Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time t = a and t = b is the change in position divided by the elapsed time: $v_{av} = \frac{s(b)-s(a)}{b-a}$

2.1.2 In order to compute the instantaneous velocity of the object at time t = a, we compute the average velocity over smaller and smaller time intervals of the form [a, t], using the formula: $v_{av} = \frac{s(t)-s(a)}{t-a}$. We let t approach a. If the quantity $\frac{s(t)-s(a)}{t-a}$ approaches a limit as t → a, then that limit is called the instantaneous velocity of the object at time t = a.

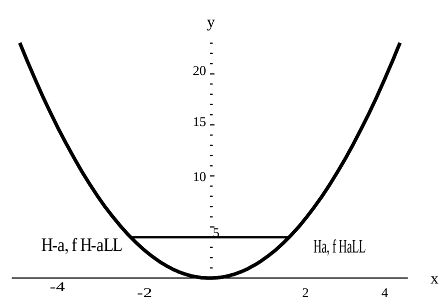
2.1.3 The slope of the secant line between points (a, f (a)) and (b, f (b)) is the ratio of the differences f (b) − f (a) and b − a. Thus $m_{sec} = \frac{f(b)-f(a)}{b-a}$

2.1.4 In order to compute the slope of the tangent line to the graph of y = f (t at (a, f (a)), we compute the slope of the secant line over smaller and smaller time intervals of the form [a, t]. Thus we consider $\frac{f(t)-f(a)}{t-a}$ and let t → a. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve y = f(t) at t = a.

2.1.5 Both problems involve the same mathematics, namely finding the limit as t → a of a quotient of differences of the form $\frac{g(t)-g(a)}{t-a}$ for some function g.

2.1.6

Because $f(x) = x^2$ is an even function, f (−a) = f (a) for all a. Thus the slope of the secant line between the points (a, f (a)) and (−a, f (−a)) is $m_{sec} = \frac{f(-a)-f(a)}{-a-a} = \frac{0}{-2a} = 0$. The slope of the tangent line at x = 0 is also zero.



2.1.7 The average velocity is $\frac{s(3)-s(2)}{3-2} = 156 - 136 = 20$.

2.1.8 The average velocity is $\frac{s(4)-s(1)}{4-1} = \frac{144-84}{3} = \frac{60}{3} = 20$.

2.1.9

a. Over [1, 4], we have $v_{av} = \frac{s(4)-s(1)}{4-1} = \frac{256-112}{3} = 48$.

b. Over [1, 3], we have $v_{av} = \frac{s(3)-s(1)}{3-1} = \frac{240-112}{2} = 64$.

c. Over [1, 2], we have $v_{av} = \frac{s(2)-s(1)}{2-1} = \frac{192-112}{1} = 80$.

d. Over [1, 1 + h], we have $v_{av} = \frac{s(1+h)-s(1)}{1+h-1} = \frac{-16(1+h)^2+128(1+h)-(112)}{h} = \frac{-16h^2-32h+128h}{h} = \frac{h(-16h+96)}{h} = 96 - 16h = 16(6 - h)$.

2.1.10

a. Over [0, 3], we have $v_{av} = \frac{s(3)-s(0)}{3-0} = \frac{65.9-20}{3} = 15.3$.

b. Over [0, 2], we have $v_{av} = \frac{s(2)-s(0)}{2-0} = \frac{60.4-20}{2} = 20.2$.

c. Over [0, 1], we have $v_{av} = \frac{s(1)-s(0)}{1-0} = \frac{45.1-20}{1} = 25.1$.

d. Over [0, h], we have $v_{av} = \frac{s(h)-s(0)}{h-0} = \frac{-4.9h^2+30h+20-20}{h} = \frac{h(-4.9h+30)}{h} = -4.9h + 30$.

2.1.11

a. $\frac{s(2)-s(0)}{2-0} = \frac{72-0}{2} = 36$.

b. $\frac{s(1.5)-s(0)}{1.5-0} = \frac{66-0}{1.5} = 44$.

c. $\frac{s(1)-s(0)}{1-0} = \frac{52-0}{1} = 52$.

d. $\frac{s(.5)-s(0)}{.5-0} = \frac{30-0}{.5} = 60$.

