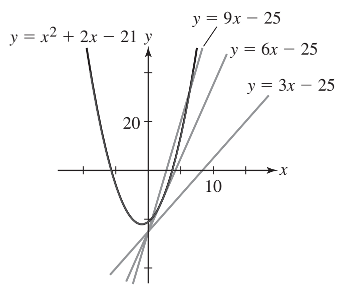
Finally, The x-coordinates of any points of intersection between the line $y = 9x - 25$ and the graph of f must satisfy the equation $x^2 + 2x - 21 = 9x - 25$, or equivalently, $x^2 - 7x + 4 = 0$. For the latter equation, the quadratic formula yields two real solutions,

$$x = \frac{7 \pm \sqrt{33}}{2}$$

so the line $y = 9x - 25$ intersects the graph of f in two points:

$$\left(\frac{7 - \sqrt{33}}{2}, \frac{13 - 9\sqrt{33}}{2}\right) \quad \text{and} \quad \left(\frac{7 + \sqrt{33}}{2}, \frac{13 + 9\sqrt{33}}{2}\right)$$

(b) The figure shows the graph of f and the three lines from (a).



(c) The lines $y = cx - 25$ have slope c and y-intercept $(0, -25)$. For $c = 0$, the line $y = cx - 25$ is horizontal and lies below the graph of f. As c increases from 0, the slope increases, but the line still lies below the graph of f for all c up to 6. When $c = 6$, the line makes contact with the graph of f at the single intersection point $(2, -13)$. As c increases beyond 6, the slope increases further, and the line intersects the graph of f in two points for all such c.

52. Let $a, b > 0$. Show that the *geometric mean* $\sqrt{ab}$ is not larger than the *arithmetic mean* $(a + b)/2$. *Hint:* Consider $(a^{1/2} - b^{1/2})^2$.

SOLUTION Let $a, b > 0$ and note

$$0 \leq \left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{ab} + b$$

Therefore,

$$\sqrt{ab} \leq \frac{a + b}{2}$$

53. If objects of weights x and w_1 are suspended from the balance in Figure 14(A), the cross-beam is horizontal if $bx = aw_1$. If the lengths a and b are known, we may use this equation to determine an unknown weight x by selecting w_1 such that the cross-beam is horizontal. If a and b are not known precisely, we might proceed as follows. First balance x by w_1 on the left, as in (A). Then switch places and balance x by w_2 on the right, as in (B). The average $\bar{x} = \frac{1}{2}(w_1 + w_2)$ gives an estimate for x. Show that $\bar{x}$ is greater than or equal to the true weight x.

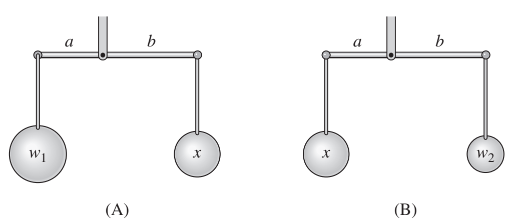


FIGURE 14

SOLUTION First, note $bx = aw_1$ and $ax = bw_2$. Thus,

$$\begin{aligned}
\bar{x} &= \frac{1}{2}(w_1 + w_2) \\
&= \frac{1}{2}\left(\frac{bx}{a} + \frac{ax}{b}\right) \\
&= \frac{x}{2}\left(\frac{b}{a} + \frac{a}{b}\right) \\
&\geq \frac{x}{2}(2) \\
&= x
\end{aligned}$$

54. Find numbers x and y with sum 10 and product 24. *Hint:* Find a quadratic polynomial satisfied by x.

SOLUTION Let x and y be numbers whose sum is 10 and product is 24. Then $x + y = 10$ and $xy = 24$. From the second equation, $y = \frac{24}{x}$. Substituting this expression for y in the first equation gives $x + \frac{24}{x} = 10$ or $x^2 - 10x + 24 = (x-4)(x-6) = 0$, when $x = 4$ or $x = 6$. If $x = 4$, then $y = \frac{24}{4} = 6$. On the other hand, if $x = 6$, then $y = \frac{24}{6} = 4$. Thus, the two numbers are 4 and 6.

55. Find a pair of numbers whose sum and product are both equal to 8.

SOLUTION Let x and y be numbers whose sum and product are both equal to 8. Then $x + y = 8$ and $xy = 8$. From the second equation, $y = \frac{8}{x}$. Substituting this expression for y in the first equation gives $x + \frac{8}{x} = 8$ or $x^2 - 8x + 8 = 0$. By the quadratic formula,

$$x = \frac{8 \pm \sqrt{64 - 32}}{2} = 4 \pm 2\sqrt{2}$$

If $x = 4 + 2\sqrt{2}$, then

$$y = \frac{8}{4 + 2\sqrt{2}} = \frac{8}{4 + 2\sqrt{2}} \cdot \frac{4 - 2\sqrt{2}}{4 - 2\sqrt{2}} = 4 - 2\sqrt{2}$$

On the other hand, if $x = 4 - 2\sqrt{2}$, then

$$y = \frac{8}{4 - 2\sqrt{2}} = \frac{8}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}} = 4 + 2\sqrt{2}$$

Thus, the two numbers are $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$.

56. Show that the parabola $y = x^2$ consists of all points P such that $d_1 = d_2$, where d_1 is the distance from P to $\left(0, \frac{1}{4}\right)$ and d_2 is the distance from P to the line $y = -\frac{1}{4}$ (Figure 15).

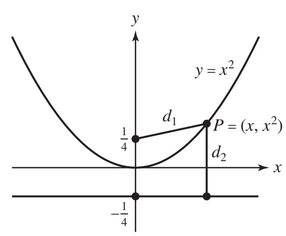


FIGURE 15

SOLUTION Let P be a point on the graph of the parabola $y = x^2$. Then P has coordinates (x, x^2) for some real number x. Now, $d_2 = x^2 + \frac{1}{4}$ and

$$d_1 = \sqrt{(x-0)^2 + \left(x^2 - \frac{1}{4}\right)^2} = \sqrt{x^2 + x^4 - \frac{1}{2}x^2 + \frac{1}{16}} = \sqrt{\left(x^2 + \frac{1}{4}\right)^2} = x^2 + \frac{1}{4} = d_2$$

Further Insights and Challenges

57. Show that if f and g are linear, then so is $f + g$. Is the same true of fg?

SOLUTION If $f(x) = mx + b$ and $g(x) = nx + d$, then

$$f(x) + g(x) = mx + b + nx + d = (m + n)x + (b + d)$$

which is linear. fg is not generally linear. Take, for example, $f(x) = g(x) = x$. Then $f(x)g(x) = x^2$.

58. Show that if f and g are linear functions such that $f(0) = g(0)$ and $f(1) = g(1)$, then $f = g$.

SOLUTION Suppose $f(x) = mx + b$ and $g(x) = nx + d$. Then $f(0) = b$ and $g(0) = d$, which implies $b = d$. Thus, $f(x) = mx + b$ and $g(x) = nx + b$. Now, $f(1) = m + b$ and $g(1) = n + b$ so $m + b = n + b$ and $m = n$. Thus, $f = g$.

59. Show that $\Delta y/\Delta x$ for the function $f(x) = x^2$ over the interval $[x_1, x_2]$ is not a constant, but depends on the interval. Determine the exact dependence of $\Delta y/\Delta x$ on x_1 and x_2.

SOLUTION For x^2, $\frac{\Delta y}{\Delta x} = \frac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1$.

60. Complete the square and use the result to derive the quadratic formula for the roots of $ax^2 + bx + c = 0$.

SOLUTION Consider the equation $ax^2 + bx + c = 0$. First, complete the square to obtain

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$$

Then

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{and} \qquad \left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a}$$

Dropping the absolute values yields

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

61. Let $a, c \neq 0$. Show that the roots of

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad cx^2 + bx + a = 0$$

are reciprocals of each other.

SOLUTION Let r_1 and r_2 be the roots of $ax^2 + bx + c$ and r_3 and r_4 be the roots of $cx^2 + bx + a$. Without loss of generality, let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{1}{r_1} = \frac{2a}{-b + \sqrt{b^2 - 4ac}} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$
$$= \frac{2a(-b - \sqrt{b^2 - 4ac})}{b^2 - b^2 + 4ac} = \frac{-b - \sqrt{b^2 - 4ac}}{2c} = r_4$$

Similarly, you can show $\frac{1}{r_2} = r_3$.

62. Show, by completing the square, that the parabola

$$y = ax^2 + bx + c$$

can be obtained from $y = ax^2$ by a vertical and horizontal translation.

SOLUTION

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Thus, the first parabola is just the second translated horizontally by $-\frac{b}{2a}$ and vertically by $\frac{4ac - b^2}{4a}$.

63. Prove **Viète's Formulas**: The quadratic polynomial with α and β as roots is $x^2 + bx + c$, where $b = -\alpha - \beta$ and $c = \alpha\beta$.

SOLUTION If a quadratic polynomial has roots α and β, then the polynomial is

$$(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = x^2 + (-\alpha - \beta)x + \alpha\beta$$

Thus, $b = -\alpha - \beta$ and $c = \alpha\beta$.

1.3 The Basic Classes of Functions

Preliminary Questions

1. Explain why both $f(x) = x^3 + 1$ and $g(x) = \frac{1}{x^3 + 1}$ are rational functions.

SOLUTION A rational function is a quotient of polynomial functions. Both $p(x) = 1$ and $q(x) = x^3 + 1$ are polynomial functions, and

$$f(x) = \frac{x^3 + 1}{1} = \frac{q(x)}{p(x)} \quad \text{while} \quad g(x) = \frac{p(x)}{q(x)}$$

Thus, both f and g are quotients of polynomial functions, so they are both rational functions.

2. Is $y = |x|$ a polynomial function? What about $y = |x^2 + 1|$?

SOLUTION $|x|$ is not a polynomial; however, because $x^2 + 1 > 0$ for all x, it follows that $|x^2 + 1| = x^2 + 1$, which is a polynomial.

3. What is unusual about the domain of the composite function $f \circ g$ for the functions $f(x) = x^{1/2}$ and $g(x) = -1 - |x|$?

SOLUTION Recall that $(f \circ g)(x) = f(g(x))$. Now, for any real number x, $g(x) = -1 - |x| \le -1 < 0$. Because we cannot take the square root of a negative number, it follows that $f(g(x))$ is not defined for any real number. In other words, the domain of $f(g(x))$ is the empty set.

4. Is $f(x) = (\frac{1}{2})^x$ increasing or decreasing?

SOLUTION The function $f(x) = (\frac{1}{2})^x$ is an exponential function with base $b = \frac{1}{2} < 1$. Therefore, f is a decreasing function.

5. Explain why both $f(x) = \frac{x}{1-x^4}$ and $g(x) = \frac{x}{\sqrt{1-x^4}}$ are algebraic functions.

SOLUTION An algebraic function is produced by taking sums, products, and quotients of roots of polynomial and rational functions. Both $p(x) = x$ and $q(x) = 1 - x^4$ are polynomial functions, and

$$f(x) = \frac{p(x)}{q(x)} \quad \text{and} \quad g(x) = \frac{p(x)}{\sqrt{q(x)}}$$

Thus, both f and g are quotients of roots of polynomial functions, so they are both algebraic functions. Note that f is also a rational function, whereas g is not.

6. We have $f(x) = (x + 1)^{1/2}$, $g(x) = x^{-2} + 1$, $h(x) = 2^x$, and $k(x) = x^2 + 1$. Identify which of the functions satisfies each of the following.

(a) Transcendental

(b) Polynomial

(c) Rational but not polynomial

(d) Algebraic but not rational

SOLUTION

(a) The function $h(x) = 2^x$ is a transcendental function.

(b) The function $k(x) = x^2 + 1$ is a polynomial function.

(c) The function $g(x) = x^{-2} + 1 = \frac{1}{x^2} + 1 = \frac{1+x^2}{x^2}$ is a rational function, but not a polynomial function.

(d) The function $f(x) = (x + 1)^{1/2}$ is an algebraic function, but not a rational function.

Exercises

In Exercises 1–12, determine the domain of the function.

1. $f(x) = x^{1/4}$

SOLUTION $x \ge 0$

2. $g(t) = t^{2/3}$

SOLUTION All reals

3. $f(x) = x^3 + 3x - 4$

SOLUTION All reals

4. $h(z) = z^3 + z^{-3}$

SOLUTION $z \ne 0$

5. $g(t) = \frac{1}{t+2}$

SOLUTION $t \ne -2$

6. $f(x) = \frac{1}{x^2+4}$

SOLUTION All reals

7. $G(u) = \frac{1}{u^2-4}$

SOLUTION $u \ne \pm 2$

8. $f(x) = \frac{\sqrt{x}}{x^2 - 9}$

SOLUTION $x \geq 0, x \neq 3$

9. $f(x) = x^{-4} + (x-1)^{-3}$

SOLUTION $x \neq 0, 1$

10. $F(s) = \sin\left(\frac{s}{s+1}\right)$

SOLUTION $s \neq -1$

11. $g(y) = 10^{\sqrt{y}+y^{-1}}$

SOLUTION $y > 0$

12. $f(x) = \frac{x + x^{-1}}{(x-3)(x+4)}$

SOLUTION $x \neq 0, 3, -4$

In Exercises 13–24, identify each of the following functions as polynomial, rational, algebraic, or transcendental.

13. $f(x) = 4x^3 + 9x^2 - 8$

SOLUTION Polynomial

14. $f(x) = x^{-4}$

SOLUTION Rational

15. $f(x) = \sqrt{x}$

SOLUTION Algebraic

16. $f(x) = \sqrt{1 - x^2}$

SOLUTION Algebraic

17. $f(x) = \frac{x^2}{x + \sin x}$

SOLUTION Transcendental

18. $f(x) = 2^x$

SOLUTION Transcendental

19. $f(x) = \frac{2x^3 + 3x}{9 - 7x^2}$

SOLUTION Rational

20. $f(x) = \frac{3x - 9x^{-1/2}}{9 - 7x^2}$

SOLUTION Algebraic

21. $f(x) = \sin(x^2)$

SOLUTION Transcendental

22. $f(x) = \frac{x}{\sqrt{x} + 1}$

SOLUTION Algebraic

23. $f(x) = x^2 + 3x^{-1}$

SOLUTION Rational

24. $f(x) = \sin(3^x)$

SOLUTION Transcendental

25. Is $f(x) = 2^{x^2}$ a transcendental function?

SOLUTION Yes

26. Show that $f(x) = x^2 + 3x^{-1}$ and $g(x) = 3x^3 - 9x + x^{-2}$ are rational functions—that is, quotients of polynomials.

SOLUTION $f(x) = x^2 + 3x^{-1} = x^2 + \dfrac{3}{x} = \dfrac{x^3+3}{x}$

$$g(x) = 3x^3 - 9x + x^{-2} = \frac{3x^5 - 9x^3 + 1}{x^2}$$

In Exercises 27–34, calculate the composite functions $f \circ g$ and $g \circ f$, and determine their domains.

27. $f(x) = \sqrt{x}, \quad g(x) = x + 1$

SOLUTION $f(g(x)) = \sqrt{x+1}$; D: $x \geq -1$, $\quad g(f(x)) = \sqrt{x} + 1$; D: $x \geq 0$

28. $f(x) = \frac{1}{x}, \quad g(x) = x^{-4}$

SOLUTION $f(g(x)) = x^4$; D: $x \neq 0$, $\quad g(f(x)) = x^4$: D: $x \neq 0$

29. $f(x) = 2^x, \quad g(x) = x^2$

SOLUTION $f(g(x)) = 2^{x^2}$; D: $\mathbf{R}$, $\quad g(f(x)) = (2^x)^2 = 2^{2x}$; D: $\mathbf{R}$

30. $f(x) = |x|, \quad g(\theta) = \sin\theta$

SOLUTION $f(g(\theta)) = |\sin\theta|$; D: $\mathbf{R}$, $\quad g(f(x)) = \sin|x|$; D: $\mathbf{R}$

31. $f(\theta) = \cos\theta, \quad g(x) = x^3 + x^2$

SOLUTION $f(g(x)) = \cos(x^3 + x^2)$; D: $\mathbf{R}$, $\quad g(f(\theta)) = \cos^3\theta + \cos^2\theta$; D: $\mathbf{R}$

32. $f(x) = \frac{1}{x^2+1}, \quad g(x) = x^{-2}$

SOLUTION $f(g(x)) = \frac{1}{(x^{-2})^2+1} = \frac{1}{x^{-4}+1}$; D: $x \neq 0$, $\quad g(f(x)) = \left(\frac{1}{x^2+1}\right)^{-2} = (x^2+1)^2$; D: $\mathbf{R}$

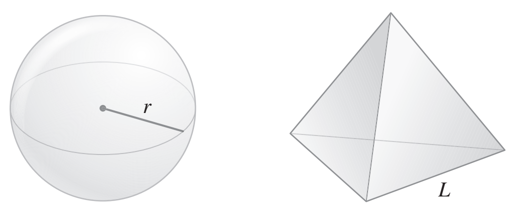
33. $f(t) = \frac{1}{\sqrt{t}}, \quad g(t) = -t^2$

SOLUTION $f(g(t)) = \frac{1}{\sqrt{-t^2}}$; D: not valid for any t, $\quad g(f(t)) = -\left(\frac{1}{\sqrt{t}}\right)^2 = -\frac{1}{t}$; D: $t > 0$

34. $f(t) = \sqrt{t}, \quad g(t) = 1 - t^3$

SOLUTION $f(g(t)) = \sqrt{1 - t^3}$; D: $t \leq 1$, $\quad g(f(t)) = 1 - t^{3/2}$; D: $t \geq 0$

35. The volume V and surface area of a sphere [Figure 6(A)] are expressed in terms of radius r by $V(r) = \frac{4}{3}\pi r^3$ and $S(r) = 4\pi r^2$, respectively. Determine $r(V)$, the radius as a function of volume. Then determine $S(V)$, the surface area as a function of volume, by computing the composite $S \circ r(V)$.



(A) (B)

FIGURE 6 A sphere (A) and tetrahedron (B).

SOLUTION Solving the equation $V = \frac{4}{3}\pi r^3$ for r yields

$$r(V) = \left(\frac{3V}{4\pi}\right)^{1/3}$$

Using this result, it follows that

$$S(V) = S \circ r(V) = 4\pi\left[\left(\frac{3V}{4\pi}\right)^{1/3}\right]^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3} = (36\pi V^2)^{1/3}$$

36. A tetrahedron is a polyhedron with four equilateral triangles as its faces [Figure 6(B)]. The volume V and surface area of a tetrahedron are expressed in terms of the side-length L of the triangles by $V(L) = \frac{\sqrt{2}L^3}{12}$ and $S(L) = \sqrt{3}L^2$, respectively. Determine $L(V)$, the side length as a function of volume. Then determine $S(V)$, the surface area as a function of volume, by computing the composite $S \circ L(V)$.

SOLUTION Solving the equation $V = \frac{\sqrt{2}L^3}{12}$ for L yields

$$L(V) = \left(\frac{12V}{\sqrt{2}}\right)^{1/3} = (6\sqrt{2}V)^{1/3} = \sqrt[6]{72}V^{1/3}$$

Using this result, it follows that

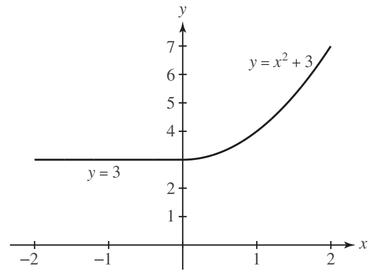
$$S(V) = S \circ L(V) = \sqrt{3}\sqrt[3]{72}V^{2/3} = 2\sqrt[6]{243}V^{2/3}$$

In Exercises 37–40, draw the graphs of each of the piecewise-defined functions.

37.

$$f(x) = \begin{cases} 3 & \text{when } x < 0 \\ x^2 + 3 & \text{when } x \geq 0 \end{cases}$$

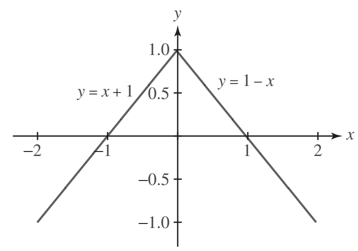
SOLUTION



38.

$$f(x) = \begin{cases} x + 1 & \text{when } x < 0 \\ 1 - x & \text{when } x \geq 0 \end{cases}$$

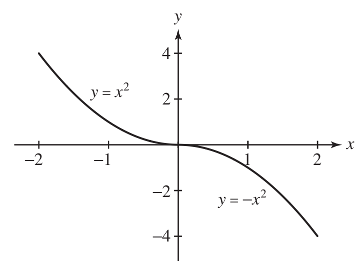
SOLUTION



39.

$$f(x) = \begin{cases} x^2 & \text{when } x < 0 \\ -x^2 & \text{when } x \geq 0 \end{cases}$$

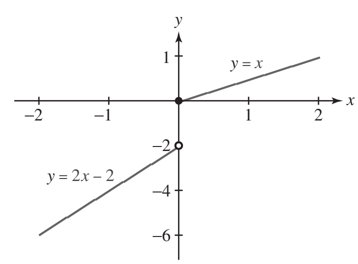
SOLUTION



40.

$$f(x) = \begin{cases} 2x - 2 & \text{when } x < 0 \\ x & \text{when } x \geq 0 \end{cases}$$

SOLUTION



41. Let $f(x) = \frac{x}{|x|}$.

(a) What are the domain and range of f?

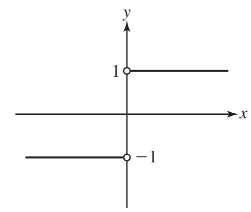
(b) Sketch the graph of f.

(c) Express f as a piecewise-defined function where each of the "pieces" is a constant.

SOLUTION

(a) The domain of f is $\{x : x \neq 0\}$, and the range of f is $\{-1, 1\}$.

(b) The graph of f is shown.



(c) For $x < 0$, $|x| = -x$ and

$$f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1$$

for $x > 0$, $|x| = x$ and

$$f(x) = \frac{x}{|x|} = \frac{x}{x} = 1$$

Thus,

$$f(x) = \begin{cases} -1 & \text{when } x < 0 \\ 1 & \text{when } x > 0 \end{cases}$$

42. The Heaviside function (named after Oliver Heaviside, 1850–1925) is defined by:

$$H(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}$$

The Heaviside function can be used to "turn on" another function at a specific value in the domain, as seen in the four examples here. For each of the following, sketch the graph of f.

(a) $f(x) = H(x)x^2$

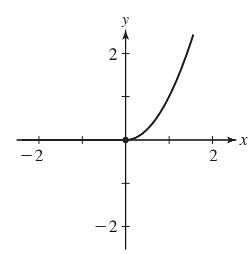
(b) $f(x) = H(x)(1 - x^2)$

(c) $f(x) = H(x - 1)x$

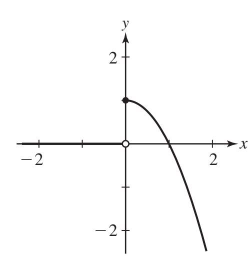
(d) $f(x) = H(x + 2)x^2$

SOLUTION

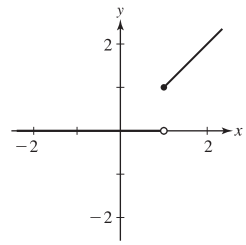
(a) Let $f(x) = H(x)x^2$. Here, the Heaviside function "turns on" the function x^2 at $x = 0$. The graph is shown.



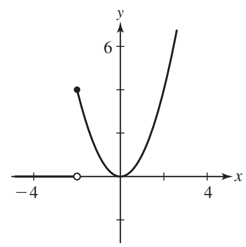
(b) Let $f(x) = H(x)(1 - x^2)$. Here, the Heaviside function "turns on" the function $1 - x^2$ at $x = 0$. The graph is shown.



(c) Let $f(x) = H(x - 1)x$. Here, the Heaviside function "turns on" the function x at $x = 1$. The graph is shown.



(d) Let $f(x) = H(x+2)x^2$. Here, the Heaviside function "turns on" the function x^2 at $x = -2$. The graph is shown.



43. The population (in millions) of Calcedonia as a function of time t (years) is $P(t) = 30 \cdot 2^{0.1t}$. Show that the population doubles every 10 years. Show more generally that for any positive constants a and k, the function $g(t) = a2^{kt}$ doubles after $1/k$ years.

SOLUTION Let $P(t) = 30 \cdot 2^{0.1t}$. Then

$$P(t+10) = 30 \cdot 2^{0.1(t+10)} = 30 \cdot 2^{0.1t+1} = 2(30 \cdot 2^{0.1t}) = 2P(t)$$

Hence, the population doubles in size every 10 years. In the more general case, let $g(t) = a2^{kt}$. Then

$$g\left(t + \frac{1}{k}\right) = a2^{k(t+1/k)} = a2^{kt+1} = 2a2^{kt} = 2g(t)$$

Hence, the function g doubles after $1/k$ years.

44. Find all values of c such that $f(x) = \frac{x+1}{x^2+2cx+4}$ has domain **R**.

SOLUTION The domain of f will consist of all real numbers provided the denominator has no real roots. The roots of $x^2 + 2cx + 4 = 0$ are

$$x = \frac{-2c \pm \sqrt{4c^2 - 16}}{2} = -c \pm \sqrt{c^2 - 4}$$

There will be no real roots when $c^2 < 4$ or when $-2 < c < 2$.

Further Insights and Challenges

In Exercises 45–51, we define the first difference δf *of a function* f *by* $\delta f(x) = f(x+1) - f(x)$.

45. Show that if $f(x) = x^2$, then $\delta f(x) = 2x + 1$. Calculate δf for $f(x) = x$ and $f(x) = x^3$.

SOLUTION $f(x) = x^2$: $\delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$
$f(x) = x$: $\delta f(x) = x + 1 - x = 1$
$f(x) = x^3$: $\delta f(x) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$

46. Show that $\delta(10^x) = 9 \cdot 10^x$ and, more generally, that $\delta(b^x) = (b-1)b^x$.

SOLUTION $\delta(10^x) = 10^{x+1} - 10^x = 10 \cdot 10^x - 10^x = 10^x(10-1) = 9 \cdot 10^x$
$\delta(b^x) = b^{x+1} - b^x = b^x(b-1)$

47. Show that for any two functions f and g, $\delta(f+g) = \delta f + \delta g$ and $\delta(cf) = c\delta(f)$, where c is any constant.

