CLASSICAL MECHANICS

SOLUTIONS MANUAL

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Contents

Chapter One

The algebra and calculus of vectors

Problem 1.1

In terms of the standard basis set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 4\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}$.

(i) Find $3\boldsymbol{a} + 2\boldsymbol{b} - 4\boldsymbol{c}$ and $|\boldsymbol{a} - \boldsymbol{b}|^2$.
(ii) Find $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $\boldsymbol{a} \cdot \boldsymbol{b}$. Deduce the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
(iii) Find the component of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ and in the direction of $\boldsymbol{b}$.
(iv) Find $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{b}\times\boldsymbol{c}$ and $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})$.
(v) Find $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ and $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$ and verify that they are equal. Is the set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ right- or left-handed?
(vi) By evaluating each side, verify the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$.

Solution

(i)

$$\begin{aligned} 3\boldsymbol{a} + 2\boldsymbol{b} - 4\boldsymbol{c} &= 3(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) + 2(3\boldsymbol{i} - 4\boldsymbol{k}) - 4(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}) \\ &= 8\boldsymbol{i} + 17\boldsymbol{j} - 26\boldsymbol{k}. \blacksquare \end{aligned}$$

$$\begin{aligned} |\boldsymbol{a} - \boldsymbol{b}|^2 &= (\boldsymbol{a} - \boldsymbol{b})\cdot(\boldsymbol{a} - \boldsymbol{b}) \\ &= (-\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})\cdot(-\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) \\ &= (-1)^2 + (-1)^2 + 2^2 = 6. \blacksquare \end{aligned}$$

(ii)

$$\begin{aligned} |\boldsymbol{a}|^2 &= \boldsymbol{a}\cdot\boldsymbol{a} \\ &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\cdot(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) \\ &= 2^2 + (-1)^2 + (-2)^2 = 9. \end{aligned}$$

Hence $|\boldsymbol{a}| = 3$. $\blacksquare$

$$\begin{aligned} |\boldsymbol{b}|^2 &= \boldsymbol{b}\cdot\boldsymbol{b} \\ &= (3\boldsymbol{i} - 4\boldsymbol{k})\cdot(3\boldsymbol{i} - 4\boldsymbol{k}) \\ &= 3^2 + (-4)^2 = 25. \end{aligned}$$

Hence $|\boldsymbol{b}| = 5$. $\blacksquare$

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b} &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\cdot(3\boldsymbol{i} - 4\boldsymbol{k}) \\ &= \big(2\times 3\big) + \big((-1)\times 0\big) + \big((-2)\times(-4)\big) \\ &= 14. \blacksquare \end{aligned}$$

The angle α between $\boldsymbol{a}$ and $\boldsymbol{b}$ is then given by

$$\begin{aligned}\cos\alpha &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\\ &= \frac{14}{3\times 5} = \frac{14}{15}.\end{aligned}$$

Thus $\alpha = \tan^{-1}\frac{14}{15}$. ■

(iii) The component of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ is

$$\begin{aligned}\boldsymbol{c}\cdot\widehat{\boldsymbol{a}} &= \boldsymbol{c}\cdot\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\right)\\ &= (\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})\cdot\left(\frac{2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}}{|2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}|}\right)\\ &= \frac{\big(1\times 2\big) + \big((-5)\times(-1)\big) + \big(3\times(-2)\big)}{3}\\ &= \frac{1}{3}. \blacksquare\end{aligned}$$

The component of $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$ is

$$\begin{aligned}\boldsymbol{c}\cdot\widehat{\boldsymbol{b}} &= \boldsymbol{c}\cdot\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\\ &= (\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})\cdot\left(\frac{3\boldsymbol{i} - 4\boldsymbol{k}}{|3\boldsymbol{i} - 4\boldsymbol{k}|}\right)\\ &= \frac{\big(1\times 3\big) + \big((-5)\times 0\big) + \big(3\times(-4)\big)}{5}\\ &= -\frac{9}{5}. \blacksquare\end{aligned}$$

(iv)

$$\begin{aligned}\boldsymbol{a}\times\boldsymbol{b} &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\times(3\boldsymbol{i} - 4\boldsymbol{k})\\ &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -1 & -2\\ 3 & 0 & -4\end{vmatrix}\\ &= \big(4-0\big)\boldsymbol{i} - \big((-8)-(-6)\big)\boldsymbol{j} + \big(0-(-3)\big)\boldsymbol{k}\\ &= 4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}. \blacksquare\end{aligned}$$

$$\begin{aligned}
\boldsymbol{b}\times\boldsymbol{c} &= (3\boldsymbol{i} - 4\boldsymbol{k})\times(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}) \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 0 & -4 \\ 1 & -5 & 3 \end{vmatrix} \\
&= \big(0 - 20\big)\boldsymbol{i} - \big(9 - (-4)\big)\boldsymbol{j} + \big((-15) - 0\big)\boldsymbol{k} \\
&= -20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k}. \blacksquare
\end{aligned}$$

Hence

$$\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c}) &= (4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})\times(-20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k}) \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 2 & 3 \\ -20 & -13 & -15 \end{vmatrix} \\
&= \big((-30) - (-39)\big)\boldsymbol{i} - \big((-60) - (-60)\big)\boldsymbol{j} + \big((-52) - (-40)\big)\boldsymbol{k} \\
&= 9\boldsymbol{i} - 12\boldsymbol{k}. \blacksquare
\end{aligned}$$

(v)

$$\begin{aligned}
\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\cdot(-20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k}) \\
&= \big(2\times(-20)\big) + \big((-1)\times(-13)\big) + \big((-2)\times(-15)\big) \\
&= 3.
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} &= (4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})\cdot(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k}) \\
&= \big(4\times 1\big) + \big(2\times(-5)\big) + \big(3\times 3\big) \\
&= 3.
\end{aligned}$$

These values are equal and this **verifies the identity**

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}.$$

Since $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is *positive*, the set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ must be **right-handed**. $\blacksquare$

(vi) The **left side** of the identity is

$$\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= (2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})\times(-20\boldsymbol{i} - 13\boldsymbol{j} - 15\boldsymbol{k}) \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & -2 \\ -20 & -13 & -15 \end{vmatrix} \\
&= \big(15 - 26\big)\boldsymbol{i} - \big((-30) - 40\big)\boldsymbol{j} + \big((-26) - 20\big)\boldsymbol{k} \\
&= -11\boldsymbol{i} + 70\boldsymbol{j} - 46\boldsymbol{k}.
\end{aligned}$$

Since

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} &= \Big(\big(2\times 1\big) + \big((-1)\times(-5)\big) + \big((-2)\times 3\big)\Big)\boldsymbol{b}\\
&= \boldsymbol{b}\\
&= 3\boldsymbol{i} - 4\boldsymbol{k},
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c} &= \Big(\big(2\times 3\big) + \big((-1)\times 0\big) + \big((-2)\times(-4)\big)\Big)\boldsymbol{c}\\
&= 14\boldsymbol{c} = 14(\boldsymbol{i} - 5\boldsymbol{j} + 3\boldsymbol{k})\\
&= 14\boldsymbol{i} - 70\boldsymbol{j} + 42\boldsymbol{k},
\end{aligned}
$$

the **right side** of the identity is

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} &= (3\boldsymbol{i} - 4\boldsymbol{k}) - (14\boldsymbol{i} - 70\boldsymbol{j} + 42\boldsymbol{k})\\
&= -11\boldsymbol{i} + 70\boldsymbol{j} - 46\boldsymbol{k}.
\end{aligned}
$$

Thus the right and left sides are equal and this **verifies the identity**. ■

Problem 1.2

Find the angle between any two diagonals of a cube.