2.1.12

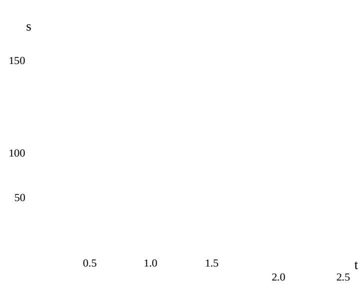
a. $\frac{s(2.5)-s(.5)}{2.5-.5} = \frac{150-46}{2} = 52$.

b. $\frac{s(2)-s(.5)}{2-.5} = \frac{136-46}{1.5} = 60$.

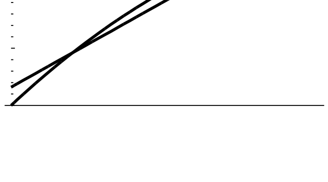
c. $\frac{s(1.5)-s(.5)}{1.5-.5} = \frac{114-46}{1} = 68$.

d. $\frac{s(1)-s(.5)}{1-.5} = \frac{84-46}{.5} = 76$.

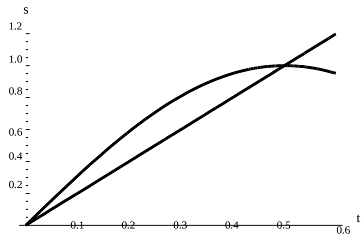
2.1.13



The slope of the secant line is given by $\frac{s(2)-s(.5)}{2-.5} = \frac{136-46}{1.5} = 60$. This represents the average velocity of the object over the time interval [.5, 2].



2.1.14



The slope of the secant line is given by $\frac{s(.5)-s(0)}{.5-0} = \frac{1}{.5} = 2$. This represents the average velocity of the object over the time interval [0, .5].

2.1.15

Time Interval	[1, 2]	[1, 1.5]	[1, 1.1]	[1, 1.01]	[1, 1.001]
Average Velocity	80	88	94.4	95.84	95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.16

Time Interval	[2, 3]	[2, 2.25]	[2, 2.1]	[2, 2.01]	[2, 2.001]
Average Velocity	5.5	9.175	9.91	10.351	10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.17 $\frac{s(1.01)-s(1)}{.01} = 47.84$, while $\frac{s(1.001)-s(1)}{.001} = 47.984$ and $\frac{s(1.0001)-s(1)}{.0001} = 47.9984$. It appears that the instantaneous velocity at $t = 1$ is approximately 48.

2.1.18 $\frac{s(2.01)-s(2)}{.01} = -4.16$, while $\frac{s(2.001)-s(2)}{.001} = -4.016$ and $\frac{s(2.0001)-s(2)}{.0001} = -4.0016$. It appears that the instantaneous velocity at $t = 2$ is approximately -4.

2.1.19

Time Interval	[2, 3]	[2.9, 3]	[2.99, 3]	[2.999, 3]	[2.9999, 3]	[2.99999, 3]
Average Velocity	20	5.6	4.16	4.016	4.0016	4.00016

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval	[π/2, π]	[π/2, π/2 + .1]	[π/2, π/2 + .01]	[π/2, π/2 + .001]	[π/2, π/2 + .0001]
Average Velocity	−1.90986	−.149875	−.0149999	−.0015	−.00015

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval	[3, 3.1]	[3, 3.01]	[3, 3.001]	[3, 3.0001]
Average Velocity	−17.6	−16.16	−16.016	−16.002

The instantaneous velocity appears to be −16 ft/s.

2.1.22

Time Interval	[π/2, π/2 + .1]	[π/2, π/2 + .01]	[π/2, π/2 + .001]	[π/2, π/2 + .0001]
Average Velocity	−19.9667	−19.9997	−20.0000	−20.0000

The instantaneous velocity appears to be −20 ft/s.