SOLUTION $\delta(f+g) = (f(x+1) + g(x+1)) - (f(x) - g(x))$

$$= (f(x+1) - f(x)) + (g(x+1) - g(x)) = \delta f(x) + \delta g(x)$$

$$\delta(cf) = cf(x+1) - cf(x) = c(f(x+1) - f(x)) = c\delta f(x)$$

48. Suppose we can find a function P such that $\delta P(x) = (x+1)^k$ and $P(0) = 0$. Prove that $P(1) = 1^k$, $P(2) = 1^k + 2^k$, and, more generally, for every whole number n,

$$P(n) = 1^k + 2^k + \cdots + n^k$$

SOLUTION Suppose we have found a function $P(x)$ such that $\delta P(x) = (x+1)^k$ and $P(0) = 0$. Taking $x = 0$, we have $\delta P(0) = P(1) - P(0) = (0+1)^k = 1^k$. Therefore, $P(1) = P(0) + 1^k = 1^k$. Next, take $x = 1$. Then $\delta P(1) = P(2) - P(1) = (1+1)^k = 2^k$, and $P(2) = P(1) + 2^k = 1^k + 2^k$.

To prove the general result, we will proceed by induction. The basis step, proving that $P(1) = 1^k$, is given in the problem, so we move on to the induction step. Assume that, for some integer j, $P(j) = 1^k + 2^k + \cdots + j^k$. Then $\delta P(j) = P(j+1) - P(j) = (j+1)^k$ and

$$P(j+1) = P(j) + (j+1)^k = 1^k + 2^k + \cdots + j^k + (j+1)^k$$

Therefore, by mathematical induction, for every whole number n, $P(n) = 1^k + 2^k + \cdots + n^k$.

49. Show that if

$$P(x) = \frac{x(x+1)}{2}$$

then $\delta P = (x+1)$. Then apply Exercise 48 to conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION Let $P(x) = x(x+1)/2$. Then

$$\delta P(x) = P(x+1) - P(x) = \frac{(x+1)(x+2)}{2} - \frac{x(x+1)}{2} = \frac{(x+1)(x+2-x)}{2} = x+1$$

Also, note that $P(0) = 0$. Thus, by Exercise 48, with $k = 1$, it follows that

$$P(n) = \frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n$$

50. Calculate $\delta(x^3)$, $\delta(x^2)$, and $\delta(x)$. Then find a polynomial P of degree 3 such that $\delta P = (x+1)^2$ and $P(0) = 0$. Conclude that $P(n) = 1^2 + 2^2 + \cdots + n^2$.

SOLUTION From Exercise 45, we know

$$\delta x = 1, \qquad \delta x^2 = 2x + 1, \qquad \text{and} \qquad \delta x^3 = 3x^2 + 3x + 1$$

Therefore,

$$\frac{1}{3}\delta x^3 + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x = x^2 + 2x + 1 = (x+1)^2$$

Now, using the properties of the first difference from Exercise 47, it follows that

$$\frac{1}{3}\delta x^3 + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x = \delta\left(\frac{1}{3}x^3\right) + \delta\left(\frac{1}{2}x^2\right) + \delta\left(\frac{1}{6}x\right) = \delta\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right) = \delta\left(\frac{2x^3 + 3x^2 + x}{6}\right)$$

Finally, let

$$P(x) = \frac{2x^3 + 3x^2 + x}{6}$$

Then $\delta P(x) = (x+1)^2$ and $P(0) = 0$, so by Exercise 48, with $k = 2$, it follows that

$$P(n) = \frac{2n^3 + 3n^2 + n}{6} = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

51. This exercise combined with Exercise 48 shows that for all whole numbers k, there exists a polynomial P satisfying Eq. (1). The solution requires the Binomial Theorem and proof by induction (see Appendix C).

(a) Show that $\delta(x^{k+1}) = (k+1)\,x^k + \cdots$, where the dots indicate terms involving smaller powers of x.

(b) Show by induction that there exists a polynomial of degree $k+1$ with leading coefficient $1/(k+1)$:

$$P(x) = \frac{1}{k+1}x^{k+1} + \cdots$$

such that $\delta P = (x+1)^k$ and $P(0) = 0$.

SOLUTION

(a) By the Binomial Theorem:

$$\delta(x^{n+1}) = (x+1)^{n+1} - x^{n+1} = \left(x^{n+1} + \binom{n+1}{1}x^n + \binom{n+1}{2}x^{n-1} + \cdots + 1\right) - x^{n+1}$$
$$= \binom{n+1}{1}x^n + \binom{n+1}{2}x^{n-1} + \cdots + 1$$

Thus,

$$\delta(x^{n+1}) = (n+1)\,x^n + \cdots$$

where the dots indicate terms involving smaller powers of x.

(b) For $k = 0$, note that $P(x) = x$ satisfies $\delta P = (x+1)^0 = 1$ and $P(0) = 0$.

Now suppose the polynomial

$$P(x) = \frac{1}{k}x^k + p_{k-1}x^{k-1} + \cdots + p_1 x$$

which clearly satisfies $P(0) = 0$, also satisfies $\delta P = (x+1)^{k-1}$. We try to prove the existence of

$$Q(x) = \frac{1}{k+1}x^{k+1} + q_k x^k + \cdots + q_1 x$$

such that $\delta Q = (x+1)^k$. Observe that $Q(0) = 0$.

If $\delta Q = (x+1)^k$ and $\delta P = (x+1)^{k-1}$, then

$$\delta Q = (x+1)^k = (x+1)\delta P = x\delta P(x) + \delta P$$

By the linearity of δ (Exercise 47), we find $\delta Q - \delta P = x\delta P$ or $\delta(Q - P) = x\delta P$. By definition,

$$Q - P = \frac{1}{k+1}x^{k+1} + \left(q_k - \frac{1}{k}\right)x^k + \cdots + (q_1 - p_1)x$$

so, by the linearity of δ,

$$\delta(Q - P) = \frac{1}{k+1}\delta(x^{k+1}) + \left(q_k - \frac{1}{k}\right)\delta(x^k) + \cdots + (q_1 - p_1) = x(x+1)^{k-1} \qquad (1)$$

By part (a),

$$\delta(x^{k+1}) = (k+1)x^k + L_{k-1,k-1}x^{k-1} + \ldots + L_{k-1,1}x + 1$$
$$\delta(x^k) = kx^{k-1} + L_{k-2,k-2}x^{k-2} + \ldots + L_{k-2,1}x + 1$$
$$\vdots$$
$$\delta(x^2) = 2x + 1$$

where the $L_{i,j}$ are real numbers for each i, j.

To construct Q, we have to group like powers of x on both sides of (1). This yields the system of equations

$$\frac{1}{k+1}\left((k+1)x^k\right) = x^k$$
$$\frac{1}{k+1}L_{k-1,k-1}x^{k-1} + \left(q_k - \frac{1}{k}\right)kx^{k-1} = (k-1)x^{k-1}$$
$$\vdots$$
$$\frac{1}{k+1} + \left(q_k - \frac{1}{k}\right) + (q_{k-1} - p_{k-1}) + \cdots + (q_1 - p_1) = 0$$

The first equation is identically true, and the second equation can be solved immediately for q_k. Substituting the value of q_k into the third equation of the system, we can then solve for q_{k-1}. We continue this process until we substitute the values of $q_k, q_{k-1}, \ldots q_2$ into the last equation, and then solve for q_1.

1.4 Trigonometric Functions

Preliminary Questions

1. How is it possible for two different rotations to define the same angle?

SOLUTION Working from the same initial radius, two rotations that differ by a whole number of full revolutions will have the same ending radius; consequently, the two rotations will define the same angle even though the measures of the rotations will be different.

2. Give two different positive rotations that define the angle $\pi/4$.

SOLUTION The angle $\pi/4$ is defined by any rotation of the form $\frac{\pi}{4} + 2\pi k$, where k is an integer. Thus, two different positive rotations that define the angle $\pi/4$ are

$$\frac{\pi}{4} + 2\pi(1) = \frac{9\pi}{4} \qquad \text{and} \qquad \frac{\pi}{4} + 2\pi(5) = \frac{41\pi}{4}$$

3. Give a negative rotation that defines the angle $\pi/3$.

SOLUTION The angle $\pi/3$ is defined by any rotation of the form $\frac{\pi}{3} + 2\pi k$, where k is an integer. Thus, a negative rotation that defines the angle $\pi/3$ is

$$\frac{\pi}{3} + 2\pi(-1) = -\frac{5\pi}{3}$$

4. The definition of $\cos\theta$ using right triangles applies when (choose the correct answer):

(a) $0 < \theta < \frac{\pi}{2}$ **(b)** $0 < \theta < \pi$ **(c)** $0 < \theta < 2\pi$

SOLUTION The correct response is **(a)**: $0 < \theta < \frac{\pi}{2}$.

5. What is the unit circle definition of $\sin\theta$?

SOLUTION Let O denote the center of the unit circle, and let P be a point on the unit circle such that the radius $\overline{OP}$ makes an angle θ with the positive x-axis. Then, $\sin\theta$ is the y-coordinate of the point P.

6. How does the periodicity of $f(\theta) = \sin\theta$ and $f(\theta) = \cos\theta$ follow from the unit circle definition?

SOLUTION Let O denote the center of the unit circle, and let P be a point on the unit circle such that the radius $\overline{OP}$ makes an angle θ with the positive x-axis. Then, $\cos\theta$ and $\sin\theta$ are the x- and y-coordinates, respectively, of the point P. The angle $\theta + 2\pi$ is obtained from the angle θ by making one full revolution around the circle. The angle $\theta + 2\pi$ will therefore have the radius $\overline{OP}$ as its terminal side. Thus,

$$\cos(\theta + 2\pi) = \cos\theta \qquad \text{and} \qquad \sin(\theta + 2\pi) = \sin\theta$$

In other words, $\sin\theta$ and $\cos\theta$ are periodic functions.

Exercises

1. Find the angle between 0 and 2π equivalent to $13\pi/4$.

SOLUTION Because $13\pi/4 > 2\pi$, we repeatedly subtract 2π until we arrive at a radian measure that is between 0 and 2π. After one subtraction, we have $13\pi/4 - 2\pi = 5\pi/4$. Because $0 < 5\pi/4 < 2\pi$, $5\pi/4$ is the angle measure between 0 and 2π that is equivalent to $13\pi/4$.

2. Describe $\theta = \pi/6$ by an angle of negative radian measure.

SOLUTION If we subtract 2π from $\pi/6$, we obtain $\theta = -11\pi/6$. Thus, the angle $\theta = \pi/6$ is equivalent to the angle $\theta = -11\pi/6$.

3. Convert from radians to degrees:

(a) 1 **(b)** $\frac{\pi}{3}$ **(c)** $\frac{5}{12}$ **(d)** $-\frac{3\pi}{4}$

SOLUTION

(a) $1\left(\frac{180^\circ}{\pi}\right) = \frac{180^\circ}{\pi} \approx 57.1^\circ$ **(b)** $\frac{\pi}{3}\left(\frac{180^\circ}{\pi}\right) = 60^\circ$ **(c)** $\frac{5}{12}\left(\frac{180^\circ}{\pi}\right) = \frac{75^\circ}{\pi} \approx 23.87^\circ$

(d) $-\frac{3\pi}{4}\left(\frac{180^\circ}{\pi}\right) = -135^\circ$

4. Convert from degrees to radians:

(a) 1° **(b)** 30° **(c)** 25° **(d)** 120°

SOLUTION

(a) $1^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{180}$ **(b)** $30^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{6}$

(c) $25^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{5\pi}{36}$ **(d)** $120^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{2\pi}{3}$

5. Find the lengths of the arcs subtended by the angles θ and ϕ radians in Figure 19.

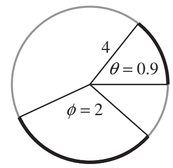


FIGURE 19 Circle of radius 4.

SOLUTION $s = r\theta = 4(0.9) = 3.6$; $s = r\phi = 4(2) = 8$

6. Calculate the values of the six standard trigonometric functions for the angle θ in Figure 20.

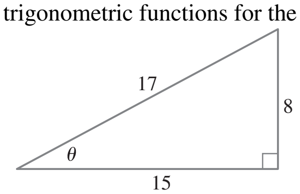


FIGURE 20

SOLUTION Using the definition of the six trigonometric functions in terms of the ratio of sides of a right triangle, we find $\sin\theta = 8/17$; $\cos\theta = 15/17$; $\tan\theta = 8/15$; $\csc\theta = 17/8$; $\sec\theta = 17/15$; $\cot\theta = 15/8$.

7. Fill in the remaining values of $(\cos\theta, \sin\theta)$ for the points in Figure 21.

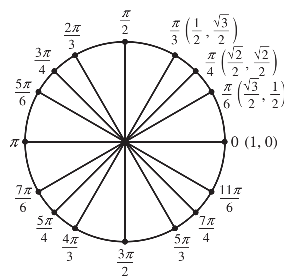


FIGURE 21

SOLUTION

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$
$(\cos\theta, \sin\theta)$	$(0, 1)$	$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$	$(-1, 0)$	$\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$

θ	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$(\cos\theta, \sin\theta)$	$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$	$\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$	$(0, -1)$	$\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

8. Find the values of the six standard trigonometric functions at $\theta = 11\pi/6$.

SOLUTION From Figure 21, we see that

$$\sin\frac{11\pi}{6} = -\frac{1}{2} \quad \text{and} \quad \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

Then

$$\tan\frac{11\pi}{6} = \frac{\sin\frac{11\pi}{6}}{\cos\frac{11\pi}{6}} = -\frac{\sqrt{3}}{3}$$

$$\cot\frac{11\pi}{6} = \frac{\cos\frac{11\pi}{6}}{\sin\frac{11\pi}{6}} = -\sqrt{3}$$

$$\csc\frac{11\pi}{6} = \frac{1}{\sin\frac{11\pi}{6}} = -2$$

$$\sec\frac{11\pi}{6} = \frac{1}{\cos\frac{11\pi}{6}} = \frac{2\sqrt{3}}{3}$$

In Exercises 9–14, use Figure 21 to find all angles between 0 *and* 2π *satisfying the given condition.*

9. $\cos\theta = \frac{1}{2}$

SOLUTION $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

10. $\tan\theta = 1$

SOLUTION $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

11. $\tan\theta = -1$

SOLUTION $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

12. $\csc\theta = 2$

SOLUTION $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

13. $\sin x = \frac{\sqrt{3}}{2}$

SOLUTION $x = \frac{\pi}{3}, \frac{2\pi}{3}$

14. $\sec t = 2$

SOLUTION $t = \frac{\pi}{3}, \frac{5\pi}{3}$

15. Fill in the following table of values:

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$							
$\sec\theta$							

SOLUTION

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$

16. Complete the following table of signs:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$0 < \theta < \frac{\pi}{2}$	+	+				
$\frac{\pi}{2} < \theta < \pi$						
$\pi < \theta < \frac{3\pi}{2}$						
$\frac{3\pi}{2} < \theta < 2\pi$						

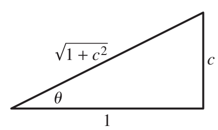
SOLUTION

θ	sin	cos	tan	cot	sec	csc
$0 < \theta < \frac{\pi}{2}$	+	+	+	+	+	+
$\frac{\pi}{2} < \theta < \pi$	+	−	−	−	−	+
$\pi < \theta < \frac{3\pi}{2}$	−	−	+	+	−	−
$\frac{3\pi}{2} < \theta < 2\pi$	−	+	−	−	+	−

17. Show that if $\tan\theta = c$ and $0 \le \theta < \pi/2$, then $\cos\theta = 1/\sqrt{1+c^2}$. *Hint:* Draw a right triangle whose opposite and adjacent sides have lengths c and 1.

SOLUTION Because $0 \le \theta < \pi/2$, we can use the definition of the trigonometric functions in terms of right triangles. $\tan\theta$ is the ratio of the length of the side opposite the angle θ to the length of the adjacent side. With $c = \frac{c}{1}$, we label the length of the opposite side as c and the length of the adjacent side as 1 (see the diagram). By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1+c^2}$. Finally, we use the fact that $\cos\theta$ is the ratio of the length of the adjacent side to the length of the hypotenuse to obtain

$$\cos\theta = \frac{1}{\sqrt{1+c^2}}$$



18. Suppose that $\cos\theta = \frac{1}{3}$.

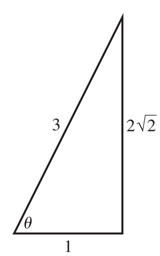
(a) Show that if $0 \le \theta < \pi/2$, then $\sin\theta = 2\sqrt{2}/3$ and $\tan\theta = 2\sqrt{2}$.

(b) Find $\sin\theta$ and $\tan\theta$ if $3\pi/2 \le \theta < 2\pi$.

SOLUTION

(a) Because $0 \le \theta < \pi/2$, we can use the definition of the trigonometric functions in terms of right triangles. $\cos\theta$ is the ratio of the length of the side adjacent to the angle θ to the length of the hypotenuse, so we label the length of the adjacent side as 1 and the length of the hypotenuse as 3 (see the diagram). By the Pythagorean Theorem, the length of the side opposite the angle θ is $\sqrt{3^2 - 1^2} = 2\sqrt{2}$. Finally, we use the definitions of $\sin\theta$ as the ratio of the length of the opposite side to the length of the hypotenuse and of $\tan\theta$ as the ratio of the length of the opposite side to the length of the adjacent side to obtain

$$\sin\theta = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$



(b) If $3\pi/2 \le \theta < 2\pi$, then θ is in the fourth quadrant and $\sin\theta$ and $\tan\theta$ are negative but have the same magnitude as found in part (a). Thus,

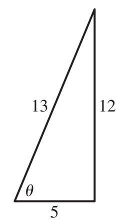
$$\sin\theta = -\frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = -2\sqrt{2}$$

In Exercises 19–24, assume that $0 \le \theta < \pi/2$.

19. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{5}{13}$.

SOLUTION Consider the triangle below. The lengths of the side adjacent to the angle θ and the hypotenuse have been labeled so that $\cos\theta = \frac{5}{13}$. The length of the side opposite the angle θ has been calculated using the Pythagorean Theorem: $\sqrt{13^2 - 5^2} = 12$. From the triangle, we see that

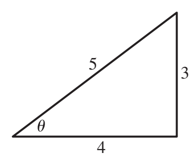
$$\sin\theta = \frac{12}{13} \qquad \text{and} \qquad \tan\theta = \frac{12}{5}$$



20. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = \frac{3}{5}$.

SOLUTION Consider the triangle shown. The lengths of the side opposite the angle θ and the hypotenuse have been labeled so that $\sin\theta = \frac{3}{5}$. The length of the side adjacent to the angle θ has been calculated using the Pythagorean Theorem: $\sqrt{5^2 - 3^2} = 4$. From the triangle, we see that

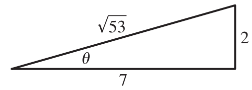
$$\cos\theta = \frac{4}{5} \qquad \text{and} \qquad \tan\theta = \frac{3}{4}$$



21. Find $\sin\theta$, $\sec\theta$, and $\cot\theta$ if $\tan\theta = \frac{2}{7}$.

SOLUTION If $\tan\theta = \frac{2}{7}$, then $\cot\theta = \frac{7}{2}$. For the remaining trigonometric functions, consider the triangle shown. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = \frac{2}{7}$. The length of the hypotenuse has been calculated using the Pythagorean Theorem: $\sqrt{2^2 + 7^2} = \sqrt{53}$. From the triangle, we see that

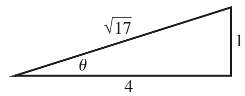
$$\sin\theta = \frac{2}{\sqrt{53}} = \frac{2\sqrt{53}}{53} \qquad \text{and} \qquad \sec\theta = \frac{\sqrt{53}}{7}$$



22. Find $\sin\theta$, $\cos\theta$, and $\sec\theta$ if $\cot\theta = 4$.

SOLUTION Consider the triangle shown. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\cot\theta = 4 = \frac{4}{1}$. The length of the hypotenuse has been calculated using the Pythagorean Theorem: $\sqrt{4^2 + 1^2} = \sqrt{17}$. From the triangle, we see that

$$\sin\theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}, \quad \cos\theta = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}, \quad \text{and} \quad \sec\theta = \frac{\sqrt{17}}{4}$$



23. Find $\cos 2\theta$ if $\sin\theta = \frac{1}{5}$.

SOLUTION Using the double-angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, we find that $\cos 2\theta = 1 - 2\sin^2\theta$. Thus, $\cos 2\theta = 1 - 2(1/25) = 23/25$.

24. Find $\sin 2\theta$ and $\cos 2\theta$ if $\tan\theta = \sqrt{2}$.

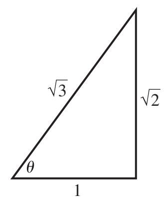
SOLUTION By the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$. We can determine $\sin\theta$ and $\cos\theta$ using the triangle shown. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = \sqrt{2}$. The hypotenuse was calculated using the Pythagorean Theorem: $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$. Thus,

$$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \qquad \text{and} \qquad \cos\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Finally,

$$\sin 2\theta = 2\frac{\sqrt{6}}{3}\cdot\frac{\sqrt{3}}{3} = \frac{2\sqrt{2}}{3}$$

$$\cos 2\theta = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$



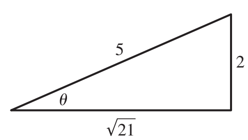
25. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = 0.4$ and $\pi/2 \le \theta < \pi$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ and $\tan\theta$ using the triangle shown. The lengths of the side opposite the angle θ and the hypotenuse have been labeled so that $\sin\theta = 0.4 = \frac{2}{5}$. The length of the side adjacent to the angle θ was calculated using the Pythagorean Theorem: $\sqrt{5^2 - 2^2} = \sqrt{21}$. From the triangle, we see that

$$|\cos\theta| = \frac{\sqrt{21}}{5} \qquad \text{and} \qquad |\tan\theta| = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

Because $\pi/2 \le \theta < \pi$, both $\cos\theta$ and $\tan\theta$ are negative; consequently,

$$\cos\theta = -\frac{\sqrt{21}}{5} \qquad \text{and} \qquad \tan\theta = -\frac{2\sqrt{21}}{21}$$



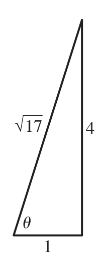
26. Find $\cos\theta$ and $\sin\theta$ if $\tan\theta = 4$ and $\pi \le \theta < 3\pi/2$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ and $\sin\theta$ using the triangle shown. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = 4 = \frac{4}{1}$. The length of the hypotenuse was calculated using the Pythagorean Theorem: $\sqrt{1^2 + 4^2} = \sqrt{17}$. From the triangle, we see that

$$|\cos\theta| = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17} \qquad \text{and} \qquad |\sin\theta| = \frac{4}{\sqrt{14}} = \frac{4\sqrt{17}}{17}$$

Because $\pi \le \theta < 3\pi/2$, both $\cos\theta$ and $\sin\theta$ are negative; consequently,

$$\cos\theta = -\frac{\sqrt{17}}{17} \qquad \text{and} \qquad \sin\theta = -\frac{4\sqrt{17}}{17}$$



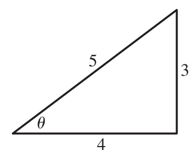
27. Find $\cos\theta$ if $\cot\theta = \frac{4}{3}$ and $\sin\theta < 0$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ using the triangle shown. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\cot\theta = \frac{4}{3}$. The length of the hypotenuse was calculated using the Pythagorean Theorem: $\sqrt{3^2+4^2} = 5$. From the triangle, we see that

$$|\cos\theta| = \frac{4}{5}$$

Because $\cot\theta > 0$ and $\sin\theta < 0$, the angle θ must be in the third quadrant so that $\cos\theta$ must be negative; consequently,

$$\cos\theta = -\frac{4}{5}$$



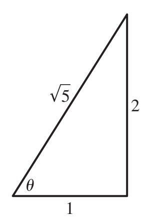
28. Find $\tan\theta$ if $\sec\theta = \sqrt{5}$ and $\sin\theta < 0$.

SOLUTION We can determine the "magnitude" of $\tan\theta$ using the triangle shown. The lengths of the side adjacent to the angle θ and the hypotenuse have been labeled so that $\sec\theta = \sqrt{5} = \frac{\sqrt{5}}{1}$. The length of the side opposite the angle θ was calculated using the Pythagorean Theorem: $\sqrt{(\sqrt{5})^2 - 1^2} = 2$. From the triangle, we see that

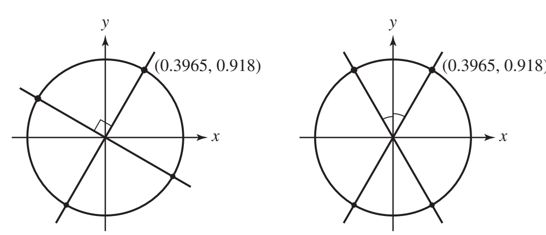
$$|\tan\theta| = \frac{2}{1} = 2$$

Because $\sec\theta > 0$ and $\sin\theta < 0$, the angle θ must be in the fourth quadrant so that $\tan\theta$ must be negative; consequently,

$$\tan\theta = -2$$



29. Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the angles corresponding to the eight points on the unit circles in Figures 22(A) and (B).



(A) (B)

FIGURE 22

SOLUTION Let's start with the four points in Figure 22(A).

- The point in the first quadrant has coordinates $(0.3965, 0.918)$. Therefore,

$$\sin\theta = 0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = 2.3153$$

- The coordinates of the point in the second quadrant are $(-0.918, 0.3965)$. Therefore,

$$\sin\theta = 0.3965, \quad \cos\theta = -0.918, \quad \text{and} \quad \tan\theta = \frac{0.3965}{-0.918} = -0.4319$$

- Because the point in the third quadrant is symmetric to the point in the first quadrant with respect to the origin, its coordinates are $(-0.3965, -0.918)$. Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{-0.3965} = 2.3153$$

- Because the point in the fourth quadrant is symmetric to the point in the second quadrant with respect to the origin, its coordinates are $(0.918, -0.3965)$. Therefore,

$$\sin\theta = -0.3965, \quad \cos\theta = 0.918, \quad \text{and} \quad \tan\theta = \frac{-0.3965}{0.918} = -0.4319$$

Now consider the four points in Figure 22(B).