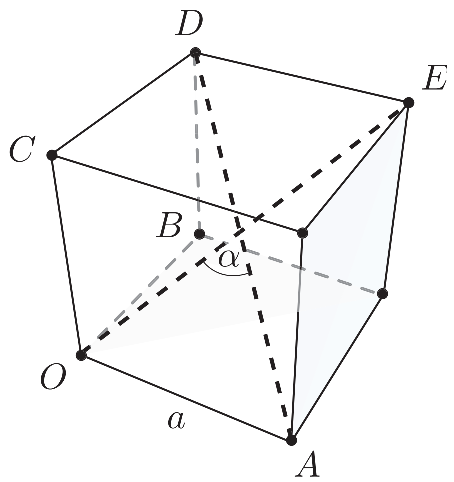

FIGURE 1.1 Two diagonals of a cube.

Solution

Figure 1.1 shows a cube of side a; OE and AD are two of its diagonals. Let O be the origin of position vectors and suppose the points A, B and C have position vectors $a\boldsymbol{i}$, $a\boldsymbol{j}$, $a\boldsymbol{k}$ respectively. Then the line segment $\overrightarrow{OE}$ represents the vector

$$a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}$$

and the line segment $\overrightarrow{AD}$ represents the vector

$$(a\boldsymbol{j} + a\boldsymbol{k}) - a\boldsymbol{i} = -a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}.$$

Let α be the angle between OE and AD. Then

$$\begin{aligned}\cos\alpha &= \frac{(a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k})\cdot(-a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k})}{|a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}||-a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k}|}\\ &= \frac{-a^2 + a^2 + a^2}{(\sqrt{3}a)(\sqrt{3}a)} = \frac{1}{3}.\end{aligned}$$

Hence the **angle between the diagonals** is $\cos^{-1}\frac{1}{3}$, which is approximately $70.5°$. ■

Problem 1.3

$ABCDEF$ is a regular hexagon with centre O which is also the origin of position vectors. Find the position vectors of the vertices C, D, E, F in terms of the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ of A and B.

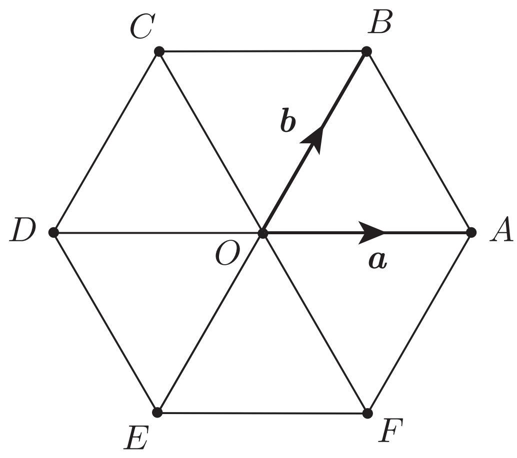

FIGURE 1.2 $ABCDEF$ is a regular hexagon.

Solution

(i) The position vector $\boldsymbol{c}$ is represented by the line segment $\overrightarrow{OC}$ which has the same magnitude and direction as the line segment $\overrightarrow{AB}$. Hence

$$\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{a}. \blacksquare$$

(ii) The position vector $\boldsymbol{d}$ is represented by the line segment $\overrightarrow{OD}$ which has the same magnitude as, but *opposite* direction to, the line segment $\overrightarrow{OA}$. Hence

$$\boldsymbol{d} = -\boldsymbol{a}. \blacksquare$$

(iii) The position vector $\boldsymbol{e}$ is represented by the line segment $\overrightarrow{OE}$ which has the same magnitude as, but *opposite* direction to, the line segment $\overrightarrow{OB}$. Hence

$$\boldsymbol{e} = -\boldsymbol{b}. \blacksquare$$

(iv) The position vector $\boldsymbol{f}$ is represented by the line segment $\overrightarrow{OF}$ which has the same magnitude as, but *opposite* direction to, the line segment $\overrightarrow{AB}$. Hence

$$\boldsymbol{e} = -(\boldsymbol{b} - \boldsymbol{a}) = \boldsymbol{a} - \boldsymbol{b}. \blacksquare$$

Problem 1.4

Let $ABCD$ be a general (skew) quadrilateral and let P, Q, R, S be the mid-points of the sides AB, BC, CD, DA respectively. Show that $PQRS$ is a parallelogram.

Solution

Let the points A, B, C, D have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ relative to some origin O. Then the position vectors of the points P, Q, R, S are given by

$$\boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}), \quad \boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}), \quad \boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d}), \quad \boldsymbol{s} = \tfrac{1}{2}(\boldsymbol{d} + \boldsymbol{a}).$$

Now the line segment $\overrightarrow{PQ}$ represents the vector

$$\boldsymbol{q} - \boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{c}) - \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) = \tfrac{1}{2}(\boldsymbol{c} - \boldsymbol{a}),$$

and the line segment $\overrightarrow{SR}$ represents the vector

$$\boldsymbol{r} - \boldsymbol{s} = \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d}) - \tfrac{1}{2}(\boldsymbol{d} + \boldsymbol{a}) = \tfrac{1}{2}(\boldsymbol{c} - \boldsymbol{a}).$$

The lines PQ and SR are therefore parallel. Similarly, the lines QR and PS are parallel. The quadrilateral $PQRS$ is therefore a **parallelogram**. ■

Problem 1.5

In a general tetrahedron, lines are drawn connecting the mid-point of each side with the mid-point of the side opposite. Show that these three lines meet in a point that bisects each of them.

Solution

Let the vertices of the tetrahedron be A, B, C, D and suppose that these points have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ relative to some origin O. Then X, the mid-point of AB, has position vector

$$\boldsymbol{x} = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}),$$

and Y, the mid-point of CD, has position vector

$$\boldsymbol{y} = \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d}).$$

Hence the mid-point of XY has position vector

$$\tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{y}) = \tfrac{1}{2}\Big(\tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) + \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{d})\Big) = \tfrac{1}{4}\big(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}\big).$$

The mid-points of the other two lines that join the mid-points of opposite sides of the tetrahedron are found to have the same position vector. These three points are therefore coincident. Hence *the three lines that join the mid-points of opposite sides of the tetrahedron meet in a point that bisects each of them*. ■

Problem 1.6

Let $ABCD$ be a general tetrahedron and let P, Q, R, S be the median centres of the faces opposite to the vertices A, B, C, D respectively. Show that the lines AP, BQ, CR, DS all meet in a point (called the *centroid* of the tetrahedron), which divides each line in the ratio 3:1.

