

Chapter 2: Basic Concepts of Probability Theory

2.1 Specifying Random Experiments

2.1 (a) $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
(b) $A = \{1, 2, 3, 4\}$ $B = \{2, 3, 4, 5, 6, 7, 8\}$ $D = \{1, 3, 5, 7, 9, 11\}$
(c) $A \cap B \cap D = \{3\}$ $A^c \cap B = \{5, 6, 7, 8\}$
 $A \cup (B \cap D^c) = \{1, 2, 3, 4, 6, 8\}$
 $(A \cup B) \cap D^c = \{2, 4, 6, 8\}$

- 2.2 The outcome of this experiment consists of a pair of numbers (x, y) where x = number of dots in first toss and y = number of dots in second toss. Therefore, S = set of ordered pairs (x, y) where $x, y \in \{1, 2, 3, 4, 5, 6\}$ which are listed in the table below:

a)

x	y	1	2	3	4	5	6
1		(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2		(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3		(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4		(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5		(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6		(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

checkmarks indicate elements of events

b)

x	y	1	2	3	4	5	6
1		✓					
2		✓					
3		✓	✓				
4		✓	✓	✓			
5		✓	✓	✓	✓		
6		✓	✓	✓	✓	✓	

$A = \{N_1 < N_2\}^c = \{N_1 \geq N_2\}$

c)

x	y	1	2	3	4	5	6
1							
2							
3							
4							
5							
6		✓	✓	✓	✓	✓	✓

$B = \{N_1 = 6\}$

d) B is a subset of A so when B occurs then A also occurs, thus B implies A

e) $A \cap B^c = \{N_2 \leq N_1 < 6\}$

x	y	1	2	3	4	5	6
1		✓					
2		✓	✓				
3		✓	✓	✓			
4		✓	✓	✓	✓		
5		✓	✓	✓	✓	✓	
6							

f) C = "number of dots differ by 2"

x	y	1	2	3	4	5	6
1				✓			
2					✓		
3		✓				✓	
4			✓				✓
5				✓			
6					✓		

Comparing the tables for A and C we see

$$A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$$

2.3

b) $A = \{0, 1, 2, 3, 4, 5\}$

⑤ $A = \{3\}$

© $\{0\} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$$\{1\} = \{(1,2), (2,3), (3,4), (4,5), (5,6), (6,5), (4,3), (3,2), (2,1)\}$$

$$\{2\} = \{(1,3), (2,4), (3,5), (4,4), (3,1), (4,2), (5,3), (6,4)\}$$

$$\{3\} = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$$

$$\{4\} = \{(1,5), (2,6), (5,1), (6,2)\}$$

$$\{5\} = \{(1, 6), (6, 1)\}$$

2.4

Y	-2	-1	0	1	2
X					
+2	-	-	(2,0)	(3,1)	(2,2)
-2	(-2,-2)	(-2,-1)	(-2,0)	-	-

⑤ "X definitely +2" (based on observed Y): $\{(2,1), (2,2)\}$

© $\{Y=0\} = \{(2,0), (-2,0)\}$

"observed output is zero"
cannot determine input

2.5

a) Each testing of a pen has two possible outcomes: "pen good" (g) or "pen bad" b . The experiment consists of testing pens until a good pen is found. Therefore each outcome of the experiment consists of a string of "b's" ended by a "g". We assume that each pen is not put back in the drawer after being tests. Thus $S = \{g, bg, bbg, bbbg, \dots, bbbbgg\}$

b) We now simply record the number of pens tested, so $\mathcal{S} = \{1, 2, 3, 4, 5\}$

c) The outcome now consists of a substring of "b's" and one "g" in any order followed by a final "g". $S = \{gg, bgg, gbg, gbbg, bbbg, gbbbg, bgbbg, bbgbg, bbbgg, gbbbbg, bgbbbg, bbgbbg, bbbbgg, gbbbbb, bbbbbb\}$

d) $S = \{2, 3, 4, 5, 6\}$

d) $S = \{2, 3, 4, 5, 6\}$

2.6

a $S = \{abc, cab, bca, acb, bac, cba\}$

b $A = \{abc, acb\}$ $B = \{abc, cba\}$ $C = \{abc, bac\}$

c $(A \cup B \cup C)^c = \{abc, acb, cba, bac\}^c = \{cab, bca\}$

d $A \cap B \cap C = \{abc\}$

e $A \cup B \cup C = \{abc, acb, cba, bac\}$

2.7

a $A = \{2, 4, 6, \dots\}$

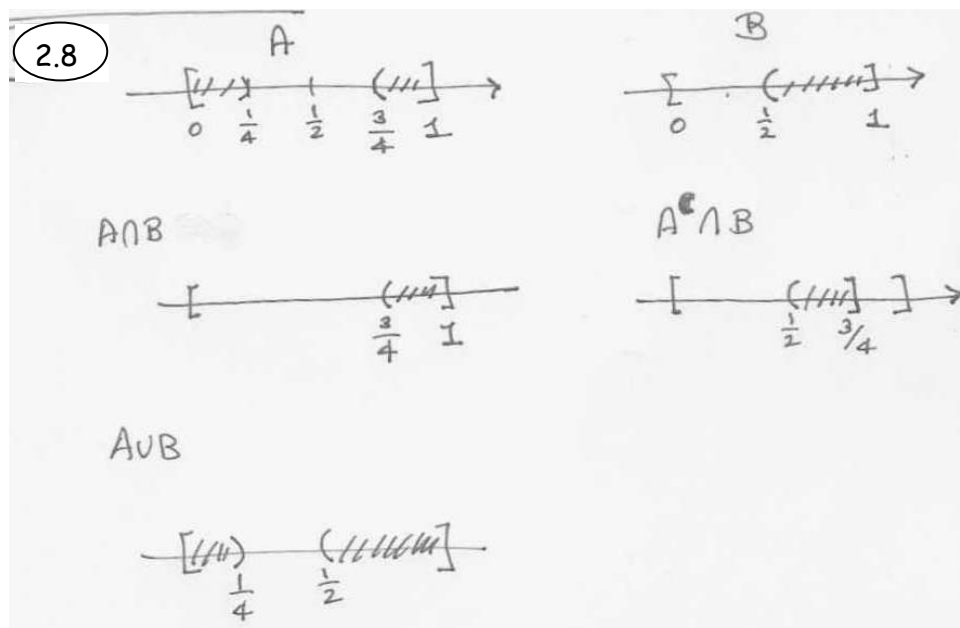
b $B = \{3, 6, 9, \dots\}$

c $C = \{1, 2, 3, 4, 5, 6\}$

d $A \cap B = \{6, 12, 18, \dots\}$ "multiple of 6"

$A - B = \text{"even positive integer and not multiple of 3"}$
 $= \{n = 2m : m \text{ positive integer, not multiple of 3}\}$

$A \cap B \cap C = \{6\}$ "even multiple of 3 less than or equal to 6"



2.9

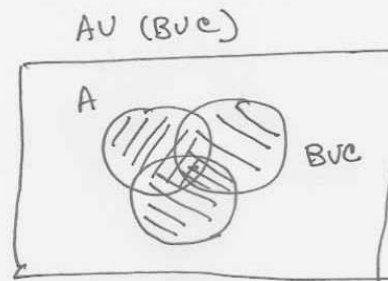
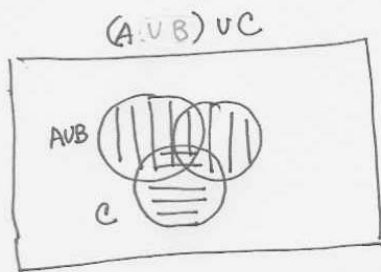
If we sketch the events A and B we see that $B = A \cup B$. We also see that the intervals corresponding to A and C have no points in common so $A \cap C = \emptyset$.



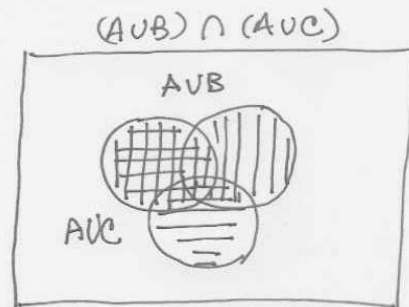
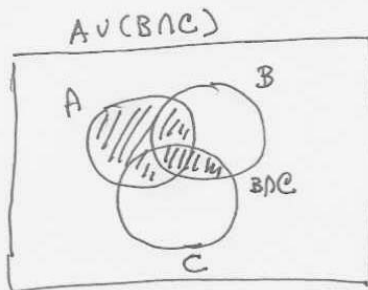
We also see that $(r, s] = (r, \infty) \cap (-\infty, s] = (-\infty, r]^c \cap (-\infty, s]$
that is $C = A^c \cap B$

2.10

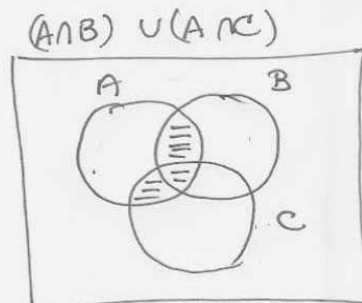
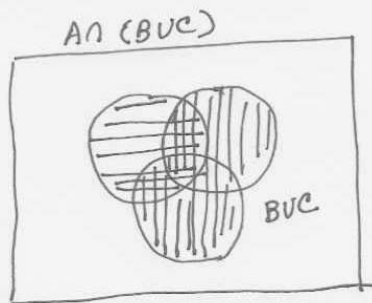
(a) $A \cup (B \cap C) = (A \cup B) \cup C$

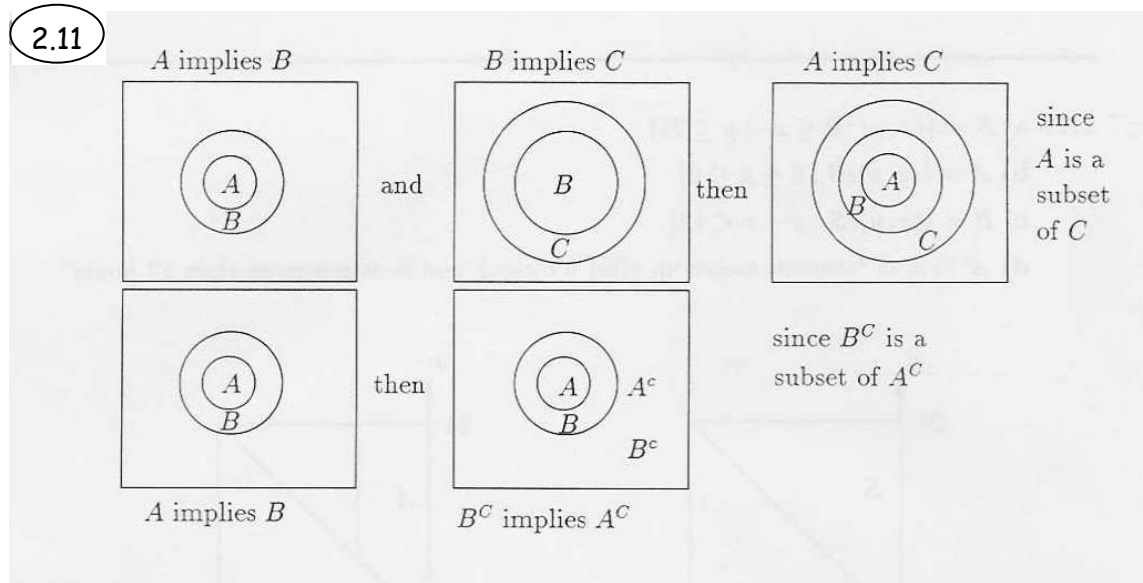


(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$





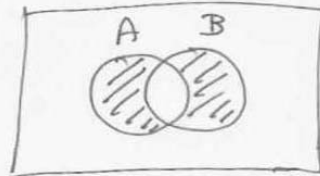
2.12 Given $A \cup B = A$ and $A \cap B = A$ claim $A = B$

Let $x \in A$, then $x \in A \cap B \Rightarrow x \in B \therefore A \subset B$

Let $x \in B$ then $x \in A \cup B \Rightarrow x \in A \therefore B \subset A$

$\therefore A = B$.

2.13 "Exactly one of A or B occurs" = $(A \cap B^c) \cup (A^c \cap B)$



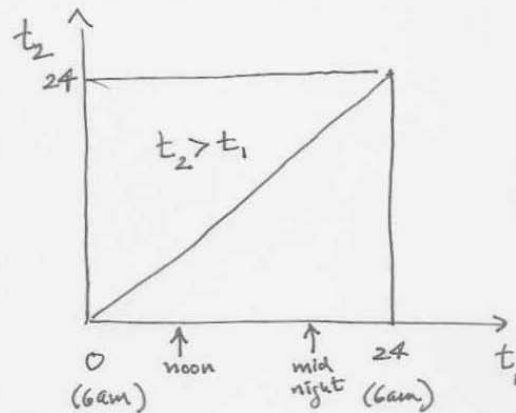
- 2.14 a) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
 b) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$
 c) $A \cup B \cup C$
 d) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$
 e) $A^c \cap B^c \cap C^c$

- 2.15 a) $D = A_1 \cap A_2 \cap A_3$
 b) $D = A_1 \cup A_2 \cup A_3$
 c) $D = (A_1 \cap A_2 \cap A_3) \cup (A_1^c \cap A_2 \cap A_3) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1 \cap A_2 \cap A_3^c)$

- 2.16 a) "System j is up" $= A_{j1} \cap A_{j2}$
 "System j is up" $= (A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})$
 b) " j th level connection active" if $A_{j1} \cap A_{j2}$
 "connection active" if any of 3 connections is active

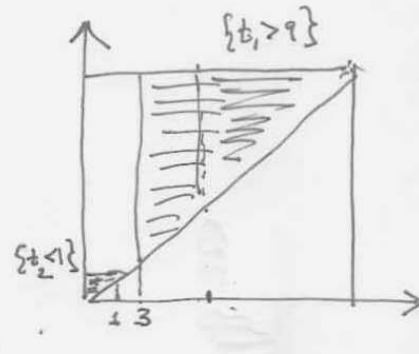
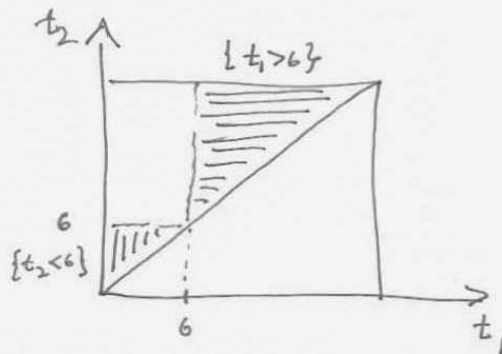
2.17