- The point in the first quadrant has coordinates $(0.3965, 0.918)$. Therefore,

$$\sin\theta = 0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = 2.3153$$

- The point in the second quadrant is a reflection through the y-axis of the point in the first quadrant. Its coordinates are therefore $(-0.3965, 0.918)$ and

$$\sin\theta = 0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = -2.3153$$

- Because the point in the third quadrant is symmetric to the point in the first quadrant with respect to the origin, its coordinates are $(-0.3965, -0.918)$. Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{-0.3965} = 2.3153$$

- Because the point in the fourth quadrant is symmetric to the point in the second quadrant with respect to the origin, its coordinates are $(0.3965, -0.918)$. Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{0.3965} = -2.3153$$

30. Refer to Figure 23(A). Express the functions $\sin\theta$, $\tan\theta$, and $\csc\theta$ in terms of c.

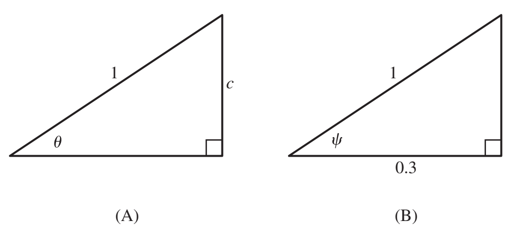


FIGURE 23

SOLUTION By the Pythagorean Theorem, the length of the side adjacent to the angle θ in Figure 23(A) is $\sqrt{1-c^2}$. Consequently,

$$\sin\theta = \frac{c}{1} = c, \quad \cos\theta = \frac{\sqrt{1-c^2}}{1} = \sqrt{1-c^2}, \quad \text{and} \quad \tan\theta = \frac{c}{\sqrt{1-c^2}}$$

31. Refer to Figure 23(B). Compute $\cos\psi$, $\sin\psi$, $\cot\psi$, and $\csc\psi$.

SOLUTION By the Pythagorean Theorem, the length of the side opposite the angle ψ in Figure 23(B) is $\sqrt{1-0.3^2} = \sqrt{0.91}$. Consequently,

$$\cos\psi = \frac{0.3}{1} = 0.3, \quad \sin\psi = \frac{\sqrt{0.91}}{1} = \sqrt{0.91}, \quad \cot\psi = \frac{0.3}{\sqrt{0.91}}, \qquad \text{and} \qquad \csc\psi = \frac{1}{\sqrt{0.91}}$$

32. Express $\cos\left(\theta + \frac{\pi}{2}\right)$ and $\sin\left(\theta + \frac{\pi}{2}\right)$ in terms of $\cos\theta$ and $\sin\theta$. *Hint:* Find the relation between the coordinates (a, b) and (c, d) in Figure 24.

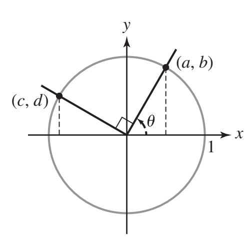


FIGURE 24

SOLUTION Note the triangle in the second quadrant in Figure 24 is congruent to the triangle in the first quadrant rotated 90° clockwise. Thus, $c = -b$ and $d = a$. But $a = \cos\theta$, $b = \sin\theta$, $c = \cos\left(\theta + \frac{\pi}{2}\right)$, and $d = \sin\left(\theta + \frac{\pi}{2}\right)$; therefore,

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \qquad \text{and} \qquad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

33. Use addition formulas and the values of $\sin\theta$ and $\cos\theta$ for $\theta = \frac{\pi}{3}, \frac{\pi}{4}$ to compute $\sin\frac{7\pi}{12}$ and $\cos\frac{7\pi}{12}$ exactly.

SOLUTION First, note that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$. Then, by the addition formula for the sine function,

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

by the addition formula for the cosine function,

$$\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

34. Use addition formulas and the values of $\sin\theta$ and $\cos\theta$ for $\theta = \frac{\pi}{3}, \frac{\pi}{4}$ to compute $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$ exactly.

SOLUTION First, note that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Then, by the addition formula for the sine function,

$$\begin{aligned}
\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left[\frac{\pi}{3} + \left(-\frac{\pi}{4}\right)\right] \\
&= \sin\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) + \cos\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

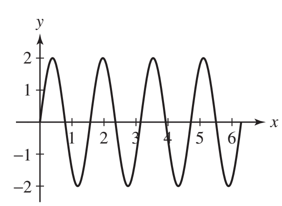
by the addition formula for the cosine function,

$$\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left[\frac{\pi}{3} + \left(-\frac{\pi}{4}\right)\right] \\
&= \cos\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) - \sin\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

In Exercises 35–38, sketch the graph over $[0, 2\pi]$.

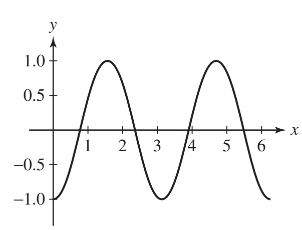
35. $f(\theta) = 2\sin 4\theta$

SOLUTION



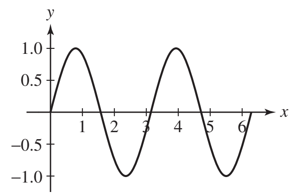
36. $f(\theta) = \cos\left(2\left(\theta - \frac{\pi}{2}\right)\right)$

SOLUTION



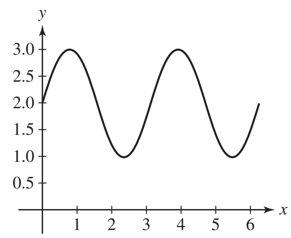
37. $f(\theta) = \cos\left(2\theta - \frac{\pi}{2}\right)$

SOLUTION



38. $f(\theta) = \sin\left(2\left(\theta - \frac{\pi}{2}\right) + \pi\right) + 2$

SOLUTION



39. Determine a function that would have a graph as in Figure 25(A), stating the period and amplitude.

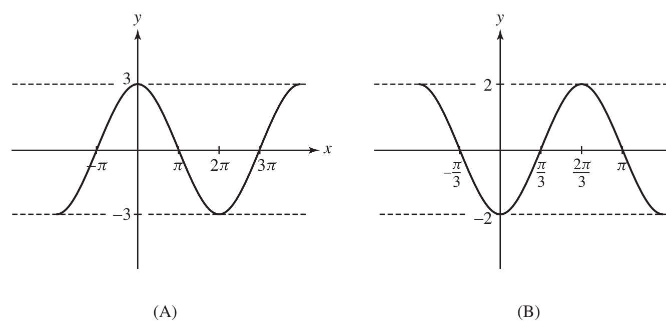


FIGURE 25

SOLUTION The graph in Figure 25(A) appears to be a cosine function with an amplitude of 3 and a period of 4π. Thus, one possible function is $y = 3\cos\frac{x}{2}$.

40. Determine a function that would have a graph as in Figure 25(B), stating the period and amplitude.

SOLUTION The graph in Figure 25(B) appears to be a cosine function with an amplitude of 2, with a period of $\frac{4\pi}{3}$, and that has been reflected across the x-axis. Thus, one possible function is $y = -2\cos\frac{3x}{2}$.

41. During a year, the length of a day, from sunrise to sunset, in Wolf Point, Montana, varies from a shortest day of approximately 8.1 hours to a longest day of approximately 15.9 hours, while in Mexico City, the day lengths vary from 10.7 hours to 13.3 hours. For each location, determine a function $L(t) = 12 + A\sin(\frac{2\pi}{365}t)$ that approximates the length of a day, in hours, where t represents the day in the year assuming $t = 0$ is the spring equinox on March 21. Compare the day lengths in each location on April 1, July 15, and November 1.

SOLUTION The length of the day in Wolf Point varies from 8.1 hours to 15.9 hours, so the amplitude of the oscillation is $\frac{15.9-8.1}{2} = 3.9$ hours. The length of the day in Wolf Point can therefore be modeled by the function

$$L_{WP}(t) = 12 + 3.9\sin\left(\frac{2\pi}{365}t\right)$$

On the other hand, the length of the day in Mexico City varies from 10.7 hours to 13.3 hours, so the amplitude of the oscillation is $\frac{13.3-10.7}{2} = 1.3$, and the length of the day can be modeled by the function

$$L_{MC}(t) = 12 + 1.3\sin\left(\frac{2\pi}{365}t\right)$$

Now, with $t = 0$ designated as March 21, it follows that April 1 corresponds to $t = 11$, July 15 corresponds to $t = 116$, and November 1 corresponds to $t = 225$. The day lengths in Wolf Point on these three dates are then given approximately by

$$L_{WP}(11) = 12 + 3.9\sin\left(\frac{22\pi}{365}\right) \approx 12.73 \text{ hours},$$

$$L_{WP}(116) = 12 + 3.9\sin\left(\frac{232\pi}{365}\right) \approx 15.55 \text{ hours, and}$$

$$L_{WP}(225) = 12 + 3.9\sin\left(\frac{450\pi}{365}\right) \approx 9.39 \text{ hours.}$$

In Mexico City, the day lengths are approximately

$$L_{MC}(11) = 12 + 1.3\sin\left(\frac{22\pi}{365}\right) \approx 12.24 \text{ hours,}$$

$$L_{MC}(116) = 12 + 1.3\sin\left(\frac{232\pi}{365}\right) \approx 13.18 \text{ hours, and}$$

$$L_{MC}(225) = 12 + 1.3\sin\left(\frac{450\pi}{365}\right) \approx 11.13 \text{ hours.}$$

42. During a year, the length of a day, from sunrise to sunset, in Taloga, Oklahoma, varies from a shortest day of approximately 9.6 hours to a longest day of approximately 14.4 hours, while in Montreal, the day lengths vary from 8.3 hours to 15.7 hours. For each location, determine a function $L(t) = 12 + A\sin(\frac{2\pi}{365}t)$ that approximates the length of a day, in hours, where t represents the day in the year assuming $t = 0$ is the spring equinox on March 21. Compare the day lengths in each location on April 15, July 30, and November 15.

SOLUTION The length of the day in Taloga varies from 9.6 hours to 14.4 hours, so the amplitude of the oscillation is $\frac{14.4-9.6}{2} = 2.4$ hours. The length of the day in Wolf Point can therefore be modeled by the function

$$L_T(t) = 12 + 2.4\sin\left(\frac{2\pi}{365}t\right)$$

On the other hand, the length of the day in Montreal varies from 8.3 hours to 15.7 hours, so the amplitude of the oscillation is $\frac{15.7-8.3}{2} = 3.7$, and the length of the day can be modeled by the function

$$L_M(t) = 12 + 3.7\sin\left(\frac{2\pi}{365}t\right)$$

Now, with $t = 0$ designated as March 21, it follows that April 15 corresponds to $t = 25$, July 30 corresponds to $t = 131$, and November 15 corresponds to $t = 239$. The day lengths in Taloga on these three dates are then given approximately by

$$L_T(25) = 12 + 2.4\sin\left(\frac{50\pi}{365}\right) \approx 13.00 \text{ hours,}$$

$$L_T(131) = 12 + 2.4\sin\left(\frac{262\pi}{365}\right) \approx 13.86 \text{ hours, and}$$

$$L_T(239) = 12 + 2.4\sin\left(\frac{478\pi}{365}\right) \approx 10.02 \text{ hours}$$

In Montreal, the day lengths are approximately

$$L_M(25) = 12 + 3.7\sin\left(\frac{50\pi}{365}\right) \approx 13.54 \text{ hours,}$$

$$L_M(131) = 12 + 3.7\sin\left(\frac{262\pi}{365}\right) \approx 14.87 \text{ hours, and}$$

$$L_M(239) = 12 + 3.7\sin\left(\frac{478\pi}{365}\right) \approx 8.94 \text{ hours}$$

43. How many points lie on the intersection of the horizontal line $y = c$ and the graph of $y = \sin x$ for $0 \le x < 2\pi$? *Hint:* The answer depends on c.

SOLUTION Recall that for any x, $-1 \le \sin x \le 1$. Thus, if $|c| > 1$, the horizontal line $y = c$ and the graph of $y = \sin x$ never intersect. If $c = +1$, then $y = c$ and $y = \sin x$ intersect at the peak of the sine curve; that is, they intersect at $x = \frac{\pi}{2}$. On the other hand, if $c = -1$, then $y = c$ and $y = \sin x$ intersect at the bottom of the sine curve; that is, they intersect at $x = \frac{3\pi}{2}$. Finally, if $|c| < 1$, the graphs of $y = c$ and $y = \sin x$ intersect twice.

44. How many points lie on the intersection of the horizontal line $y = c$ and the graph of $y = \tan x$ for $0 \le x < 2\pi$?

SOLUTION Recall that the graph of $y = \tan x$ consists of an infinite collection of "branches," each between two consecutive vertical asymptotes. Because each branch is increasing and has a range of all real numbers, the graph of the horizontal line $y = c$ will intersect each branch of the graph of $y = \tan x$ once, regardless of the value of c. The interval $0 \le x < 2\pi$ covers the equivalent of two branches of the tangent function, so over this interval, there are two points of intersection for each value of c.

In Exercises 45–46, solve for $0 \le \theta < 2\pi$.

45. $\sin\theta = \sin 2\theta$ *Hint*: Use the double-angle formula for sine.

SOLUTION Using the double-angle formula for the sine function, we rewrite the equation as $\sin 2\theta = 2\sin\theta\cos\theta$ or $\sin\theta(1 - 2\cos\theta) = 0$. Thus, either $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

46. $\sin\theta = \cos 2\theta$ *Hint*: Use appropriate identities to express $\cos 2\theta$ in terms of the sine function.

SOLUTION Solving the double-angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ for $\cos 2\theta$ yields $\cos 2\theta = 1 - 2\sin^2\theta$. We can therefore rewrite the original equation as $\sin\theta = 1 - 2\sin^2\theta$ or $2\sin^2\theta + \sin\theta - 1 = 0$. The left-hand side of this latter equation factors as $(2\sin\theta - 1)(\sin\theta + 1)$, so we have either $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$. The solutions on the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

In Exercises 47–56, derive the identity using the identities listed in this section.

47. $\cos 2\theta = 2\cos^2\theta - 1$

SOLUTION Starting from the double-angle formula for cosine, $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we solve for $\cos 2\theta$. This gives $2\cos^2\theta = 1 + \cos 2\theta$ and then $\cos 2\theta = 2\cos^2\theta - 1$.

48. $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

SOLUTION Substitute $x = \theta/2$ into the double-angle formula for cosine, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, to obtain $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$.

49. $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

SOLUTION Substitute $x = \theta/2$ into the double-angle formula for sine, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, to obtain $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$.

50. $\sin(\theta + \pi) = -\sin\theta$

SOLUTION From the addition formula for the sine function, we have

$$\sin(\theta + \pi) = \sin\theta\cos\pi + \cos\theta\sin\pi = -\sin\theta$$

51. $\cos(\theta + \pi) = -\cos\theta$

SOLUTION From the addition formula for the cosine function, we have

$$\cos(\theta + \pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = \cos\theta(-1) = -\cos\theta$$

52. $\tan x = \cot\left(\frac{\pi}{2} - x\right)$

SOLUTION Using the Complementary Angle Identity,

$$\cot\left(\frac{\pi}{2} - x\right) = \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x)} = \frac{\sin x}{\cos x} = \tan x$$

53. $\tan(\pi - \theta) = -\tan\theta$

SOLUTION Using Exercises 50 and 51,

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin(\pi + (-\theta))}{\cos(\pi + (-\theta))} = \frac{-\sin(-\theta)}{-\cos(-\theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

The second to last equality occurs because $\sin x$ is an odd function and $\cos x$ is an even function.

54. $\tan 2x = \frac{2\tan x}{1 - tan^2 x}$

SOLUTION Using the definition of the tangent function and the double-angle formulas for sine and cosine, we find

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} \cdot \frac{1/\cos^2 x}{1/\cos^2 x} = \frac{2\tan x}{1 - \tan^2 x}$$

55. $\tan x = \frac{\sin 2x}{1 + \cos 2x}$

SOLUTION Using the addition formula for the sine function, we find

$$\sin 2x = \sin(x + x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$$

By Exercise 47, we know that $\cos 2x = 2\cos^2 x - 1$. Therefore,

$$\frac{\sin 2x}{1 + \cos 2x} = \frac{2\sin x\cos x}{1 + 2\cos^2 x - 1} = \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin x}{\cos x} = \tan x$$

56. $\sin^2 x\cos^2 x = \frac{1 - \cos 4x}{8}$

SOLUTION Using the double-angle formulas for sine and cosine, we find

$$\sin^2 x\cos^2 x = \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) = \frac{1}{4}(1 - \cos^2 2x)$$
$$= \frac{1}{4}\left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right) = \frac{1}{8}(1 - \cos 4x)$$

57. Use Exercises 50 and 51 to show that $\tan\theta$ and $\cot\theta$ are periodic with period π.

SOLUTION By Exercises 50 and 51,

$$\tan(\theta + \pi) = \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

and

$$\cot(\theta + \pi) = \frac{\cos(\theta + \pi)}{\sin(\theta + \pi)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta$$

Thus, both $\tan\theta$ and $\cot\theta$ are periodic with period π.

58. Use the double-angle formulas to show that $\sin^2\theta$ and $\cos^2\theta$ are periodic with period π.

SOLUTION By the double-angle formula for the sine function and the fact that $\cos\theta$ is periodic with period 2π, we have

$$\sin^2(\theta + \pi) = \frac{1 - \cos(2(\theta + \pi))}{2} = \frac{1 - \cos(2\theta + 2\pi)}{2} = \frac{1 - \cos 2\theta}{2} = \sin^2\theta$$

Because $\sin^2(\theta + \pi) = \sin^2\theta$, it follows that $\sin^2\theta$ is periodic with period π.

By the double-angle formula for the cosine function and the fact that $\cos\theta$ is periodic with period 2π, we have

$$\cos^2(\theta + \pi) = \frac{1 + \cos(2(\theta + \pi))}{2} = \frac{1 + \cos(2\theta + 2\pi)}{2} = \frac{1 + \cos 2\theta}{2} = \cos^2\theta$$

Because $\cos^2(\theta + \pi) = \cos^2\theta$, it follows that $\cos^2\theta$ is periodic with period π.

59. Use the identity of Exercise 48 to show that $\cos\frac{\pi}{8}$ is equal to $\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}$.

SOLUTION Using the identity of Exercise 48 with $\theta = \frac{\pi}{4}$ yields

$$\cos^2\frac{\pi}{8} = \frac{1 + \cos\frac{\pi}{4}}{2} = \frac{1}{2} + \frac{\sqrt{2}}{4}$$

Because $\frac{\pi}{8}$ is a first quadrant angle, $\cos\frac{\pi}{8} > 0$; therefore,

$$\cos\frac{\pi}{8} = \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}$$

60. Use Exercise 55 to compute $\tan\frac{\pi}{8}$.

SOLUTION By Exercise 55,

$$\tan\frac{\pi}{8} = \frac{\sin 2\frac{\pi}{8}}{1+\cos 2\frac{\pi}{8}} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1$$

61. Use the Law of Cosines to find the distance from P to Q in Figure 26.

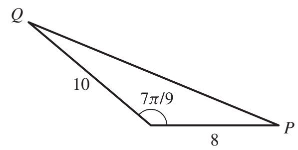


FIGURE 26

SOLUTION By the Law of Cosines, the distance from P to Q is

$$\sqrt{10^2 + 8^2 - 2(10)(8)\cos\frac{7\pi}{9}} = 16.928$$

Further Insights and Challenges

62. Use Figure 27 to derive the Law of Cosines from the Pythagorean Theorem.

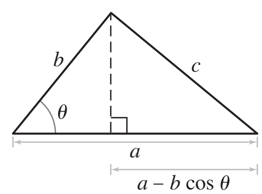


FIGURE 27

SOLUTION Applying the Pythagorean Theorem to the right triangle on the right-hand side of Figure 27 and noting that the length of the vertical leg of this triangle is $b\sin\theta$ yields

$$\begin{aligned} c^2 &= (b\sin\theta)^2 + (a - b\cos\theta)^2 \\ &= b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta \\ &= a^2 + b^2(\sin^2\theta + \cos^2\theta) - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

63. Use the addition formula to prove

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

SOLUTION

$$\begin{aligned} \cos 3\theta = \cos(2\theta+\theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= \cos\theta(2\cos^2\theta - 1 - 2\sin^2\theta) = \cos\theta(2\cos^2\theta - 1 - 2(1-\cos^2\theta)) \\ &= \cos\theta(2\cos^2\theta - 1 - 2 + 2\cos^2\theta) = 4\cos^3\theta - 3\cos\theta \end{aligned}$$

64. Use the addition formulas for sine and cosine to prove

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\cot(a-b) = \frac{\cot a\cot b + 1}{\cot b - \cot a}$$

SOLUTION

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a\cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} = \frac{\frac{\sin a \cos b}{\cos a \cos b} + \frac{\cos a \sin b}{\cos a \cos b}}{\frac{\cos a \cos b}{\cos a \cos b} - \frac{\sin a \sin b}{\cos a \cos b}} = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\cot(a-b) = \frac{\cos(a-b)}{\sin(a-b)} = \frac{\cos a\cos b + \sin a \sin b}{\sin a \cos b - \cos a \sin b} = \frac{\frac{\cos a \cos b}{\sin a \sin b} + \frac{\sin a \sin b}{\sin a \sin b}}{\frac{\sin a \cos b}{\sin a \sin b} - \frac{\cos a \sin b}{\sin a \sin b}} = \frac{\cot a \cot b + 1}{\cot b - \cot a}$$

65. Let θ be the angle between the line $y = mx + b$ and the x-axis [Figure 28(A)]. Prove that $m = \tan\theta$.

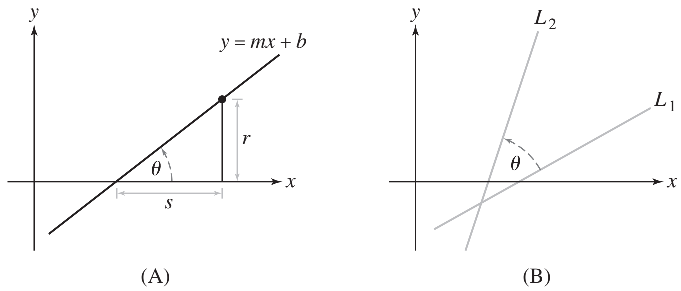


FIGURE 28

SOLUTION Using the distances labeled in Figure 28(A), we see that the slope of the line is given by the ratio r/s. The tangent of the angle θ is given by the same ratio. Therefore, $m = \tan\theta$.

66. Let L_1 and L_2 be the lines of slope m_1 and m_2 [Figure 28(B)]. Show that the angle θ between L_1 and L_2 satisfies $\cot\theta = \frac{m_2 m_1 + 1}{m_2 - m_1}$.

SOLUTION Measured from the positive x-axis, let α and β satisfy $\tan\alpha = m_1$ and $\tan\beta = m_2$. Without loss of generality, let $\beta \geq \alpha$. Then the angle between the two lines will be $\theta = \beta - \alpha$. Then from Exercise 64,

$$\cot\theta = \cot(\beta - \alpha) = \frac{\cot\beta\cot\alpha + 1}{\cot\alpha - \cot\beta} = \frac{(\frac{1}{m_1})(\frac{1}{m_2}) + 1}{\frac{1}{m_1} - \frac{1}{m_2}} = \frac{1 + m_1 m_2}{m_2 - m_1}$$

67. Perpendicular Lines Use Exercise 66 to prove that two lines with nonzero slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

SOLUTION If lines are perpendicular, then the angle between them is $\theta = \pi/2$. Substituting $\theta = \pi/2$ into the result from Exercise 66 yields

$$\cot(\pi/2) = \frac{1 + m_1 m_2}{m_1 - m_2} \quad \text{or} \quad 0 = \frac{1 + m_1 m_2}{m_1 - m_2}$$

Thus, $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$.

68. Apply the double-angle formula to prove:

(a) $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

(b) $\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

Guess the values of $\cos\frac{\pi}{32}$ and of $\cos\frac{\pi}{2^n}$ for all n.

SOLUTION

(a) $\cos\dfrac{\pi}{8} = \cos\dfrac{\pi/4}{2} = \sqrt{\dfrac{1 + \cos\frac{\pi}{4}}{2}} = \sqrt{\dfrac{1 + \frac{\sqrt{2}}{2}}{2}} = \dfrac{1}{2}\sqrt{2 + \sqrt{2}}$

(b) $\cos\dfrac{\pi}{16} = \sqrt{\dfrac{1 + \cos\frac{\pi}{8}}{2}} = \sqrt{\dfrac{1 + \frac{1}{2}\sqrt{2 + \sqrt{2}}}{2}} = \dfrac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

Observe that $8 = 2^3$ and $\cos\frac{\pi}{8}$ involves two nested square roots of 2; further, $16 = 2^4$ and $\cos\frac{\pi}{16}$ involves three nested square roots of 2. Since $32 = 2^5$, it seems plausible that

$$\cos\frac{\pi}{32} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

and that $\cos\frac{\pi}{2^n}$ involves $n - 1$ nested square roots of 2. Note that the general case can be proven by induction.

1.5 Inverse Functions

Preliminary Questions

1. Which of the following satisfy $f^{-1}(x) = f(x)$?

(a) $f(x) = x$ **(b)** $f(x) = 1 - x$ **(c)** $f(x) = 1$
(d) $f(x) = \sqrt{x}$ **(e)** $f(x) = |x|$ **(f)** $f(x) = x^{-1}$

SOLUTION The functions **(a)** $f(x) = x$, **(b)** $f(x) = 1 - x$ and **(f)** $f(x) = x^{-1}$ satisfy $f^{-1}(x) = f(x)$.

2. The function f maps teenagers in the United States to their last names. Explain why the inverse function f^{-1} does not exist.