Solution

Let the vertices of the tetrahedron be A, B, C, D and suppose that these points have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ respectively, relative to some origin O. Then P, the median centre of the face BCD has position vector

$$\boldsymbol{p} = \tfrac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}).$$

The point that divides the line AP in the ratio 3:1 therefore has position vector

$$\frac{\boldsymbol{a} + 3\,\boldsymbol{p}}{4} = \tfrac{1}{4}\big(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}\big).$$

The corresponding points on the other three lines that join the vertices of the tetrahedron to the median centres of the opposite faces are all found to have the same position vector. These four points are therefore coincident. Hence *the four lines that join the vertices of the tetrahedron to the median centres of the opposite faces meet in a point that divides each line in the ratio 3:1*. It is the same point as was constructed in Problem 1.5. ■

Problem 1.7

A number of particles with masses $m_1, m_2, m_3, \ldots$ are situated at the points with position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots$ relative to an origin O. The centre of mass G of the particles is defined to be the point of space with position vector

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

Show that if a different origin O' were used, this definition would still place G at the same point of space.

Solution

Suppose the line segment $\overrightarrow{OO'}$ (that connects the two origins) represents the vector $\boldsymbol{a}$. Then $\boldsymbol{r}'_1$, $\boldsymbol{r}'_2$, $\boldsymbol{r}'_3, \ldots$, the position vectors of the masses relative to the origin O', are given by the triangle law of addition to be

$$\boldsymbol{r}'_i = \boldsymbol{r}_i - \boldsymbol{a}.$$

The position vector of the centre of mass measured relative to O' is defined to be

$$\begin{aligned}
\boldsymbol{R}' &= \frac{m_1\boldsymbol{r}'_1 + m_2\boldsymbol{r}'_2 + m_3\boldsymbol{r}'_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \\
&= \frac{m_1(\boldsymbol{r}_1 - \boldsymbol{a}) + m_2(\boldsymbol{r}_2 - \boldsymbol{a}) + m_3(\boldsymbol{r}_3 - \boldsymbol{a}) + \cdots}{m_1 + m_2 + m_3 + \cdots} \\
&= \left(\frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}\right) - \boldsymbol{a} \\
&= \boldsymbol{R} - \boldsymbol{a}.
\end{aligned}$$

By the triangle law of addition, this defines the **same point of space** as before. ■

Problem 1.8

Prove that the three perpendiculars of a triangle are concurrent.

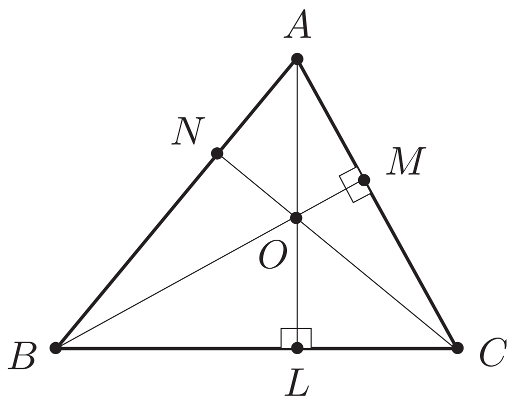

FIGURE 1.3 AL and BM are two of the perpendiculars of the triangle ABC.

Solution

Let ABC be the triangle and construct the perpendiculars AL and BM from A and B; let O be their point of intersection. Now construct the line CO and extend it to meet AB in the point N. We wish to show that CN is perpendicular to AB.

Suppose the points A, B, C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ relative to O. Then, since AL is perpendicular to BC, we have

$$\boldsymbol{a} \cdot (\boldsymbol{c} - \boldsymbol{b}) = 0,$$

and, since BM is perpendicular to CA, we have

$$\boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) = 0.$$

On adding these equalities, we obtain

$$\boldsymbol{c} \cdot (\boldsymbol{a} - \boldsymbol{b}) = 0,$$

which shows that the line CN is **perpendicular** to the side AB. ■

Problem 1.9

If $\boldsymbol{a}_1 = \lambda_1\boldsymbol{i} + \mu_1\boldsymbol{j} + \nu_1\boldsymbol{k}$, $\boldsymbol{a}_2 = \lambda_2\boldsymbol{i} + \mu_2\boldsymbol{j} + \nu_2\boldsymbol{k}$, $\boldsymbol{a}_3 = \lambda_3\boldsymbol{i} + \mu_3\boldsymbol{j} + \nu_3\boldsymbol{k}$, where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis, show that

$$\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

Deduce that cyclic rotation of the vectors in a triple scalar product leaves the value of the product unchanged.

Solution

Since

$$\begin{aligned} \boldsymbol{a}_2 \times \boldsymbol{a}_3 &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} \\ &= \boldsymbol{i}\begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \lambda_2 & \nu_2 \\ \lambda_3 & \nu_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix}, \end{aligned}$$

it follows that

$$\begin{aligned} \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) &= \big(\lambda_1\boldsymbol{i} + \mu_1\boldsymbol{j} + \nu_1\boldsymbol{k}\big) \cdot \left(\boldsymbol{i}\begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \lambda_2 & \nu_2 \\ \lambda_3 & \nu_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix}\right) \\ &= \lambda_1\begin{vmatrix} \mu_2 & \nu_2 \\ \mu_3 & \nu_3 \end{vmatrix} - \mu_1\begin{vmatrix} \lambda_2 & \nu_2 \\ \lambda_3 & \nu_3 \end{vmatrix} + \nu_1\begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix} \\ &= \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}. \blacksquare \end{aligned}$$

Since the value of this determinant is unchanged a cyclic rotation of its rows, it follows that *the value of a triple scalar product is unchanged by a cyclic rotation of its vectors*. ■

Problem 1.10

By expressing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ in terms of a suitable standard basis, prove the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}$.

Solution

The algebra in this solution is reduced by selecting a special basis set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ so that

$$\begin{aligned}
\boldsymbol{a} &= a_1\boldsymbol{i}\,,\\
\boldsymbol{b} &= b_1\boldsymbol{i} + b_2\boldsymbol{j}\,,\\
\boldsymbol{c} &= c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}\,.
\end{aligned}$$

Such a choice is always possible. Then

$$\begin{aligned}
\boldsymbol{b}\times\boldsymbol{c} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix}\\
&= \big(b_2c_3 - 0\big)\boldsymbol{i} - \big(b_1c_3 - 0\big)\boldsymbol{j} + \big(b_1c_2 - b_2c_1\big)\boldsymbol{k}\\
&= b_2c_3\,\boldsymbol{i} - b_1c_3\,\boldsymbol{j} + \big(b_1c_2 - b_2c_1\big)\,\boldsymbol{k}
\end{aligned}$$

and hence the **left side** of the identity is

$$\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & 0 & 0 \\ b_2c_3 & -b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}\\
&= \big(0-0\big)\boldsymbol{i} - \big(a_1(b_1c_2 - b_2c_1) - 0\big)\boldsymbol{j} + \big(a_1(-b_1c_3) - 0\big)\boldsymbol{k}\\
&= a_1(b_2c_1 - b_1c_2)\,\boldsymbol{j} - a_1b_1c_3\,\boldsymbol{k}\,.
\end{aligned}$$