(a)



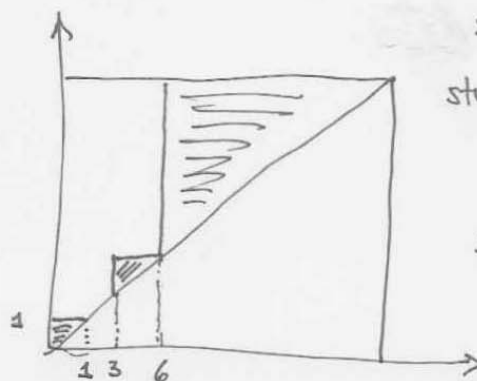
wake at t_1
sleep at $t_2 > t_1$

- (b) "Asleep at noon" = wake up after noon
or wake up & go to sleep before noon
 $= \{t_1 > 6\} \cup \{t_2 < 6\}$



- (c) "Student sleeps through breakfast" = $\{t_1 > 3\} \cup \{t_2 < 1\}$

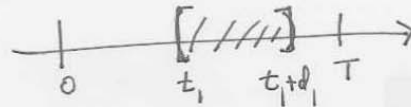
- (d) A ∩ B



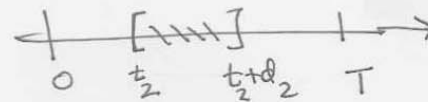
student wakes up after noon
or
student wakes up after 9am
and goes back to sleep
before noon
or
student wakes up and goes
back to sleep before 7am

(2.18) (a) $A = \{(t_1, t_2) : 0 < t_1 < T, 0 < t_2 < T\}$

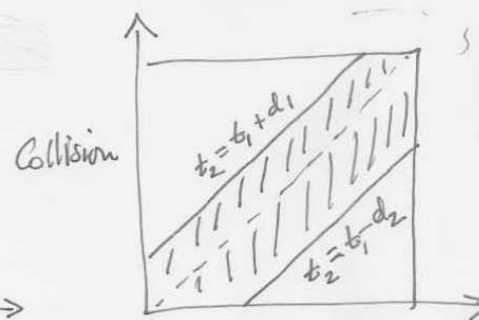
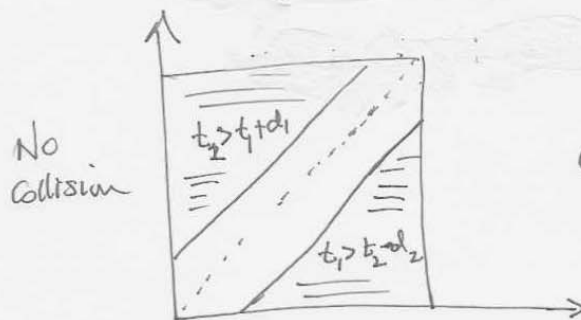
(b) $A = \text{train in crossing}$
 $= \{t_1 < t < t_1 + d_1\}$



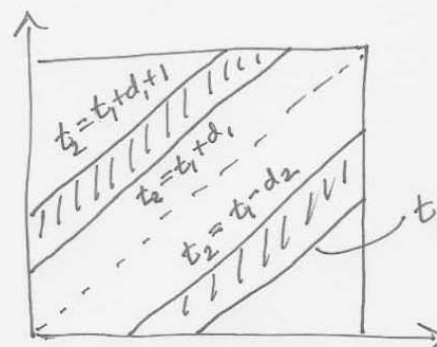
$B = \text{car in crossing}$
 $= \{t_2 < t < t_2 + d_2\}$



No Collision occurs if $A \cap B$ is empty $\Leftrightarrow t_1 > t_2 + d_2$
 or $t_2 > t_1 + d_1$



(c) Collision Missed by 1 second or less.
 $= \{\text{No Collision}\} \cap \{\text{Within 1 second of collision}\}$



$t_2 = t_1 - d_2 - 1$
 $\{t_1 + d_1 < t_2 < t_1 + d_1 + 1\}$
 $\cup \{t_2 + d_2 < t_1 < t_2 + d_2 + 1\}$

2.19 (a) $\phi, A = \{-1, 0, +1\}, \{-1\}, \{0\}, \{+1\}, \{-1, 0\}, \{-1, +1\}, \{0, +1\}$

(b) $A = \{(-1, 0), (-1, +1), (0, \neq 1), (0, +1), (+1, \neq 1), (+1, 0)\}$

power set has $2^6 = 64$ ~~subsets~~ subsets.

2.20 $A = \{ \overset{HH}{\cancel{HH}}, \overset{HT}{\cancel{HT}}, \overset{TH}{\cancel{TH}}, \overset{TT}{\cancel{TT}} \}$

(a) $\phi, A, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, TH, TT\}, \{HH, HT, TT\}, \{HT, TH, TT\}$

(b) $A' = \{0, 1, 2\}$

$\phi, A, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}$

(c) A has 2^{10} elements & its power set has $2^{2^{10}} = 2^{1024}$ subsets

A' has 11 elements & its power set has 2^{11} subsets

2.2 The Axioms of Probability

2.21 The sample space in tossing a die is $S = \{1, 2, 3, 4, 5, 6\}$. Let $p_i = P[\{i\}] = p$ since all faces are equally likely. By Axiom 1

$$\begin{aligned} 1 &= P[S] \\ &= P[\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}] \end{aligned}$$

The elementary events $\{i\}$ are mutually exclusive so by Corollary 4:

$$1 = p_1 + p_2 + \dots + p_6 = 6p \Rightarrow p_i = p = \frac{1}{6} \text{ for } i = 1, \dots, 6$$

2.21

(b) $P[A] = P[\dots] = P[\{4, 5, 6\}] = P[\{4\}] + P[\{5\}] + P[\{6\}] = \frac{3}{6}$
 $P[B] = P[\text{odd\#}] = P[\{1, 3, 5\}] = P[\{1\}] + P[\{3\}] + P[\{5\}] = \frac{3}{6}$

(c) $P[A \cup B] = P[\{1, 3, 4, 5, 6\}] = \frac{5}{6}$
 $P[A \cap B] = P[\{5\}] = \frac{1}{6}$
 $P[A^c] = 1 - P[A] = \frac{3}{6}$

2.22

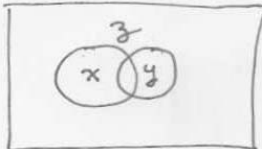
(a) In first toss, each face occurs with relative frequency $\frac{1}{6}$
 Each first toss outcome is followed by each possible face $\frac{1}{6}$ of the time
 $\therefore$ Each pair occurs with relative frequency $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

(b) $P[A] = \frac{21}{36}$ $P[B] = \frac{6}{36}$ $P[C] = \frac{8}{36}$ $P[A \cap B^c] = \frac{15}{36}$ $P[A^c] = \frac{15}{36}$

2.23 $P[a, d] = P_c + P_d = \frac{3}{8}$ by expressing each event in terms of elementary events
 $P[b, c] = P_b + P_c = \frac{6}{8}$ solving this set of linear equations
 $P[a, d] = P_a + P_d = \frac{1}{8}$
 $P[b, c] = P_b + P_c = \frac{6}{8}$
 $1 - P[a, d] = P_a + P_b + P_c + P_d = 1$
 Solving this set of linear equations gives
 $P_a = \frac{1}{8} \quad P_b = \frac{4}{8} \quad P_c = \frac{2}{8} \quad P_d = \frac{1}{8}$

2.24 ④ $P[A \cap B^c] = P[A] - P[A \cap B]$
 $P[A^c \cap B] = P[B] - P[A \cap B]$
 ⑤ $P[A \cap B^c \cup A^c \cap B] = P[A] + P[B] - 2P[A \cap B]$
 ⑥ $P[(A \cup B)^c] = 1 - P[A \cup B] = 1 - P[A] - P[B] + P[A \cap B]$

2.25 $z = P[A \cup B] = P[A] + P[B] - P[A \cap B] = x + y - P[A \cap B]$
 $P[A \cap B] = x + y - z$
 $P[A^c \cap B^c] = 1 - P[(A \cap B)^c] = 1 - P[A \cup B] = 1 - z$
 $P[A^c \cup B^c] = 1 - P[(A^c \cup B^c)^c] = 1 - P[A \cap B] = 1 - x - y + z$
 $P[A \cap B^c] = P[A] - P[A \cap B] = x - (x + y - z) = z - y$
 $P[A^c \cap B] = 1 - P[A \cap B^c] = 1 - z + y$



2.26 Identities of this type are shown by application of the axioms. We begin by treating $(A \cup B)$ as a single event, then

$$\begin{aligned}
 P[A \cup B \cup C] &= P[(A \cup B) \cup C] \\
 &= P[A \cup B] + P[C] - P[(A \cup B) \cap C] && \text{by Cor. 5} \\
 &= P[A] + P[B] - P[A \cap B] + P[C] && \text{by Cor. 5 on } A \cup B \\
 &\quad - P[(A \cap C) \cup (B \cap C)] && \text{and by distributive property} \\
 &= P[A] + P[B] + P[C] - P[A \cap B] \\
 &\quad - P[A \cap C] - P[B \cap C] && \text{by Cor. 5 on} \\
 &\quad + P[(A \cap B) \cap (B \cap C)] && (A \cap C) \cup (B \cap C) \\
 &= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] && \text{since} \\
 &\quad - P[B \cap C] + P[A \cap B \cap C]. && (A \cap B) \cap (B \cap C) = A \cap B \cap C
 \end{aligned}$$

2.27 Corollary 5 implies that the result is true for $n = 2$. Suppose the result is true for n , that is,

$$P\left[\bigcup_{k=1}^n A_k\right] = \sum_{j=1}^n P[A_j] - \sum_{j < k \leq n} P[A_j \cap A_k] + \sum_{j < k < l \leq n} P[A_j \cap A_k \cap A_l] + \dots + (-1)^{n+1} P[A_1 \cap A_2 \cap \dots \cap A_n] \quad (*)$$

Consider the $n + 1$ case and use the argument applied in Prob. 2.18:

$$\begin{aligned} P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^n A_k\right) \cup A_{n+1}\right] \\ &= P\left[\bigcup_{k=1}^n A_k\right] + P[A_{n+1}] - P\left[\left(\bigcup_{k=1}^n A_k\right) \cap A_{n+1}\right] \\ &= \sum_{j=1}^n P[A_j] - \sum_{j < k \leq n} P[A_j \cap A_k] + \dots + (-1)^{n+1} P[A_1 \cap \dots \cap A_n] \\ &\quad + P[A_{n+1}] - P\left[\bigcup_{k=1}^n (A_k \cap A_{n+1})\right] \text{ from } (*) \end{aligned}$$

Apply Equation (*) to the last term in the previous expression

$$P\left[\bigcup_{k=1}^n (A_k \cap A_{n+1})\right] = \sum_{j=1}^n P[A_k \cap A_{n+1}] - \sum_{j < k \leq n} P[A_j \cap A_k \cap A_{n+1}] + \dots + (-1)^{n+1} P[A_1 \cap A_2 \cap \dots \cap A_{n+1}]$$

Thus

$$\begin{aligned} P\left[\bigcup_{k=1}^{n+1} A_k\right] &= \sum_{j=1}^n P[A_j] + P[A_{n+1}] + \\ &\quad - \sum_{j < k \leq n} P[A_j \cap A_k] - \sum_{j=1}^n P[A_k \cap A_{n+1}] \\ &\quad + \sum_{j < k \leq n} P[A_j \cap A_k \cap A_l] + \sum_{j < k \leq n} P[A_j \cap A_k \cap A_{n+1}] \\ &\quad + \dots + (-1)^{n+2} P[A_1 \cap A_2 \cap \dots \cap A_{n+1}] \\ &= \sum_{j=1}^{n+1} P[A_j] - \sum_{j < k \leq n+1} P[A_j \cap A_k] \\ &\quad + \sum_{j < k < l \leq n+1} P[A_j \cap A_k \cap A_l] \\ &\quad + \dots + (-1)^{n+2} P[A_1 \cap A_2 \cap \dots \cap A_{n+1}] \end{aligned}$$

which shows that the $n + 1$ case holds. This completes the induction argument, and the result holds for $n \geq 2$.