SOLUTION Many different teenagers will have the same last name, so this function will not be one-to-one. Consequently, the function does not have an inverse.

3. The following fragment of a train schedule for the New Jersey Transit System defines a function f from towns to times. Is f one-to-one? What is $f^{-1}(6{:}27)$?

Trenton	6:21
Hamilton Township	6:27
Princeton Junction	6:34
New Brunswick	6:38

SOLUTION This function is one-to-one, and $f^{-1}(6{:}27) =$ Hamilton Township.

4. A homework problem asks for a sketch of the graph of the *inverse* of $f(x) = x + \cos x$. Frank, after trying but failing to find a formula for $f^{-1}(x)$, says it's impossible to graph the inverse. Bianca hands in an accurate sketch without solving for f^{-1}. How did Bianca complete the problem?

SOLUTION The graph of the inverse function is the reflection of the graph of $y = f(x)$ through the line $y = x$.

5. Which of the following quantities is undefined?

(a) $\sin^{-1}(-\frac{1}{2})$
(b) $\cos^{-1}(2)$
(c) $\csc^{-1}(\frac{1}{2})$
(d) $\csc^{-1}(2)$

SOLUTION **(b)** and **(c)** are undefined. $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$ and $\csc^{-1}(2) = \frac{\pi}{6}$.

6. Give an example of an angle θ such that $\cos^{-1}(\cos\theta) \neq \theta$. Does this contradict the definition of inverse function?

SOLUTION Any angle $\theta < 0$ or $\theta > \pi$ will work. No, this does not contradict the definition of inverse function.

Exercises

1. Show that $f(x) = 7x - 4$ is invertible and find its inverse.

SOLUTION Solving $y = 7x - 4$ for x yields $x = \frac{y+4}{7}$. Thus, $f^{-1}(x) = \frac{x+4}{7}$.

2. Is $f(x) = x^2 + 2$ one-to-one? If not, describe a domain on which it is one-to-one.

SOLUTION f is not one-to-one because $f(-1) = f(1) = 3$. However, if the domain is restricted to $x \geq 0$ or $x \leq 0$, then f is one-to-one.

3. What is the largest interval containing zero on which $f(x) = \sin x$ is one-to-one?

SOLUTION Looking at the graph of $\sin x$, the function is one-to-one on the interval $[-\pi/2, \pi/2]$.

4. Show that $f(x) = \frac{x-2}{x+3}$ is invertible and find its inverse.

(a) What is the domain of f? The range of f^{-1}?
(b) What is the domain of f^{-1}? The range of f?

SOLUTION We solve $y = f(x)$ for x as follows:

$$\begin{aligned} y &= \frac{x-2}{x+3} \\ yx + 3y &= x - 2 \\ yx - x &= -3y - 2 \\ x &= \frac{-3y-2}{y-1} = \frac{3y+2}{1-y} \end{aligned}$$

Therefore,

$$f^{-1}(x) = \frac{3x+2}{1-x}$$

(a) Domain of $f(x) = \{x|x \neq -3\}$ = range of $f^{-1}(x)$
(b) Domain of $f^{-1}(x) = \{x|x \neq 1\}$ = range of $f(x)$

5. Verify that $f(x) = x^3 + 3$ and $g(x) = (x-3)^{1/3}$ are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

SOLUTION

- $f(g(x)) = \left((x-3)^{1/3}\right)^3 + 3 = x - 3 + 3 = x$
- $g(f(x)) = \left(x^3 + 3 - 3\right)^{1/3} = \left(x^3\right)^{1/3} = x$

6. Repeat Exercise 5 for $f(t) = \frac{t+1}{t-1}$ and $g(t) = \frac{t+1}{t-1}$.

SOLUTION

$$f(g(t)) = \frac{\frac{t+1}{t-1}+1}{\frac{t+1}{t-1}-1} = \frac{t+1+t-1}{t+1-(t-1)} = t$$

The calculations for $g(f(t))$ are identical.

7. The escape velocity from a planet of radius R is $v(R) = \sqrt{\frac{2GM}{R}}$, where G is the universal gravitational constant and M is the mass. Find the inverse of $v(R)$ expressing R in terms of v.

SOLUTION To find the inverse, we solve

$$y = \sqrt{\frac{2GM}{R}}$$

for R. This yields

$$R = \frac{2GM}{y^2}$$

Therefore,

$$v^{-1}(R) = \frac{2GM}{R^2}$$

8. Show that the power law relationship $P(Q) = kQ^r$, for $Q \geq 0$ has an inverse that is also a power law, $Q(P) = mP^s$, where $m = k^{-1/r}$ and $s = 1/r$.

SOLUTION With $r \neq 0$ and $Q \geq 0$, the power law relationship $P(Q) = kQ^r$ is one-to-one and therefore has an inverse. To obtain the inverse, we solve $P = kQ^r$ for Q, which yields $Q = k^{-1/r}P^{1/r}$. Thus, the inverse is $Q(P) = k^{-1/r}P^{1/r}$, which has the form of a power law, $Q(P) = mP^s$, with $m = k^{-1/r}$ and $s = 1/r$.

9. The volume V of a cone that has height equal to its radius r is given by $V(r) = \frac{1}{3}\pi r^3$. Find the inverse of $V(r)$, expressing r as a function of V.

SOLUTION The function $V(r) = \frac{1}{3}\pi r^3$ is one-to-one, so it has an inverse. To obtain the inverse, we solve $V = \frac{1}{3}\pi r^3$ for r. This yields

$$r(V) = \left(\frac{3V}{\pi}\right)^{1/3}$$

10. The surface area S of a sphere of radius r is given by $S(r) = 4\pi r^2$. Explain why, in the given context, $S(r)$ has an inverse function. Find the inverse of $S(r)$, expressing r as a function of S.

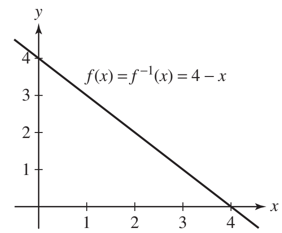
SOLUTION Because $S(r)$ is defined only for nonnegative r in this context, S is an increasing function. Therefore, $S(r)$ is one-to-one and has an inverse function. To obtain the inverse function, we solve $S = 4\pi r^2$ for r, remembering that r must be nonnegative:

$$r(S) = \sqrt{\frac{S}{4\pi}} = \frac{1}{2}\sqrt{\frac{S}{\pi}}$$

In Exercises 11–17, find a domain on which f is one-to-one and a formula for the inverse of f restricted to this domain. Sketch the graphs of f and f^{-1}.

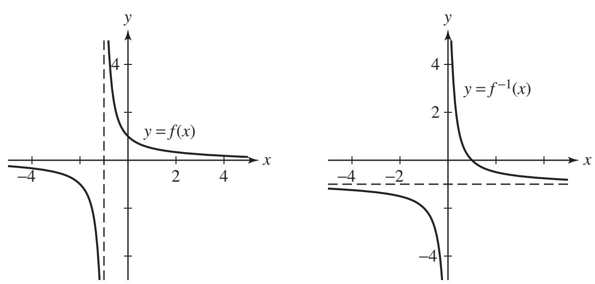
11. $f(x) = 4 - x$

SOLUTION The linear function $f(x) = 4 - x$ is one-to-one for all real numbers. Solving $y = x - 4$ for x gives $x = 4 - y$. Thus, $f^{-1}(x) = 4 - x$.



12. $f(x) = \frac{1}{x+1}$

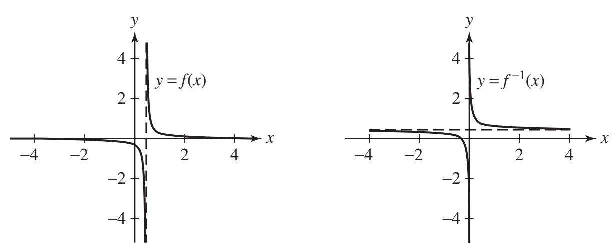
SOLUTION The graph of $f(x) = 1/(x+1)$ shown here shows that f passes the horizontal line test, and is therefore one-to-one, on its entire domain $\{x : x \neq -1\}$. Solving $y = \frac{1}{x+1}$ for x gives $x = \frac{1}{y} - 1$. Thus, $f^{-1}(x) = \frac{1}{x} - 1$.



13. $f(x) = \frac{1}{7x-3}$

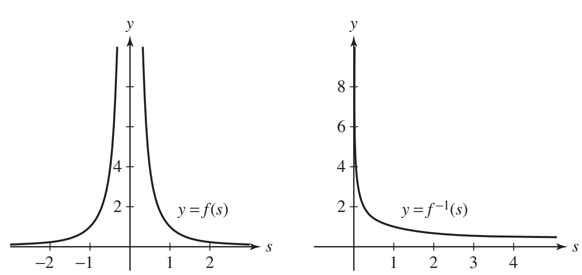
SOLUTION The graph of $f(x) = 1/(7x-3)$ shown here shows that f passes the horizontal line test, and is therefore one-to-one, on its entire domain $\{x : x \neq \frac{3}{7}\}$. Solving $y = 1/(7x-3)$ for x gives

$$x = \frac{1}{7y} + \frac{3}{7}; \quad \text{thus,} \quad f^{-1}(x) = \frac{1}{7x} + \frac{3}{7}$$



14. $f(s) = \frac{1}{s^2}$

SOLUTION To make $f(s) = s^{-2}$ one-to-one, we must restrict the domain to either $\{s : s > 0\}$ or $\{s : s < 0\}$. If we choose the domain $\{s : s > 0\}$, then solving $y = \frac{1}{s^2}$ for s yields $s = \frac{1}{\sqrt{y}}$. Hence, $f^{-1}(s) = \frac{1}{\sqrt{s}}$. Had we chosen the domain $\{s : s < 0\}$, the inverse would have been $f^{-1}(s) = -\frac{1}{\sqrt{s}}$.



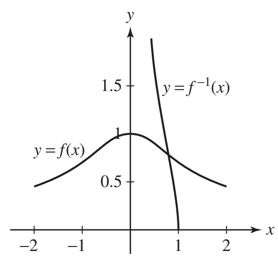
15. $f(x) = \frac{1}{\sqrt{x^2+1}}$

SOLUTION To make the function $f(x) = \frac{1}{\sqrt{x^2+1}}$ one-to-one, we must restrict the domain to either $\{x : x \geq 0\}$ or $\{x : x \leq 0\}$. If we choose the domain $\{x : x \geq 0\}$, then solving $y = \frac{1}{\sqrt{x^2+1}}$ for x yields

$$x = \frac{\sqrt{1-y^2}}{y}; \quad \text{hence,} \quad f^{-1}(x) = \frac{\sqrt{1-x^2}}{x}$$

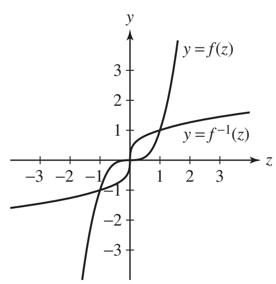
Had we chosen the domain $\{x : x \leq 0\}$, the inverse would have been

$$f^{-1}(x) = -\frac{\sqrt{1-x^2}}{x}$$



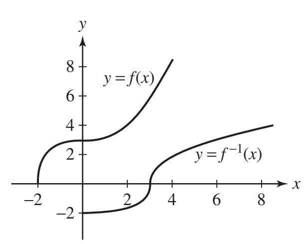
16. $f(z) = z^3$

SOLUTION The function $f(z) = z^3$ is one-to-one over its entire domain (see the graph). Solving $y = z^3$ for z yields $y^{1/3} = z$. Thus, $f^{-1}(z) = z^{1/3}$.



17. $f(x) = \sqrt{x^3+9}$

SOLUTION The graph of $f(x) = \sqrt{x^3+9}$ shown here shows that f passes the horizontal line test, and therefore is one-to-one, on its entire domain $\{x : x \geq -9^{1/3}\}$. Solving $y = \sqrt{x^3+9}$ for x yields $x = (y^2-9)^{1/3}$. Thus, $f^{-1}(x) = (x^2-9)^{1/3}$.



18. For each function shown in Figure 19, sketch the graph of the inverse (restrict the function's domain if necessary).

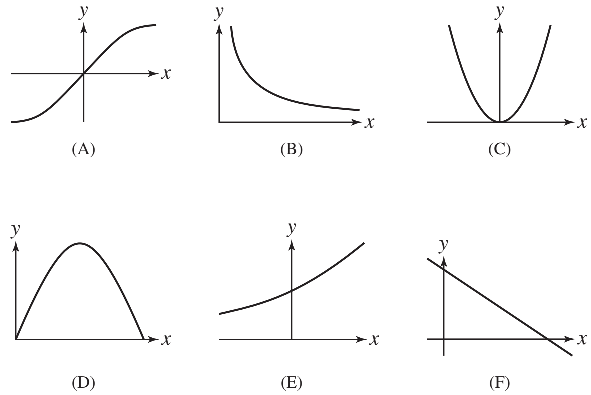
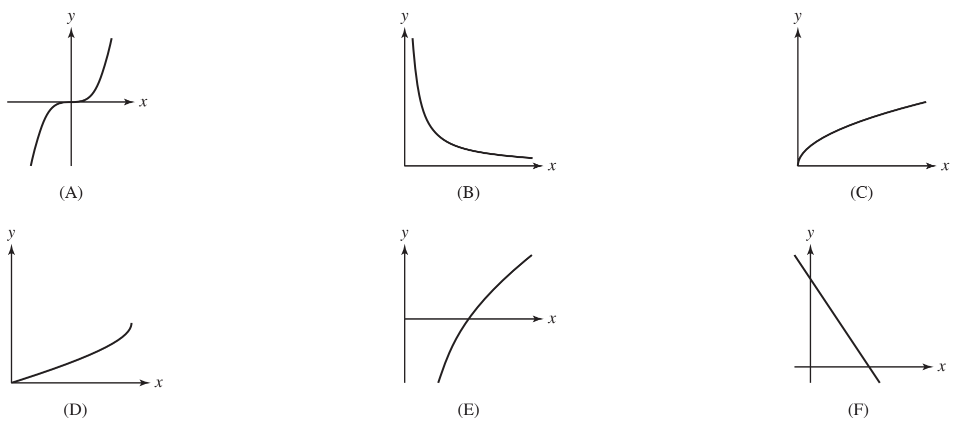


FIGURE 19

SOLUTION Here, we apply the rule that the graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$. For (C) and (D), we must restrict the domain of f to make f one-to-one.



19. Which of the graphs in Figure 20 is the graph of a function satisfying $f^{-1} = f$?

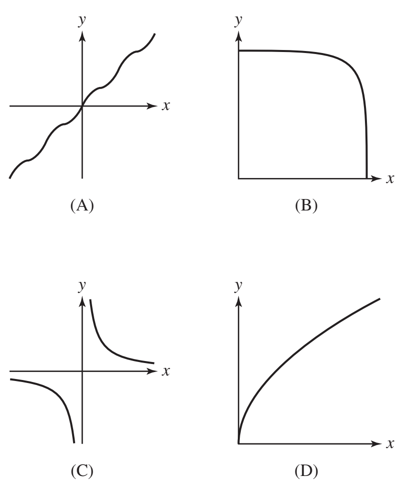


FIGURE 20

SOLUTION Figures (B) and (C) would not change when reflected around the line $y = x$. Therefore, these two satisfy $f^{-1} = f$.

20. Let n be a nonzero integer. Find a domain on which $f(x) = (1 - x^n)^{1/n}$ coincides with its inverse. *Hint:* The answer depends on whether n is even or odd.

SOLUTION First, note

$$f(f(x)) = \left(1 - ((1 - x^n)^{1/n})^n\right)^{1/n} = (1 - (1 - x^n))^{1/n} = (x^n)^{1/n} = x$$

so $f(x)$ coincides with its inverse. For the domain and range of f, let's first consider the case when $n > 0$. If n is even, then $f(x)$ is defined only when $1 - x^n \geq 0$. Hence, the domain is $-1 \leq x \leq 1$. The range is $0 \leq y \leq 1$. If n is odd, then $f(x)$ is defined for all real numbers, and the range is also all real numbers. Now, suppose $n < 0$. Then $-n > 0$, and

$$f(x) = \left(1 - \frac{1}{x^{-n}}\right)^{-1/-n} = \left(\frac{x^{-n}}{x^{-n} - 1}\right)^{1/-n}$$

If n is even, then $f(x)$ is defined only when $x^{-n} - 1 > 0$. Hence, the domain is $|x| > 1$. The range is $y > 1$. If n is odd, then $f(x)$ is defined for all real numbers except $x = 1$. The range is all real numbers except $y = 1$.

21. Let $f(x) = x^7 + x + 1$.

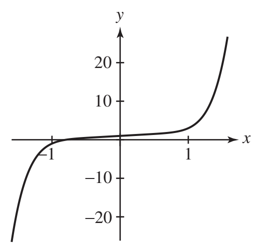
(a) Show that f^{-1} exists (but do not attempt to find it). *Hint:* Show that f is increasing.

(b) What is the domain of f^{-1}?

(c) Find $f^{-1}(3)$.

SOLUTION

(a) The graph of $f(x) = x^7 + x + 1$ is shown here. From this graph, we see that $f(x)$ is a strictly increasing function; by Example 3, it is therefore one-to-one. Because f is one-to-one, by Theorem 3, f^{-1} exists.

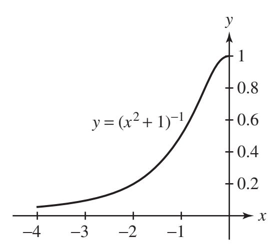


(b) The domain of $f^{-1}(x)$ is the range of $f(x)$: $(-\infty, \infty)$.

(c) Note that $f(1) = 1^7 + 1 + 1 = 3$; therefore, $f^{-1}(3) = 1$.

22. Show that $f(x) = (x^2 + 1)^{-1}$ is one-to-one on $(-\infty, 0]$, and find a formula for f^{-1} for this domain of f.

SOLUTION



Notice that the graph of $f(x) = (x^2 + 1)^{-1}$ over the interval $(-\infty, 0]$ (shown here) passes the horizontal line test. Thus, $f(x)$ is one-to-one on $(-\infty, 0]$. To find a formula for f^{-1}, we solve $y = (x^2 + 1)^{-1}$ for x, which yields $x = \pm\sqrt{\frac{1}{y} - 1}$. Because the domain of f was restricted to $x \leq 0$, we choose the negative sign in front of the radical. Therefore, $f^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$.

23. Let $f(x) = x^2 - 2x$. Determine a domain on which f^{-1} exists, and find a formula for f^{-1} for this domain of f.

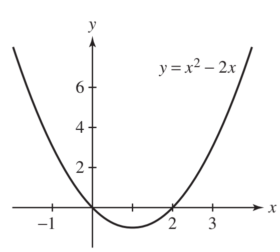
SOLUTION From the graph of $y = x^2 - 2x$ shown here, we see that if the domain of f is restricted to either $x \leq 1$ or $x \geq 1$, then f is one-to-one and f^{-1} exists. To find a formula for f^{-1}, we solve $y = x^2 - 2x$ for x as follows:

$$y + 1 = x^2 - 2x + 1 = (x - 1)^2$$

$$x - 1 = \pm\sqrt{y + 1}$$

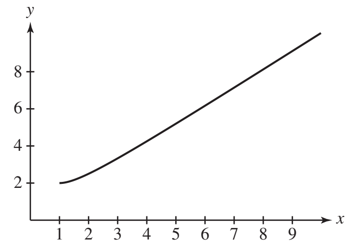
$$x = 1 \pm \sqrt{y + 1}$$

If the domain of f is restricted to $x \leq 1$, then we choose the negative sign in front of the radical and $f^{-1}(x) = 1 - \sqrt{x + 1}$. If the domain of f is restricted to $x \geq 1$, we choose the positive sign in front of the radical and $f^{-1}(x) = 1 + \sqrt{x + 1}$.



24. Show that $f(x) = x + x^{-1}$ is one-to-one on $[1, \infty)$, and find the corresponding inverse f^{-1}. What is the domain of f^{-1}?

SOLUTION The graph of $f(x) = x + x^{-1}$ on $[1, \infty)$ is shown here. From this graph, we see that for $x > 1$, the function is increasing, which implies that the function is one-to-one. Also, note that since f is increasing for $x > 1$, $f(x) \geq f(1) = 2$ for $x > 1$.



To find a formula for f^{-1}, let $y = x + x^{-1}$. Then $xy = x^2 + 1$ or $x^2 - xy + 1 = 0$. Using the quadratic formula, we find

$$x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

To have $x \geq 1$ for $y \geq 2$, we must choose the positive sign in front of the radical. Thus,

$$f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

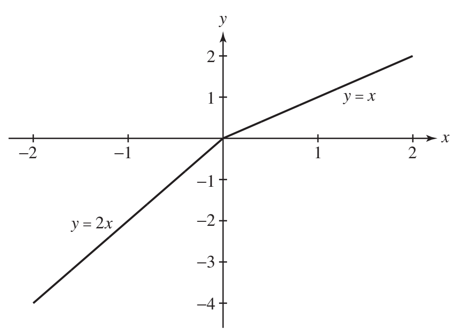
for $x \geq 2$.

For each of the piecewise-defined functions in Exercises 25–28, determine whether or not the function is one-to-one, and if it is, determine its inverse function.

25.

$$f(x) = \begin{cases} x & \text{when } x < 0 \\ 2x & \text{when } x \geq 0 \end{cases}$$

SOLUTION



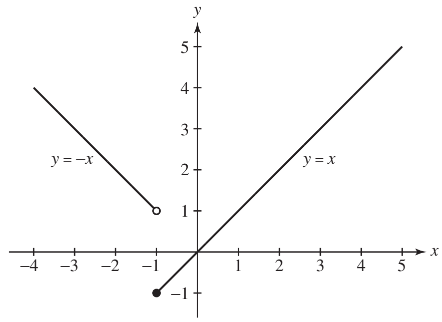
Because the graph of f, which is shown here, passes the horizontal line test, f is a one-to-one function and has an inverse function. For $x < 0$, $f(x) = x$. Solving $y = x$ for x yields $x = y$, so this piece of the inverse function is $f^{-1}(x) = x$. On the other hand, for $x \geq 0$, $f(x) = 2x$. Solving $y = 2x$ for x yields $x = \frac{1}{2}y$, so this piece of the inverse function is $f^{-1}(x) = \frac{1}{2}x$. Bringing these two pieces together, we have

$$f^{-1}(x) = \begin{cases} x & \text{when} \quad x < 0 \\ \frac{1}{2}x & \text{when} \quad x \geq 0 \end{cases}$$

26.

$$g(x) = \begin{cases} -x & \text{when } x < -1 \\ x & \text{when } x \geq -1 \end{cases}$$

SOLUTION

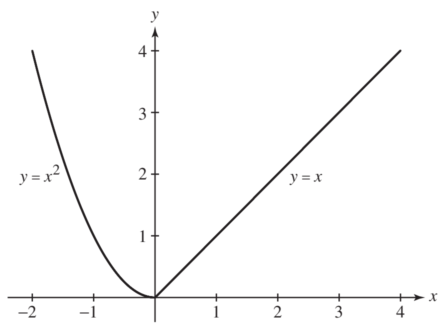


Because the graph of g, which is shown here, does not pass the horizontal line test, g is not a one-to-one function.

27.

$$f(x) = \begin{cases} x^2 & \text{when } x < 0 \\ x & \text{when } x \geq 0 \end{cases}$$

SOLUTION

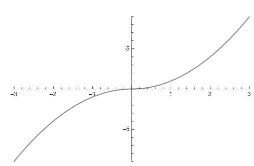


Because the graph of f, which is shown here, does not pass the horizontal line test, f is not a one-to-one function.

28.

$$g(x) = \begin{cases} -x^2 & \text{when } x < 0 \\ x^2 & \text{when } x \geq 0 \end{cases}$$

SOLUTION



Because the graph of g, which is shown here, passes the horizontal line test, g is a one-to-one function and has an inverse function. For $x < 0$, $g(x) = -x^2$. Solving $y = -x^2$ for x (remembering that we are interested in only $x < 0$) yields $x = -\sqrt{-y}$, so this piece of the inverse function is $g^{-1}(x) = -\sqrt{-x}$. On the other hand, for $x \geq 0$, $g(x) = x^2$. Solving $y = x^2$ for x (remembering that we are interested in only $x \geq 0$) yields $x = \sqrt{y}$, so this piece of the inverse function is $g^{-1}(x) = \sqrt{x}$. Bringing these two pieces together, we have

$$g^{-1}(x) = \begin{cases} -\sqrt{-x} & \text{when} \quad x < 0 \\ \sqrt{x} & \text{when} \quad x \geq 0 \end{cases}$$

In Exercises 29–34, evaluate without using a calculator.

29. $\cos^{-1} 1$

SOLUTION $\cos^{-1} 1 = 0$

30. $\sin^{-1} \frac{1}{2}$

SOLUTION $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$

31. $\cot^{-1} 1$

SOLUTION $\cot^{-1} 1 = \frac{\pi}{4}$

32. $\sec^{-1} \frac{2}{\sqrt{3}}$

SOLUTION $\sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6}$

33. $\tan^{-1} \sqrt{3}$

SOLUTION $\tan^{-1} \sqrt{3} = \tan^{-1}(\frac{\sqrt{3}/2}{1/2}) = \frac{\pi}{3}$

34. $\sin^{-1}(-1)$

SOLUTION $\sin^{-1}(-1) = -\frac{\pi}{2}$

In Exercises 35–44, compute without using a calculator.

35. $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$

SOLUTION $\sin^{-1}(\sin \frac{\pi}{3}) = \frac{\pi}{3}$

36. $\sin^{-1}\left(\sin \frac{4\pi}{3}\right)$

SOLUTION $\sin^{-1}(\sin \frac{4\pi}{3}) = \sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$. The answer is not $\frac{4\pi}{3}$ because $\frac{4\pi}{3}$ is not in the range of the inverse sine function.

37. $\cos^{-1}\left(\cos \frac{3\pi}{2}\right)$

SOLUTION $\cos^{-1}(\cos \frac{3\pi}{2}) = \cos^{-1}(0) = \frac{\pi}{2}$. The answer is not $\frac{3\pi}{2}$ because $\frac{3\pi}{2}$ is not in the range of the inverse cosine function.