The **right side** of the identity is

$$\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c} &= (a_1c_1)\,\boldsymbol{b} - (a_1b_1)\,\boldsymbol{c}\\
&= a_1c_1\big(b_1\boldsymbol{i} + b_2\boldsymbol{j}\big) - a_1b_1\,(c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k})\\
&= a_1(b_2c_1 - b_1c_2)\,\boldsymbol{j} - (a_1b_1c_3)\boldsymbol{k}\,.
\end{aligned}$$

Thus the right and left sides are equal and **this proves the identity**. ■

Problem 1.11

Prove the identities

(i) $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$

(ii) $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = [\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]\boldsymbol{d}$

(iii) $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{0}$ (Jacobi's identity)

Solution

(i)

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) &= \boldsymbol{a}\cdot\big(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d})\big)\\ &= \boldsymbol{a}\cdot\big((\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{d}\big)\\ &= (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}). \blacksquare\end{aligned}$$

(ii)

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) &= \big((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{d}\big)\boldsymbol{c} - \big((\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\big)\boldsymbol{d}\\ &= [\boldsymbol{a},\boldsymbol{b},\boldsymbol{d}]\boldsymbol{c} - [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]\boldsymbol{d}. \blacksquare\end{aligned}$$

(iii)

$$\begin{aligned}&\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})\\ &= \big((\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\big) + \big((\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}\big) + \big((\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}\big)\\ &= \big(\boldsymbol{c}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{c}\big)\boldsymbol{a} + \big(\boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{c}\cdot\boldsymbol{a}\big)\boldsymbol{b} + \big(\boldsymbol{b}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{b}\big)\boldsymbol{c}\\ &= \boldsymbol{0}. \blacksquare\end{aligned}$$

Problem 1.12 *Reciprocal basis*

Let $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ be any basis set. Then the corresponding **reciprocal basis** $\{\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*\}$ is defined by

$$\boldsymbol{a}^* = \frac{\boldsymbol{b}\times\boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}, \quad \boldsymbol{b}^* = \frac{\boldsymbol{c}\times\boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}, \quad \boldsymbol{c}^* = \frac{\boldsymbol{a}\times\boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}.$$

(i) If $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis, show that $\{\boldsymbol{i}^*, \boldsymbol{j}^*, \boldsymbol{k}^*\} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$.

(ii) Show that $[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*] = 1/[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$. Deduce that if $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a right handed set then so is $\{\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*\}$.

(iii) Show that $\{(\boldsymbol{a}^*)^*, (\boldsymbol{b}^*)^*, (\boldsymbol{c}^*)^*] = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$.

(iv) If a vector $\boldsymbol{v}$ is expanded in terms of the basis set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ in the form

$$\boldsymbol{v} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c},$$

show that the coefficients λ, μ, ν are given by $\lambda = \boldsymbol{v}\cdot\boldsymbol{a}^*$, $\mu = \boldsymbol{v}\cdot\boldsymbol{b}^*$, $\nu = \boldsymbol{v}\cdot\boldsymbol{c}^*$.

Solution

(i) If $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis, then

$$\begin{aligned}\boldsymbol{i}^* &= \frac{\boldsymbol{j}\times\boldsymbol{k}}{\boldsymbol{i}\cdot(\boldsymbol{j}\times\boldsymbol{k})}\\ &= \frac{\boldsymbol{i}}{\boldsymbol{i}\cdot\boldsymbol{i}} = \frac{\boldsymbol{i}}{1}\\ &= \boldsymbol{i}.\end{aligned}$$

Similar arguments hold for $\boldsymbol{j}^*$ and $\boldsymbol{k}^*$ and hence $\{\boldsymbol{i}^*, \boldsymbol{j}^*, \boldsymbol{k}^*\} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. ■

(ii)

$$\begin{aligned}[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*] &= \boldsymbol{a}^*\cdot\big(\boldsymbol{b}^*\times\boldsymbol{c}^*\big)\\ &= \boldsymbol{a}^*\cdot\left(\frac{\boldsymbol{c}\times\boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\times\frac{\boldsymbol{a}\times\boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\right)\\ &= \frac{\boldsymbol{a}^*}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]^2}\cdot\Big(\big((\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}\big)\boldsymbol{a} - \big((\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{a}\big)\boldsymbol{b}\Big)\\ &= \frac{\boldsymbol{b}\times\boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]^3}\cdot\Big([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} - \boldsymbol{0}\Big)\\ &= \frac{1}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}.\end{aligned}$$

If $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a right-handed basis set, then $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is positive. It follows that $[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*]$ must also be positive and therefore also **right-handed**. ■

(iii)

$$\begin{aligned}
\left(\boldsymbol{a}^*\right)^* &= \frac{\boldsymbol{b}^* \times \boldsymbol{c}^*}{[\boldsymbol{a}^*, \boldsymbol{b}^*, \boldsymbol{c}^*]}\\
&= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] \left(\frac{\boldsymbol{c} \times \boldsymbol{a}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \times \frac{\boldsymbol{a} \times \boldsymbol{b}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right)\\
&= \frac{1}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \Big(((\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b})\, \boldsymbol{a} - ((\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{a})\, \boldsymbol{b} \Big)\\
&= \frac{1}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \Big([\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]\boldsymbol{a} - \boldsymbol{0} \Big)\\
&= \boldsymbol{a}.
\end{aligned}$$

Similar arguments hold for $\left(\boldsymbol{b}^*\right)^*$ and $(\boldsymbol{c}^*)^*$ and hence $\{(\boldsymbol{a}^*)^*, (\boldsymbol{b}^*)^*, (\boldsymbol{c}^*)^*\} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$. ■

(iv) Suppose $\boldsymbol{v}$ is expanded in terms of the basis set $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ in the form

$$\boldsymbol{v} = \lambda \boldsymbol{a} + \mu \boldsymbol{b} + \nu \boldsymbol{c}.$$

On taking the scalar product of this equation with $\boldsymbol{a}^*$, we obtain

$$\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{a}^* &= \lambda \boldsymbol{a} \cdot \boldsymbol{a}^* + \mu \boldsymbol{b} \cdot \boldsymbol{a}^* + \nu \boldsymbol{c} \cdot \boldsymbol{a}^*\\
&= \lambda \boldsymbol{a} \cdot \left(\frac{\boldsymbol{b} \times \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right) + \mu \boldsymbol{b} \cdot \left(\frac{\boldsymbol{b} \times \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right) + \nu \boldsymbol{c} \cdot \left(\frac{\boldsymbol{b} \times \boldsymbol{c}}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]} \right)\\
&= \lambda + 0 + 0\\
&= \lambda.
\end{aligned}$$

Hence $\lambda = \boldsymbol{v} \cdot \boldsymbol{a}^*$, and, by similar arguments, $\mu = \boldsymbol{v} \cdot \boldsymbol{b}^*$ and $\mu = \boldsymbol{v} \cdot \boldsymbol{c}^*$. ■

Problem 1.13

Lamé's equations The directions in which X-rays are strongly scattered by a crystal are determined from the solutions $\boldsymbol{x}$ of Lamé's equations, namely

$$\boldsymbol{x}\cdot\boldsymbol{a}=L, \qquad \boldsymbol{x}\cdot\boldsymbol{b}=M, \qquad \boldsymbol{x}\cdot\boldsymbol{c}=N,$$

where $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$ are the basis vectors of the crystal lattice, and L, M, N are *any* integers. Show that the solutions of Lamé's equations are

$$\boldsymbol{x}=L\,\boldsymbol{a}^*+M\,\boldsymbol{b}^*+N\,\boldsymbol{c}^*,$$

where $\{\boldsymbol{a}^*,\boldsymbol{b}^*,\boldsymbol{c}^*\}$ is the reciprocal basis to $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$.