2.28

This experiment is equivalent to tossing a coin 3 times and noting the sequence of heads and tails. There are 8 outcomes and each outcome has probability $\frac{1}{8}$.

$$S = \{000, 001, 010, 100, 011, 101, 110, 111\}$$

(a)

$$P[A_1] = P[\{100, 101, 110, 111\}] = \frac{4}{8} = \frac{1}{2}$$

$$P[A_1 \cap A_3] = P[\{101, 111\}] = \frac{2}{8} = \frac{1}{4}$$

$$P[A_1 \cap A_2 \cap A_3] = P[\{111\}] = \frac{1}{8}$$

$$\begin{aligned} P[A_1 \cup A_2 \cup A_3] &= 1 - P[(A_1 \cup A_2 \cup A_3)^c] = 1 - P[A_1^c \cap A_2^c \cap A_3^c] \\ &= 1 - P[\{000\}] = \frac{7}{8}. \end{aligned}$$

(b) Let $p = P[\text{"1"}]$

$$\begin{aligned} P[A_1] &= P[\{100\}] + P[\{101\}] + P[\{110\}] + P[\{111\}] \\ &= p(1-p)^2 + 2p^2(1-p) + p^3 \end{aligned}$$

$$P[A_1 \cap A_3] = p^2(1-p) + p^3$$

$$P[A_1 \cap A_2 \cap A_3] = p^3$$

$$P[A_1 \cup A_2 \cup A_3] = 1 - (1-p)^3$$

2.29

Each transmission is equivalent to tossing a fair coin.
If the outcome is heads, then the transmission is successful.
If tails, then another transmission is required.
As in Example 2.11 the probability that j transmissions
are required is:

$$P[A_j] = \left(\frac{1}{2}\right)^j$$

$$P[A] = P[j \text{ even}] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k - 1$$

$$= \frac{1}{1 - \frac{1}{4}} - 1 = \frac{1}{3}$$

$$P[B] = P[j \text{ multiple of } 3] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{3k} = \frac{1}{1 - \frac{1}{8}} - 1 = \frac{1}{7}$$

$$P[C] = \sum_{k=1}^6 \left(\frac{1}{2}\right)^k = \frac{1}{2} \sum_{k=0}^5 \left(\frac{1}{2}\right)^k = \frac{1}{2} \frac{1 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}} = \frac{63}{64}$$

$$P[C^c] = 1 - P[C] = \frac{1}{64}$$

$$P[A \cap B] = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{6k} = \frac{1}{1 - \frac{1}{64}} - 1 = \frac{1}{63}$$

$$P[A - B] = P[A] - P[A \cap B] = \frac{1}{3} - \frac{1}{63} = \frac{20}{63}$$

$$P[A \cap B \cap C] = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

2.30 a) Corollary 7 implies $P[A \cup B] \leq P[A] + P[B]$. (Eqn. 2.8). Applying this inequality twice, we have

$$P[(A \cup B) \cup C] \leq P[A \cup B] + P[C] \leq P[A] + P[B] + P[C]$$

b) Eqn. 2.8 implies the $n = 2$ case.
Suppose the result is true for n :

$$P\left[\bigcup_{k=1}^n A_k\right] \leq \sum_{k=1}^n P[A_k] \quad (*)$$

Then

$$\begin{aligned} P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^n A_k\right) \cup A_{n+1}\right] \\ &\leq P\left[\bigcup_{k=1}^n A_k\right] + P[A_{n+1}] \text{ by Eqn. 2.8} \\ &\leq \sum_{k=1}^n P[A_k] + P[A_{n+1}] \text{ by } (*) \\ &= \sum_{k=1}^{n+1} P[A_k] \end{aligned}$$

which completes the induction argument.

(c)
$$P\left[\bigcap_{k=1}^n A_k\right] = 1 - P\left[\left(\bigcap_{k=1}^n A_k\right)^c\right] = 1 - P\left[\bigcup_{k=1}^n A_k^c\right]$$

$$\geq 1 - \sum_{k=1}^n P[A_k^c] \text{ using the result of part b.}$$

2.31 Let $A_i = \{\text{ith character is in error}\}$

$$P[\text{any error in document}] = P\left[\bigcup_{i=1}^n A_i\right] \leq \sum_{i=1}^n P[A_i] = np$$

2.32

$$① \quad p_1 = p_3 = p_5 = p \quad p_2 = p_4 = p_6 = 2p$$

$$1 = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 9p \quad p = \frac{1}{9}$$

$$② \quad P[A] = p_4 + p_5 + p_6 = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}$$

$$P[B] = p_1 + p_3 + p_5 = \frac{3}{9}$$

$$③ \quad P[A \cup B] = p_1 + p_3 + p_4 + p_5 + p_6 = 1 - p_2 = \frac{7}{9}$$

$$P[A \cap B] = p_5 = \frac{1}{9}$$

$$P[A^c] = 1 - \frac{5}{9} = \frac{4}{9}$$

2.33

$$A = \{1, 2, \dots, 59, 60\}$$

$$① \quad P[k] = \frac{1}{60} \quad k \in A$$

$$② \quad p_2 = \frac{1}{2} p_1 \quad p_3 = \frac{1}{3} p_1 \quad \dots \quad p_{60} = \frac{1}{60} p_1$$

$$1 = p_1 + p_2 + \dots + p_{60} = p_1 \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{60}\right) = 4.68 p_1$$

$$p_1 = 0.2137$$

$$③ \quad p_2 = \frac{1}{2} p_1 \quad p_3 = \frac{1}{4} p_1 \quad p_4 = \frac{1}{8} p_1 \quad \dots \quad p_{60} = \left(\frac{1}{2}\right)^{59} p_1$$

$$1 = p_1 \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \left(\frac{1}{2}\right)^{59}\right) \approx 2 p_1$$

$$p_1 = \frac{1}{2}$$

$$④ \quad \text{for } k: p[60] = \frac{1}{60} \quad b: p[60] = \frac{0.2137}{60} = 0.00356 \quad c: p[60] = 0.86 \times 10^{-18}$$

2.34

Assume that the probability of any subinterval I of $[-1, 2]$ is proportional to its length, then

$$P[I] = k \text{ length}(I).$$

If we let $I = [-1, 2]$ then we must have that

$$1 = P[S] = P[[-1, 2]] = k \text{ length}([-1, 2]) = 3k \Rightarrow k = \frac{1}{3}.$$

$$\begin{aligned} \text{a) } P[A] &= \frac{1}{3} \text{ length}([-1, 0]) = \frac{1}{3}(1) = \frac{1}{3} \\ P[B] &= \frac{1}{3} \text{ length}((-0.5, 1)) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2} \\ P[C] &= \frac{1}{3} \text{ length}((0.75, 2)) = \frac{1}{3} \cdot \frac{5}{4} = \frac{5}{12} \\ P[A \cap B] &= \frac{1}{3} \text{ length}((-0.5, 0)) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \\ P[A \cap C] &= P[\emptyset] = 0 \end{aligned}$$

$$\text{b) } P[A \cup B] = \frac{1}{3} \text{ length}([-1, 1]) = \frac{2}{3}$$

$$\begin{aligned} P[A \cup C] &= \frac{1}{3} \text{ length}(A \cup C) \\ &= \frac{1}{3} \left(1 + \frac{5}{4} \right) = \frac{9}{4} \end{aligned}$$

$$P[A \cup B \cup C] = P[S] = 1$$

Now use axioms and corollaries:

$$\begin{aligned} P[A \cup B] &= P[A] + P[B] - P[A \cap B] \quad \text{by Cor. 5} \\ &= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \quad \checkmark \end{aligned}$$

$$P[A \cup C] = P[A] + P[C] - P[A \cap C] = \frac{1}{3} + \frac{5}{12} = \frac{9}{12} = \frac{3}{4} \quad \checkmark \quad \text{by Cor. 5}$$

$$\begin{aligned} P[A \cup B \cup C] &= P[A] + P[B] + P[C] \\ &\quad - P[A \cap B] - P[A \cap C] - P[B \cap C] \\ &\quad + P[A \cap B \cap C] \quad \text{by Eq. (2.7)} \\ &= \frac{1}{3} + \frac{1}{2} + \frac{5}{12} - \frac{1}{6} - 0 - \frac{1}{12} + 0 \\ &= 1 \quad \checkmark \end{aligned}$$

2.35 a) Let I be a subinterval of $[-1, 2]$ then

$$P[I] = \text{length}(I \cap [0, 2]) + \text{length}(I \cap [-1, 0])$$

Letting $I = [-1, 2]$ we have

$$1 = P[[-1, 2]] = 2k + 2k = 4k \Rightarrow k = \frac{1}{4}$$

$$\text{b) } P[A] = \frac{2}{4}(1) = \frac{1}{2}$$

$$P[B] = \frac{2}{4}\left(\frac{1}{2}\right) + \frac{1}{4}(1) = \frac{5}{8}$$

$$P[C] = \frac{1}{4}\left(\frac{5}{4}\right) = \frac{5}{16}$$

$$P[A \cap B] = \frac{2}{4}\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$P[A \cap C] = P[\emptyset] = 0$$

$$P[A \cup B] = \frac{2}{4}(1) + \frac{1}{4}(1) = \frac{3}{4}$$

$$P[A \cup C] = \frac{2}{4}(1) + \frac{1}{4}\left(\frac{5}{4}\right) = \frac{13}{16}$$

$$P[A \cup B \cup C] = P[S] = 1$$

Now use axioms and corollaries

$$\begin{aligned} P[A \cup B] &= P[A] + P[B] - P[A \cap B] \\ &= \frac{1}{2} + \frac{5}{8} - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

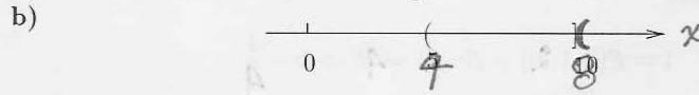
$$\begin{aligned} P[A \cup C] &= P[A] + P[C] - P[A \cap C] \\ &= \frac{1}{2} + \frac{5}{16} - 0 = \frac{13}{16} \end{aligned}$$

$$\begin{aligned} P[A \cup B \cup C] &= P[A] + P[B] + P[C] + \\ &\quad - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C] \\ &= \frac{1}{2} + \frac{5}{8} + \frac{5}{16} - \frac{1}{4} - 0 - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = 1 \checkmark \end{aligned}$$

2.36 Let x denote the lifetime, then

$$A = \{x > 4\} \text{ and } B = \{x > 8\}$$

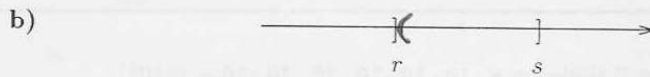
$$\begin{aligned} \text{a) } P[A \cap B] &= P[\{x > 4\} \cap \{x > 8\}] = P[\{x > 8\}] = \frac{1}{8} \\ P[A \cup B] &= P[\{x > 4\} \cup \{x > 8\}] = P[\{x > 4\}] = \frac{1}{4} \end{aligned}$$



$$\begin{aligned} P[\{x > 5\}] &= P[\{5 < x \leq 10\} \cup \{x > 10\}] \\ &= P[\{5 < x \leq 10\}] + P[\{x > 10\}] \\ \Rightarrow P[\{5 < x \leq 10\}] &= P[\{x > 5\}] - P[\{x > 10\}] = \frac{1}{6} - \frac{1}{12} = \frac{1}{12} \end{aligned}$$

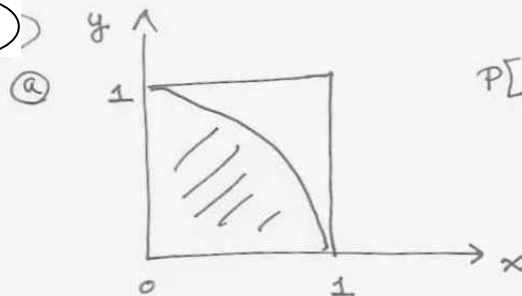
2.37 a) Since $(-\infty, r] \subset (-\infty, s]$ when $r < s$

$$P[(-\infty, r]] \leq P[(-\infty, s]] \text{ by Corollary 7.}$$



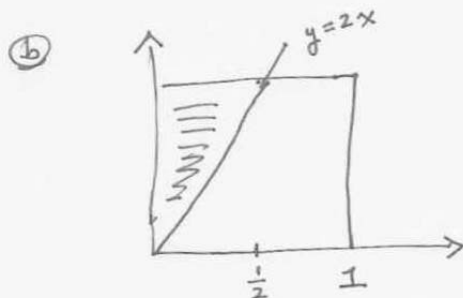
$$\begin{aligned} P[(-\infty, s]] &= P[(-\infty, r] \cup (r, s]] \\ &= P[(-\infty, r]] + P[(r, s]] \\ \Rightarrow P[(r, s]] &= P[(-\infty, s]] - P[(-\infty, r]] \end{aligned}$$

2.38



$$P[x^2 + y^2 < 1] = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$$

Area inside circle



$$P[y > 2x] = \frac{1}{4}$$

Area in right triangle

2.3 *Computing Probabilities Using Counting Methods

2.39 The number of distinct ordered triplets $= 60 \cdot 60 \cdot 60 = 60^3$

2.40 The number of distinct 7-tuples $= 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8(10^6)$

2.41 The number of distinct ordered triplets $= 6 \cdot 2 \cdot 52 = 624$

2.42 #sequences of length 8 $= 2^8 = 256$
 $P[\text{arbitrary sequence} = \text{correct sequence}] = \frac{1}{256}$
 $P[\text{success in two tries}] = 1 - P[\text{failure in both tries}]$
 $= 1 - \frac{255}{256} \cdot \frac{255}{256}$

2.43 8, 9, or 10 characters long
 - at least 1 special character from set of size 24
 - numbers from size 10
 - upper & lower case letters $26 \times 2 = 52$ } 62 choices
 For length n :
 - pick position of required special character & pick character
 n positions $\times$ 24 characters.
 - pick number/letter/special character for remaining $n-1$ positions
 86^{n-1}
 Total # passwords $= n \cdot 24 \cdot 86^{n-1}$
 Length 8, 9, or 10 $= 8 \cdot 24 \cdot 86^7 + 9 \cdot 24 \cdot 86^8 + 10 \cdot 24 \cdot 86^9 = 6.24 \times 10^{12}$
 Time to try all passwords $= 6.24 \times 10^{13}$ seconds $= 2(10^6)$ years

2.44 $3^{10} = 59049$ possible answers
 Assuming each paper selects answers at random
 $P[\text{two papers are identical}] = \frac{1}{3^{10}} \times \frac{1}{3^{10}} = \frac{1}{3^{20}} = 2.87 \times 10^{-10}$

2.45 (a) # combinations = $5 \times 3 = 15$

(b) The table below shows the 15 combinations and a schedule that allows all combinations without using the same t-shirt on consecutive days.

jeans	t-shirts				
	1	2	3	4	5
1	1	4	7	10	13
2	14	2	5	8	11
3	12	15	3	6	9

2.46 The order in which the 4 toppings are selected does not matter so we have sampling without ordering.

If toppings may not be repeated, Eqn. (2.22) gives

$$\binom{15}{4} = 1365 \text{ possible deluxe pizzas}$$

If toppings may be repeated, we have sampling with replacement and without ordering. The number of such arrangements is

$$\binom{14 + 4}{4} = 3060 \text{ possible deluxe pizzas.}$$

2.47 # student seat selection = $60 \cdot 59 \cdot 58 \cdot \dots \cdot 16 = \frac{60!}{15!}$

2.48 $a b \quad b a \Rightarrow 2 = 2!$

$a b c \quad \cancel{b a c} \quad b c a \quad a c b \quad b a c \quad c b a \Rightarrow 6 = 3!$

$a b c d \quad d a b c \quad c d a b \quad b c d a$
 $a c b d \quad d a c b \quad b d a c \quad c b d a$
 $a d b c \quad c a d b \quad b c a d \quad d b c a$
 $a b d c \quad c a b d \quad d c a b \quad b d c a$
 $a c d b \quad b a c d \quad d b a c \quad c d b a$
 $a d c b \quad b a d c \quad c b a d \quad d c b a$

} $\Rightarrow 24 = 4!$

- 2.49 There are $3!$ permutations of which only one corresponds to the correct order; assuming equiprobable permutations:

$$P[\text{correct order}] = \frac{1}{3!} = \frac{1}{6}$$

2.50

ways to cover all buckets $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$
placement of 5 balls w 5 buckets $= 5^5$
probability all buckets covered $= 5! / 5^5 = 0.0384$

2.51

Combinations of 2 from 2 objects : $ab \quad \binom{2}{2} = 1$
Combinations of 2 " 3 objects : $ab \quad ac \quad bc \quad \binom{3}{2} = \frac{3!}{2!} = 3$
Combinations of 2 " 4 objects : $ab \quad ac \quad ad \quad bc \quad bd \quad cd \quad \binom{4}{2} = \frac{4!}{2!2!} = 6$

2.52

$8!$ arrangements of people around a table $= 40320$
Experiment: Select male or female for first spot: 2
Select first spot gender $\times$ 4
" 2nd spot gender $\times$ 4
" 3rd spot gender $\times$ 3
:
:
 $2 \times 4! \times 4! = 1152$

2.53

Number ways of picking one out of 6 $= \binom{6}{1} = 6$
Number ways of picking two out of 6 $= \binom{6}{2} = 15$
Number ways of picking none, some or all of 6 $= \sum_{j=0}^6 \binom{6}{j} = 2^6 = 64$

2.54a The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which m of the chosen items are defective and $M - m$ are nondefective.