38. $\sin^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right)$

SOLUTION $\sin^{-1}(\sin(-\frac{5\pi}{6})) = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$. The answer is not $-\frac{5\pi}{6}$ because $-\frac{5\pi}{6}$ is not in the range of the inverse sine function.

39. $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

SOLUTION $\tan^{-1}(\tan \frac{3\pi}{4}) = \tan^{-1}(-1) = -\frac{\pi}{4}$. The answer is not $\frac{3\pi}{4}$ because $\frac{3\pi}{4}$ is not in the range of the inverse tangent function.

40. $\tan^{-1}(\tan \pi)$

SOLUTION $\tan^{-1}(\tan \pi) = \tan^{-1}(0) = 0$. The answer is not π because π is not in the range of the inverse tangent function.

41. $\sec^{-1}(\sec 3\pi)$

SOLUTION $\sec^{-1}(\sec 3\pi) = \sec^{-1}(-1) = \pi$. The answer is not 3π because 3π is not in the range of the inverse secant function.

42. $\sec^{-1}\left(\sec \frac{3\pi}{2}\right)$

SOLUTION No inverse since $\sec \frac{3\pi}{2} = \frac{1}{\cos \frac{3\pi}{2}} = \frac{1}{0} \longrightarrow \infty$.

43. $\csc^{-1}(\csc(-\pi))$

SOLUTION No inverse since $\csc(-\pi) = \frac{1}{\sin(-\pi)} = \frac{1}{0} \longrightarrow \infty$.

44. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

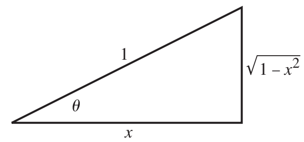
SOLUTION $\cot^{-1}\left(\cot(-\frac{\pi}{4})\right) = \cot^{-1}(-1) = \frac{3\pi}{4}$. The answer is not $-\frac{\pi}{4}$ because $-\frac{\pi}{4}$ is not in the range of the inverse cotangent function.

In Exercises 45–48, simplify by referring to the appropriate triangle or trigonometric identity.

45. $\tan(\cos^{-1} x)$

SOLUTION Let $\theta = \cos^{-1} x$. Then $\cos\theta = x$ and we generate the triangle shown here. From the triangle,

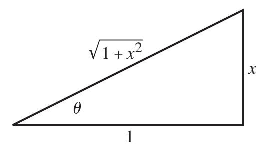
$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$$



46. $\cos(\tan^{-1} x)$

SOLUTION Let $\theta = \tan^{-1} x$. Then $\tan\theta = x$ and we generate the triangle shown here. From the triangle,

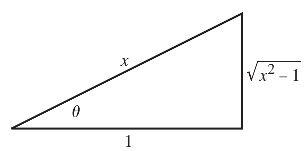
$$\cos(\tan^{-1} x) = \cos\theta = \frac{1}{\sqrt{x^2+1}}$$



47. $\cot(\sec^{-1} x)$

SOLUTION Let $\theta = \sec^{-1} x$. Then $\sec\theta = x$ and we generate the triangle shown here. From the triangle,

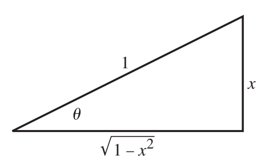
$$\cot(\sec^{-1} x) = \cot\theta = \frac{1}{\sqrt{x^2-1}}$$



48. $\cot(\sin^{-1} x)$

SOLUTION Let $\theta = \sin^{-1} x$. Then $\sin\theta = x$ and we generate the triangle shown here. From the triangle,

$$\cot(\sin^{-1} x) = \cot\theta = \frac{\sqrt{1-x^2}}{x}$$

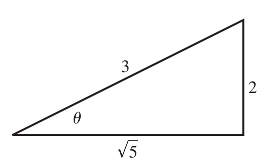


In Exercises 49–56, refer to the appropriate triangle or trigonometric identity to compute the given value.

49. $\cos(\sin^{-1} \frac{2}{3})$

SOLUTION Let $\theta = \sin^{-1} \frac{2}{3}$. Then $\sin\theta = \frac{2}{3}$ and we generate the triangle shown here. From the triangle,

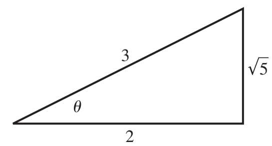
$$\cos\left(\sin^{-1}\frac{2}{3}\right) = \cos\theta = \frac{\sqrt{5}}{3}$$



50. $\tan(\cos^{-1} \frac{2}{3})$

SOLUTION Let $\theta = \cos^{-1}\frac{2}{3}$. Then $\cos\theta = \frac{2}{3}$ and we generate the triangle shown here. From the triangle,

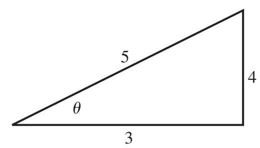
$$\tan\left(\cos^{-1}\frac{2}{3}\right) = \tan\theta = \frac{\sqrt{5}}{2}$$



51. $\tan(\sin^{-1} 0.8)$

SOLUTION Let $\theta = \sin^{-1} 0.8$. Then $\sin\theta = 0.8 = \frac{4}{5}$ and we generate the triangle shown here. From the triangle,

$$\tan(\sin^{-1} 0.8) = \tan\theta = \frac{4}{3}$$



52. $\cos(\cot^{-1} 1)$

SOLUTION $\cot^{-1} 1 = \frac{\pi}{4}$; hence, $\cos(\cot^{-1} 1) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

53. $\cot(\csc^{-1} 2)$

SOLUTION $\csc^{-1} 2 = \frac{\pi}{6}$; hence, $\cot(\csc^{-1} 2) = \cot\frac{\pi}{6} = \sqrt{3}$.

54. $\tan(\sec^{-1}(-2))$

SOLUTION $\sec^{-1}(-2) = \frac{2\pi}{3}$; hence, $\tan(\sec^{-1}(-2)) = \tan\frac{2\pi}{3} = -\sqrt{3}$.

55. $\cot(\tan^{-1} 20)$

SOLUTION Let $\theta = \tan^{-1} 20$. Then $\tan\theta = 20$, so $\cot(\tan^{-1} 20) = \cot\theta = \frac{1}{\tan\theta} = \frac{1}{20}$.

56. $\sin(\csc^{-1} 20)$

SOLUTION Let $\theta = \csc^{-1} 20$. Then $\csc\theta = 20$, so $\sin(\csc^{-1} 20) = \sin\theta = \frac{1}{\csc\theta} = \frac{1}{20}$.

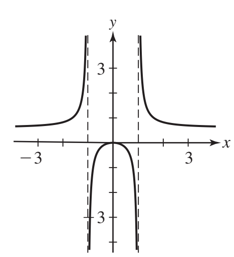
57. $\boxed{\text{GU}}$ Let $f(x) = \frac{x^2}{x^2-1}$

(a) From a graph of f, explain why f is not invertible. Furthermore, explain why f is invertible if we restrict the domain to $[0, \infty)$.

(b) Find the inverse function for f restricted to $[0, \infty)$.

SOLUTION

(a) Because the graph of f, shown here, does not pass the horizontal line test, f is not a one-to-one function and therefore does not have an inverse. If the domain is restricted to $[0, \infty)$, however, that portion of the graph does pass the horizontal line test, so f is one-to-one and therefore does have an inverse.



(b) Solving $y = \frac{x^2}{x^2-1}$ for x yields

$$x^2y - y = x^2$$

$$x^2y - x^2 = y$$

$$x^2 = \frac{y}{y-1}$$

$$x = \sqrt{\frac{y}{y-1}}$$

where we have taken the positive square root because we are interested in only $x \geq 0$. Thus,

$$f^{-1}(x) = \sqrt{\frac{x}{x-1}}$$

58. GU Let $f(x) = \frac{x}{1-x^2}$

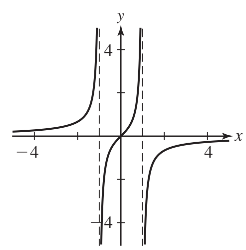
(a) From a graph of f, explain why f is not invertible. Furthermore, explain why f is invertible if we restrict the domain to $(-1, 1)$.

(b) Let h be the inverse function for f restricted to $(-1, 1)$. What are $h(-10)$, $h(-2)$, $h(0)$, $h(3)$, $h(9)$?

(c) Use the quadratic formula to find an expression for $h(x)$.

SOLUTION

(a) Because the graph of f, shown here, does not pass the horizontal line test, f is not a one-to-one function and therefore does not have an inverse. If the domain is restricted to $(-1, 1)$, however, that portion of the graph does pass the horizontal line test, so f is one-to-one and therefore does have an inverse.



(b) Let h be the inverse function for f restricted to $(-1, 1)$. By the inverse relationship between the functions f and h, $h(-10)$ is the value of x such that $f(x) = -10$. Thus, to determine $h(-10)$, we need to find the solution of the equation

$$\frac{x}{1-x^2} = -10 \quad \text{or} \quad 10x^2 - x - 10 = 0$$

that lies on the interval $(-1, 1)$. By the quadratic formula,

$$x = \frac{1 \pm \sqrt{1+400}}{20} = \frac{1 \pm \sqrt{401}}{20}$$

Of these two solutions, $\frac{1-\sqrt{401}}{20}$ is between -1 and 1, so $h(-10) = \frac{1-\sqrt{401}}{20}$. We can proceed in a similar manner to obtain the remaining requested values for h. To determine $h(-2)$, we need to find the solution of the equation

$$\frac{x}{1-x^2} = -2 \quad \text{or} \quad 2x^2 - x - 2 = 0$$

that lies on the interval $(-1, 1)$. By the quadratic formula,

$$x = \frac{1 \pm \sqrt{1+16}}{4} = \frac{1 \pm \sqrt{17}}{4}$$

Of these two solutions, $\frac{1-\sqrt{17}}{4}$ is between -1 and 1, so $h(-2) = \frac{1-\sqrt{17}}{4}$. To determine $h(0)$, the equation we need to solve is

$$\frac{x}{1-x^2} = 0 \quad \text{or} \quad x = 0$$

Thus, $h(0) = 0$. To determine $h(3)$, the equation we need to solve is

$$\frac{x}{1-x^2} = 3 \quad \text{or} \quad 3x^2 + x - 3 = 0$$

By the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1+36}}{6} = \frac{-1 \pm \sqrt{37}}{6}$$

Of these two solutions, $\frac{-1+\sqrt{37}}{6}$ is between -1 and 1, so $h(3) = \frac{-1+\sqrt{37}}{6}$. Finally, to determine $h(9)$, the equation we need to solve is

$$\frac{x}{1-x^2} = 9 \quad \text{or} \quad 9x^2 + x - 9 = 0$$

By the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1+324}}{18} = \frac{-1 \pm \sqrt{325}}{18}$$

Of these two solutions, $\frac{-1+\sqrt{325}}{18}$ is between -1 and 1, so $h(9) = \frac{-1+\sqrt{325}}{18}$.

(c) Solving $y = \frac{x}{1-x^2}$ for x yields

$$y - x^2y = x \quad \text{or} \quad x^2y + x - y = 0$$

If $y = 0$, this last equation reduces to $x = 0$. For $y \neq 0$, we use the quadratic formula to obtain

$$x = \frac{-1 \pm \sqrt{1+4y^2}}{2y}$$

Of these two solutions, $\frac{-1+\sqrt{1+4y^2}}{2y}$ is between -1 and 1. Thus,

$$h(x) = \begin{cases} \frac{-1+\sqrt{1+4x^2}}{2x} & \text{when} \quad x \neq 0 \\ 0 & \text{when} \quad x = 0 \end{cases}$$

Further Insights and Challenges

59. Show that if f is odd and f^{-1} exists, then f^{-1} is odd. Show, on the other hand, that an even function does not have an inverse.

SOLUTION Suppose $f(x)$ is odd and $f^{-1}(x)$ exists. Because $f(x)$ is odd, $f(-x) = -f(x)$. Let $y = f^{-1}(x)$; then $f(y) = x$. Since $f(x)$ is odd, $f(-y) = -f(y) = -x$. Thus, $f^{-1}(-x) = -y = -f^{-1}(x)$. Hence, f^{-1} is odd.

On the other hand, if $f(x)$ is even, then $f(-x) = f(x)$. Hence, f is not one-to-one and f^{-1} does not exist.

60. A cylindrical tank of radius R and length L lying horizontally, as in Figure 21, is filled with oil to height h. Show that the volume $V(h)$ of oil in the tank as a function of height h is

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{2hR - h^2}\right)$$

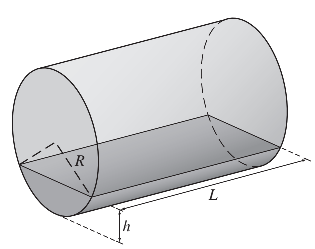


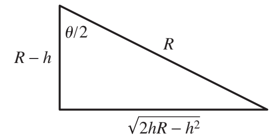
FIGURE 21 Oil in the tank has level h.

SOLUTION From Figure 21, we see that the volume of oil in the tank, $V(h)$, is equal to L times $A(h)$, the area of that portion of the circular cross section occupied by the oil. Now,

$$A(h) = \text{area of sector} - \text{area of triangle} = \frac{R^2\theta}{2} - \frac{R^2\sin\theta}{2}$$

where θ is the central angle of the sector. Referring to the diagram here,

$$\cos\frac{\theta}{2} = \frac{R-h}{R} \quad \text{and} \quad \sin\frac{\theta}{2} = \frac{\sqrt{2hR-h^2}}{R}$$



Thus,

$$\theta = 2\cos^{-1}\left(1 - \frac{h}{R}\right),$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{(R-h)\sqrt{2hR-h^2}}{R^2}, \text{ and}$$

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{2hR-h^2}\right)$$

1.6 Exponential and Logarithmic Functions

Preliminary Questions

1. When is $\ln x$ negative?

SOLUTION $\ln x$ is negative for $0 < x < 1$.

2. What is $\ln(-3)$? Explain.

SOLUTION $\ln(-3)$ is not defined.

3. Explain the phrase "The logarithm converts multiplication into addition."

SOLUTION This phrase is a verbal description of the general property of logarithms that states

$$\log(ab) = \log a + \log b$$

4. What are the domain and range of $f(x) = \ln x$?

SOLUTION The domain of $\ln x$ is $x > 0$ and the range is all real numbers.

5. Explain why $f(x) = x^3$ does not grow exponentially.

SOLUTION In exponential growth, an increase in x by 1 produces an increase in the function value by the same fixed percent, regardless of the value of x involved. For $f(x) = x^3$, note that $f(1) = 1$, $f(2) = 8$, and $f(3) = 27$. When x increases by 1 from 1 to 2, the value of f increases 700%, whereas when x increases by 1 from 2 to 3, the value of f increases 237.5%. Because the percent increase in f is not the same, it follows that $f(x) = x^3$ does not exhibit exponential growth.

6. Compute $\log_{b^2}(b^4)$.

SOLUTION Because $b^4 = (b^2)^2$, $\log_{b^2}(b^4) = 2$.

7. Which hyperbolic functions take on only positive values?

SOLUTION $\cosh x$ and $\operatorname{sech} x$ take on only positive values.

8. Which hyperbolic functions are increasing on their domains?

SOLUTION $\sinh x$ and $\tanh x$ are increasing on their domains.

9. Describe three properties of hyperbolic functions that have trigonometric analogs.

SOLUTION Hyperbolic functions have the following analogs with trigonometric functions: parity, identities, and derivative formulas.

Exercises

In Exercises 1–8, solve for the unknown variable.

1. $e^{2x} = e^{x+1}$

SOLUTION If $e^{2x} = e^{x+1}$, then $2x = x + 1$, and $x = 1$.

2. $e^{t^2} = e^{4t-3}$

SOLUTION If $e^{t^2} = e^{4t-3}$, then $t^2 = 4t - 3$ or $t^2 - 4t + 3 = (t-3)(t-1) = 0$. Thus, $t = 1$ or $t = 3$.

3. $3^x = \left(\frac{1}{3}\right)^{x+1}$

SOLUTION Rewrite $\left(\frac{1}{3}\right)^{x+1}$ as $(3^{-1})^{x+1} = 3^{-x-1}$. Then $3^x = 3^{-x-1}$, which requires $x = -x - 1$. Thus, $x = -1/2$.

4. $(\sqrt{5})^x = 125$

SOLUTION Rewrite $(\sqrt{5})^x$ as $(5^{1/2})^x = 5^{x/2}$ and 125 as 5^3. Then $5^{x/2} = 5^3$, so $x/2 = 3$ and $x = 6$.

5. $4^{-x} = 2^{x+1}$

SOLUTION Rewrite 4^{-x} as $(2^2)^{-x} = 2^{-2x}$. Then $2^{-2x} = 2^{x+1}$, which requires $-2x = x + 1$. Solving for x gives $x = -1/3$.

6. $b^4 = 10^{12}$

SOLUTION $b^4 = 10^{12}$ is equivalent to $b^4 = (10^3)^4$ so $b = 10^3$. Alternately, raise both sides of the equation to the one-fourth power. This gives $b = (10^{12})^{1/4} = 10^3$.

7. $k^{3/2} = 27$

SOLUTION Raise both sides of the equation to the two-thirds power. This gives $k = (27)^{2/3} = (27^{1/3})^2 = 3^2 = 9$.

8. $(b^2)^{x+1} = b^{-6}$

SOLUTION Rewrite $(b^2)^{x+1}$ as $b^{2(x+1)}$. Then $2(x + 1) = -6$, and $x = -4$.

In Exercises 9–24, calculate without using a calculator.

9. $\log_3 27$

SOLUTION $\log_3 27 = \log_3 3^3 = 3\log_3 3 = 3$

10. $\log_5 \frac{1}{25}$

SOLUTION $\log_5 \dfrac{1}{25} = \log_5 5^{-2} = -2\log_5 5 = -2$

11. $\ln 1$

SOLUTION $\ln 1 = 0$

12. $\log_5(5^4)$

SOLUTION $\log_5(5^4) = 4\log_5 5 = 4$

13. $\log_2(2^{5/3})$

SOLUTION $\log_2 2^{5/3} = \dfrac{5}{3}\log_2 2 = \dfrac{5}{3}$

14. $\log_2(8^{5/3})$

SOLUTION $\log_2(8^{5/3}) = \dfrac{5}{3}\log_2 2^3 = 5\log_2 2 = 5$

15. $\log_{64} 4$

SOLUTION $\log_{64} 4 = \log_{64} 64^{1/3} = \dfrac{1}{3}\log_{64} 64 = \dfrac{1}{3}$

16. $\log_7(49^2)$

SOLUTION $\log_7 49^2 = 2\log_7 7^2 = 2 \cdot 2 \cdot \log_7 7 = 4$

17. $\log_8 2 + \log_4 2$

SOLUTION $\log_8 2 + \log_4 2 = \log_8 8^{1/3} + \log_4 4^{1/2} = \dfrac{1}{3} + \dfrac{1}{2} = \dfrac{5}{6}$

18. $\log_{25} 30 + \log_{25} \frac{5}{6}$

SOLUTION $\log_{25} 30 + \log_{25} \dfrac{5}{6} = \log_{25}\left(30 \cdot \dfrac{5}{6}\right) = \log_{25} 25 = 1$

19. $\log_4 48 - \log_4 12$

SOLUTION $\log_4 48 - \log_4 12 = \log_4 \dfrac{48}{12} = \log_4 4 = 1$

20. $\ln(\sqrt{e} \cdot e^{7/5})$

SOLUTION $\ln(\sqrt{e} \cdot e^{7/5}) = \ln(e^{1/2} \cdot e^{7/5}) = \ln(e^{1/2+7/5}) = \ln(e^{19/10}) = \dfrac{19}{10}$

21. $\ln(e^3) + \ln(e^4)$

SOLUTION $\ln(e^3) + \ln(e^4) = 3 + 4 = 7$

22. $\log_2 \frac{4}{3} + \log_2 24$

SOLUTION $\log_2 \frac{4}{3} + \log_2 24 = \log_2 \left(\frac{4}{3} \cdot 24\right) = \log_2 32 = \log_2 2^5 = 5 \log_2 2 = 5$

23. $7^{\log_7(29)}$

SOLUTION $7^{\log_7(29)} = 29$

24. $8^{3 \log_8(2)}$

SOLUTION $8^{3 \log_8(2)} = \left(8^{\log_8(2)}\right)^3 = 2^3 = 8$

25. Write as the natural log of a single expression:

(a) $2 \ln 5 + 3 \ln 4$ **(b)** $5 \ln(x^{1/2}) + \ln(9x)$

SOLUTION

(a) $2 \ln 5 + 3 \ln 4 = \ln 5^2 + \ln 4^3 = \ln 25 + \ln 64 = \ln(25 \cdot 64) = \ln 1600$

(b) $5 \ln x^{1/2} + \ln 9x = \ln x^{5/2} + \ln 9x = \ln(x^{5/2} \cdot 9x) = \ln(9x^{7/2})$

26. Solve for x: $\ln(x^2 + 1) - 3 \ln x = \ln(2)$

SOLUTION Combining terms on the left-hand side gives

$$\ln(x^2 + 1) - 3 \ln x = \ln(x^2 + 1) - \ln x^3 = \ln \frac{x^2 + 1}{x^3}$$

Therefore, $\frac{x^2+1}{x^3} = 2$ or $2x^3 - x^2 - 1 = 0$; $x = 1$ is the only real root to this equation. Substituting $x = 1$ into the original equation, we find

$$\ln 2 - 3 \ln 1 = \ln 2 - 0 = \ln 2$$

as needed. Hence, $x = 1$ is the only solution.

In Exercises 27–32, solve for the unknown.

27. $7e^{5t} = 100$

SOLUTION Divide the equation by 7 and then take the natural logarithm of both sides. This gives

$$5t = \ln\left(\frac{100}{7}\right) \quad \text{or} \quad t = \frac{1}{5} \ln\left(\frac{100}{7}\right)$$

28. $6e^{-4t} = 2$

SOLUTION Divide the equation by 6 and then take the natural logarithm of both sides. This gives

$$-4t = \ln\left(\frac{1}{3}\right) \quad \text{or} \quad t = \frac{\ln 3}{4}$$

29. $2^{x^2-2x} = 8$

SOLUTION Since $8 = 2^3$, we have $x^2 - 2x - 3 = 0$ or $(x - 3)(x + 1) = 0$. Thus, $x = -1$ or $x = 3$.

30. $e^{2t+1} = 9e^{1-t}$

SOLUTION Taking the natural logarithm of both sides of the equation gives

$$2t + 1 = \ln\left(9e^{1-t}\right) = \ln 9 + \ln e^{1-t} = \ln 9 + (1 - t)$$

Thus, $3t = \ln 9$ or $t = \frac{1}{3} \ln 9$.

31. $\ln(x^4) - \ln(x^2) = 2$

SOLUTION $\ln(x^4) - \ln(x^2) = \ln\left(\frac{x^4}{x^2}\right) = \ln(x^2) = 2 \ln x$. Thus, $2 \ln x = 2$ or $\ln x = 1$. Hence, $x = e$.

32. $\log_3 y + 3 \log_3(y^2) = 14$

SOLUTION $14 = \log_3 y + 3 \log_3(y^2) = \log_3 y + \log_3 y^6 = \log_3 y^7$. Thus, $y^7 = 3^{14}$ or $y = 3^2 = 9$.

33. Find the inverse of $y = e^{2x-3}$.

SOLUTION Solving $y = e^{2x-3}$ for x yields

$$\ln y = \ln(e^{2x-3}) = 2x - 3 \qquad \text{or} \qquad x = \frac{1}{2}(3 + \ln y)$$

Thus, the inverse of $y = e^{2x-3}$ is $y = \frac{1}{2}(3 + \ln x)$.

34. Find the inverse of $y = \ln(x^2 - 2)$ for $x > \sqrt{2}$.

SOLUTION Solving $y = \ln(x^2 - 2)$ for x yields

$$e^y = x^2 - 2 \qquad \text{or} \qquad x = \pm\sqrt{e^y + 2}$$

Because we are interested in $x > \sqrt{2}$, we choose the positive sign on the square root. Thus, the inverse of $y = \ln(x^2 - 2)$ for $x > \sqrt{2}$ is $y = \sqrt{e^x + 2}$.

35. Use a calculator to compute $\sinh x$ and $\cosh x$ for $x = -3, 0, 5$.

SOLUTION

x	-3	0	5
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\dfrac{e^{-3} - e^3}{2} = -10.0179$	$\dfrac{e^0 - e^0}{2} = 0$	$\dfrac{e^5 - e^{-5}}{2} = 74.203$
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\dfrac{e^{-3} + e^3}{2} = 10.0677$	$\dfrac{e^0 + e^0}{2} = 1$	$\dfrac{e^5 + e^{-5}}{2} = 74.210$

36. Compute $\sinh(\ln 5)$ and $\tanh(3\ln 5)$ without using a calculator.

SOLUTION

$$\sinh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 - 1/5}{2} = \frac{24/5}{2} = 12/5$$

$$\tanh(3\ln 5) = \frac{\sinh(3\ln 5)}{\cosh(3\ln 5)} = \frac{\frac{e^{3\ln 5} - e^{-3\ln 5}}{2}}{\frac{e^{3\ln 5} + e^{-3\ln 5}}{2}} = \frac{5^3 - 1/5^3}{5^3 + 1/5^3} = \frac{5^6 - 1}{5^6 + 1}$$

37. Show, by producing a counterexample, that $\ln(ab)$ is not equal to $(\ln a)(\ln b)$.

SOLUTION Take $a = \sqrt{e} = e^{1/2}$ and $b = e^2$. Then $ab = e^{5/2}$ so that

$$\ln(ab) = \ln(e^{5/2}) = \frac{5}{2}\ln e = \frac{5}{2}$$

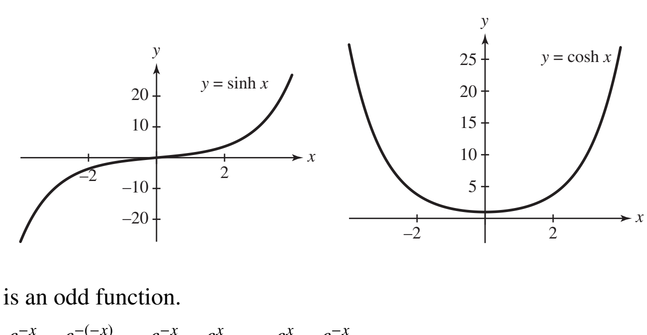
however,

$$(\ln a)(\ln b) = \ln(e^{1/2}) \cdot \ln(e^2) = \frac{1}{2}\ln e \cdot 2\ln e = \frac{1}{2}(2) = 1$$

Thus, $\ln(ab)$ is not equal to $(\ln a)(\ln b)$ in general.