Solution

Let us seek solutions of Lamé's equations in the form

$$\boldsymbol{x}=\lambda\,\boldsymbol{a}^*+\mu\,\boldsymbol{b}^*+\nu\,\boldsymbol{c}^*,$$

where $\{\boldsymbol{a}^*,\boldsymbol{b}^*,\boldsymbol{c}^*\}$ is the **reciprocal basis** corresponding to the lattice basis $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$. On substituting this expansion into Lamé's equations, we find that $\lambda=L$, $\mu=M$ and $\nu=N$. The only **solution of Lamé's equations** (corresponding to given values of L, M, N) is therefore

$$\boldsymbol{x}=L\,\boldsymbol{a}^*+M\,\boldsymbol{b}^*+N\,\boldsymbol{c}^*. \blacksquare$$

Problem 1.14

If $\boldsymbol{r}(t) = (3t^2 - 4)\boldsymbol{i} + t^3\boldsymbol{j} + (t+3)\boldsymbol{k}$, where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a constant standard basis, find $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$. Deduce the time derivative of $\boldsymbol{r} \times \dot{\boldsymbol{r}}$.

Solution

If $\boldsymbol{r}(t) = (3t^2 - 4)\boldsymbol{i} + t^3\boldsymbol{j} + (t+3)\boldsymbol{k}$, then

$$\begin{aligned}
\dot{\boldsymbol{r}} &= 6t\,\boldsymbol{i} + 3t^2\boldsymbol{j} + \boldsymbol{k},\\
\ddot{\boldsymbol{r}} &= 6\,\boldsymbol{i} + 6t\,\boldsymbol{j}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) &= \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}}\\
&= \boldsymbol{0} + \boldsymbol{r} \times \ddot{\boldsymbol{r}}\\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3t^2 - 4 & t^3 & t+3 \\ 6 & 6t & 0 \end{vmatrix}\\
&= -6t(t+3)\,\boldsymbol{i} + 6(t+3)\,\boldsymbol{j} + 12t(t^2-2)\,\boldsymbol{k}. \blacksquare
\end{aligned}$$

Problem 1.15

The vector $\boldsymbol{v}$ is a function of the time t and $\boldsymbol{k}$ is a constant vector. Find the time derivatives of (i) $|\,\boldsymbol{v}\,|^2$, (ii) $(\boldsymbol{v}\cdot\boldsymbol{k})\,\boldsymbol{v}$, (iii) $[\,\boldsymbol{v},\dot{\boldsymbol{v}},\boldsymbol{k}\,]$.

Solution

(i)

$$\begin{aligned}\frac{d}{dt}|\,\boldsymbol{v}\,|^2 &= \frac{d}{dt}\big(\boldsymbol{v}\cdot\boldsymbol{v}\big)\\ &= \dot{\boldsymbol{v}}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\dot{\boldsymbol{v}}\\ &= 2\,\boldsymbol{v}\cdot\dot{\boldsymbol{v}}.\ \blacksquare\end{aligned}$$

(ii)

$$\begin{aligned}\frac{d}{dt}\big((\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{v}\big) &= \big(\dot{\boldsymbol{v}}\cdot\boldsymbol{k}+\boldsymbol{v}\cdot\dot{\boldsymbol{k}}\big)\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{k})\dot{\boldsymbol{v}}\\ &= (\dot{\boldsymbol{v}}\cdot\boldsymbol{k})\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{k})\dot{\boldsymbol{v}}.\ \blacksquare\end{aligned}$$

(iii)

$$\begin{aligned}\frac{d}{dt}[\,\boldsymbol{v},\dot{\boldsymbol{v}},\boldsymbol{k}\,] &= [\,\dot{\boldsymbol{v}},\dot{\boldsymbol{v}},\boldsymbol{k}\,]+[\,\boldsymbol{v},\ddot{\boldsymbol{v}},\boldsymbol{k}\,]+[\,\boldsymbol{v},\dot{\boldsymbol{v}},\dot{\boldsymbol{k}}\,]\\ &= 0+[\,\boldsymbol{v},\ddot{\boldsymbol{v}},\boldsymbol{k}\,]+0\\ &= [\,\boldsymbol{v},\ddot{\boldsymbol{v}},\boldsymbol{k}\,].\ \blacksquare\end{aligned}$$

Problem 1.16

Find the unit tangent vector, the unit normal vector and the curvature of the circle $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$ at the point with parameter θ.

Solution

Let $\boldsymbol{i}$, $\boldsymbol{j}$ be unit vectors in the directions Ox, Oy respectively. Then the vector form of the equation for the circle is

$$\boldsymbol{r} = a\cos\theta\,\boldsymbol{i} + a\sin\theta\,\boldsymbol{j}.$$

Then

$$\frac{d\boldsymbol{r}}{d\theta} = -a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j}$$

and so

$$\left|\frac{d\boldsymbol{r}}{d\theta}\right| = a.$$

The **unit tangent vector** to the circle is therefore

$$\boldsymbol{t}(\theta) = \frac{d\boldsymbol{r}}{d\theta}\Bigg/\left|\frac{d\boldsymbol{r}}{d\theta}\right| = -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j}. \blacksquare$$

By the chain rule,

$$\frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}/d\theta}{ds/d\theta} = \frac{d\boldsymbol{t}/d\theta}{|d\boldsymbol{r}/d\theta|} = \frac{-\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}}{a}.$$

Hence the **unit normal vector** and **curvature** of the circle are given by

$$\boldsymbol{n}(\theta) = -\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}, \qquad \kappa(\theta) = \frac{1}{a}. \blacksquare$$

The **radius of curvature** of the circle is a. ■

Problem 1.17

Find the unit tangent vector, the unit normal vector and the curvature of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = b\theta$ at the point with parameter θ.