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M - m$ nondefectives out of $100 - k$ is $\binom{100 - k}{M - m}$.

The number of ways of choosing m defectives out of k and $M - m$ non-defectives out of $100 - k$ is

$$\binom{k}{m} \binom{100 - k}{M - m}$$

$$\begin{aligned} P[m \text{ defectives in } M \text{ samples}] &= \frac{\# \text{ outcomes with } k \text{ defective}}{\text{Total } \# \text{ of outcomes}} \\ &= \frac{\binom{k}{m} \binom{100 - k}{M - m}}{\binom{100}{M}} \end{aligned}$$

This is called the Hypergeometric distribution.

(b) $P[\text{lot accepted}] = P[m=0 \text{ or } m=1] = \frac{\binom{100-k}{M}}{\binom{100}{M}} + \frac{k \binom{100-k-1}{M-1}}{\binom{100}{M}}$

2.55

Number ways of picking 20 raccoons out of $N = \binom{N}{20}$

Number ways of picking 4 tagged raccoons out of 8 and 16 untagged raccoons out of $N - 16 = \binom{8}{4} \binom{N - 16}{16}$

$$P[5 \text{ tagged out of } 20 \text{ samples}] = \frac{\binom{8}{5} \binom{N - 16}{15}}{\binom{N}{20}} \triangleq p(N)$$

$p(N)$ increases with N as long as $p(N)/p(N-1) > 1$

$$\frac{p(N)}{p(N-1)} = \frac{\binom{N-16}{15} \binom{N-1}{20}}{\binom{N}{20} \binom{N-17}{15}} = \frac{(N-16)(N-20)}{(N-25)N} \geq 1$$

$$(N-16)(N-20) \geq (N-25)N \Rightarrow 40 \geq N$$

$$p(40) = p(39) = \text{maxima of } p(N).$$

$$0.305$$

2.56

⑥ $P[X=k] = \frac{\binom{10}{k} \binom{40}{5-k}}{\binom{50}{5}} \quad k=0,1,\dots,5 \quad \text{without replacement}$
Hypergeometric probabilities

⑦ With replacement:
pick k defective balls then pick $5-k$ nondefective balls
 $\underbrace{\hspace{10em}}_{10^k} \quad \underbrace{\hspace{10em}}_{40^{5-k}}$

There are $\binom{5}{k}$ arrangements of this composition

ways of obtaining k defective in 5 tested $= \frac{\binom{50}{k} 10^k 40^{5-k}}{50^5}$

$= \binom{5}{k} \left(\frac{10}{50}\right)^k \left(\frac{40}{50}\right)^{5-k} \quad k=0,1,\dots,5$

Binomial probabilities.

2.57

$\frac{9!}{4!2!3!} = 1260$

2.58

forward combinations $\binom{6}{3}$
defense combinations $\binom{4}{2}$
goalie combinations $\binom{2}{1}$ } assuming forwards do not have assigned position (left, center, right) and similarly for defensemen

teams $= \binom{6}{3} \binom{4}{2} \binom{2}{1} = 240$

∴ forwards + defensemen have assigned position

teams $= \binom{6}{3} \times 3! \times \binom{4}{2} \times 2! \times \binom{2}{1} = 4760$

2.59

Suppose each student is viewed as selecting one of the 7 days (e.g. placing a ball in one of 7 urns) then there are 7^{28} possible sequences of choices. Of the sequences that have 4 choices for each day there are

$$\frac{28!}{4!4!4!4!4!4!4!} \text{ such sequences.}$$

$$\therefore P[4 \text{ students at each day}] = \frac{28!}{(4!)^7} \frac{1}{7^{28}}$$

2.60

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!}$$

2.61

a) Since N_i denotes the number of possible outcomes of the i th subset after $i-1$ subsets have been selected, it can be considered as the number of subpopulations of size k_i from a population of size $n - k_1 - k_2 - \dots - k_{i-1}$, hence

$$N_i = \binom{n - k_1 - \dots - k_{i-1}}{k_i} \quad i = 1, \dots, J-1$$

Note that after $J-1$ subsets are selected, the set B_J is determined, i.e. $N_J = 1$.

b) The number of possible outcomes for B_1 is N_1 , B_2 is N_2 , etc. hence

$$\# \text{ partitions} = N_1 N_2 \dots N_{J-1} = \prod_{i=1}^{J-1} \frac{(n - k_1 - \dots - k_{i-1})!}{k_i!(n - k_1 - \dots - k_i)!} = \frac{n!}{k_1! k_2! \dots k_J!}$$

2.4 Conditional Probability

2.62 $A = \{N_1 \geq N_2\}$ $B = \{N_1 = 6\}$

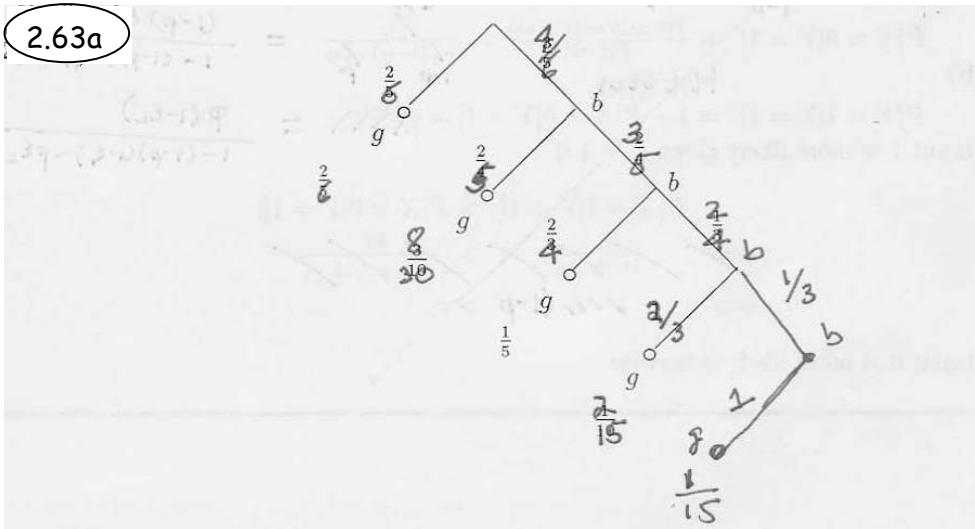
From problem 2.2 we have that $A \supset B$, therefore

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$$

and

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[B]}{P[A]} = \frac{4/36}{24/36} = \frac{2}{7}$$

2.63a



$$P[g] = \frac{2}{5}$$

$$P[b]g = P[b]P[g|b] = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$$

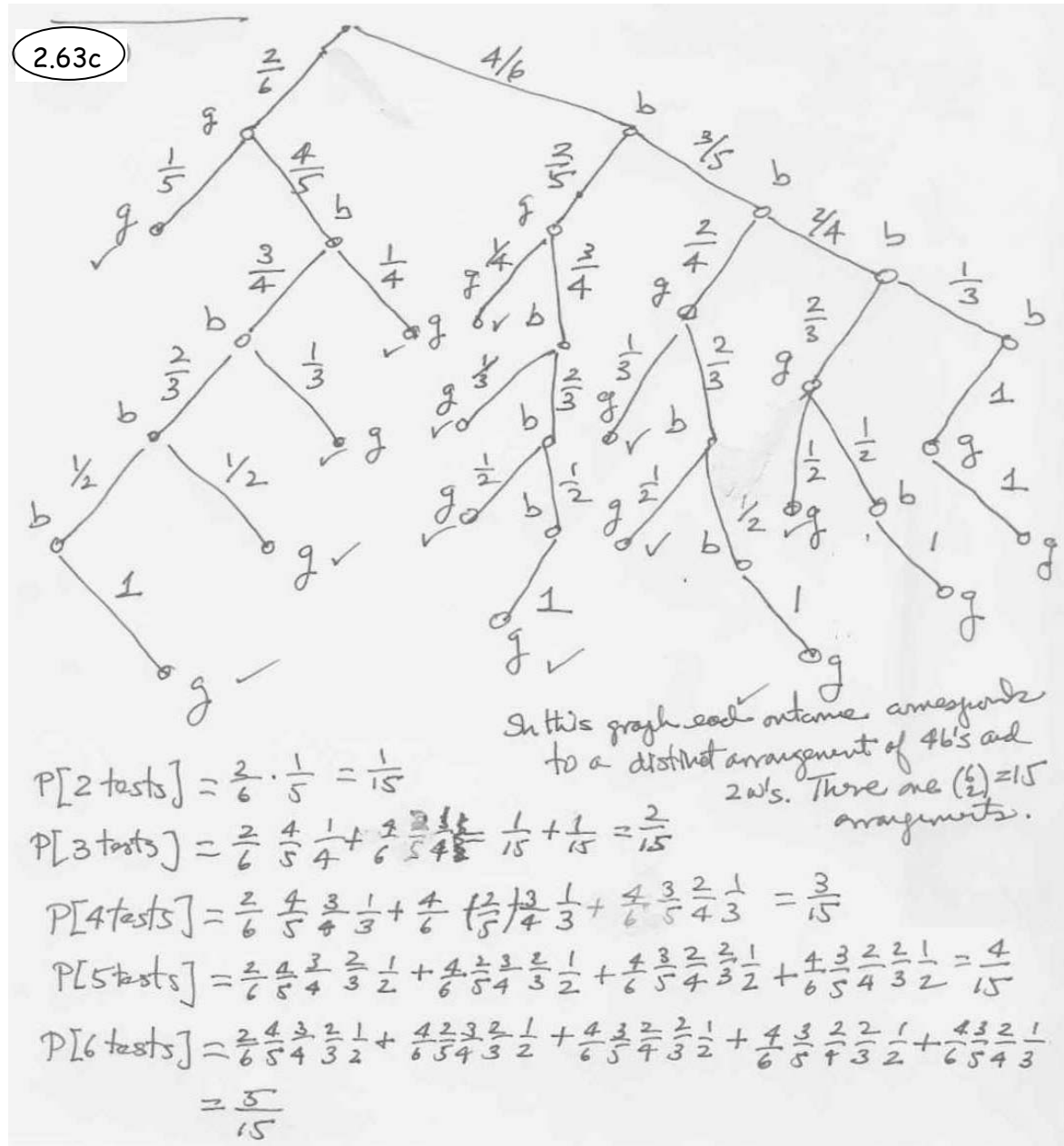
$$P[bb]g = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}$$

$$P[bbb]g = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{15}$$

$$P[bbbb]g = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{15}$$

b) $P[1 \text{ pen tested}] = P[g] = \frac{2}{5}$

$P[2] = P[b]g$ $P[3] = P[bb]g$ $P[4] = P[bbb]g$ $P[5] = P[bbbb]g$



2.64

$$P[B \cap C | A] = P[\text{Bob \& Chris pick their names} | \text{Al picked his name}]$$

$$= \frac{P[B \cap C \cap A]}{P[A]} = \frac{P[\{abc\}]}{P[\{abc, acb\}]} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$$

$$P[C | A \cap B] = P[\text{Chris picks his name} | \text{Al \& Bob picked their names}]$$

$$= \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{P[\{abc\}]}{P[\{abc\}]} = 1$$

2.65
$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[\text{multiple of 6}]}{P[\text{even}]} = \frac{1/6}{1/3} = \frac{1}{2}$$
$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[\text{multiple of 6}]}{P[\text{multiple of 6}]} = 1.$$

2.66 From problem 2.8:

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[\frac{3}{4} < U \leq 1]}{P[|U - \frac{1}{2}| > \frac{1}{4}]} = \frac{1/4}{1/2} = \frac{1}{2}$$
$$P[B|A] = \frac{P[A \cap B]}{P[B]} = \frac{P[\frac{3}{4} < U \leq 1]}{P[\frac{1}{2} < U \leq 1]} = \frac{1/4}{1/2} = \frac{1}{2}.$$

2.67 From problem 2.36

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[X > 8]}{P[X > 4]} = \frac{1/8}{1/4} = \frac{1}{2}$$
$$P[A|B] = \frac{P[X > 8]}{P[X > 8]} = 1.$$

2.68

(a)

$$P[A] = P[\text{hand rests in last 10 minutes}]$$

$$P[A] = P_{51} + P_{52} + \dots + P_{60} = \frac{10}{60} = \frac{1}{6}$$

$$P[B] = P_{52} + P_{57} + P_{58} + P_{59} + P_{60} = \frac{5}{60} = \frac{1}{12}$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{1/12}{1/6} = \frac{1}{2}$$

(b)

$$P[A] = P_1 \left(\frac{1}{51} + \frac{1}{52} + \dots + \frac{1}{60} \right)$$

$$P[B] = P_1 \left(\frac{1}{56} + \frac{1}{57} + \dots + \frac{1}{60} \right)$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{\frac{1}{56} + \frac{1}{57} + \dots + \frac{1}{60}}{\frac{1}{51} + \frac{1}{52} + \dots + \frac{1}{60}} = 0.477$$

(c)

$$P[A] = \frac{1}{2} \left(\left(\frac{1}{2}\right)^{50} + \left(\frac{1}{2}\right)^{56} + \dots + \left(\frac{1}{2}\right)^{59} \right)$$

$$P[B] = \frac{1}{2} \left(\left(\frac{1}{2}\right)^{55} + \dots + \left(\frac{1}{2}\right)^{59} \right)$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{\left(\frac{1}{2}\right)^{56} + \dots + \left(\frac{1}{2}\right)^{60}}{\left(\frac{1}{2}\right)^{51} + \dots + \left(\frac{1}{2}\right)^{60}} = 0.030$$

2.69 Proceeding as in Problem 2.84

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[(-0.5, 0)]}{P[(-0.5, 1)]} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$P[B|C] = \frac{P[B \cap C]}{P[C]} = \frac{P[(0.75, 1)]}{P[(0.75, 2)]} = \frac{1/12}{5/12} = \frac{1}{5}$$

$$P[A|C^c] = \frac{P[A \cap C^c]}{P[C^c]} = \frac{P[(-1, 0)]}{P[[-1, 0.75]]} = \frac{1/3}{7/12} = \frac{4}{7}$$

$$P[B|C^c] = \frac{P[B \cap C^c]}{P[C^c]} = \frac{P[(-0.5, 0.75)]}{P[[-1, 0.75]]} = \frac{5/12}{7/12} = \frac{5}{7}$$

2.70

$$P[x > 2t / x > t] = \frac{P[\{x > 2t\} \cap \{x > t\}]}{P[x > t]} = \frac{P[x > 2t]}{P[x > t]}$$

$$= \frac{1/2t}{1/t} = \frac{1}{2} \quad t > 1$$

This conditional probability does not depend on t .
The corresponding probability law is said to be scale-invariant.