38. For which values of x are $y = \sinh x$ and $y = \cosh x$ increasing and decreasing?

SOLUTION The graph of $y = \sinh x$ is shown below on the left. From this graph, we see that $\sinh x$ is increasing for all x. On the other hand, from the graph of $y = \cosh x$ shown below on the right, we see that $\cosh x$ is decreasing for $x < 0$ and is increasing for $x > 0$.



39. Show that $y = \tanh x$ is an odd function.

SOLUTION $\tanh(-x) = \dfrac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = \dfrac{e^{-x} - e^x}{e^{-x} + e^x} = -\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = -\tanh x$

40. The population of Integraton (in millions) at time t (years) is $P(t) = 2.4e^{0.06t}$, where $t = 0$ is the year 2000.

(a) What is the population at time $t = 0$?

(b) When will the population double from its size at $t = 0$?

SOLUTION

(a) The initial population is $P(0) = 2.4e^{0.06(0)} = 2.4e^0 = 2.4$ million.

(b) The population doubles when $4.8 = 2.4e^{0.06t}$. Thus, $0.06t = \ln 2$ or $t = \frac{\ln 2}{0.06} \approx 11.55$ years.

41. The decibel level for the intensity of a sound is a logarithmic scale defined by $D = 10\log_{10} I + 120$, where I is the intensity of the sound in watts per square meter.

(a) Express I as a function of D.

(b) Show that when D increases by 20, the intensity increases by a factor of 100.

SOLUTION

(a) Solving $D = 10\log_{10} I + 120$ for I yields

$$\log_{10} I = \frac{D - 120}{10} \quad \text{or} \quad I = 10^{\frac{D-120}{10}}$$

(b) Using the result from part (a), we see that

$$I(D+20) = 10^{\frac{D+20-120}{10}} = 10^{2+\frac{D-20}{10}} = 10^2 10^{\frac{D-120}{10}} = 100I(D)$$

Thus, when D increases by 20, the intensity increases by a factor of 100.

42. Consider the equation $M_w = \frac{2}{3}\log_{10} E - 10.7$ relating the moment magnitude of an earthquake and the energy E (in ergs) released by it.

(a) Express E as a function of M_w.

(b) Show that when M_w increases by 1, the energy increases by a factor of approximately 31.6.

SOLUTION

(a) Solving $M_w = \frac{2}{3}\log_{10} E - 10.7$ for E yields

$$\log_{10} E = \frac{3}{2}(M_w + 10.7) \quad \text{or} \quad E = 10^{\frac{3}{2}(M_w+10.7)}$$

(b) Using the result from part (a), we see that

$$E(M_w + 1) = 10^{\frac{3}{2}(M_w+1+10.7)} = 10^{\frac{3}{2}(M_w+10.7)} = 10^{\frac{3}{2}} 10^{\frac{3}{2}(M_w+10.7)} = 10^{\frac{3}{2}} E(M_w)$$

Thus, when M_w increases by 1, the energy increases by a factor of $10^{\frac{3}{2}} \approx 31.6$.

43. Refer to the graphs to explain why the equation $\sinh x = t$ has a unique solution for every t and why $\cosh x = t$ has two solutions for every $t > 1$.

SOLUTION From its graph, we see that $\sinh x$ is a one-to-one function with $\lim_{x\to-\infty} \sinh x = -\infty$ and $\lim_{x\to\infty} \sinh x = \infty$. Thus, for every real number t, the equation $\sinh x = t$ has a unique solution. On the other hand, from its graph, we see that $\cosh x$ is not one-to-one. Rather, it is an even function with a minimum value of $\cosh 0 = 1$. Thus, for every $t > 1$, the equation $\cosh x = t$ has two solutions: one positive, the other negative.

44. Compute $\cosh x$ and $\tanh x$, assuming that $\sinh x = 0.8$.

SOLUTION Using the identity $\cosh^2 x - \sinh^2 x = 1$, it follows that $\cosh^2 x - (\frac{4}{5})^2 = 1$, so that $\cosh^2 x = \frac{41}{25}$ and

$$\cosh x = \frac{\sqrt{41}}{5}$$

Then, by definition,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{4}{5}}{\frac{\sqrt{41}}{5}} = \frac{4}{\sqrt{41}}$$

45. Prove the addition formula for $\cosh x$ given by $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

SOLUTION

$$\begin{aligned}
\cosh(x+y) &= \frac{e^{x+y} + e^{-(x+y)}}{2} = \frac{2e^{x+y} + 2e^{-(x+y)}}{4} \\
&= \frac{e^{x+y} + e^{-x+y} + e^{x-y} + e^{-(x+y)}}{4} + \frac{e^{x+y} - e^{-x+y} - e^{x-y} + e^{-(x+y)}}{4} \\
&= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \cosh x \cosh y + \sinh x \sinh y
\end{aligned}$$

46. Use the addition formulas to prove

$$\sinh(2x) = 2\cosh x \sinh x$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

SOLUTION $\sinh(2x) = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2\cosh x \sinh x$ and $\cosh(2x) = \cosh(x + x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x$

47. A train moves along a track at velocity v. Bionica walks down the aisle of the train with velocity u in the direction of the train's motion. Compute the velocity w of Bionica relative to the ground using the laws of both Galileo and Einstein in the following cases:

(a) $v = 500$ m/s and $u = 10$ m/s. Is your calculator accurate enough to detect the difference between the two laws?

(b) $v = 10^7$ m/s and $u = 10^6$ m/s.

SOLUTION The speed of light is $c \approx 3 \times 10^8$ m/s.

(a) By Galileo's law, $w = 500 + 10 = 510$ m/s. Using Einstein's law and a calculator,

$$\tanh^{-1}\frac{w}{c} = \tanh^{-1}\frac{500}{c} + \tanh^{-1}\frac{10}{c} = 1.7 \times 10^{-6}$$

so $w = c \cdot \tanh(1.7 \times 10^{-6}) \approx 510$ m/s. No, the calculator was not accurate enough to detect the difference between the two laws.

(b) By Galileo's law, $u + v = 10^7 + 10^6 = 1.1 \times 10^7$ m/s. By Einstein's law,

$$\tanh^{-1}\frac{w}{c} = \tanh^{-1}\frac{10^7}{3 \times 10^8} + \tanh^{-1}\frac{10^6}{3 \times 10^8} \approx 0.036679$$

so $w \approx 3 \times 10^8 \tanh(0.036679) \approx 1.09988 \times 10^7$ m/s.

Further Insights and Challenges

48. Show that $\log_a b \log_b a = 1$ for all $a, b > 0$ such that $a \neq 1$ and $b \neq 1$.

SOLUTION $\log_a b = \frac{\ln b}{\ln a}$ and $\log_b a = \frac{\ln a}{\ln b}$; thus, $\log_a b \cdot \log_b a = \frac{\ln b}{\ln a} \cdot \frac{\ln a}{\ln b} = 1$

49. Verify that for all x, the change-of-base formula holds:
$\log_b x = \frac{\log_a x}{\log_a b}$ for $a, b > 0$ such that $a \neq 1$, $b \neq 1$.

SOLUTION Let $y = \log_b x$. Then $x = b^y$ and $\log_a x = \log_a b^y = y \log_a b$. Thus, $y = \frac{\log_a x}{\log_a b}$.

50. Verify that for all x, the change-of-base formula holds: $b^x = a^{x \log_a b}$ for $a, b > 0$ such that $a \neq 1$, $b \neq 1$.

SOLUTION By the inverse relationship between the exponential function base a and the logarithmic function base a, we know that $b = a^{\log_a b}$. Raising both sides of this expression to the power x yields

$$b^x = (a^{\log_a b})^x = a^{x \log_a b}$$

51. Use the addition formulas for $\sinh x$ and $\cosh x$ to prove

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

SOLUTION

$$\begin{aligned}\tanh(u + v) &= \frac{\sinh(u + v)}{\cosh(u + v)} = \frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v} \\ &= \frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v} \cdot \frac{1/(\cosh u \cosh v)}{1/(\cosh u \cosh v)} = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}\end{aligned}$$

52. Use Exercise 51 to show that Einstein's Law of Velocity Addition [Eq. (3)] is equivalent to

$$w = \frac{u + v}{1 + \frac{uv}{c^2}}$$

SOLUTION Einstein's law states: $\tanh^{-1}(w/c) = \tanh^{-1}(u/c) + \tanh^{-1}(v/c)$. Thus,

$$\frac{w}{c} = \tanh\left(\tanh^{-1}(u/c) + \tanh^{-1}(v/c)\right) = \frac{\tanh(\tanh^{-1}(v/c)) + \tanh(\tanh^{-1}(u/c))}{1 + \tanh(\tanh^{-1}(v/c))\tanh(\tanh^{-1}(u/c))}$$

$$= \frac{\frac{v}{c} + \frac{u}{c}}{1 + \frac{v}{c}\cdot\frac{u}{c}} = \frac{(1/c)(u+v)}{1 + \frac{uv}{c^2}}$$

Hence,

$$w = \frac{u+v}{1 + \frac{uv}{c^2}}$$

53. Prove that every function f can be written as a sum $f(x) = f_+(x) + f_-(x)$ of an even function $f_+(x)$ and an odd function $f_-(x)$. Express $f(x) = 5e^x + 8e^{-x}$ in terms of $\cosh x$ and $\sinh x$. *Hint:* $y = f(x) + f(-x)$ is an even function, and $y = f(x) - f(-x)$ is an odd function.

SOLUTION Let $f_+(x) = \frac{f(x)+f(-x)}{2}$ and $f_-(x) = \frac{f(x)-f(-x)}{2}$. Then $f_+ + f_- = \frac{2f(x)}{2} = f(x)$. Moreover,

$$f_+(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_+(x)$$

so $f_+(x)$ is an even function, while

$$f_-(-x) = \frac{f(-x) - f(-(-x))}{2}$$
$$= \frac{f(-x) - f(x)}{2} = -\left(\frac{f(x) - f(-x)}{2}\right) = -f_-(x)$$

so $f_-(x)$ is an odd function.

For $f(x) = 5e^x + 8e^{-x}$, we have

$$f_+(x) = \frac{5e^x + 8e^{-x} + 5e^{-x} + 8e^x}{2} = 8\cosh x + 5\cosh x = 13\cosh x$$

and

$$f_-(x) = \frac{5e^x + 8e^{-x} - 5e^{-x} - 8e^x}{2} = 5\sinh x - 8\sinh x = -3\sinh x$$

Therefore, $f(x) = f_+(x) + f_-(x) = 13\cosh x - 3\sinh x$.

1.7 Technology: Calculators and Computers

Preliminary Questions

1. Is there a definite way of choosing the optimal viewing rectangle, or is it best to experiment until you find a viewing rectangle appropriate to the problem at hand?

SOLUTION It is best to experiment with the window size until one is found that is appropriate for the problem at hand.

2. Describe the calculator screen produced when the function $y = 3 + x^2$ is plotted with a viewing rectangle:

(a) $[-1, 1] \times [0, 2]$ **(b)** $[0, 1] \times [0, 4]$

SOLUTION

(a) Using the viewing rectangle $[-1, 1]$ by $[0, 2]$, the screen will display nothing, as the minimum value of $y = 3 + x^2$ is $y = 3$.

(b) Using the viewing rectangle $[0, 1]$ by $[0, 4]$, the screen will display the portion of the parabola between the points $(0, 3)$ and $(1, 4)$.

3. According to the evidence in Example 4, it appears that $f(n) = (1 + 1/n)^n$ never takes on a value greater than 3 for $n > 0$. Does this evidence *prove* that $f(n) \le 3$ for $n > 0$?

SOLUTION No, this evidence does not constitute a proof that $f(n) \le 3$ for $n \ge 0$.

4. How can a graphing calculator be used to find the minimum value of a function?

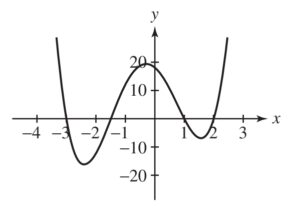
SOLUTION Experiment with the viewing window to zoom in on the lowest point on the graph of the function. The y-coordinate of the lowest point on the graph is the minimum value of the function.

Exercises

The exercises in this section should be done using a graphing calculator or computer algebra system.

1. Plot $f(x) = 2x^4 + 3x^3 - 14x^2 - 9x + 18$ in the appropriate viewing rectangles and determine its roots.

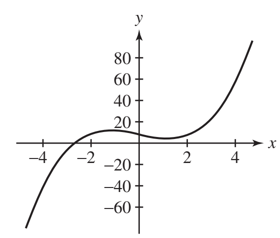
SOLUTION Using a viewing rectangle of $[-4, 3]$ by $[-20, 20]$, we obtain the plot shown here.



Now, the roots of $f(x)$ are the x-intercepts of the graph of $y = f(x)$. From the plot, we can identify the x-intercepts as -3, -1.5, 1, and 2. The roots of $f(x)$ are therefore $x = -3$, $x = -1.5$, $x = 1$, and $x = 2$.

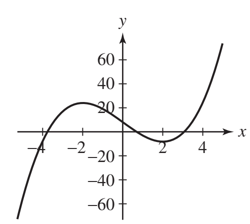
2. How many solutions does $x^3 - 4x + 8 = 0$ have?

SOLUTION Solutions to the equation $x^3 - 4x + 8 = 0$ are the x-intercepts of the graph of $y = x^3 - 4x + 8$. From the figure shown here, we see that the graph has one x-intercept (between $x = -4$ and $x = -2$), so the equation has one solution.



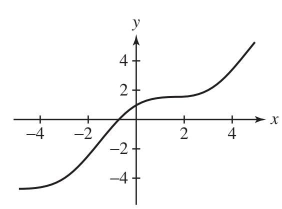
3. How many *positive* solutions does $x^3 - 12x + 8 = 0$ have?

SOLUTION The graph of $y = x^3 - 12x + 8$ shown here has two x-intercepts to the right of the origin; therefore, the equation $x^3 - 12x + 8 = 0$ has two positive solutions.



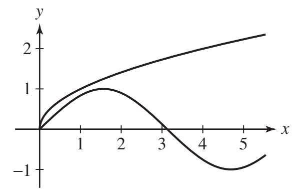
4. Does $\cos x + x = 0$ have a solution? A positive solution?

SOLUTION The graph of $y = \cos x + x$ shown here has one x-intercept; therefore, the equation $\cos x + x = 0$ has one solution. The lone x-intercept is to the left of the origin, so the equation has no positive solutions.



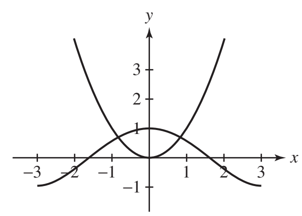
5. Find all the solutions of $\sin x = \sqrt{x}$ for $x > 0$.

SOLUTION Solutions to the equation $\sin x = \sqrt{x}$ correspond to points of intersection between the graphs of $y = \sin x$ and $y = \sqrt{x}$. The two graphs are shown here; the only point of intersection is at $x = 0$. Therefore, there are no solutions of $\sin x = \sqrt{x}$ for $x > 0$.



6. How many solutions does $\cos x = x^2$ have?

SOLUTION Solutions to the equation $\cos x = x^2$ correspond to points of intersection between the graphs of $y = \cos x$ and $y = x^2$. The two graphs are shown here; there are two points of intersection. Thus, the equation $\cos x = x^2$ has two solutions.

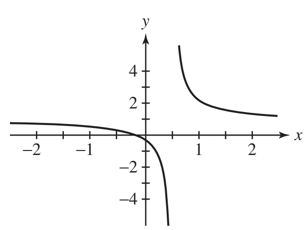


7. Let $f(x) = (x - 100)^2 + 1000$. What will the display show if you graph f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$? Find an appropriate viewing rectangle.

SOLUTION Because $(x - 100)^2 \geq 0$ for all x, it follows that $f(x) = (x - 100)^2 + 1000 \geq 1000$ for all x. Thus, a viewing rectangle of $[-10, 10]$ by $[-10, 10]$ will display nothing. The minimum value of the function occurs when $x = 100$, so an appropriate viewing rectangle would be $[50, 150]$ by $[1000, 2000]$.

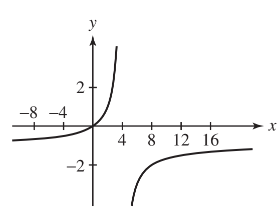
8. Plot $f(x) = \frac{8x+1}{8x-4}$ in an appropriate viewing rectangle. What are the vertical and horizontal asymptotes?

SOLUTION From the graph of $y = \frac{8x+1}{8x-4}$ shown here, we see that the vertical asymptote is $x = \frac{1}{2}$ and the horizontal asymptote is $y = 1$.



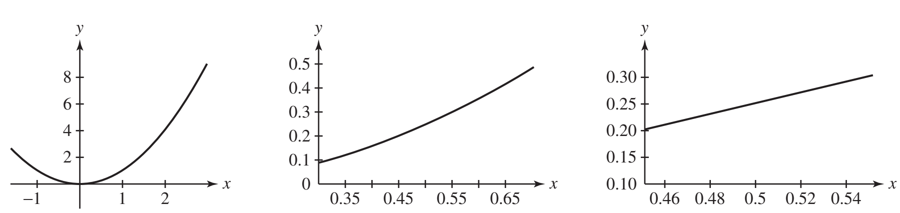
9. Plot the graph of $f(x) = x/(4 - x)$ in a viewing rectangle that clearly displays the vertical and horizontal asymptotes.

SOLUTION From the graph of $y = \frac{x}{4-x}$ shown here, we see that the vertical asymptote is $x = 4$ and the horizontal asymptote is $y = -1$.



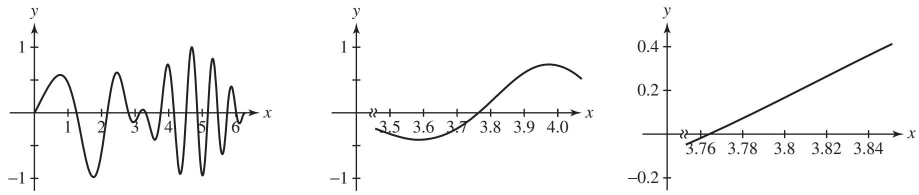
10. Illustrate local linearity for $f(x) = x^2$ by zooming in on the graph at $x = 0.5$ (see Example 6).

SOLUTION The following three graphs display $f(x) = x^2$ over the intervals $[-1, 3]$, $[0.3, 0.7]$, and $[0.45, 0.55]$. The final graph looks like a straight line.



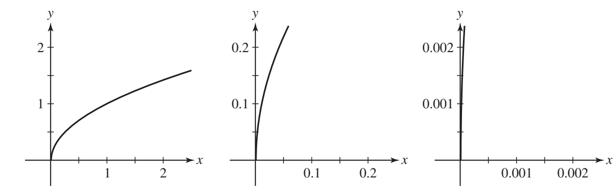
11. Plot $f(x) = \cos(x^2)\sin x$ for $0 \le x \le 2\pi$. Then illustrate local linearity at $x = 3.8$ by choosing appropriate viewing rectangles.

SOLUTION The following three graphs display $f(x) = \cos(x^2)\sin x$ over the intervals $[0, 2\pi]$, $[3.5, 4.1]$, and $[3.75, 3.85]$. The final graph looks like a straight line.



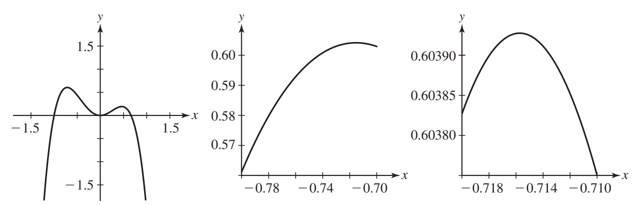
12. By zooming in on the graph of $f(x) = \sqrt{x}$ at $x = 0$ examine the local linearity. How does the resulting "line" appear?

SOLUTION The following three graphs display $f(x) = \sqrt{x}$ on the rectangles $[-0.1, 0.1] \times [-0.5, 0.5]$, $[-0.01, 0.01] \times [-0.05, 0.05]$, and $[-0.001, 0.001] \times [-0.005, 0.005]$. The final graph looks like a half-life nearly coinciding with the non-negative y-axis.



13. By examining the graph of $f(x) = 2x^2 - x^3 - 3x^4$ in appropriate viewing rectangles, approximate the maximum value of $f(x)$ and the value of x at which it occurs.

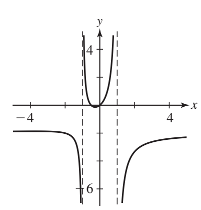
SOLUTION The following three graphs display $f(x) = 2x^2 - x^3 - 3x^4$ over the intervals $[-1.5, 1.5]$, $[-0.8, -0.7]$, and $[-0.72, -0.71]$. From the final graph, we see that the maximum value of $f(x)$ is approximately 0.60393, and this value occurs for $x \approx -0.716$.



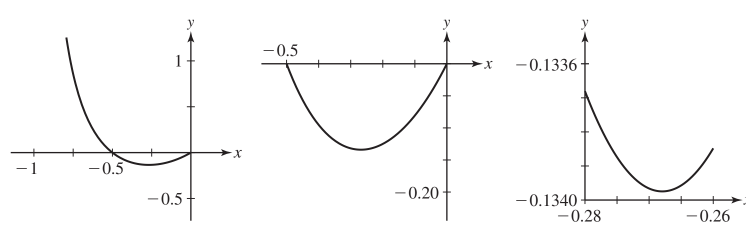
14. (a) Plot the graph of $f(x) = \frac{2x^2+x}{1-x^2}$ in a viewing rectangle that clearly shows the two vertical asymptotes.
(b) By examining the graph of f in appropriate viewing rectangles, approximate the minimum value of $f(x)$ between the vertical asymptotes, and approximate the value of x at which the minimum occurs.

SOLUTION

(a) A plot of the graph of f over the interval $[-3, 3]$, as given here, clearly shows the vertical asymptotes at $x = -1$ and $x = 1$.

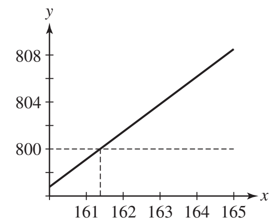


(b) The following three graphs display $f(x) = \frac{2x^2+x}{1-x^2}$ over the intervals $[-0.8, 0]$, $[-0.4, -0.2]$, and $[-0.3, -0.25]$. From the final graph, we see that the minimum value of $f(x)$ is approximately -0.134, and this value occurs for $x \approx -0.268$.



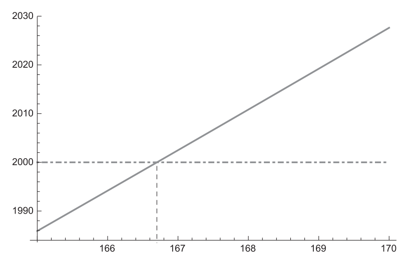
15. If \$500 is deposited in a bank account paying 3.5% interest compounded monthly, then the account has value $V(N) = 500\left(1 + \frac{0.035}{12}\right)^N$ dollars after N months. By examining the graph of $V(N)$ find, to the nearest integer N, the number of months it takes for the account value to reach \$800.

SOLUTION Let $V(N) = 500\left(1 + \dfrac{0.035}{12}\right)^N$. To determine when the account value reaches \$800, we graph $y = V(N)$ and $y = 800$ to locate the point of intersection between the graphs. From the figure below, we see that, to the nearest integer, the account value will reach \$800 after 161 months.



16. If \$1000 is deposited in a bank account paying 5% interest compounded monthly, then the account has value $V(N) = 1000\left(1 + \frac{0.05}{12}\right)^N$ dollars after N months. By examining the graph of $V(N)$ find, to the nearest integer N, the number of months it takes for the account value to double.

SOLUTION Let $V(N) = 1000\left(1 + \dfrac{0.05}{12}\right)^N$. To determine when the account value doubles, we graph $y = V(N)$ and $y = 2000$ to locate the point of intersection between the graphs. From the figure below, we see that, to the nearest integer, the account value will double after 167 months.

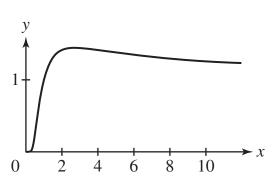


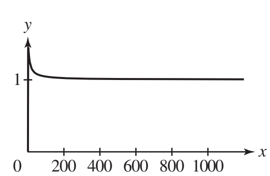
In Exercises 17–22, investigate the behavior of the function as n or x grows large by making a table of function values and plotting a graph (see Example 4). Describe the behavior in words.