Solution

Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors in the directions Ox, Oy, Oz respectively. Then the vector form of the equation for the helix is

$$\boldsymbol{r} = a\cos\theta\,\boldsymbol{i} + a\sin\theta\,\boldsymbol{j} + b\theta\,\boldsymbol{k}.$$

Then

$$\frac{d\boldsymbol{r}}{d\theta} = -a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j} + b\,\boldsymbol{k}$$

and so

$$\left|\frac{d\boldsymbol{r}}{d\theta}\right| = \left(a^2 + b^2\right)^{1/2}.$$

The **unit tangent vector** to the helix is therefore

$$\begin{aligned} \boldsymbol{t}(\theta) &= \frac{d\boldsymbol{r}}{d\theta}\bigg/\left|\frac{d\boldsymbol{r}}{d\theta}\right| \\ &= \frac{-a\sin\theta\,\boldsymbol{i} + a\cos\theta\,\boldsymbol{j} + b\,\boldsymbol{k}}{\left(a^2+b^2\right)^{1/2}}. \blacksquare \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{d\boldsymbol{t}}{ds} &= \frac{d\boldsymbol{t}/d\theta}{ds/d\theta} = \frac{d\boldsymbol{t}/d\theta}{|d\boldsymbol{r}/d\theta|} \\ &= \frac{-a\cos\theta\,\boldsymbol{i} - a\sin\theta\,\boldsymbol{j}}{a^2+b^2}. \end{aligned}$$

Hence the **unit normal vector** and **curvature** of the helix are given by

$$\boldsymbol{n}(\theta) = -\cos\theta\,\boldsymbol{i} - \sin\theta\,\boldsymbol{j}, \qquad \kappa(\theta) = \frac{a}{a^2+b^2}\ \blacksquare$$

The **radius of curvature** of the helix is $\left(a^2+b^2\right)/a$. $\blacksquare$

Problem 1.18

Find the unit tangent vector, the unit normal vector and the curvature of the parabola $x = ap^2$, $y = 2ap$, $z = 0$ at the point with parameter p.

Solution

Let $\boldsymbol{i}$, $\boldsymbol{j}$ be unit vectors in the directions Ox, Oy respectively. Then the vector form of the equation for the parabola is

$$\boldsymbol{r} = ap^2\boldsymbol{i} + 2ap\,\boldsymbol{j}\,.$$

Then

$$\frac{d\boldsymbol{r}}{dp} = 2ap\,\boldsymbol{i} + 2a\,\boldsymbol{j} \qquad \text{and} \qquad \left|\frac{d\boldsymbol{r}}{dp}\right| = 2a\left(p^2+1\right)^{1/2}.$$

The **unit tangent vector** to the parabola is therefore

$$\begin{aligned}\boldsymbol{t}(p) &= \frac{d\boldsymbol{r}}{dp}\Big/\left|\frac{d\boldsymbol{r}}{dp}\right|\\ &= \frac{p\,\boldsymbol{i} + \boldsymbol{j}}{\left(p^2+1\right)^{1/2}}.\ \blacksquare\end{aligned}$$

By the chain rule,

$$\begin{aligned}\frac{d\boldsymbol{t}}{ds} &= \frac{d\boldsymbol{t}/dp}{ds/dp} = \frac{d\boldsymbol{t}/dp}{|d\boldsymbol{r}/dp|}\\ &= \frac{1}{2a\left(p^2+1\right)^{1/2}}\left(\frac{\boldsymbol{i}}{\left(p^2+1\right)^{1/2}} - \frac{p(p\,\boldsymbol{i} + \boldsymbol{j})}{\left(p^2+1\right)^{3/2}}\right)\\ &= \frac{\boldsymbol{i} - p\,\boldsymbol{j}}{2a\left(p^2+1\right)^{2}}.\end{aligned}$$

Hence the **unit normal vector** and **curvature** of the parabola are given by

$$\boldsymbol{n}(\theta) = \frac{\boldsymbol{i} - p\,\boldsymbol{j}}{\left(p^2+1\right)^{1/2}} \qquad\qquad \kappa(\theta) = \frac{1}{2a\left(p^2+1\right)^{3/2}}.\ \blacksquare$$

The **radius of curvature** of the parabola is $2a\left(p^2+1\right)^{3/2}$. ■

Chapter Two

Velocity, acceleration and scalar angular velocity

Problem 2.1

A particle P moves along the x-axis with its displacement at time t given by $x = 6t^2 - t^3 + 1$, where x is measured in metres and t in seconds. Find the velocity and acceleration of P at time t. Find the times at which P is at rest and find its position at these times.

Solution

Since the displacement of P at time t is

$$x = 6t^2 - t^3 + 1,$$

the **velocity** of P at time t is given by

$$v = \frac{dx}{dt} = 12t - 3t^2 \text{ m s}^{-1},$$

and the **acceleration** of P at time t is given by

$$a = \frac{dv}{dt} = 12 - 6t \text{ m s}^{-2}.$$

P is instantaneously **at rest** when $v = 0$, that is, when

$$12t - 3t^2 = 0.$$

This equation can be written in the form

$$3t(4 - t) = 0$$

and its solutions are therefore $t = 0$ s and $t = 4$ s.

When $t = 0$ s, the displacement of P is $x = 6(0^2) - 0^3 + 1 = 1$ m and when $t = 4$ s, the displacement of P is $x = 6(4^2) - 4^3 + 1 = 33$ m. ■

Problem 2.2

A particle P moves along the x-axis with its acceleration a at time t given by

$$a = 6t - 4\ \mathrm{m\,s^{-2}}.$$

Initially P is at the point $x = 20$ m and is moving with speed $15\ \mathrm{m\,s^{-1}}$ in the negative x-direction. Find the velocity and displacement of P at time t. Find when P comes to rest and its displacement at this time.

Solution

Since the acceleration of P at time t is given to be

$$a = 6t - 4,$$

the velocity v of P at time t must satisfy the ODE

$$\frac{dv}{dt} = 6t - 4.$$

Integrating with respect to t gives

$$v = 3t^2 - 4t + C,$$

where C is a constant of integration. The initial condition that $v = -15$ when $t = 0$ gives

$$-15 = 3(0^2) - 4(0) + C,$$

from which $C = -15$. Hence the **velocity** of P at time t is

$$v = 3t^2 - 4t - 15\ \mathrm{m\,s^{-1}}.$$

By writing $v = dx/dt$ and integrating again, we obtain

$$x = t^3 - 2t^2 - 15t + D,$$

where D is a second constant of integration. The initial condition that $x = 20$ when $t = 0$ gives

$$20 = 0^3 - 2(0^2) - 15(0) + D,$$

from which $D = 20$. Hence the **displacement** of P at time t is

$$x = t^3 - 2t^2 - 15t + 20 \text{ m}.$$

P comes to rest when $v = 0$, that is, when

$$3t^2 - 4t - 15 = 0.$$

This equation can be written in the form

$$(t - 3)(3t + 5) = 0$$

and its solutions are therefore $t = 3$ s and $t = -\frac{5}{3}$ s. The time $t = -\frac{5}{3}$ s is *before* the motion started and is therefore not a permissible solution. It follows that P **comes to rest** only when $t = 3$ s. The **displacement** of P at this time is

$$x = 3^3 - 2(3^2) - 15(3) + 20 = -16 \text{ m.} \blacksquare$$

Problem 2.3 *Constant acceleration formulae*

A particle P moves along the x-axis with *constant* acceleration a in the positive x-direction. Initially P is at the origin and is moving with velocity u in the positive x-direction. Show that the velocity v and displacement x of P at time t are given by

$$v = u + at, \qquad x = ut + \tfrac{1}{2}at^2,$$

and deduce that

$$v^2 = u^2 + 2ax.$$

In a standing quarter mile test, the Suzuki Bandit 1200 motorcycle covered the quarter mile (from rest) in 11.4 seconds and crossed the finish line doing 116 miles per hour. Are these figures consistent with the assumption of constant acceleration?