2.1.23

Time Interval	[0, 0.1]	[0, 0.01]	[0, 0.001]	[0, 0.0001]
Average Velocity	79.4677	79.9947	79.9999	80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval	[0, 1]	[0, 0.1]	[0, 0.01]	[0, 0.001]
Average Velocity	−10	−18.1818	−19.802	−19.98

The instantaneous velocity appears to be −20 ft/s.

2.1.25

x Interval	[2, 2.1]	[2, 2.01]	[2, 2.001]	[2, 2.0001]
Slope of Secant Line	8.2	8.02	8.002	8.0002

The slope of the tangent line appears to be 8.

2.1.26

x Interval	[π/2, π/2 + .1]	[π/2, π/2 + .01]	[π/2, π/2 + .001]	[π/2, π/2 + .0001]
Slope of Secant Line	−2.995	−2.99995	−3.0000	−3.0000

The slope of the tangent line appears to be −3.

2.1.27

x Interval	[−1, −.9]	[−1, −.99]	[−1, −.999]	[−1, −.9999]
Slope of the Secant Line	.524862	.5025	.50025	.500025

The slope of the tangent line appears to be .5.

2.1.28

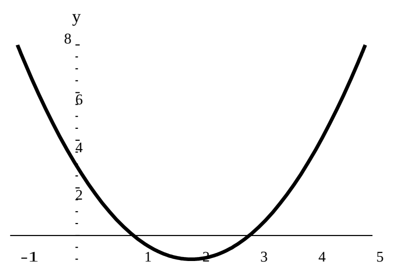
x Interval	[1, 1.1]	[1, 1.01]	[1, 1.001]	[1, 1.0001]
Slope of the Secant Line	2.31	2.0301	2.003	2.0003

The slope of the tangent line appears to be 2.

2.1.29

a. Note that the graph is a parabola with vertex (2, −1).

b. At (2, −1) the function has tangent line with slope 0.



c.

x Interval	[2, 2.1]	[2, 2.01]	[2, 2.001]	[2, 2.0001]
Slope of the Secant Line	.1	.01	.001	.0001

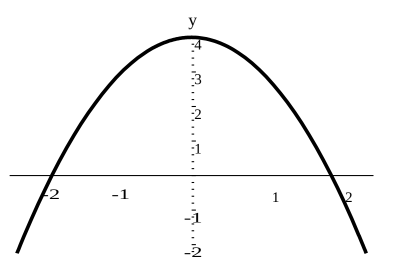
The slope of the tangent line at (2, −1) appears to be 0.

2.1.30

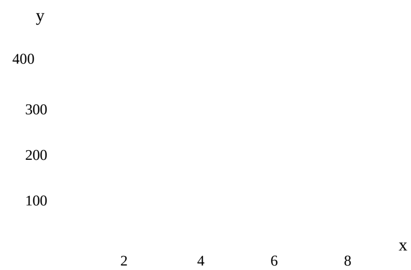
a. Note that the graph is a parabola with vertex (0, 4).

b. At (0, 4) the function has a tangent line with slope 0.

c. This is true for this function - because the function is symmetric about the y-axis and we are taking pairs of points symmetrically about the y axis. Thus f (0 + h) = 4 − (0 + h)^2 = 4 − (−h)^2 = f (0 − h). So the slope of any such secant line is $\frac{4-h^2-(4-h^2)}{h-(-h)} = \frac{0}{2h} = 0$.



2.1.31



a. Note that the graph is a parabola with vertex (4, 448).

b. At (4, 448) the function has tangent line with slope 0, so a = 4.

c.

x Interval	[4, 4.1]	[4, 4.01]	[4, 4.001]	[4, 4.0001]
Slope of the Secant Line	−1.6	−.16	−.016	−.0016

The slopes of the secant lines appear to be approaching zero.

d. On the interval [0, 4) the instantaneous velocity of the projectile is positive.

e. On the interval (4, 9] the instantaneous velocity of the projectile is negative.

2.1.32

a. The rock strikes the water when s(t) = 96. This occurs when $16t^2 = 96$, or $t^2 = 6$, whose only positive solution is $t = \sqrt{6} \approx 2.45$ seconds.