2.71

$$P[2 \text{ or more students have same birthday}]$$

$$= 1 - P[\text{all students have different birthdays}]$$

$$P[\text{all students have different birthdays}]$$

$$= \frac{365}{365} \frac{364}{365} \frac{363}{365} \cdots \frac{346}{365} = 0.588$$

$$P[2 \text{ or more have same birthday}] = 0.412$$

$P[2 \text{ or more have same birthday in class of } 23] = 0.507$

2.72

of fingerprints = 2^L $L=64$ or $L=128$

Pick hashes at random until we find a repeat.

Same as birthday problem (problem 2.71)

$$P[\text{all hashes different given } N \text{ tries}] = \frac{2^L}{2^L} \frac{2^L-1}{2^L} \dots \frac{2^L-N+1}{2^L}$$

Find N so that

$$\frac{1}{2} = 1 - \prod_{j=0}^{N-1} \frac{2^L-j}{2^L} = 1 - p(N)$$

$$\ln p(N) = \sum_{j=0}^{N-1} \ln \left(1 - \frac{j}{2^L} \right) \approx \sum_{j=0}^{N-1} -\frac{j}{2^L} = -\frac{1}{2^L} \sum_{j=0}^{N-1} j$$

$$\approx -\frac{1}{2^L} \frac{N(N-1)}{2}$$

$$p(N) = e^{-\frac{N(N-1)}{2} \frac{1}{2^L}} \approx e^{-\frac{N^2}{2} \frac{1}{2^L}} = \frac{1}{2}$$

$$N \approx \sqrt{(2 \ln 2) 2^L} = 1.17 \cdot 2^{L/2}$$

For $L=64$ 2^{32} attempts required
 $L=128$ 2^{64} "

2.73 a) The results follow directly from the definition of conditional probability. $P[A|B] = \frac{P[A \cap B]}{P[B]}$

If $A \cap B = \emptyset$ then $P[A \cap B] = 0$ by Corollary 3 and thus $P[A|B] = 0$.

If $A \subset B$ then $A \cap B = A$ and $P[A|B] = \frac{P[A]}{P[B]}$.

If $A \supset B \Rightarrow A \cap B = B$ and $P[A|B] = \frac{P[B]}{P[B]} = 1$.

b) If $P[A|B] = \frac{P[A \cap B]}{P[B]} > P[A]$ then multiplying both sides by $P[B]$ we have:
 $P[A \cap B] > P[A]P[B]$

We then also have that $P[B|A] = \frac{P[A \cap B]}{P[A]} > \frac{P[A]P[B]}{P[A]} = P[B]$.

We conclude that if $P[A|B] > P[A]$ then B and A tend to occur jointly.

2.74

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \text{ for } P[B] > 0.$$

$$(i) \quad P[A \cap B] \geq 0 \Rightarrow P[A|B] \geq 0. \quad \checkmark$$

$$A \cap B \subset B \Rightarrow P[A \cap B] \leq P[B] \Rightarrow P[A|B] \leq 1. \quad \checkmark$$

$$(ii) \quad P[A|B] = \frac{P[B \cap A]}{P[B]} = \frac{P[B]}{P[B]} = 1 \quad \checkmark$$

(iii) If $A \cap C = \emptyset$ then

$$P[A \cup C|B] = \frac{P[(A \cup C) \cap B]}{P[B]} = \frac{P[(A \cap B) \cup (C \cap B)]}{P[B]}$$

$$= \frac{P[A \cap B] + P[C \cap B]}{P[B]} \quad \text{since } (A \cap B) \cap (C \cap B) = A \cap B \cap C = \emptyset$$

$$= P[A|B] + P[C|B] \quad \checkmark$$

$$\begin{aligned} 2.75 \quad P[A \cap B \cap C] &= P[A|B \cap C]P[B \cap C] \\ &= P[A|B \cap C]P[B|C]P[C] \end{aligned}$$

2.76 a) We use conditional probability to solve this problem. Let $A_i = \{\text{nondefective item found in } i\text{th test}\}$. A lot is accepted if the items in tests 1 and 2 are nondefective, that is, if $A_1 \cap A_2$ occurs. Therefore

$$\begin{aligned} P[\text{lot accepted}] &= P[A_2 \cap A_1] \\ &= P[A_2|A_1]P[A_1] \quad \text{by Eqn. 2.28} \end{aligned}$$

This equation simply states that we must have A_1 occur, and then A_2 occur given that A_1 already occurred. If the lot of 100 items contains k defective items then

$$\begin{aligned} P[A_1] &= \frac{100-k}{100} \quad \text{and} \\ P[A_2|A_1] &= \frac{99-k}{99} \quad \text{since } 99 \text{ of the many } 99 \text{ items are non-defective.} \end{aligned}$$

Thus

$$P[\text{lot accepted}] = \frac{94}{99} \cdot \frac{95}{100} \cdot \frac{99-k}{99} \cdot \frac{100-k}{100}$$

(b) $P[1 \text{ or more items in } m \text{ tested are defective}] > 99\%$
 $\Leftrightarrow P[\text{no items in } m \text{ are defective}] < 1\%$

$$P[A_m A_{m-1} \dots A_1] = \frac{50}{100} \cdot \frac{49}{99} \dots \frac{50-m+1}{100-m+1} = 0.01$$

For $m=6$ we have

$$P[A_6 A_5 A_4 A_3 A_2 A_1] = \frac{50}{100} \dots \frac{45}{95} = 0.0133$$

2.77 Let X denote the input and Y the output

$$\begin{aligned} \textcircled{a} \quad P[Y=0] &= P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=1] \\ &= (1-\varepsilon_1)p + \varepsilon_1 p. \end{aligned}$$

similarly

$$P[Y=1] = (1-\varepsilon_2)p + \varepsilon_2 p$$

$$\textcircled{b} \quad P[X=0|Y=1] = \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\varepsilon_1 p}{(1-\varepsilon_2)p + \varepsilon_1 p}$$

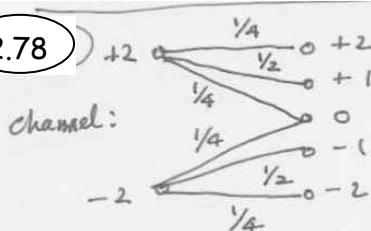
$$P[X=1|Y=1] = \frac{(1-\varepsilon_2)p}{(1-\varepsilon_2)p + \varepsilon_1 p}$$

$$P[X=1|Y=1] > P[X=0|Y=1]$$

$$\Leftrightarrow (1-\varepsilon_2)p > \varepsilon_1 p = \varepsilon_1(1-p)$$

$$\Leftrightarrow p > \frac{\varepsilon_1}{1-\varepsilon_2+\varepsilon_1}$$

2.78



$$\textcircled{b} \quad P[X=+2, Y=+2] = P[Y=+2|X=+2]P[X=+2] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

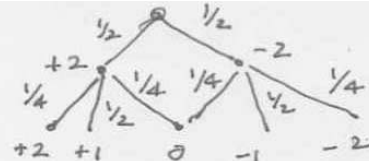
$$P[X=+2, Y=+1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P[X=+2, Y=0] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

$$P[X=-2, Y=0] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

$$P[X=-2, Y=+1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P[X=-2, Y=-2] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$



$$P[Y=+2] = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} = P[Y=-2]$$

$$P[Y=+1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P[Y=-1]$$

$$P[Y=0] = 2 \left(\frac{1}{2} \cdot \frac{1}{4} \right) = \frac{1}{2} = P[Y=0]$$

$$\textcircled{c} \quad P[X=2|Y=k] = \frac{P[Y=k|X=2]P[X=2]}{P[Y=k]}$$

$$= \begin{cases} \frac{1/8}{1/8} = 1 & k=2 \\ \frac{1/4}{1/4} = 1 & k=1 \\ \frac{1/8}{1/8} = 1/2 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

2.79

$$\textcircled{a} P[N=k] = P[N=k|\text{coin 1}]P[\text{coin 1}] + P[N=k|\text{coin 2}]P[\text{coin 2}]$$

$$= \binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

$$\textcircled{b} P[\text{coin 1} | N=k] = \frac{P[N=k|\text{coin 1}]P[\text{coin 1}]}{P[N=k]} \quad k=0,1,2,3$$

$$= \frac{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2}}{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}}$$

$$\textcircled{c} \text{Coin 1 is more probable if}$$

$$\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} > \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{3-k} = 2^k \left(\frac{1}{2}\right)^{3-k} = \left(\frac{1}{8}\right) 4^k$$

$$0 > \ln \frac{1}{8} + k \ln 4$$

$$1.5 = \frac{-\ln 8}{\ln 4} > k$$

Coin 1 more probable if $N=0$ or 1
Coin 2 more probable otherwise.

$$\textcircled{d} \text{In general coin 1 is more probable if}$$

$$\binom{n}{k} p_1^k (1-p_1)^{n-k} \frac{1}{2} > \binom{n}{k} p_2^k (1-p_2)^{n-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{n-k} = \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^n$$

$$T = \frac{n \ln \left(\frac{1-p_1}{1-p_2}\right)}{\ln \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)} > k$$

$$\textcircled{e} \text{If } p_2 = 1 \text{ then } P[N=k|\text{coin 2}] = \begin{cases} 1 & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$$

We cannot determine coin with certainty only if all tosses are heads.
 $P[\text{coin 1} | n \text{ heads}] = (1-p_1)^n / [1 + (1-p_1)^n]$

2.80

$$P[\text{chip defective}] = P[\text{def.}|A]P[A] + P[\text{def.}|B]P[B] + P[\text{def.}|C]P[C] \\ = 5(10^{-3})p_A + 10(10^{-3})p_B + 10(10^{-3})p_C = 6.6 \times 10^{-3}$$

$$P[A|\text{chip defective}] = \frac{P[\text{def.}|A]P[A]}{P[\text{def.}]} = \frac{5 \cdot 10^{-3} p_A}{10^{-3} p_A + 5(10^{-3})p_B + 10(10^{-3})p_C} = 0.3788$$

$$= \frac{p_A}{p_A + 5p_B + 10p_C}$$

Similarly

$$P[C|\text{chip defective}] = \frac{10(10^{-3})(0.4)}{10^{-3} p_A + 5(10^{-3})p_B + 10(10^{-3})p_C} = 0.6061$$

2.81

Let X denote the input and Y the output.

$$\begin{aligned} \text{a) } P[Y = 0] &= P[Y = 0|X = 0]P[X = 0] + P[Y = 0|X = 1]P[X = 1] \\ &\quad + P[Y = 0|X = 2]P[X = 2] \\ &= (1 - \varepsilon) \frac{1}{2} + \varepsilon \frac{1}{4} + 0 \cdot \frac{1}{4} = (1 - \varepsilon) \frac{1}{2} + \varepsilon \frac{1}{4} = \frac{1}{4} \end{aligned}$$

Similarly

$$P[Y = 1] = \varepsilon \frac{1}{2} + (1 - \varepsilon) \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{1}{4} + \frac{\varepsilon}{4}$$

$$P[Y = 2] = 0 \cdot \frac{1}{2} + \varepsilon \frac{1}{4} + (1 - \varepsilon) \frac{1}{4} = \frac{1}{4}$$

b) Using Bayes' Rule

$$P[X = 0|Y = 1] = \frac{P[Y = 1|X = 0]P[X = 0]}{P[Y = 1]} = \frac{\frac{1}{4} \varepsilon}{\frac{1}{4} + \frac{\varepsilon}{4}} = \frac{\varepsilon}{1 + \varepsilon}$$

$$P[X = 1|Y = 1] = \frac{P[Y = 1|X = 1]P[X = 1]}{P[Y = 1]} = \frac{(1 - \varepsilon) \frac{1}{2}}{\frac{1}{4} + \frac{\varepsilon}{4}} = \frac{2(1 - \varepsilon)}{1 + \varepsilon}$$

$$P[X = 2|Y = 1] = 0$$

2.5 Independence of Events

2.82

$$P[A \cap B] = P[\{1\}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2} \cdot \frac{1}{2} \quad \checkmark$$

$$P[A \cap C] = P[\{1\}] = \frac{1}{4} = P[A]P[C] = \frac{1}{2} \cdot \frac{1}{2} \quad \checkmark$$

$$P[B \cap C] = P[\{1\}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2} \quad \checkmark$$

$$P[A \cap B \cap C] = P[\{1\}] = \frac{1}{4} \neq P[A]P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$\Rightarrow$ Not independent