Solution

When the acceleration a is a *constant*, the ODE

$$\frac{dv}{dt} = a$$

integrates to give

$$v = at + C,$$

where C is a constant of integration. The initial condition $v = u$ when $t = 0$ gives

$$u = 0 + C,$$

from which $C = u$. Hence the **velocity** of P at time t is given by

$$v = u + at. \tag{1}$$

On writing $v = dx/dt$ and integrating again, we obtain

$$x = ut + \tfrac{1}{2}at^2 + D,$$

where D is a second constant of integration. The initial condition $x = 0$ when $t = 0$ gives $D = 0$ so that the **displacement** of P at time t is given by

$$x = ut + \tfrac{1}{2}at^2. \tag{2}$$

From equation (1),

$$\begin{aligned} v^2 &= (u+at)^2 \\ &= u^2 + 2uat + a^2t^2 \\ &= u^2 + 2a\left(ut + \tfrac{1}{2}at^2\right) \\ &= u^2 + 2ax, \end{aligned}$$

on using equation (1). We have thus obtained the relation

$$v^2 = u^2 + 2ax. \tag{3}$$

In the notation used above, the results of the Bandit test run were

$$\begin{aligned} u &= 0, & v &= 116 \text{ mph } (= 170 \text{ ft s}^{-1}), \\ x &= 1320 \text{ ft}, & t &= 11.4 \text{ s}, \end{aligned}$$

in Imperial units.

Suppose that the Bandit does have constant acceleration a. Then formula (1) gives

$$170 = 0 + 11.4\,a,$$

from which $a = 14.9$ ft s^{-2}. However, formula (2) gives

$$1320 = 0 + \tfrac{1}{2}a(11.4)^2$$

from which $a = 20.3$ ft s^{-2}. These two values for a do not agree and so the Bandit must have had **non constant acceleration**. ■

Problem 2.4

The trajectory of a charged particle moving in a magnetic field is given by

$$\boldsymbol{r} = b\cos\Omega t\,\boldsymbol{i} + b\sin\Omega t\,\boldsymbol{j} + ct\,\boldsymbol{k},$$

where b, Ω and c are positive constants. Show that the particle moves with constant speed and find the magnitude of its acceleration.

Solution

Since the position vector of P at time t is

$$\boldsymbol{r} = b\cos\Omega t\,\boldsymbol{i} + b\sin\Omega t\,\boldsymbol{j} + ct\,\boldsymbol{k},$$

the **velocity** of P at time t is given by

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -\Omega b\sin\Omega t\,\boldsymbol{i} + \Omega b\cos\Omega t\,\boldsymbol{j} + c\,\boldsymbol{k},$$

and the **acceleration** of P at time t is given by

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -\Omega^2 b\cos\Omega t\,\boldsymbol{i} - \Omega^2 b\sin\Omega t\,\boldsymbol{j}.$$

It follows that

$$\begin{aligned}|\boldsymbol{v}|^2 &= (-\Omega b\sin\Omega t)^2 + (\Omega b\cos\Omega t)^2 + c^2\\ &= \Omega^2 b^2\left(\sin^2\Omega t + \cos^2\Omega t\right) + c^2\\ &= \Omega^2 b^2 + c^2.\end{aligned}$$

Hence $|\boldsymbol{v}| = \left(\Omega^2 b^2 + c^2\right)^{1/2}$, which is a constant.

Furthermore,

$$\begin{aligned}|\boldsymbol{a}|^2 &= (-\Omega^2 b\cos\Omega t)^2 + (-\Omega^2 b\sin\Omega t)^2\\ &= \Omega^4 b^2\left(\cos^2\Omega t + \sin^2\Omega t\right)\\ &= \Omega^4 b^2.\end{aligned}$$

Hence $|\boldsymbol{a}| = \Omega^2 b$, which is also a constant. ■

Problem 2.5 *Acceleration due to rotation and orbit of the Earth*

A body is at rest at a location on the Earth's equator. Find its acceleration due to the Earth's rotation. [Take the Earth's radius at the equator to be 6400 km.]

Find also the acceleration of the Earth in its orbit around the Sun. [Take the Sun to be fixed and regard the Earth as a particle following a circular path with centre the Sun and radius 15×10^{10} m.

Solution

(i) The distance travelled by a body on the equator in one rotation of the Earth is $2\pi R$, where R is the Earth's radius. This distance is traversed in one day. The **speed** of the body is therefore

$$v = \frac{2\pi \times 6,400,000}{24 \times 60 \times 60} = 465 \text{ m s}^{-1},$$

in S.I. units. The **acceleration** of the body is directed towards the centre of the Earth and has magnitude

$$a = \frac{v^2}{R} = 0.034 \text{ m s}^{-2}. \blacksquare$$

(ii) The distance travelled by the Earth in one orbit of the Sun is $2\pi R$, where R is now the radius of the Earth's *orbit*. This distance is traversed in one year. The **speed** of the Earth in its orbit is therefore

$$v = \frac{2\pi \left(15 \times 10^{10}\right)}{365 \times 24 \times 60 \times 60} = 3.0 \times 10^4 \text{ m s}^{-1},$$

in S.I. units. The **acceleration** of the Earth is directed towards the Sun and has magnitude

$$a = \frac{v^2}{R} = 0.0060 \text{ m s}^{-2}. \blacksquare$$

Problem 2.6

An insect flies on a spiral trajectory such that its polar coordinates at time t are given by

$$r = be^{\Omega t}, \qquad \theta = \Omega t,$$

where b and Ω are positive constants. Find the velocity and acceleration vectors of the insect at time t, and show that the angle between these vectors is always $\pi/4$.

Solution

The **velocity** of the insect at time t is given by

$$\begin{aligned} \boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= \left(\Omega b e^{\Omega t}\right)\widehat{\boldsymbol{r}} + \left(\Omega b e^{\Omega t}\right)\widehat{\boldsymbol{\theta}} \end{aligned}$$

and the **acceleration** of the insect at time t is given by

$$\begin{aligned} \boldsymbol{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= \left(\Omega^2 b e^{\Omega t} - \Omega^2 b e^{\Omega t}\right)\widehat{\boldsymbol{r}} + \left(0 + 2\Omega^2 b e^{\Omega t}\right)\widehat{\boldsymbol{\theta}} \\ &= 2\Omega^2 b e^{\Omega t}\,\widehat{\boldsymbol{\theta}}. \end{aligned}$$

It follows that

$$|\boldsymbol{v}| = \sqrt{2}\Omega b e^{\Omega t} \qquad \text{and} \qquad |\boldsymbol{a}| = 2\Omega^2 b e^{\Omega t}.$$

The **angle** α between $\boldsymbol{v}$ and $\boldsymbol{a}$ is then given by

$$\begin{aligned} \cos\alpha &= \frac{\boldsymbol{v}\cdot\boldsymbol{a}}{|\boldsymbol{v}||\boldsymbol{a}|} \\ &= \frac{2\Omega^3 b^2 e^{2\Omega t}}{\left(\sqrt{2}\Omega b e^{\Omega t}\right)\left(2\Omega^2 b e^{\Omega t}\right)} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Hence the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ is always $\pi/4$. ■

Problem 2.7

A racing car moves on a circular track of radius b. The car starts from rest and its *speed* increases at a constant rate α. Find the angle between its velocity and acceleration vectors at time t.