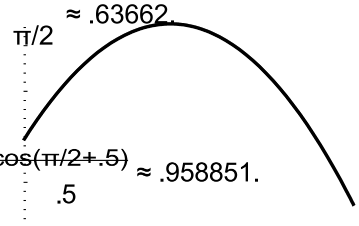
b.

t Interval	$[\sqrt{6} - .1, \sqrt{6}]$	$[\sqrt{6} - .01, \sqrt{6}]$	$[\sqrt{6} - .001, \sqrt{6}]$	$[\sqrt{6} - .0001, \sqrt{6}]$
Average Velocity	76.7837	78.2237	78.3677	78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

$$m_{AD} = \frac{y_D - y_A}{x_D - x_A} = \frac{f(\pi) - f(\pi/2)}{\pi - (\pi/2)} = \frac{1}{\pi/2} \approx .63662.$$



For line AC, we have

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{f(\pi/2 + .5) - f(\pi/2)}{(\pi/2 + .5) - (\pi/2)} = -\frac{\cos(\pi/2 + .5)}{.5} \approx .958851.$$

For line AB, we have

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(\pi/2 + .05) - f(\pi/2)}{(\pi/2 + .05) - (\pi/2)} = -\frac{\cos(\pi/2 + .05)}{.05} \approx .999583.$$

Computing one more slope of a secant line:

$$m_{sec} = \frac{f(\pi/2 + .01) - f(\pi/2)}{(\pi/2 + .01) - (\pi/2)} = -\frac{\cos(\pi/2 + .01)}{.01} \approx .999983.$$

Conjecture: The slope of the tangent line to the graph of f at $x = \pi/2$ is 1.

2.2 Definitions of Limits

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If f (x) is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write $\lim_{x\to a} f(x) = L$.

2.2.2 False. For example, consider the function $f(x) = \begin{cases} x^2 & \text{if } x = 0 \\ 4 & \text{if } x = 0. \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$, but $f(0) = 4$.

2.2.3 Suppose the function f is defined for all x near a but greater than a. If f (x) is arbitrarily close to L for x sufficiently close to (but strictly greater than) a, then $\lim_{x\to a^+} f(x) = L$.

2.2.4 Suppose the function f is defined for all x near a but less than a. If f (x) is arbitrarily close to L for x sufficiently close to (but strictly less than) a, then $\lim_{x\to a^-} f(x) = L$.

2.2.5 It must be true that L = M .

2.2.6 Because graphing utilities generally just plot a sampling of points and "connect the dots," they can sometimes mislead the user investigating the subtleties of limits.

2.2.7

a. h(2) = 5.

b. $\lim_{x\to 2} h(x) = 3$.

c. h(4) does not exist.

d. $\lim_{x\to 4} f(x) = 1$.

e. $\lim_{x\to 5} h(x) = 2$.

2.2.8

a. g(0) = 0.

b. $\lim_{x\to 0} g(x) = 1$.

c. g(1) = 2.

d. $\lim_{x\to 1} g(x) = 2$.

2.2.9

a. f (1) = −1.

b. $\lim_{x\to 1} f(x) = 1$.

c. f (0) = 2.

d. $\lim_{x\to 0} f(x) = 2$.

2.2.10

a. f (2) = 2.

b. $\lim_{x\to 2} f(x) = 4$.

c. $\lim_{x\to 4} f(x) = 4$.

d. $\lim_{x\to 5} f(x) = 2$.

2.2.11

a.

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x) = \frac{x^2-4}{x-2}$	3.9	3.99	3.999	3.9999	undefined	4.0001	4.001	4.01	4.1

b. $\lim_{x\to 2} f(x) = 4$.