2.83

$$P[A \cap B] = P[\frac{1}{4} < U < \frac{1}{2}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2} \cdot \frac{1}{2} \quad \checkmark \quad A \text{ and } B \text{ indep}$$

$$P[A \cap C] = 0 \neq P[A]P[C] = \frac{1}{2} \cdot \frac{1}{2} \Rightarrow \text{Not indep.}$$

$$P[B \cap C] = P[\frac{1}{2} < U < \frac{3}{4}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2} \quad \checkmark \quad B \text{ and } C \text{ indep.}$$

2.84

Let $A = \{\text{Alice makes shot}\}$ $M = \{\text{Mary makes shot}\}$

We assume that A and M are independent

$$P[A] = p_a$$

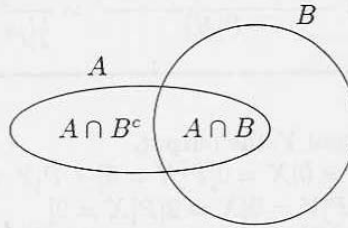
$$P[\text{one makes a shot}] = P[A^c \cup A^c M] = P[A^c] + P[A^c M] \quad \leftarrow \text{since } A^c M \cap A^c M^c = \emptyset$$

$$= p_a(1-p_m) + (1-p_a)p_m \quad \text{by independence}$$

$$P[AM] = p_a p_m$$

$$P[A^c M^c] = (1-p_a)(1-p_m).$$

2.85 The event A is the union of the mutually exclusive events $A \cap B$ and $A \cap B^c$, thus



$$\begin{aligned}
 P[A] &= P[A \cap B] + P[A \cap B^c] \quad \text{by Corollary 1} \\
 \Rightarrow P[A \cap B^c] &= P[A] - P[A \cap B] \\
 &= P[A] - P[A]P[B] \quad \text{since } A \text{ and } B \text{ are independent} \\
 &= P[A](1 - P[B]) \\
 &= P[A]P[B^c] \Rightarrow \quad A \text{ and } B^c \text{ are independent}
 \end{aligned}$$

Similarly

$$\begin{aligned}
 P[B] &= P[A \cap B] + P[A^c \cap B] = P[A]P[B] + P[A^c \cap B] \\
 \Rightarrow P[A^c \cap B] &= P[B](1 - P[A]) = P[B]P[A^c] \\
 &\Rightarrow A \text{ and } B \text{ are independent}
 \end{aligned}$$

Finally

$$P[A^c] = P[A^c \cap B] + P[A^c \cap B^c] = P[A^c]P[B] + P[A^c \cap B^c]$$

$$\begin{aligned}
 \Rightarrow P[A^c \cap B^c] &= P[A^c](1 - P[B]) = P[A^c]P[B^c] \\
 &\Rightarrow A^c \text{ and } B^c \text{ are independent}
 \end{aligned}$$

2.86

$$P[A|B] = P[A|B^c] \Rightarrow \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B^c]}{P[B^c]}$$

$$\begin{aligned}
 \Rightarrow P[A \cap B]P[B^c] &= P[A \cap B^c]P[B] \\
 &= (P[A] - P[A \cap B])P[B] \quad \text{see Prob. 2.58 solution}
 \end{aligned}$$

$$\Rightarrow P[A \cap B] \underbrace{(P[B^c] + P[B])}_1 = P[A]P[B]$$

$$\Rightarrow P[A \cap B] = P[A]P[B]$$

2.87

$$(a) P[A \cup B] = P[A] + P[B] - P[AB] = P_A + P_B - P_A P_B$$

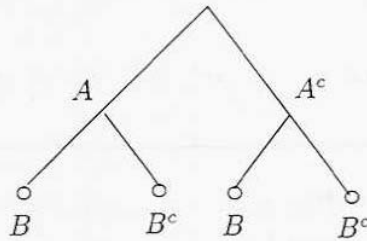
$$(b) P[A \cup B] = P[A] + P[B] = P_A + P_B$$

$$(c) P[A \cup B \cup C] = P[A] + P[B] + P[C] + P[AB] - P[AC] - P[BC] + P[ABC] \\ = P_A + P_B + P_C - P_A P_B - P_A P_C - P_B P_C + P_A P_B P_C$$

$$(d) P[A \cup B \cup C] = P_A + P_B + P_C$$

2.88

We use a tree diagram to show the sequence of events. First we choose an urn, so A or A^c occurs. We then select a ball, so B or B^c occurs:



Now A and B are independent events if

$$P[B|A] = P[B]$$

But

$$P[B|A] = P[B] = P[B|A]P[A] + P[B|A^c]P[A^c]$$

$$\Rightarrow P[B|A](1 - P[A]) = P[B|A^c]P[A^c]$$

$\Rightarrow P[B|A] = P[B|A^c]$ prob. of B is the same given A or A^c , that is,
the probability of B is the same for both urns.

2.89

- a) $P[A]P[B^c]P[C^c] + P[A^c]P[B]P[C^c] + P[A^c]P[B^c]P[C]$
 b) $P[A]P[B]P[C^c] + P[A^c]P[B]P[C] + P[A]P[B^c]P[C]$
 c) $1 - P[A^c]P[B^c]P[C^c]$
 d) $P[A]P[B]P[C^c] + P[A]P[B^c]P[C] + P[A^c]P[B]P[C] + P[A]P[B]P[C]$
 e) $P[A^c]P[B^c]P[C^c]$

2.90

Series $P[D_a] = P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$
 Parallel $P[D_a] = P[A_1 \cup A_2 \cup A_3]$

$$= P[A_1] + P[A_2] + P[A_3] - P[A_1 \cap A_2] - P[A_1 \cap A_3] - P[A_2 \cap A_3] + P[A_1 \cap A_2 \cap A_3]$$

$$= P_{A_1} + P_{A_2} + P_{A_3} - P_{A_1}P_{A_2} - P_{A_1}P_{A_3} - P_{A_2}P_{A_3} + P_{A_1}P_{A_2}P_{A_3}$$

 2-of-3 $P[D_a] = P[A_1 \cap A_2 \cap A_3] + P[A_1^c \cap A_2 \cap A_3] + P[A_1 \cap A_2^c \cap A_3] + P[A_1 \cap A_2 \cap A_3^c]$

$$= P_{A_1}P_{A_2}P_{A_3} + (1 - P_{A_1})P_{A_2}P_{A_3} + P_{A_1}(1 - P_{A_2})P_{A_3} + P_{A_1}P_{A_2}(1 - P_{A_3})$$

2.91

$$P[\text{System Up}] = P[(A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})]$$

$$= P[A_{11} \cap A_{12}] + P[A_{21} \cap A_{22}] + P[A_{31} \cap A_{32}] - P[A_{11} \cap A_{12} \cap A_{21} \cap A_{22}]$$

$$- P[A_{11} \cap A_{12} \cap A_{31} \cap A_{32}] - P[A_{21} \cap A_{22} \cap A_{31} \cap A_{32}]$$

$$+ P[A_{11} \cap A_{12} \cap A_{21} \cap A_{22} \cap A_{31} \cap A_{32}]$$

$$= P_{A_{11}}P_{A_{12}} + P_{A_{21}}P_{A_{22}} + P_{A_{31}}P_{A_{32}} - P_{A_{11}}P_{A_{12}}P_{A_{21}}P_{A_{22}}$$

$$- P_{A_{11}}P_{A_{12}}P_{A_{31}}P_{A_{32}} - P_{A_{21}}P_{A_{22}}P_{A_{31}}P_{A_{32}}$$

$$+ P_{A_{11}}P_{A_{12}}P_{A_{21}}P_{A_{22}}P_{A_{31}}P_{A_{32}}$$

2.92 Events A and B are independent iff

$$P[A \cap B] = P[A]P[B]$$

In terms of relative frequencies we expect

$$\underbrace{f_{A \cap B}^n}_{\text{rel. freq. if joint occurrence of } A \text{ and } B} = f_A(n)f_B(n)$$

rel. freq. if joint occurrence of A and B rel. freq.'s of A and B

2.93

Let the j th bit in the hex character be B_j
To test independence we need:

All pairs should satisfy $f_{B_j \cap B_k} \approx f_{B_j} f_{B_k}$

All triplets should satisfy $f_{B_j \cap B_k \cap B_l} \approx f_{B_j} f_{B_k} f_{B_l}$

Quadruplets should satisfy $f_{B_1 \cap B_2 \cap B_3 \cap B_4} \approx f_{B_1} f_{B_2} f_{B_3} f_{B_4}$

Note Relative frequencies for different B_j need not be the same.

2.94

$$P[\text{System Up}] = P[\text{at least one controller is working}] \times P[\text{at least two peripherals are working}]$$

$$\begin{aligned} P[\text{at least one controller working}] &= 1 - P[\text{both not working}] \\ &= 1 - p^2 \end{aligned}$$

$$\therefore P[\text{System Up}] = (1 - p^2)\{(1 - a)^3 + 3(1 - a)^2 a\}$$

2.95

$$P[A_0 \cap B_0] = (1-p)(1-\epsilon)$$

$$P[B_0] = (1-p)(1-\epsilon) + p\epsilon$$

$$P[A_0] = (1-p)$$

$$P[A_0 \cap B_0] = P[B_0]P[A_0]$$

$$\Leftrightarrow (1-p)(1-\epsilon) = [(1-p)(1-\epsilon) + p\epsilon](1-p)$$

$$\Leftrightarrow (1-\epsilon) = (1-p)(1-\epsilon) + p\epsilon$$

$$\Leftrightarrow (1-\epsilon)p = p\epsilon$$

$$\Leftrightarrow \epsilon = \frac{1}{2}$$

Channel cannot transmit information of output is independent of the input.

2.96

Regardless of the value of ϵ , we always have

$$P[X=2 | Y=1] = 0 \neq P[X=2] = \frac{1}{3}$$

$\therefore$ the output cannot be independent of the input.

2.6 Sequential Experiments

2.97

$$\textcircled{a} P[0 \text{ or } 1 \text{ errors}] = (1-p)^{100} + 100(1-p)^{99}p \quad p=10^{-2}$$

$$= 0.3660 + 0.3697$$

$$= 0.7357$$

$$\textcircled{b} p_R = P[\text{Retransmission required}] = 1 - P[0 \text{ or } 1 \text{ error}] = 0.2642$$

$$P[M \text{ transmissions in total}] = (1-p)^m p_R^m \quad m=1, 2, \dots$$

$$P[M \text{ or more transmissions required}] = \sum_{j=m}^{\infty} (1-p)^j p_R^j = \sum_{j=0}^{\infty} (1-p)^j p_R^j$$

$$= p_R^m$$

2.98

$$\textcircled{a} P[N > 1] = 1 - P[N=0 \text{ or } N=1]$$

$$= 1 - (1-p)^n - n(1-p)^{n-1}p$$

$$\textcircled{b} P[N > 0] = 0.99 = 1 - (1-0.1)^n$$

$$0.01 = (0.9)^n$$

$$n = \frac{\ln 100}{\ln 1/0.9} = 44$$

2.99

$p = \text{prob. of success} = \frac{95}{100} = \frac{19}{20}$

Pick n so that $P[k \geq 8] \geq 0.9$

$$P[k \geq 8] = \sum_{k=8}^n \binom{n}{k} p^k (1-p)^{n-k}$$

for

$$n = 11 \quad P[k \geq 8] = 0.8811 \quad 0.95^8 + 8(0.95)^7(0.05) = 0.92$$

$$n = 12 \quad P[k \geq 10] = 0.98093 \Rightarrow \text{pick } n = 12$$

1 extra drop is enough.

2.100

(a) $P[1 \text{ of } n \text{ terminals transmit}] = n(1-p)p^{n-1}$

(b) Take derivative with respect to p :

$$0 = -n(n-1)(1-p)^{n-2}p + n(1-p)^{n-1}$$

$$\Rightarrow (n-1)p = (1-p) \Rightarrow np = 1-p+p \Rightarrow p = \frac{1}{n}$$

(c) $P_{\text{success}} = n \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n} = \left(1 - \frac{1}{n}\right)^{n-1} \rightarrow e^{-1} = \frac{1}{e} \text{ as } n \rightarrow \infty$
 $= 0.3678$

2.101

$$P[N \geq 2] = 1 - P[N=0] - P[N=1]$$

$$P[X \leq \frac{2}{\lambda}] = 1 - e^{-(\lambda \frac{2}{\lambda})^2} = 1 - e^{-4} = 0.9816$$

$$P[N \geq 2] = 1 - (1 - e^{-4})^8 - 8(1 - e^{-4})^7 e^{-4}$$

$$= 1 - 0.8625 - 0.1287 = 0.7 \times 10^{-3}$$

2.102

a) $P[k \text{ errors}] = \binom{n}{k} p^k (1-p)^{n-k}$

b) Type 1 errors occur with problem $p\alpha$ and do not occur with problem $1-p\alpha$

$$P[k_1 \text{ type 1 errors}] = \binom{n}{k_1} (p\alpha)^{k_1} (1-p\alpha)^{n-k_1}$$

c) $P[k_2 \text{ type 2 errors}] = \binom{n}{k_2} (p(1-\alpha))^{k_2} (1-p(1-\alpha))^{n-k_2}$

d) Three outcomes: type 1 error, type 2 error, no error

$$P[k_1, k_2, n - k_1 - k_2] = \frac{n!}{k_1! k_2! (n - k_1 - k_2)!} (p\alpha)^{k_1} (p(1-\alpha))^{k_2} (1-p)^{n-k_1-k_2}$$

2.103

$$P[EF] = 0.10 \quad P[AF] = 0.30 \quad P[BE] = 0.60$$

a) $P[k \text{ are not EF}] = P[N-k \text{ are EF}] = \binom{N}{N-k} (0.10)^{N-k} (0.90)^k$

b) $P[k \text{ until EF}] = (1 - P(EF))^{k-1} P[EF] = 0.9^{k-1} (0.1)$

c) $P[k_{EF} = 4, k_{AF} = 6, k_{BE} = 10] = \frac{20!}{4! 6! 10!} (0.1)^4 (0.3)^6 (0.6)^{10}$

2.104

2.78 a)

$$P[k = 0] = p$$

$$P[k = 1] = (1 - p)p$$

$$P[k = 2] = (1 - p)^2 p$$

$$P[k = 3] = 1 - P[k = 0] - P[k = 1] - P[k = 2] = (1 - p)^3$$

b)

$$P[k] = (1 - p)^k p \quad 0 \leq k < m$$

$$\begin{aligned} P[m] &= 1 - \sum_{k=0}^{m-1} P[k] \\ &= 1 - \sum_{k=0}^{m-1} (1 - p)^k p \\ &= 1 - p \frac{1 - (1 - p)^m}{1 - (1 - p)} = (1 - p)^m \end{aligned}$$

2.105

a)

$$P[k \text{ half hours}] = \left(\frac{1}{2}\right)^k \quad k = 1, 2, \dots$$

$$P[k \text{ dollars paid}] = \left(\frac{1}{2}\right)^k$$

b) $P[k \text{ dollars}] = \left(\frac{1}{2}\right)^k \quad k = 1, 2, 3, 4, 5$

$$P[6] = 1 - \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

2.106

2.80 $P[k \text{ tosses required until heads comes up twice}] = P[\text{heads in } k\text{th toss} \mid \text{heads in } k-1 \text{ tosses}] P[\text{head in } k-1 \text{ tosses}] = P[A|B]P[B]$.