17. $f(n) = n^{1/n}$

SOLUTION The table and graphs shown here suggest that as n gets large, $n^{1/n}$ approaches 1.

n	$n^{1/n}$
10	1.258925412
10^2	1.047128548
10^3	1.006931669
10^4	1.000921458
10^5	1.000115136
10^6	1.000013816

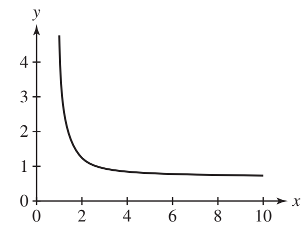


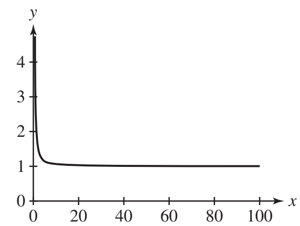


18. $f(n) = \frac{4n+1}{6n-5}$

SOLUTION The table and graphs shown here suggest that as n gets large, $\frac{4n+1}{6n-5}$ approaches $\frac{2}{3}$.

n	$\dfrac{4n+1}{6n-5}$
10	0.7454545455
10^2	0.6739495798
10^3	0.6673894912
10^4	0.6667388949
10^5	0.6666738889
10^6	0.6666673889

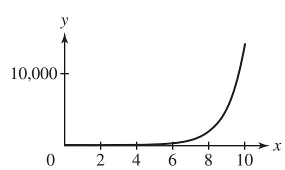


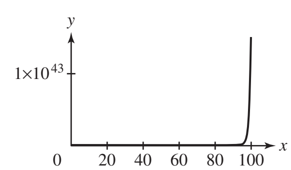


19. $f(n) = \left(1 + \frac{1}{n}\right)^{n^2}$

SOLUTION The table and graphs shown here suggest that as n gets large, $f(n)$ tends toward ∞.

n	$\left(1 + \dfrac{1}{n}\right)^{n^2}$
10	13780.61234
10^2	$1.635828711 \times 10^{43}$
10^3	$1.195306603 \times 10^{434}$
10^4	$5.341783312 \times 10^{4342}$
10^5	$1.702333054 \times 10^{43,429}$
10^6	$1.839738749 \times 10^{434,294}$

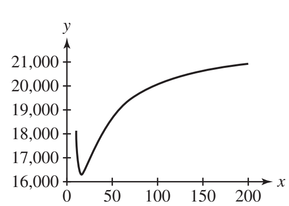


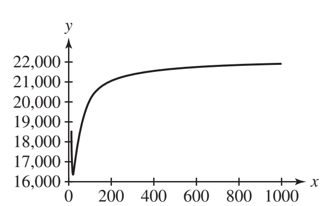


20. $f(x) = \left(\frac{x+6}{x-4}\right)^{x}$

SOLUTION The table and graphs shown here suggest that as x gets large, $f(x)$ roughly tends toward 22,026.

x	$\left(\dfrac{x+6}{x-4}\right)^{x}$
10	18,183.91210
10^2	20,112.36934
10^3	21,809.33633
10^4	22,004.43568
10^5	22,024.26311
10^6	22,025.36451
10^7	22,026.35566

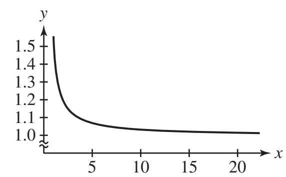


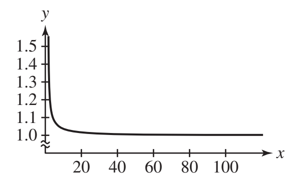


21. $f(x) = \left(x \tan \frac{1}{x}\right)^x$

SOLUTION The table and graphs shown here suggest that as x gets large, $f(x)$ approaches 1.

x	$\left(x \tan \dfrac{1}{x}\right)^x$
10	1.033975759
10^2	1.003338973
10^3	1.000333389
10^4	1.000033334
10^5	1.000003333
10^6	1.000000333

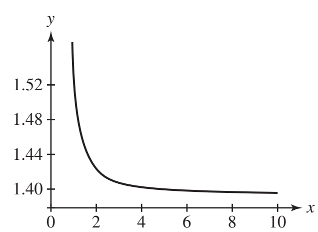


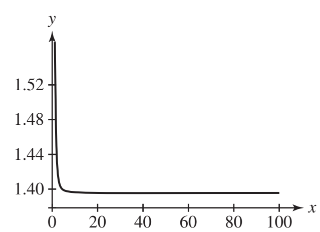


22. $f(x) = \left(x \tan \frac{1}{x}\right)^{x^2}$

SOLUTION The table and graphs shown here suggest that as x gets large, $f(x)$ approaches 1.39561.

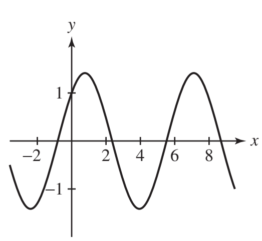
x	$\left(x \tan \dfrac{1}{x}\right)^{x^2}$
10	1.396701388
10^2	1.395623280
10^3	1.395612534
10^4	1.395612426
10^5	1.395612425
10^6	1.395612425



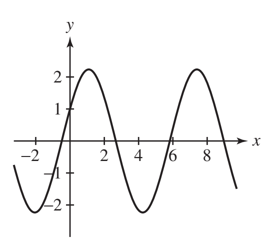


23. The graph of $f(\theta) = A\cos\theta + B\sin\theta$ is a sinusoidal wave for any constants A and B. Confirm this for $(A, B) = (1, 1)$, $(1, 2)$, and $(3, 4)$ by plotting f.

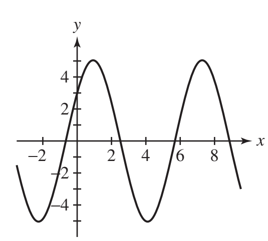
SOLUTION The graphs of $f(\theta) = \cos\theta + \sin\theta$, $f(\theta) = \cos\theta + 2\sin\theta$ and $f(\theta) = 3\cos\theta + 4\sin\theta$ are shown here.



$(A, B) = (1, 1)$



$(A, B) = (1, 2)$



$(A, B) = (3, 4)$

24. Find the maximum value of f for the graphs produced in Exercise 23. Can you guess the formula for the maximum value in terms of A and B?

SOLUTION For $A = 1$ and $B = 1$, max $\approx 1.4 \approx \sqrt{2}$

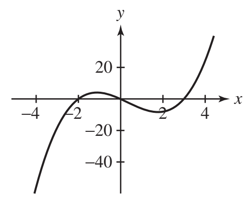
For $A = 1$ and $B = 2$, max $\approx 2.25 \approx \sqrt{5}$

For $A = 3$ and $B = 4$, max $\approx 5 = \sqrt{3^2 + 4^2}$

Max $= \sqrt{A^2 + B^2}$

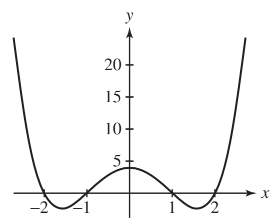
25. Find the intervals on which $f(x) = x(x+2)(x-3)$ is positive by plotting a graph.

SOLUTION The function $f(x) = x(x+2)(x-3)$ is positive when the graph of $y = x(x+2)(x-3)$ lies above the x-axis. The graph of $y = x(x+2)(x-3)$ is shown here. Clearly, the graph lies above the x-axis and the function is positive for $x \in (-2, 0) \cup (3, \infty)$.



26. Find the set of solutions to the inequality $(x^2-4)(x^2-1) < 0$ by plotting a graph.

SOLUTION To solve the inequality $(x^2-4)(x^2-1) < 0$, we can plot the graph of $y = (x^2-4)(x^2-1)$ and identify when the graph lies below the x-axis. The graph of $y = (x^2-4)(x^2-1)$ is shown here. The solution set for the inequality $(x^2-4)(x^2-1) < 0$ is clearly $x \in (-2, -1) \cup (1, 2)$.



Further Insights and Challenges

27. CAS Let $f_1(x) = x$ and define a sequence of functions by $f_{n+1}(x) = \frac{1}{2}(f_n(x) + x/f_n(x))$. For example, $f_2(x) = \frac{1}{2}(x+1)$. Use a computer algebra system to compute $f_n(x)$ for $n = 3, 4, 5$ and plot $y = f_n(x)$ together with $y = \sqrt{x}$ for $x \geq 0$. What do you notice?

SOLUTION With $f_1(x) = x$ and $f_2(x) = \frac{1}{2}(x+1)$, we calculate

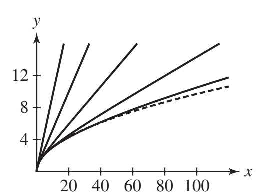
$$f_3(x) = \frac{1}{2}\left(\frac{1}{2}(x+1) + \frac{x}{\frac{1}{2}(x+1)}\right) = \frac{x^2 + 6x + 1}{4(x+1)}$$

$$f_4(x) = \frac{1}{2}\left(\frac{x^2+6x+1}{4(x+1)} + \frac{x}{\frac{x^2+6x+1}{4(x+1)}}\right) = \frac{x^4 + 28x^3 + 70x^2 + 28x + 1}{8(1+x)(1+6x+x^2)}$$

and

$$f_5(x) = \frac{1 + 120x + 1820x^2 + 8008x^3 + 12870x^4 + 8008x^5 + 1820x^6 + 120x^7 + x^8}{16(1+x)(1+6x+x^2)(1+28x+70x^2+28x^3+x^4)}$$

A plot of $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, $f_5(x)$, and $\sqrt{x}$ is shown here, with the graph of $\sqrt{x}$ shown as a dashed curve. It seems as if the f_n are asymptotic to $\sqrt{x}$.



28. Set $P_0(x) = 1$ and $P_1(x) = x$. The **Chebyshev polynomials** (useful in approximation theory) are defined inductively by the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$.

(a) Show that $P_2(x) = 2x^2 - 1$.

(b) Compute $P_n(x)$ for $3 \leq n \leq 6$ using a computer algebra system or by hand, and plot $y = P_n(x)$ over $[-1, 1]$.

(c) Check that your plots confirm two interesting properties: (A) $y = P_n(x)$ has n real roots in $[-1, 1]$, and (B) for $x \in [-1, 1]$, $P_n(x)$ lies between -1 and 1.

SOLUTION

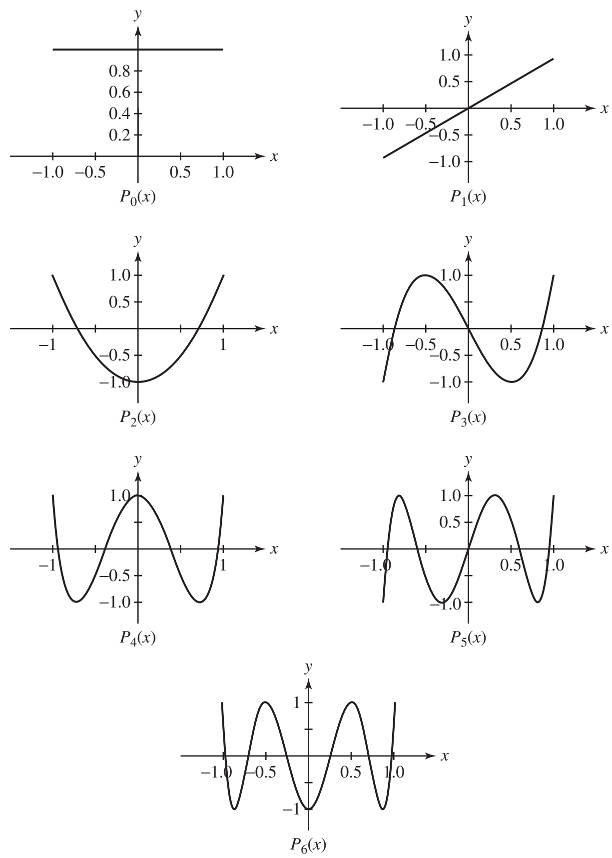
(a) With $P_0(x) = 1$ and $P_1(x) = x$, we calculate

$$P_2(x) = 2x(P_1(x)) - P_0(x) = 2x(x) - 1 = 2x^2 - 1$$

(b) Using the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$ with $n = 2, 3, 4$, and 5, we find

$$P_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$$
$$P_4(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1$$
$$P_5(x) = 16x^5 - 20x^3 + 5x$$
$$P_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

The graphs of the functions $P_n(x)$ for $0 \le n \le 6$ are shown here.



(c) From the graphs shown in part (b), it is clear that for each n, the polynomial $P_n(x)$ has precisely n roots on the interval $[-1, 1]$ and that $-1 \le P_n(x) \le 1$ for $x \in [-1, 1]$.

CHAPTER REVIEW EXERCISES

1. Match each quantity (a)–(d) with (i), (ii), or (iii) if possible, or state that no match exists.

(a) $2^a 3^b$

(b) $\frac{2^a}{3^b}$

(c) $(2^a)^b$

(d) $2^{a-b}3^{b-a}$

(i) 2^{ab}

(ii) 6^{a+b}

(iii) $(\frac{2}{3})^{a-b}$

SOLUTION

(a) No match

(b) No match

(c) **(i)**: $(2^a)^b = 2^{ab}$

(d) **(iii)**: $2^{a-b}3^{b-a} = 2^{a-b}\left(\frac{1}{3}\right)^{a-b} = \left(\frac{2}{3}\right)^{a-b}$

2. Indicate which of the following are correct, and correct the ones that are not.

(a) $5^2 \cdot 5^{1/2} = 5$ **(b)** $(\sqrt{8})^{4/3} = 8^{2/3}$

(c) $\frac{3^8}{3^4} = 3^2$ **(d)** $(2^4)^{-2} = 2^2$

SOLUTION

(a) Incorrect; $5^2 \cdot 5^{1/2} = 5^{2+1/2} = 5^{5/2}$

(b) Correct

(c) Incorrect; $\frac{3^8}{3^4} = 3^{8-4} = 3^4$

(d) Incorrect: $(2^4)^{-2} = 2^{4(-2)} = 2^{-8}$

3. Express $(4, 10)$ as a set $\{x : |x - a| < c\}$ for suitable a and c.

SOLUTION The center of the interval $(4, 10)$ is $\frac{4+10}{2} = 7$ and the radius is $\frac{10-4}{2} = 3$. Therefore, the interval $(4, 10)$ is equivalent to the set $\{x : |x - 7| < 3\}$.

4. Express as an interval:

(a) $\{x : |x - 5| < 4\}$ **(b)** $\{x : |5x + 3| \leq 2\}$

SOLUTION

(a) Upon dropping the absolute value, the inequality $|x - 5| < 4$ becomes $-4 < x - 5 < 4$ or $1 < x < 9$. The set $\{x : |x - 5| < 4\}$ can therefore be expressed as the interval $(1, 9)$.

(b) Upon dropping the absolute value, the inequality $|5x + 3| \leq 2$ becomes $-2 \leq 5x + 3 \leq 2$ or $-1 \leq x \leq -\frac{1}{5}$. The set $\{x : |5x + 3| \leq 2\}$ can therefore be expressed as the interval $[-1, -\frac{1}{5}]$.

5. Express $\{x : 2 \leq |x - 1| \leq 6\}$ as a union of two intervals.

SOLUTION The set $\{x : 2 \leq |x - 1| \leq 6\}$ consists of those numbers that are at least 2 but at most 6 units from 1. The numbers larger than 1 that satisfy these conditions are $3 \leq x \leq 7$, while the numbers smaller than 1 that satisfy these conditions are $-5 \leq x \leq -1$. Therefore, $\{x : 2 \leq |x - 1| \leq 6\} = [-5, -1] \cup [3, 7]$.

6. Give an example of numbers x, y such that $|x| + |y| = x - y$.

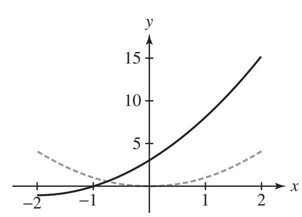
SOLUTION Let $x = 3$ and $y = -1$. Then $|x| + |y| = 3 + 1 = 4$ and $x - y = 3 - (-1) = 4$.

7. Describe the pairs of numbers x, y such that $|x + y| = x - y$.

SOLUTION First, consider the case when $x + y \geq 0$. Then $|x + y| = x + y$ and we obtain the equation $x + y = x - y$. The solution of this equation is $y = 0$. Thus, the pairs $(x, 0)$ with $x \geq 0$ satisfy $|x + y| = x - y$. Next, consider the case when $x + y < 0$. Then $|x + y| = -(x + y) = -x - y$ and we obtain the equation $-x - y = x - y$. The solution of this equation is $x = 0$. Thus, the pairs $(0, y)$ with $y < 0$ also satisfy $|x + y| = x - y$.

8. Sketch the graph of $y = f(x + 2) - 1$, where $f(x) = x^2$ for $-2 \leq x \leq 2$.

SOLUTION The graph of $y = f(x + 2) - 1$ is obtained by shifting the graph of $y = f(x)$ 2 units to the left and 1 unit down. In the figure shown here, the graph of $y = f(x)$ is shown as the dashed curve, and the graph of $y = f(x + 2) - 1$ is shown as the solid curve.



In Exercises 9–12, let $f(x)$ be the function shown in Figure 1.

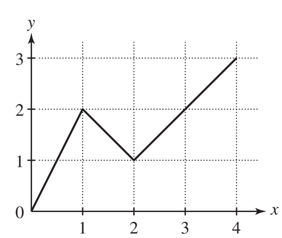
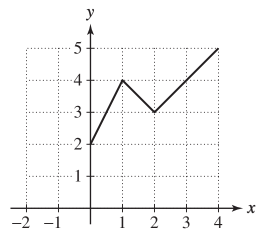


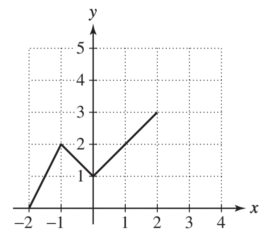
FIGURE 1

9. Sketch the graphs of $y = f(x) + 2$ and $y = f(x + 2)$.

SOLUTION The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ up 2 units (see the graph below at the left). The graph of $y = f(x + 2)$ is obtained by shifting the graph of $y = f(x)$ to the left 2 units (see the graph below at the right).



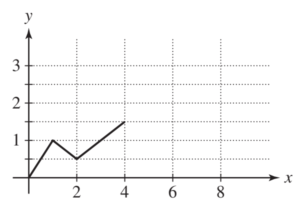
$f(x) + 2$



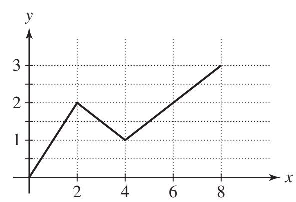
$f(x + 2)$

10. Sketch the graphs of $y = \frac{1}{2}f(x)$ and $y = f(\frac{1}{2}x)$.

SOLUTION The graph of $y = \frac{1}{2}f(x)$ is obtained by compressing the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below at the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below at the right).



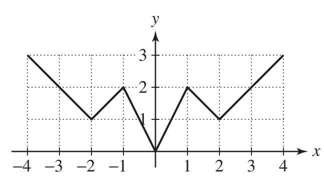
$f(x)/2$



$f(x/2)$

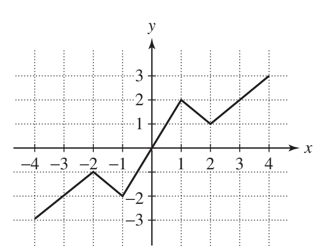
11. Continue the graph of f to the interval $[-4, 4]$ as an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph).



12. Continue the graph of f to the interval $[-4, 4]$ as an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph).



In Exercises 13–16, find the domain and range of the function.

13. $f(x) = \sqrt{x + 1}$

SOLUTION The domain of the function $f(x) = \sqrt{x + 1}$ is $\{x : x \geq -1\}$ and the range is $\{y : y \geq 0\}$.

14. $f(x) = \frac{4}{x^4 + 1}$

SOLUTION The domain of the function $f(x) = \frac{4}{x^4+1}$ is the set of all real numbers and the range is $\{y : 0 < y \leq 4\}$.

15. $f(x) = \frac{2}{3 - x}$

SOLUTION The domain of the function $f(x) = \frac{2}{3-x}$ is $\{x : x \neq 3\}$ and the range is $\{y : y \neq 0\}$.

16. $f(x) = \sqrt{x^2 - x + 5}$

SOLUTION Because

$$x^2 - x + 5 = \left(x^2 - x + \frac{1}{4}\right) + 5 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{19}{4}$$

$x^2 - x + 5 \geq \frac{19}{4}$ for all x. It follows that the domain of the function $f(x) = \sqrt{x^2 - x + 5}$ is all real numbers and the range is $\{y : y \geq \sqrt{19}/2\}$.

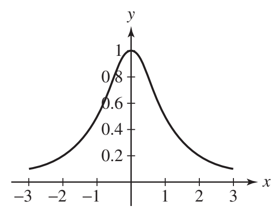
17. Determine whether the function is increasing, decreasing, or neither:

(a) $f(x) = 3^{-x}$
(b) $f(x) = \frac{1}{x^2+1}$
(c) $g(t) = t^2 + t$
(d) $g(t) = t^3 + t$

SOLUTION

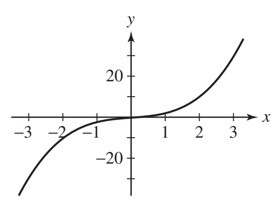
(a) The function $f(x) = 3^{-x}$ can be rewritten as $f(x) = (\frac{1}{3})^x$. This is an exponential function with a base less than 1; therefore, this is a decreasing function.

(b) From the graph of $y = 1/(x^2 + 1)$ shown here, we see that this function is neither increasing nor decreasing for all x (though it is increasing for $x < 0$ and decreasing for $x > 0$).



(c) The graph of $y = t^2 + t$ is an upward opening parabola; therefore, this function is neither increasing nor decreasing for all t. By completing the square we find $y = (t + \frac{1}{2})^2 - \frac{1}{4}$. The vertex of this parabola is then at $t = -\frac{1}{2}$, so the function is decreasing for $t < -\frac{1}{2}$ and increasing for $t > -\frac{1}{2}$.

(d) From the graph of $y = t^3 + t$ shown here, we see that this is an increasing function.



18. Determine whether the function is even, odd, or neither:

(a) $f(x) = x^4 - 3x^2$
(b) $g(x) = \sin(x + 1)$
(c) $f(x) = 2^{-x^2}$

SOLUTION

(a) $f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2 = f(x)$, so this function is even.

(b) $g(-x) = \sin(-x + 1)$, which is neither equal to $g(x)$ nor to $-g(x)$, so this function is neither even nor odd.

(c) $f(-x) = 2^{-(-x)^2} = 2^{-x^2} = f(x)$, so this function is even.

In Exercises 19–26, find the equation of the line.

19. Line passing through $(-1, 4)$ and $(2, 6)$

SOLUTION The slope of the line passing through $(-1, 4)$ and $(2, 6)$ is

$$m = \frac{6 - 4}{2 - (-1)} = \frac{2}{3}$$

The equation of the line passing through $(-1, 4)$ and $(2, 6)$ is therefore $y - 4 = \frac{2}{3}(x + 1)$ or $2x - 3y = -14$.

20. Line passing through $(-1, 4)$ and $(-1, 6)$

SOLUTION The line passing through $(-1, 4)$ and $(-1, 6)$ is vertical with an x-coordinate of -1. Therefore, the equation of the line is $x = -1$.

21. Line of slope 6 through $(9, 1)$

SOLUTION Using the point-slope form for the equation of a line, the equation of the line of slope 6 and passing through $(9, 1)$ is $y - 1 = 6(x - 9)$ or $6x - y = 53$.

22. Line of slope $-\frac{3}{2}$ through $(4, -12)$

SOLUTION Using the point-slope form for the equation of a line, the equation of the line of slope $-\frac{3}{2}$ and passing through $(4, -12)$ is $y + 12 = -\frac{3}{2}(x - 4)$ or $3x + 2y = -12$.

23. Line through $(2, 1)$ perpendicular to the line given by $y = 3x + 7$

SOLUTION The equation $y = 3x + 7$ is in slope-intercept form; it follows that the slope of this line is 3. Perpendicular lines have slopes that are negative reciprocals of one another, so we are looking for the equation of the line of slope $-\frac{1}{3}$ and passing through $(2, 1)$. The equation of this line is $y - 1 = -\frac{1}{3}(x - 2)$ or $x + 3y = 5$.

24. Line through $(3, 4)$ perpendicular to the line given by $y = 4x - 2$

SOLUTION The equation $y = 4x - 2$ is in slope-intercept form; it follows that the slope of this line is 4. Perpendicular lines have slopes that are negative reciprocals of one another, so we are looking for the equation of the line of slope $-\frac{1}{4}$ and passing through $(3, 4)$. The equation of this line is $y - 4 = -\frac{1}{4}(x - 3)$ or $x + 4y = 19$.

25. Line through $(2, 3)$ parallel to $y = 4 - x$

SOLUTION The equation $y = 4 - x$ is in slope-intercept form; it follows that the slope of this line is -1. Any line parallel to $y = 4 - x$ will have the same slope, so we are looking for the equation of the line of slope -1 and passing through $(2, 3)$. The equation of this line is $y - 3 = -(x - 2)$ or $x + y = 5$.

26. Horizontal line through $(-3, 5)$

SOLUTION A horizontal line has a slope of 0; the equation of the specified line is therefore $y - 5 = 0(x + 3)$ or $y = 5$.

27. Does the following table of market data suggest a linear relationship between price and number of homes sold during a one-year period? Explain.

Price (thousands of \$)	180	195	220	240
No. of homes sold	127	118	103	91

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{118 - 127}{195 - 180} = -\frac{9}{15} = -\frac{3}{5}$$

while the second pair of data points yields a slope of

$$\frac{103 - 118}{220 - 195} = -\frac{15}{25} = -\frac{3}{5}$$

and the last pair of data points yields a slope of

$$\frac{91 - 103}{240 - 220} = -\frac{12}{20} = -\frac{3}{5}$$

Because all three slopes are equal, the data do suggest a linear relationship between price and the number of homes sold.

28. Does the following table of revenue data for a computer manufacturer suggest a linear relation between revenue and time? Explain.

Year	2005	2009	2011	2014
Revenue (billions of \$)	13	18	15	11

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{18 - 13}{2009 - 2005} = \frac{5}{4}$$

while the second pair of data points yields a slope of

$$\frac{15 - 18}{2011 - 2009} = -\frac{3}{2}$$

and the last pair of data points yields a slope of

$$\frac{11 - 15}{2014 - 2011} = -\frac{4}{3}$$

Because the three slopes are not equal, the data do not suggest a linear relationship between revenue and time.

29. Suppose that a cell phone plan that is offered at a price of P dollars per month attracts C customers, where $C(P)$ is a linear demand function for $\$100 \le P \le \$500$. Assume $C(100) = 1{,}000{,}000$ and $C(500) = 100{,}000$.

(a) Determine the demand function $C(P)$.

(b) What is the slope of the graph of $C(P)$? Describe what the slope represents.

(c) What is the decrease in the number of customers for each increase of \$100 in the price?