2.2.12

a.

x	.9	.99	.999	.9999	1	1.0001	1.001	1.01	1.1
$f(x) = \frac{x^3-1}{x-1}$	2.71	2.9701	2.997	2.9997	undefined	3.0003	3.003	3.0301	3.31

b. $\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = 3$

2.2.13

a.

t	8.9	8.99	8.999	9	9.001	9.01	9.1
$g(t) = \frac{t-9}{\sqrt{t}-3}$	5.98329	5.99833	5.99983	undefined	6.00017	6.00167	6.01662

b. $\lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} = 6.$

2.2.14

a.

x	.01	.001	.0001	.00001
$f(x) = (1+x)^{1/x}$	2.70481	2.71692	2.71815	2.71827

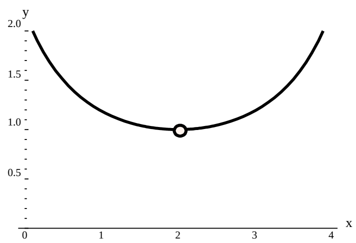
x	−.01	−.001	−.0001	−.00001
$f(x) = (1+x)^{1/x}$	2.732	2.71964	2.71842	2.71830

b. $\lim_{x\to 0} (1+x)^{1/x} \approx 2.718.$

c. $\lim_{x\to 0} (1+x)^{1/x} = e.$

2.2.15

a.



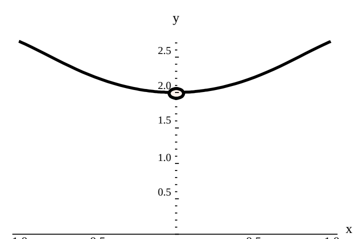
b.

x	1.8	1.9	1.99	2.01	2.1	2.2
$f(x)$	1.0067	1.00167	1.00002	1.00002	1.00167	1.0067

From both the graph and the table, the limit appears to be 1.

2.2.16

a.



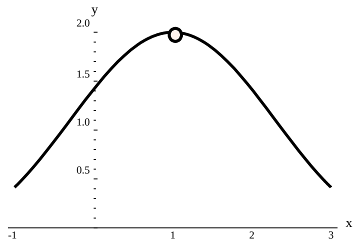
b.

x	−.2	−0.1	−0.01	.01	0.1	0.2
$f(x)$	2.03336	2.00834	2.00008	2.00008	2.00834	2.03336

From both the graph and the table, the limit appears to be 2.

2.2.17

a.



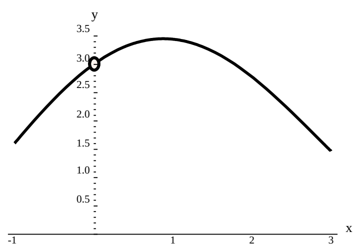
b.

x	0.9	0.99	0.999	1.001	1.01	1.1
f (x)	1.993342	1.999933	1.999999	1.999999	1.999933	1.993342

From both the graph and the table, the limit appears to be 2.

2.2.18

a.



b.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
f (x)	2.8951	2.99	2.999	3.001	3.0099	3.0949

From both the graph and the table, the limit appears to be 3.

2.2.19

x	4.9	4.99	4.999	4.9999	5	5.0001	5.001	5.01	5.1
f (x) = $\frac{x^2-25}{x-5}$	9.9	9.99	9.999	9.9999	undefined	10.0001	10.001	10.01	10.1

$\lim_{x\to5^+} \frac{x^2-25}{x-5} = 10$, $\lim_{x\to5^-} \frac{x^2-25}{x-5} = 10$, and thus $\lim_{x\to5} \frac{x^2-25}{x-5} = 10$.

2.2.20

x	99.9	99.99	99.999	99.9999	100	100.0001	100.001	100.01	100.1
f (x) = $\frac{x-100}{\sqrt{x}-10}$	19.995	19.9995	19.99995	≈ 20	undefined	≈ 20	20.0005	20.00005	20.005

$\lim_{x\to100^+} \frac{x-100}{\sqrt{x}-10} = 20$, $\lim_{x\to100^-} \frac{x-100}{\sqrt{x}-10} = 20$, and thus $\lim_{x\to100} \frac{x-100}{\sqrt{x}-10} = 20$.

2.2.21

a. $f(1) = 0$.

b. $\lim_{x\to1^-} f(x) = 1$.

c. $\lim_{x\to1^+} f(x) = 0$.

d. $\lim_{x\to1} f(x)$ does not exist, since the two one-sided limits aren't equal.