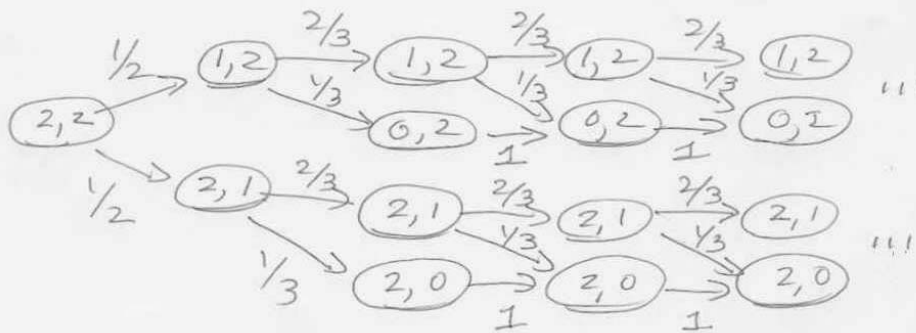
Now $P[A|B] = P[\text{head in first } k-1 \text{ tosses}] = \binom{k-1}{2} p^2 (1-p)^{k-3}$

Thus $P[A|B]P[B] = P[A|B]p = \binom{k-1}{2} p^3 (1-p)^{k-3} \quad k=3, 4, \dots$

2.107

The first draw is key since that ball is not put back.

Let (j, k) be a state where $j = \# \text{ black balls in urn}$ $k = \# \text{ white balls in urn}$



(b) $P[bb] = \frac{1}{2} \frac{1}{3} = \frac{1}{6}$ $P[bw] = \frac{1}{2} \frac{2}{3} = \frac{1}{3}$
 $P[ww] = \frac{1}{2} \frac{1}{3} = \frac{1}{6}$ $P[wb] = \frac{1}{2} \frac{2}{3} = \frac{1}{3}$
 $P[bbw] = \frac{1}{2} \frac{1}{3} 1 = \frac{1}{6}$ $P[bww] = \frac{1}{2} \frac{2}{3} \frac{2}{3} = \frac{2}{9}$ $P[bwb] = \frac{1}{2} \frac{2}{3} \frac{1}{3} = \frac{1}{9}$
 $P[wbw] = \frac{1}{2} \frac{1}{3} 1 = \frac{1}{6}$ $P[wbb] = \frac{1}{2} \frac{2}{3} \frac{2}{3} = \frac{2}{9}$ $P[wbw] = \frac{1}{2} \frac{2}{3} \frac{1}{3} = \frac{1}{9}$

(c) $P[(0, 2) \text{ after 3 draws}] = P[bbw] + P[bwb] = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$
 Similarly $P[(2, 0) \text{ after 3 draws}] = \frac{5}{18}$

(2.101)

(a) $P[(2, 0) \text{ after } n] = P[\text{1st draw is white and at least one white in } n-1]$
 $= \frac{1}{2} \left[1 - \underbrace{\left(\frac{2}{3} \right)^{n-1}}_{R \text{ all blacks}} \right]$

2.108

a) $p_0(1) = \frac{1}{2}$ $p_1(1) = \frac{1}{2}$

b) $p_0(n+1) = \frac{2}{3}p_0(n) + \frac{1}{6}p_1(n)$

$p_1(n+1) = \frac{1}{3}p_0(n) + \frac{5}{6}p_1(n)$

In matrix notation, we have

$$[p_0(n+1), p_1(n+1)] = [p_0(n), p_1(n)] \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

or

$$\underline{p}(n+1) = \underline{p}(n)\mathbb{P}$$

c) $\underline{p}(0) = \left[\frac{1}{2}, \frac{1}{2} \right]$

$\underline{p}(1) = \underline{p}(0)\mathbb{P}$

$\underline{p}(2) = \underline{p}(1)\mathbb{P} = \underline{p}(0)\mathbb{P}^2$

in general

$$\underline{p}(n) = \underline{p}(0)\mathbb{P}^n$$

To find $\mathbb{P}^n$ we note that if $\mathbb{P}$ has eigenvalues λ_1, λ_2 and eigenvectors $\underline{e}_1, \underline{e}_2$ then

$$\mathbb{P} = \underbrace{\mathbb{E} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbb{E}^{-1}}_{\Lambda} \quad \text{where } \mathbb{E} \text{ has } \underline{e}_1 \text{ and } \underline{e}_2 \text{ as columns}$$

and

$$\begin{aligned} \mathbb{P}^n &= (\mathbb{E}\Lambda\mathbb{E}^{-1})(\mathbb{E}\Lambda\mathbb{E}^{-1})\dots(\mathbb{E}\Lambda\mathbb{E}^{-1}) \quad n \text{ times} \\ &= \mathbb{E}\Lambda(\mathbb{E}^{-1}\mathbb{E})\Lambda\dots(\mathbb{E}^{-1}\mathbb{E})\Lambda\mathbb{E}^{-1} \\ &= \mathbb{E}\Lambda^n\mathbb{E}^{-1} \end{aligned}$$

Now $\mathbb{P} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$ and eigenvector $\underline{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{e}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Thus

$$\begin{aligned} \mathbb{P}^n &= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^n & \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^n \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \underline{p}(n) &= \underline{p}(0)\mathbb{P}^n \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^n & \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^n \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n+1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n+1} \end{bmatrix} \end{aligned}$$

c) $\underline{p}(n) \rightarrow \left[\frac{1}{3}, \frac{2}{3} \right]$ as $n \rightarrow \infty$

2.7 *Synthesizing Randomness: Random Number Generators

2.109

$$p_1 = \frac{1}{3} \quad p_2 = \frac{1}{5} \quad p_3 = \frac{1}{4} \quad p_4 = \frac{1}{7} \quad p_5 = 1 - \sum_{i=1}^4 p_i = 1 - \frac{140+84+105+60}{420} = \frac{31}{420}$$

Use an urn with 420 ^{identical} balls labeled as follows

140 labeled 1
84 " 2
105 " 3
60 " 4
31 " 5

By finding least common multiple of denominators of rational probabilities we can define an equivalent urn experiment.

2.110

2.84 Three tosses of a fair coin result in eight equiprobable outcomes:

000	→	0	100	→	4
001	→	1	101	→	5
010	→	2	101	} → No output	
011	→	3	111		

a)

$$\begin{aligned} P[\text{a number is output in step 1}] &= 1 - P[\text{no output}] \\ &= 1 - \frac{2}{8} = \frac{3}{4} \end{aligned}$$

b) Let $A_i = \{\text{output number } i\}$ $i = 0, \dots, 5$
and $B = \{\text{a number is output in step 1}\}$
then

$$\begin{aligned} P[A_i|B] &= \frac{P[A_i \cap B]}{P[B]} = \frac{P[\text{binary string corresponds to } i]}{\frac{3}{4}} \\ &= \frac{\frac{1}{8}}{\frac{3}{4}} = \frac{1}{6} \end{aligned}$$

c) Suppose we want to an urn experiment with N equiprobable outcomes. Let n be the smallest integer such that $2^n \geq N$. We can simulate the urn experiment by tossing a fair coin n times and outputting a number when the binary string for the numbers $0, \dots, N-1$ occur and not outputting a number otherwise.

2.111

```
> X=rand(1000,1)
> Y=rand(1000,1)
> plot(X,Y,"+")
```

This program will produce a 2D scattergram in unit square

2.112

```
> X=rand(1000,1);
> Y=rand(1000,1);
> Xacc=zeros(500,1);
> Yacc=zeros(500,1);
> n=0
> j=0
> while n<500
    j=j+1
    if X(j)<Y(j)
        n=n+1
        Xacc(n)=X(j);
        Yacc(n)=Y(j);
    end
```

```
end
plot(Xacc,Yacc,"+")
```

This program will plot 500 points in the upper diagonal region of the unit square.

2.113

a) Assume that $X(j)$ assumes values from the sample space $S = \{x_1, x_2, \dots, x_K\}$, and let $N_k(n)$ be the number of times x_k occurs in n repetitions of the experiment, then

$$\begin{aligned} \langle X^2 \rangle_n &= \frac{1}{n} \sum_{j=1}^n X^2(j) \\ &= \frac{1}{n} \sum_{k=1}^K x_k^2 N_k(n) \\ &\rightarrow \sum_{k=1}^K x_k^2 f_k(n) \end{aligned}$$

Thus we expect that $\langle x^2 \rangle_n \rightarrow \sum_{k=1}^K x_k^2 p_k$.

b) The same derivation of Problem 1.9, gives

$$\langle X^2 \rangle_n = \langle X^2 \rangle_{n-1} + \frac{X_n^2 - \langle X^2 \rangle_{n-1}}{n}$$

2.114

$$\begin{aligned}
 \text{a) } \langle V^2 \rangle_n &= \frac{1}{n} \sum_{j=1}^n \{X(j) - \langle X \rangle_n\}^2 \\
 &= \frac{1}{n} \sum_{j=1}^n \{X^2(j) - 2X(j)\langle X \rangle_n + \langle X \rangle_n^2\} \\
 &= \frac{1}{n} \sum_{j=1}^n X^2(j) - 2 \left(\frac{1}{n} \sum_{j=1}^n X(j) \right) \langle X \rangle_n + \langle X \rangle_n^2 \\
 &= \langle X^2 \rangle_n - \langle X \rangle_n^2
 \end{aligned}$$

b) From the next to last line in solution to Problem 1.7, we have:

$$\begin{aligned}
 \langle V^2 \rangle_n &= \langle X^2 \rangle_n - \langle X \rangle_n^2 \\
 &= \frac{n-1}{n} \langle X^2 \rangle_{n-1} + \frac{X^2(n)}{n} - \left\{ \frac{n-1}{n} \langle X \rangle_{n-1} + \frac{X(n)}{n} \right\}^2 \\
 &= \frac{n-1}{n} (\langle V^2 \rangle_{n-1} + \langle X \rangle_{n-1}^2) + \frac{X^2(n)}{n} \\
 &\quad - \left(\frac{n-1}{n} \right)^2 \langle X \rangle_{n-1}^2 - 2 \frac{1}{n} \left(\frac{n-1}{n} \right) \langle X \rangle_{n-1} X(n) \\
 &\quad - \frac{X^2(n)}{n^2} \\
 &= \frac{n-1}{n} \langle V^2 \rangle_{n-1} + \frac{n-1}{n} \left(1 - \frac{n-1}{n} \right) \langle X \rangle_{n-1}^2 \\
 &\quad - 2 \frac{1}{n} \left(\frac{n-1}{n} \right) \langle X \rangle_{n-1} X(n) + \frac{1}{n} \left(1 - \frac{1}{n} \right) X^2(n) \\
 &= \left(1 - \frac{1}{n} \right) \langle V^2 \rangle_{n-1} + \frac{1}{n} \left(1 - \frac{1}{n} \right) \{ \langle X \rangle_{n-1}^2 \\
 &\quad - 2 \langle X \rangle_{n-1} X(n) + X^2(n) \} \\
 &= \left(1 - \frac{1}{n} \right) \langle V^2 \rangle_{n-1} + \frac{1}{n} \left(1 - \frac{1}{n} \right) \{ X(n) - \langle X \rangle_{n-1} \}^2
 \end{aligned}$$

2.115) $Y_n = \alpha V_n + \beta$ should map onto $[a, b]$

$$\begin{aligned} \textcircled{a} \quad & \left. \begin{array}{l} \text{when } V_n = 0 \text{ we want } Y_n = \beta = a \\ \text{when } V_n = 1 \text{ we want } Y_n = \alpha + \beta = b \end{array} \right\} \Rightarrow \alpha = b - \beta = b - a \\ & \alpha = b - a \quad \beta = a \\ & \Rightarrow Y_n = (b - a) V_n + a \end{aligned}$$

$\textcircled{b}$
 $> a = -5$
 $> b = 15$
 $> Y = (b - a) * \text{rand}(1000, 1) + a * \text{ones}(1000, 1);$
 $> \text{mean}(Y)$ % computes sample mean
 $> \text{cov}(Y, Y)$ % computes sample variance

In a test we obtained

$$\begin{aligned} \text{mean}(Y) &= 5.2670 \quad \text{vs} \quad \frac{b-a}{2} = 5 \\ \text{cov}(Y, Y) &= 34.065 \quad \text{vs} \quad \frac{(b-a)^2}{12} = 33.333 \end{aligned}$$

2.116) $\textcircled{a}$ This problem uses the code in Example 2.47

$\textcircled{b}$ The ^{histogram} ~~plot~~ will change with different values of p .