SOLUTION

(a) We first determine the slope of the line:

$$m = \frac{1000000 - 100000}{100 - 500} = \frac{900000}{-400} = -2250$$

Knowing that $C(100) = 1{,}000{,}000$, it follows that

$$C - 1000000 = -2250(P - 100), \quad \text{or} \quad C(P) = -2250P + 1225000$$

(b) The slope of the graph of $C(P)$ is -2250 customers/dollar. This tells us that for every \$1 increase in the monthly price, the number of customers decreases by 2250.

(c) Because the slope of the demand function is -2250 customers/dollar, a \$100 increase in price will lead to a decrease in the number of customers of $2250(100) = 225{,}000$ customers.

30. Suppose that Internet domain names are sold at a price of \$$P$ per month for $\$2 \le P \le \$100$. The number of customers C who buy the domain names is a linear function of the price. Assume that 10,000 customers buy a domain name when the price is \$2 per month, and 1000 customers buy when the price is \$100 per month.

(a) Determine the demand function $C(P)$.

(b) What is the slope of the graph of $C(P)$? Describe what the slope represents.

SOLUTION

(a) We first determine the slope of the line:

$$m = \frac{10000 - 1000}{2 - 100} = \frac{9000}{-98} = -\frac{4500}{49}$$

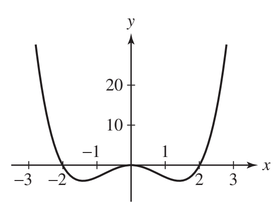
Knowing that $C(2) = 10{,}000$, it follows that

$$C - 10000 = -\frac{4500}{49}(P - 2), \quad \text{or} \quad C(P) = -\frac{4500}{49}P + \frac{499000}{49}$$

(b) The slope of the graph of $C(P)$ is $-\frac{4500}{49} \approx 91.84$ customers/dollar. This tells us that for every \$1 increase in the cost of domain names, the number of customers decreases by approximately 92.

31. Find the roots of $f(x) = x^4 - 4x^2$ and sketch its graph. On which intervals is f decreasing?

SOLUTION The roots of $f(x) = x^4 - 4x^2$ are obtained by solving the equation $x^4 - 4x^2 = x^2(x - 2)(x + 2) = 0$, which yields $x = -2$, $x = 0$ and $x = 2$. The graph of $y = f(x)$ is shown here. From this graph, we see that $f(x)$ is decreasing for x less than approximately -1.4 and for x between 0 and approximately 1.4.



32. Let $h(z) = -2z^2 + 12z + 3$. Complete the square and find the maximum value of h.

SOLUTION Let $h(z) = -2z^2 + 12z + 3$. Completing the square yields

$$h(z) = -2(z^2 - 6z) + 3 = -2(z^2 - 6z + 9) + 3 + 18 = -2(z - 3)^2 + 21$$

Because $(z - 3)^2 \ge 0$ for all z, it follows that $h(z) = -2(z - 3)^2 + 21 \le 21$ for all z. Thus, the maximum value of h is 21.

33. Let $f(x)$ be the square of the distance from the point $(2, 1)$ to a point $(x, 3x + 2)$ on the line $y = 3x + 2$. Show that f is a quadratic function, and find its minimum value by completing the square.

SOLUTION Let $f(x)$ denote the square of the distance from the point $(2, 1)$ to a point $(x, 3x + 2)$ on the line $y = 3x + 2$. Then

$$f(x) = (x - 2)^2 + (3x + 2 - 1)^2 = x^2 - 4x + 4 + 9x^2 + 6x + 1 = 10x^2 + 2x + 5$$

which is a quadratic function. Completing the square, we find

$$f(x) = 10\left(x^2 + \frac{1}{5}x + \frac{1}{100}\right) + 5 - \frac{1}{10} = 10\left(x + \frac{1}{10}\right)^2 + \frac{49}{10}$$

Because $(x + \frac{1}{10})^2 \geq 0$ for all x, it follows that $f(x) \geq \frac{49}{10}$ for all x. Hence, the minimum value of $f(x)$ is $\frac{49}{10}$.

34. Prove that $x^2 + 3x + 3 \geq 0$ for all x.

SOLUTION Observe that

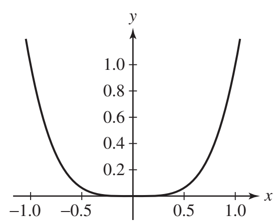
$$x^2 + 3x + 3 = \left(x^2 + 3x + \frac{9}{4}\right) + 3 - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 + \frac{3}{4}$$

Thus, $x^2 + 3x + 3 \geq \frac{3}{4} > 0$ for all x.

In Exercises 35–40, sketch the graph by hand.

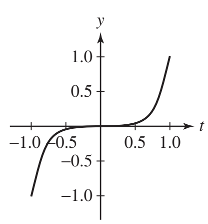
35. $y = t^4$

SOLUTION



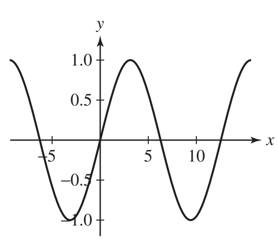
36. $y = t^5$

SOLUTION



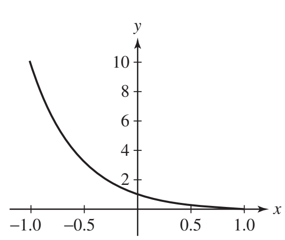
37. $y = \sin \frac{\theta}{2}$

SOLUTION



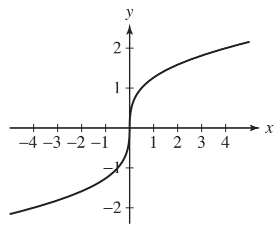
38. $y = 10^{-x}$

SOLUTION



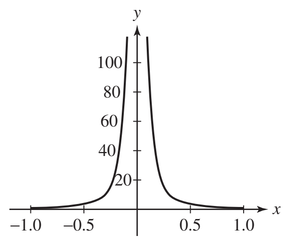
39. $y = x^{1/3}$

SOLUTION



40. $y = \frac{1}{x^2}$

SOLUTION



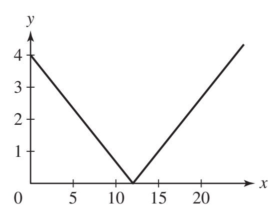
41. Show that the graph of $y = f(\frac{1}{3}x - b)$ is obtained by shifting the graph of $y = f(\frac{1}{3}x)$ to the right $3b$ units. Use this observation to sketch the graph of $y = \left|\frac{1}{3}x - 4\right|$.

SOLUTION Let $g(x) = f(\frac{1}{3}x)$. Then

$$g(x - 3b) = f\left(\frac{1}{3}(x - 3b)\right) = f\left(\frac{1}{3}x - b\right)$$

Thus, the graph of $y = f(\frac{1}{3}x - b)$ is obtained by shifting the graph of $y = f(\frac{1}{3}x)$ to the right $3b$ units.

The graph of $y = |\frac{1}{3}x - 4|$ is the graph of $y = |\frac{1}{3}x|$ shifted right 12 units (see the graph).



42. Let $h(x) = \cos x$ and $g(x) = x^{-1}$. Compute the composite functions $h \circ g$ and $g \circ h$, and find their domains.

SOLUTION Let $h(x) = \cos x$ and $g(x) = x^{-1}$. Then

$$h(g(x)) = h(x^{-1}) = \cos x^{-1}$$

The domain of this function is $x \neq 0$. On the other hand,

$$g(h(x)) = g(\cos x) = \frac{1}{\cos x} = \sec x$$

The domain of this function is

$$x \neq \frac{(2n + 1)\pi}{2} \text{ for any integer } n$$

43. Find functions f and g such that the function

$$f(g(t)) = (12t + 9)^4$$

SOLUTION One possible choice is $f(t) = t^4$ and $g(t) = 12t + 9$. Then

$$f(g(t)) = f(12t + 9) = (12t + 9)^4$$

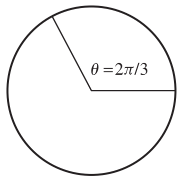
as desired.

44. Sketch the points on the unit circle corresponding to the following three angles, and find the values of the six standard trigonometric functions at each angle:

(a) $\frac{2\pi}{3}$ **(b)** $\frac{7\pi}{4}$ **(c)** $\frac{7\pi}{6}$

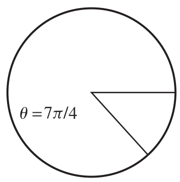
SOLUTION

(a)



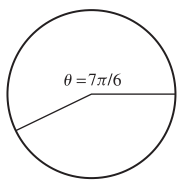
$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}$$
$$\tan\frac{2\pi}{3} = -\sqrt{3} \qquad \cot\frac{2\pi}{3} = -\frac{\sqrt{3}}{3}$$
$$\sec\frac{2\pi}{3} = -2 \qquad \csc\frac{2\pi}{3} = \frac{2\sqrt{3}}{3}$$

(b)



$$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$$
$$\tan\frac{7\pi}{4} = -1 \qquad \cot\frac{7\pi}{4} = -1$$
$$\sec\frac{7\pi}{4} = \sqrt{2} \qquad \csc\frac{7\pi}{4} = -\sqrt{2}$$

(c)



$$\sin\frac{7\pi}{6} = -\frac{1}{2} \qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3} \qquad \cot\frac{7\pi}{6} = \sqrt{3}$$
$$\sec\frac{7\pi}{6} = -\frac{2\sqrt{3}}{2} \qquad \csc\frac{7\pi}{6} = -2$$

45. What are the periods of these functions?

(a) $y = \sin 2\theta$ **(b)** $y = \sin\frac{\theta}{2}$

(c) $y = \sin 2\theta + \sin\frac{\theta}{2}$

SOLUTION

(a) π

(b) 4π

(c) The function $\sin 2\theta$ has a period of π, and the function $\sin(\theta/2)$ has a period of 4π. Because 4π is a multiple of π, the period of the function $g(\theta) = \sin 2\theta + \sin\theta/2$ is 4π.

46. Determine A, B, and C so that $f(x) = A\cos(Bx) + C$ cycles once from 8 to -2 and back to 8 as x goes from 0 to 2.

SOLUTION The amplitude of the motion is $\frac{8-(-2)}{2} = 5$ about a central value of $\frac{8+(-2)}{2} = 3$, so $A = 5$ and $C = 3$. For one cycle to be completed as x goes from 0 to 2, the period must be 2. Thus, $\frac{2\pi}{B} = 2$, or $B = \pi$.

47. $H(t) = A\sin(Bt) + C$ models the height (in meters) of the tide in Happy Harbor at time t (hours since midnight) in a day. Determine A, B, and C if the high tide of 18 m occurs at 6:00 A.M. and the subsequent low tide of 15 m occurs at 6:00 P.M.

SOLUTION The amplitude of the motion is $\frac{18-15}{2} = 1.5$ about a central value of $\frac{18+15}{2} = 16.5$, so $A = 1.5$ and $C = 16.5$. For one cycle to be completed over 24 h, the period must be 24. Thus, $\frac{2\pi}{B} = 24$, or $B = \frac{\pi}{12}$.

48. Assume that $\sin\theta = \frac{4}{5}$, where $\pi/2 < \theta < \pi$. Find:

(a) $\tan\theta$ **(b)** $\sin 2\theta$ **(c)** $\csc\frac{\theta}{2}$

SOLUTION If $\sin\theta = 4/5$, then by the fundamental trigonometric identity,

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$$

Because $\pi/2 < \theta < \pi$, it follows that $\cos\theta$ must be negative. Hence, $\cos\theta = -3/5$.

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{4/5}{-3/5} = -\frac{4}{3}$

(b) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{4}{5}\cdot -\frac{3}{5} = -\frac{24}{25}$

(c) We first note that

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-3/5)}{2}} = 2\frac{\sqrt{5}}{5}$$

Thus,

$$\csc\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{2}$$

49. Give an example of values a, b such that

(a) $\cos(a+b) \neq \cos a + \cos b$ **(b)** $\cos\frac{a}{2} \neq \frac{\cos a}{2}$

SOLUTION

(a) Take $a = b = \pi/2$. Then $\cos(a+b) = \cos\pi = -1$ but

$$\cos a + \cos b = \cos\frac{\pi}{2} + \cos\frac{\pi}{2} = 0 + 0 = 0$$

(b) Take $a = \pi$. Then

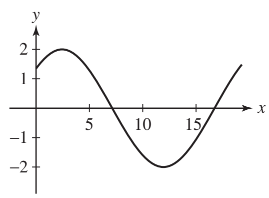
$$\cos\left(\frac{a}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

but

$$\frac{\cos a}{2} = \frac{\cos\pi}{2} = \frac{-1}{2} = -\frac{1}{2}$$

50. Let $f(x) = \cos x$. Sketch the graph of $y = 2f\left(\frac{1}{3}x - \frac{\pi}{4}\right)$ for $0 \le x \le 6\pi$.

SOLUTION



51. Solve $\sin 2x + \cos x = 0$ for $0 \le x < 2\pi$.

SOLUTION Using the double-angle formula for the sine function, we rewrite the equation as $2\sin x\cos x + \cos x = \cos x(2\sin x + 1) = 0$. Thus, either $\cos x = 0$ or $\sin x = -1/2$. From here, we see that the solutions are $x = \pi/2$, $x = 7\pi/6$, $x = 3\pi/2$, and $x = 11\pi/6$.

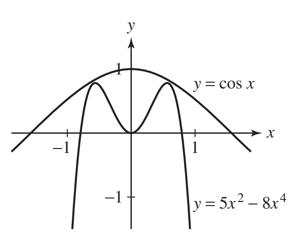
52. How does $h(n) = n^2/2^n$ behave for large whole-number values of n? Does $h(n)$ tend to infinity?

SOLUTION The table suggests that for large whole number values of n, $h(n) = \frac{n^2}{2^n}$ tends toward 0.

n	$h(n) = n^2/2^n$
10	0.09765625000
10^2	$7.888609052 \times 10^{-27}$
10^3	$9.332636185 \times 10^{-296}$
10^4	$5.012372749 \times 10^{-3003}$
10^5	$1.000998904 \times 10^{-30,093}$
10^6	$1.010034059 \times 10^{-301,018}$

53. $\boxed{\text{GU}}$ Use a graphing calculator to determine whether the equation $\cos x = 5x^2 - 8x^4$ has any solutions.

SOLUTION The graphs of $y = \cos x$ and $y = 5x^2 - 8x^4$ are shown here. Because the graphs do not intersect, there are no solutions to the equation $\cos x = 5x^2 - 8x^4$.



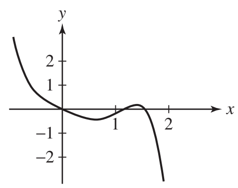
54. GU Using a graphing calculator, find the number of real roots and estimate the largest root to two decimal places:

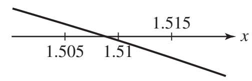
(a) $f(x) = 1.8x^4 - x^5 - x$

(b) $g(x) = 1.7x^4 - x^5 - x$

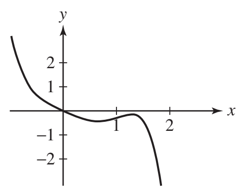
SOLUTION

(a) The graph of $y = 1.8x^4 - x^5 - x$ is shown below at the left. Because the graph has three x-intercepts, the function $f(x) = 1.8x^4 - x^5 - x$ has three real roots. From the graph shown below at the right, we see that the largest root of $f(x) = 1.8x^4 - x^5 - x$ is approximately $x = 1.51$.





(b) The graph of $y = 1.7x^4 - x^5 - x$ is shown. Because the graph has only one x-intercept, the function $f(x) = 1.7x^4 - x^5 - x$ has only one real root. From the graph, we see that the largest root of $f(x) = 1.7x^4 - x^5 - x$ is approximately $x = 0$.



55. Match each quantity (a)–(d) with (i), (ii), or (iii) if possible, or state that no match exists.

(a) $\ln\left(\frac{a}{b}\right)$

(b) $\frac{\ln a}{\ln b}$

(c) $e^{\ln a - \ln b}$

(d) $(\ln a)(\ln b)$

(i) $\ln a + \ln b$

(ii) $\ln a - \ln b$

(iii) $\frac{a}{b}$

SOLUTION

(a) **(ii)**: $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$

(b) No match

(c) **(iii)**: $e^{\ln a - \ln b} = e^{\ln a}\dfrac{1}{e^{\ln b}} = \dfrac{a}{b}$

(d) No match

56. Indicate which of the following are correct, and correct the ones that are not.

(a) $\ln e^x = x$

(b) $e^3 e^{1/3} = e$

(c) $\frac{\ln 6}{\ln 3} = \ln 2$

(d) $\ln 2 + \ln 4 = \ln 8$

SOLUTION

(a) Correct

(b) Incorrect; $e^3 e^{1/3} = e^{3+1/3} = e^{10/3}$

(c) Incorrect; $\ln 6 - \ln 3 = \ln \frac{6}{3} = \ln 2$

(d) Correct

57. The decibel level for the intensity of a sound is a logarithmic scale defined by $D = 10\log_{10} I + 120$, where I is the intensity of the sound in watts per square meter. If the intensity of one sound is 5000 times greater than the intensity of another, how much greater is the decibel level of the more intense sound?

SOLUTION Let $D(I) = 10\log_{10} I + 120$. Then

$$D(5000I) = 10\log_{10}(5000I) + 120 = 10\log_{10} 5000 + 10\log_{10} I + 120 = 10\log_{10} 5000 + D(I)$$

Thus, if the intensity of one sound is 5000 times greater than the intensity of another, the decibel level of the more intense sound is $10\log_{10} 5000 \approx 36.99$ greater than the decibel level of the less intense sound.

58. Consider the equation $M_w = \frac{2}{3}\log_{10} E - 10.7$ relating the moment magnitude of an earthquake and the energy E (in ergs) released by it. If M_w increases by 2, by what factor does the energy increase?

SOLUTION Solving the equation $M_w = \frac{2}{3}\log_{10} E - 10.7$ for E yields

$$\log_{10} E = \frac{3}{2}(M_w + 10.7) \quad \text{or} \quad E = 10^{\frac{3}{2}(M_w+10.7)}$$

Now,

$$E(M_w + 2) = 10^{\frac{3}{2}(M_w+2+10.7)} = 10^{3+\frac{3}{2}(M_w+10.7)} = 10^3 10^{\frac{3}{2}(M_w+10.7)} = 1000E(M_w)$$

Thus, if M_w increases by 2, the energy increases by a factor of 1000.

59. Find the inverse of $f(x) = \sqrt{x^3 - 8}$ and determine its domain and range.

SOLUTION To find the inverse of $f(x) = \sqrt{x^3 - 8}$, we solve $y = \sqrt{x^3 - 8}$ for x as follows:

$$y^2 = x^3 - 8$$
$$x^3 = y^2 + 8$$
$$x = \sqrt[3]{y^2 + 8}$$

Therefore, $f^{-1}(x) = \sqrt[3]{x^2 + 8}$. The domain of f^{-1} is the range of f, namely $\{x : x \ge 0\}$; the range of f^{-1} is the domain of f, namely $\{y : y \ge 2\}$.

60. Find the inverse of $f(x) = \frac{x-2}{x-1}$ and determine its domain and range.

SOLUTION To find the inverse of $f(x) = \frac{x-2}{x-1}$, we solve $y = \frac{x-2}{x-1}$ for x as follows:

$$x - 2 = y(x - 1) = yx - y$$
$$x - yx = 2 - y$$
$$x = \frac{2 - y}{1 - y}$$

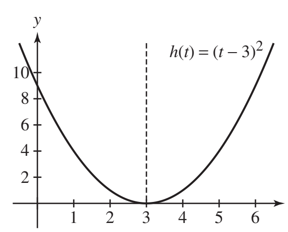
Therefore,

$$f^{-1}(x) = \frac{2 - x}{1 - x} = \frac{x - 2}{x - 1}$$

The domain of f^{-1} is the range of f, namely $\{x : x \ne 1\}$; the range of f^{-1} is the domain of f, namely $\{y : y \ne 1\}$.

61. Find a domain on which $h(t) = (t - 3)^2$ is one-to-one and determine the inverse on this domain.

SOLUTION From the graph of $h(t) = (t - 3)^2$ shown here, we see that h is one-to-one on each of the intervals $t \ge 3$ and $t \le 3$.



We find the inverse of $h(t) = (t - 3)^2$ on the domain $\{t : t \le 3\}$ by solving $y = (t - 3)^2$ for t. First, we find

$$\sqrt{y} = \sqrt{(t - 3)^2} = |t - 3|$$

Having restricted the domain to $\{t : t \le 3\}$, $|t - 3| = -(t - 3) = 3 - t$. Thus,

$$\sqrt{y} = 3 - t$$
$$t = 3 - \sqrt{y}$$

The inverse function is $h^{-1}(t) = 3 - \sqrt{t}$. For $t \ge 3$, $h^{-1}(t) = 3 + \sqrt{t}$.

62. Show that $g(x) = \frac{x}{x-1}$ is equal to its inverse on the domain $\{x : x \ne 1\}$.

SOLUTION To show that $g(x) = \frac{x}{x-1}$ is equal to its inverse, we need to show that for $x \ne 1$,

$$g(g(x)) = x$$

First, we notice that for $x \ne 1$, $g(x) \ne 1$. Therefore,

$$g(g(x)) = g\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x - (x - 1)} = \frac{x}{1} = x$$

63. Let

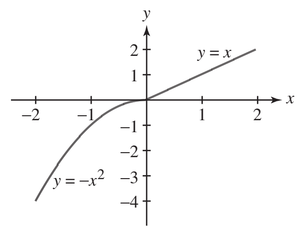
$$f(x) = \begin{cases} -x^2 & \text{when } x < 0 \\ x & \text{when } x \geq 0 \end{cases}$$

(a) Is f increasing?

(b) Does f have an inverse? If so, what is it?

SOLUTION

(a) From the graph of f shown here, we see that f is an increasing function over its domain.



(b) Because the graph of f, which is shown in part (a), passes the horizontal line test, f is a one-to-one function and has an inverse function. For $x < 0$, $f(x) = -x^2$. Solving $y = -x^2$ for x (remembering that we are interested in only $x < 0$) yields $x = -\sqrt{-y}$, so this piece of the inverse function is $f^{-1}(x) = -\sqrt{-x}$. On the other hand, for $x \geq 0$, $f(x) = x$. Solving $y = x$ for x yields $x = y$, so this piece of the inverse function is $f^{-1}(x) = x$. Bringing these two pieces together, we have

$$f^{-1}(x) = \begin{cases} -\sqrt{-x} & \text{when} \quad x < 0 \\ x & \text{when} \quad x \geq 0 \end{cases}$$

64. Let

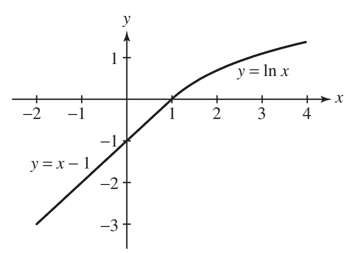
$$f(x) = \begin{cases} x - 1 & \text{when } x < 1 \\ \ln x & \text{when } x \geq 1 \end{cases}$$

(a) Is f increasing?

(b) Does f have an inverse? If so, what is it?

SOLUTION

(a) From the graph of f shown here, we see that f is an increasing function over its domain.



(b) Because the graph of f, which is shown in part (a), passes the horizontal line test, f is a one-to-one function and has an inverse function. For $x < 1$, $f(x) = x - 1$. Solving $y = x - 1$ for x yields $x = y + 1$, so this piece of the inverse function is $f^{-1}(x) = x + 1$. On the other hand, for $x \geq 1$, $f(x) = \ln x$. Solving $y = \ln x$ for x yields $x = e^y$, so this piece of the inverse function is $f^{-1}(x) = e^x$. Bringing these two pieces together, we have

$$f^{-1}(x) = \begin{cases} x + 1 & \text{when} \quad x < 1 \\ e^x & \text{when} \quad x \geq 1 \end{cases}$$

65. Suppose that g is the inverse of f. Match the functions (a)–(d) with their inverses (i)–(iv).

(a) $f(x) + 1$ **(b)** $f(x + 1)$ **(c)** $4f(x)$ **(d)** $f(4x)$

(i) $g(x)/4$ **(ii)** $g(x/4)$ **(iii)** $g(x - 1)$ **(iv)** $g(x) - 1$

SOLUTION

(a) **(iii)**: $f(x) + 1$ and $g(x - 1)$ are inverse functions:

$$f(g(x - 1)) + 1 = (x - 1) + 1 = x$$

$$g(f(x) + 1 - 1) = g(f(x)) = x$$

(b) **(iv)**: $f(x+1)$ and $g(x)-1$ are inverse functions:

$$f(g(x)-1+1) = f(g(x)) = x$$
$$g(f(x+1)) - 1 = (x+1) - 1 = x$$

(c) **(ii)**: $4f(x)$ and $g(x/4)$ are inverse functions:

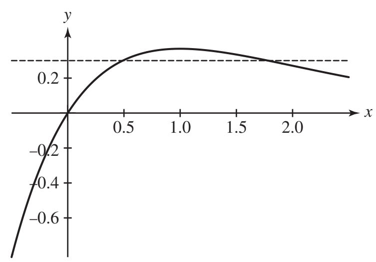
$$4f(g(x/4)) = 4(x/4) = x$$
$$g(4f(x)/4) = g(f(x)) = x$$

(d) **(i)**: $f(4x)$ and $g(x)/4$ are inverse functions:

$$f(4 \cdot g(x)/4) = f(g(x)) = x$$
$$\frac{1}{4}g(f(4x)) = \frac{1}{4}(4x) = x$$

66. $\boxed{\text{GU}}$ Plot $f(x) = xe^{-x}$ and use the zoom feature to find two solutions of $f(x) = 0.3$.

SOLUTION The graph of $f(x) = xe^{-x}$ is shown here. Based on this graph, we should zoom in near $x = 0.5$ and near $x = 1.75$ to find solutions of $f(x) = 0.3$.



From the figure below at the left, we see that one solution of $f(x) = 0.3$ is approximately $x = 0.49$; from the figure below at the right, we see that a second solution of $f(x) = 0.3$ is approximately $x = 1.78$.