2.8 *Fine Points: Event Classes

2.117

$$f(r) = R \quad f(g) = G \quad f(t) = G$$

Home's events are quite simple:

$$\phi, \{R\}, \{G\}, \{R, G\} = \mathcal{A}_H$$

$$\textcircled{a} \quad f^{-1}(\{R\} \cup \{G\}) = f^{-1}(\{R, G\}) = \{r, g, t\}$$

and $f^{-1}(\{R\}) \cup f^{-1}(\{G\}) = \{r\} \cup \{g, t\} = \{r, g, t\}$ same.

$$\textcircled{b} \quad f^{-1}(\{R\} \cap \{R, G\}) = f^{-1}(\{R\}) = \{r\}$$

$$f^{-1}(\{R\}) \cap f^{-1}(\{R, G\}) = \{r\} \cap \{r, g, t\} = \{r\}$$
 same

$$\textcircled{c} \quad f^{-1}(\{G\}^c) = f^{-1}(\{R\}) = \{r\}$$

$$f^{-1}(\{G\})^c = \{g, t\}^c = \{r\}$$
 same

$$\textcircled{d} \quad (i) \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$\text{If } \xi \in f^{-1}(A \cup B) \text{ then } f(\xi) \in A \cup B \Rightarrow f(\xi) \in A \text{ and/or } f(\xi) \in B$$

$$\Rightarrow \xi \in f^{-1}(A) \text{ and/or } \xi \in f^{-1}(B)$$

$$\Rightarrow \xi \in f^{-1}(A) \cup f^{-1}(B) \quad \therefore f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B).$$

$$\text{If } \xi \in f^{-1}(A) \cup f^{-1}(B) \Rightarrow \xi \in f^{-1}(A) \text{ and/or } \xi \in f^{-1}(B)$$

$$\Rightarrow f(\xi) \in A \text{ and/or } f(\xi) \in B$$

$$\Rightarrow f(\xi) \in A \cup B$$

$$\Rightarrow \xi \in f^{-1}(A \cup B) \Rightarrow f^{-1}(A \cup B) \supset f^{-1}(A) \cup f^{-1}(B)$$

$$\Rightarrow \text{equality.}$$

(d) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$$\begin{aligned} \text{If } \xi \in f^{-1}(A \cap B) &\Rightarrow f(\xi) \in A \cap B \Rightarrow f(\xi) \in A \text{ and } f(\xi) \in B \\ &\Rightarrow \xi \in f^{-1}(A) \text{ and } \xi \in f^{-1}(B) \Rightarrow \xi \in f^{-1}(A) \cap f^{-1}(B). \\ &\Rightarrow f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

$$\begin{aligned} \text{If } \xi \in f^{-1}(A) \cap f^{-1}(B) &\Rightarrow \xi \in f^{-1}(A) \text{ and } \xi \in f^{-1}(B) \\ &\Rightarrow f(\xi) \in A \text{ and } f(\xi) \in B \Rightarrow f(\xi) \in A \cap B \\ &\Rightarrow \xi \in f^{-1}(A \cap B) \\ &\Rightarrow f^{-1}(A \cap B) \supset f^{-1}(A) \cap f^{-1}(B) \quad \checkmark \end{aligned}$$

$$f^{-1}(A^c) = f^{-1}(A)^c$$

$$\begin{aligned} \text{If } \xi \in f^{-1}(A^c) &\Rightarrow f(\xi) \in A^c \Rightarrow f(\xi) \notin A \Rightarrow \xi \notin f^{-1}(A) \\ &\Rightarrow \xi \in f^{-1}(A)^c \\ &\Rightarrow f^{-1}(A^c) \subset f^{-1}(A)^c \end{aligned}$$

$$\begin{aligned} \text{If } \xi \in f^{-1}(A)^c &\Rightarrow \xi \notin f^{-1}(A) \Rightarrow f(\xi) \notin A \\ &\Rightarrow f(\xi) \in A^c \\ &\Rightarrow \xi \in f^{-1}(A^c) \\ &\Rightarrow f^{-1}(A)^c \subset f^{-1}(A^c) \quad \checkmark \end{aligned}$$

2.118

(a) Show that $A_1, \dots, A_n$ forms a partition of S , that is,
 $A_i \cap A_j = \emptyset$ $i \neq j$ and $\bigcup_{i=1}^n A_i = S$

(i) For $i \neq j$ consider $A_i \cap A_j$

$$A_i \cap A_j = \{s: s \in A_i \text{ and } s \in A_j\} = \{s: f(s) = y_i \text{ and } f(s) = y_j\}$$

but if $y_i \neq y_j$ then we cannot have $f(s) = y_i$ and $f(s) = y_j$
 since each $s \in S$ mapped into a single value

$$\therefore A_i \cap A_j = \emptyset.$$

(ii) Now consider $\bigcup_{i=1}^n A_i$

Suppose $s \in S$, then $f(s) \in S' = \{y_1, \dots, y_n\}$

$$\Rightarrow s \in A_j \text{ for some } j$$

$$\Rightarrow s \in \bigcup_{i=1}^n A_i \Rightarrow \bigcup_{i=1}^n A_i \supset S.$$

But S contains all subsets

$$\Rightarrow \bigcup_{i=1}^n A_i \subset S \quad \checkmark.$$

(b) Any $B \subset S'$ has form $B = \{y_{i_1}\} \cup \{y_{i_2}\} \cup \dots \cup \{y_{i_m}\}$

From problem 2.117 (d)

$$\begin{aligned} f^{-1}(B) &= f^{-1}(\{y_{i_1}\} \cup \{y_{i_2}\} \cup \dots \cup \{y_{i_m}\}) \\ &= f^{-1}(\{y_{i_1}\}) \cup f^{-1}(\{y_{i_2}\}) \cup \dots \cup f^{-1}(\{y_{i_m}\}) \\ &= A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_m}. \end{aligned}$$

$\therefore$ Inverse image of B is a union of sets from the partition.

2.119

$$\mathcal{F} = \{\emptyset, A, A^c, S\}$$

(i) $\emptyset \in \mathcal{F}$ ✓

(ii) if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$?

$$A \cup A^c = S \in \mathcal{F}$$

and any other union of events in $\mathcal{F}$ yields an event in $\mathcal{F}$ ✓

(iii) if $B \in \mathcal{F}$ then $B^c \in \mathcal{F}$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$A^c \in \mathcal{F} \Rightarrow A \in \mathcal{F}$$

and similarly for other events in $\mathcal{F}$ ✓

∴ $\mathcal{F}$ is a field.

2.9 *Fine Points: Probabilities of Sequences of Events

2.120

$$\begin{aligned} \textcircled{a} \quad \bigcup_n A_n &= \bigcup_n \left[a + \frac{1}{n}, b - \frac{1}{n} \right] = (a, b) \\ \textcircled{b} \quad \bigcup_n B_n &= \bigcup_n \left(-\infty, b - \frac{1}{n} \right] = (-\infty, b) \\ \textcircled{c} \quad \bigcup_n C_n &= \bigcup_n \left[a + \frac{1}{n}, b \right) = (a, b) \end{aligned}$$

2.121

$$\begin{aligned} \textcircled{a} \quad \bigcap_n \left(a - \frac{1}{n}, b + \frac{1}{n} \right) &= [a, b] \\ \textcircled{b} \quad \bigcap_n \left[a, b + \frac{1}{n} \right) &= [a, b] \\ \textcircled{c} \quad \bigcap_n \left[a - \frac{1}{n}, b \right] &= [a, b] \end{aligned}$$

2.122

$\textcircled{a}$ Startly with open sets (a, b)
 $(-\infty, b)^c = [b, \infty) \in \mathcal{B}$
 then $(-\infty, b) \cap [a, \infty) = [a, b)$ for $a \leq b$
 $\therefore$ We can use semi-infinite intervals as in the chapter
 to show that all elements in the Borel field can be generated.
 $\textcircled{b}$ Closed interval of the form $[a, b]$ can also be used
 to generate the Borel field.

2.123

$$\begin{aligned} \textcircled{a} \quad \lim_{n \rightarrow \infty} P[A_n] &= P[\lim_{n \rightarrow \infty} A_n] = P[a < x < b] \\ \textcircled{b} \quad \lim_{n \rightarrow \infty} P[B_n] &= P[\lim_{n \rightarrow \infty} B_n] = P[-\infty < x < b] \\ \textcircled{c} \quad \lim_{n \rightarrow \infty} P[C_n] &= P[\lim_{n \rightarrow \infty} C_n] = P[a < x < b] \end{aligned}$$

2.124

$$\begin{aligned} \textcircled{a} \quad \lim_{n \rightarrow \infty} P[A_n] &= P[\lim_{n \rightarrow \infty} A_n] = P[a \leq x \leq b] \\ \textcircled{b} \quad \lim_{n \rightarrow \infty} P[B_n] &= P[\lim_{n \rightarrow \infty} B_n] = P[a \leq x \leq b] \\ \textcircled{c} \quad \lim_{n \rightarrow \infty} P[C_n] &= P[\lim_{n \rightarrow \infty} C_n] = P[a \leq x \leq b] \end{aligned}$$

Problems Requiring Cumulative Knowledge

2.125

(a)
$$P_H[k \text{ defective of } 10 \text{ tested}] = \begin{cases} \frac{\binom{5}{k} \binom{15}{10-k}}{\binom{20}{10}} & k=0, 1, 2, 3, 4, 5 \\ 0 & k > 5 \end{cases}$$

(b)
$$P_B[k \text{ defective}] = \binom{10}{k} (0.25)^k (0.75)^{10-k} \quad k=0, 1, 2, \dots, 10$$

See Table of values:

Probabilities for hypergeometric and binomial are very different.

k	Hypergeometric	Binomial
0	0.01625	0.18771
1	0.13545	0.28157
2	0.34830	0.25028
3	0.34830	0.14600
4	0.13545	0.058399
5	0.01625	0.016226
6	0	0.003089
7	0	0.00003

(b)
$$P_H[k \text{ defective}] = \frac{\binom{250}{k} \binom{750}{10-k}}{\binom{1000}{10}} \quad k=0, 1, \dots, 10$$

$$P_B[k \text{ defective}] = \binom{10}{k} (0.25)^k (0.75)^{10-k} \quad k=0, \dots, 10$$

See Table:

k	Hypergeometric
0	0.18714
1	0.28260
2	0.25154
3	0.14614
4	0.057907
5	0.015848
6	0.0029581
7	0.00036

These are very close to the binomial probabilities.

Because of the large population size sampling without replacement is almost the same as sampling with replacement.

2.126

$$P[\text{both in error}] = q_1 q_2$$

$$(a) \quad P[k \text{ transmissions needed}] = (q_1 q_2)^{k-1} (1 - q_1 q_2) \quad k=1, 2, \dots$$

$$P[\text{more than } k \text{ transmissions required}]$$

$$\begin{aligned} &= \sum_{j=k+1}^{\infty} (q_1 q_2)^{j-1} (1 - q_1 q_2) = (q_1 q_2)^k \sum_{j=0}^{\infty} (1 - q_1 q_2)^j \\ &= (q_1 q_2)^k \end{aligned}$$

$$\begin{aligned} (b) \quad &P[\text{link 2 error-free} \mid \text{one or more error-free}] \\ &= \frac{P[\text{one or more error-free, link 2 error-free}]}{1 - q_1 q_2} \\ &= \frac{q_1 (1 - q_2) + (1 - q_1) (1 - q_2)}{1 - q_1 q_2} = \frac{1 - q_2}{1 - q_1 q_2} \end{aligned}$$

2.127

$$(a) \quad P_b = P[N_c \geq 7] = P[N=7] + P[N=8] = 7(1-p)^7 p + (1-p)^8$$

$$(b) \quad P[N_b \geq 1] = 1 - P[N_b = 0] = 1 - (1 - P_b)^n = 0.99$$

$$0.01 = (1 - P_b)^n \Rightarrow \ln 100 = n \ln \frac{1}{1 - P_b}$$

$$n = \frac{\ln 100}{\ln \frac{1}{1 - P_b}} = \frac{\ln 100}{-\ln (1 - 7(1-p)^7 p - (1-p)^8)}$$

2.128

(a) $P[\text{ace}] = \frac{4}{52} = \frac{1}{13}$

(b) Let $A = \text{ace in 1st draw}$
 $B = \text{ace in 2nd draw}$

$$P[A] = \frac{4}{52} \quad P[A^c] = \frac{48}{52}$$

if we look at 1st draw:

$$P[B|A] = \frac{3}{51} \quad P[B|A^c] = \frac{4}{51}$$

Suppose we don't look

$$\begin{aligned} P[B] &= P[B|A]P[A] + P[B|A^c]P[A^c] \\ &= \frac{3}{51} \frac{1}{13} + \frac{4}{51} \frac{12}{13} = \frac{3+48}{51(13)} = \frac{1}{13} \end{aligned}$$

$\Rightarrow$ 2nd Draw has same probability of ace as 1st draw

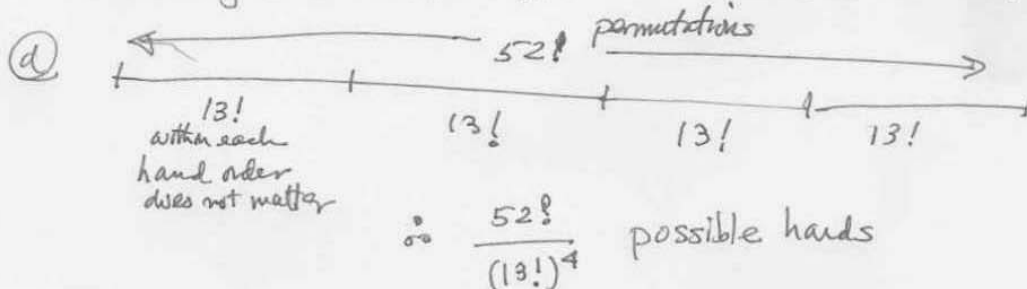
(c)
$$P[\underbrace{3 \text{ aces in 7 cards}}_A] = \frac{\binom{4}{3} \binom{48}{4}}{\binom{52}{7}} = 0.00582$$

$$P[\underbrace{2 \text{ kings in 7 cards}}_B] = \frac{\binom{4}{2} \binom{48}{5}}{\binom{52}{7}} = 0.07679$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cap B] = \frac{\binom{4}{3} \binom{4}{2} \binom{44}{2}}{\binom{52}{7}} = 0.00017$$

$$P[A \cup B] = 0.00582 + 0.07679 - 0.00017 = 0.0824$$



2.128 (d) - continued -

There are $4! = 24$ of arranging the 4 aces and allotting one to each player.

There are $\frac{48!}{(12!)^4}$ ways of distributing the other 48 cards

$$\therefore P[\text{1 ace to each player}] = \frac{4! \frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = \frac{24(48!)}{52!} 13^4$$
$$= 0.1055$$