Solution

Since the car has speed $v = \alpha t$ at time t, its **velocity** is

$$\boldsymbol{v} = v\,\widehat{\boldsymbol{\theta}} = \alpha t\,\widehat{\boldsymbol{\theta}}$$

and its **acceleration** is

$$\boldsymbol{a} = \left(-\frac{v^2}{b}\right)\widehat{\boldsymbol{r}} + \dot{v}\,\widehat{\boldsymbol{\theta}} = \left(-\frac{\alpha^2 t^2}{b}\right)\widehat{\boldsymbol{r}} + \alpha\,\widehat{\boldsymbol{r}}.$$

The **angle** β between $\boldsymbol{v}$ and $\boldsymbol{a}$ is then given by

$$\begin{aligned}\cos\beta &= \frac{\boldsymbol{v}\cdot\boldsymbol{a}}{|\boldsymbol{v}||\boldsymbol{a}|}\\ &= \frac{\alpha^2 t}{\alpha t\left(\dfrac{\alpha^4 t^4}{b^2} + \alpha^2\right)^{1/2}}\\ &= \frac{b}{\left(b^2 + \alpha^2 t^4\right)^{1/2}}.\end{aligned}$$

The angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$ at time t is therefore

$$\beta = \cos^{-1}\left(\frac{b}{\left(b^2 + \alpha^2 t^4\right)^{1/2}}\right). \blacksquare$$

Problem 2.8

A particle P moves on a circle with centre O and radius b. At a certain instant the speed of P is v and its acceleration vector makes an angle α with PO. Find the magnitude of the acceleration vector at this instant.

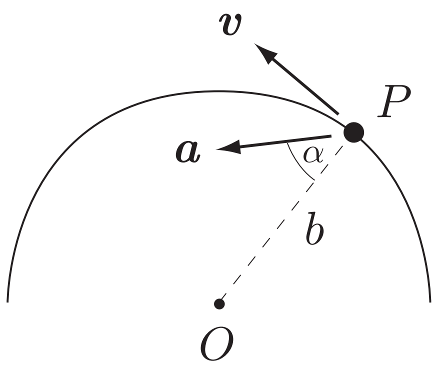

FIGURE 2.1 The velocity and acceleration vectors of the particle P.

Solution

In the standard notation, the **velocity** and **acceleration** vectors of P have the form

$$\boldsymbol{v} = v\widehat{\boldsymbol{\theta}},$$
$$\boldsymbol{a} = -\frac{v^2}{b}\widehat{\boldsymbol{r}} + \dot{v}\widehat{\boldsymbol{\theta}},$$

where v is the *circumferential velocity* of P.

Consider the component of $\boldsymbol{a}$ in the direction $\overrightarrow{PO}$. This can be written in the geometrical form $|\boldsymbol{a}|\cos\alpha$ and also in the algebraic form $\boldsymbol{a}\cdot(-\widehat{\boldsymbol{r}})$. Hence

$$\begin{aligned}|\boldsymbol{a}|\cos\alpha &= \boldsymbol{a}\cdot(-\widehat{\boldsymbol{r}})\\ &= \left(-\frac{v^2}{b}\widehat{\boldsymbol{r}} + \dot{v}\widehat{\boldsymbol{\theta}}\right)\cdot(-\widehat{\boldsymbol{r}})\\ &= \frac{v^2}{b}.\end{aligned}$$

It follows that

$$|\boldsymbol{a}| = \frac{v^2}{b\cos\alpha}. \blacksquare$$

Problem 2.9 ✻

A bee flies on a trajectory such that its polar coordinates at time t are given by

$$r = \frac{bt}{\tau^2}(2\tau - t) \qquad \theta = \frac{t}{\tau} \qquad (0 \le t \le 2\tau),$$

where b and τ are positive constants. Find the velocity vector of the bee at time t.

Show that the least speed achieved by the bee is b/τ. Find the acceleration of the bee at this instant.

Solution

The **velocity** vector of the bee is given by

$$\begin{aligned} \boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\ &= \frac{2b}{\tau^2}(\tau - t)\,\widehat{\boldsymbol{r}} + \frac{bt}{\tau^3}(2\tau - t)\,\widehat{\boldsymbol{\theta}}. \end{aligned}$$

It follows that

$$\begin{aligned} |\boldsymbol{v}|^2 &= \frac{4b^2}{\tau^4}(\tau - t)^2 + \frac{b^2t^2}{\tau^6}(2\tau - t)^2 \\ &= \frac{b^2}{\tau^6}\left(t^4 - 4\tau t^3 + 8\tau^2t^2 - 8\tau^3 t + 4\tau^4\right), \end{aligned}$$

after some simplification.

To find the maximum value of $|\boldsymbol{v}|$, consider the time derivative of $|\boldsymbol{v}|^2$.

$$\begin{aligned} \frac{d}{dt}|\boldsymbol{v}|^2 &= \frac{b^2}{\tau^6}\left(4t^3 - 12\tau t^2 + 16\tau^2 t - 8\tau^3\right) \\ &= \frac{4b^2}{\tau^6}(t - \tau)\left(t^2 - 2\tau t + 2\tau^2\right). \end{aligned}$$

The expression $t^2 - 2\tau t + 2\tau^2$ is always positive and hence

$$\frac{d}{dt}|\boldsymbol{v}|^2 \begin{cases} < 0 & \text{for} \quad t < \tau, \\ = 0 & \text{for} \quad t = \tau, \\ > 0 & \text{for} \quad t > \tau. \end{cases}$$

Hence $|\boldsymbol{v}|$ achieves its minimum value when $t = \tau$. At this instant,

$$|\boldsymbol{v}| = \frac{b}{\tau},$$

which is therefore the **minimum speed** of the bee.

The **acceleration** vector of the bee at time t is given by

$$
\begin{aligned}
\boldsymbol{a} &= \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\boldsymbol{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\widehat{\boldsymbol{\theta}} \\
&= \left(-\frac{2b}{\tau^2} - \frac{bt}{\tau^4}(2\tau - t)\right)\widehat{\boldsymbol{r}} + \left(0 + \frac{4b}{\tau^3}(\tau - t)\right)\widehat{\boldsymbol{\theta}} \\
&= -\frac{3b}{\tau^2}\widehat{\boldsymbol{r}},
\end{aligned}
$$

when $t = \tau$. Hence, when the speed of the bee is a minimum,

$$
|\boldsymbol{a}| = \frac{3b}{\tau^2}. \blacksquare
$$

Problem 2.10 ✻ *A pursuit problem: Daniel and the Lion*

The luckless Daniel (D) is thrown into a circular arena of radius a containing a lion (L). Initially the lion is at the centre O of the arena while Daniel is at the perimeter. Daniel's strategy is to run with his maximum speed u around the perimeter. The lion responds by running at its maximum speed U in such a way that it remains on the (moving) radius OD. Show that r, the distance of L from O, satisfies the differential equation

$$\dot{r}^2 = \frac{u^2}{a^2}\left(\frac{U^2a^2}{u^2} - r^2\right).$$

Find r as a function of t. If $U \geq u$, show that Daniel will be caught, and find how long this will take.

Show that the path taken by the lion is a circle. For the special case in which $U = u$, sketch the path taken by the lion and find the point of capture.

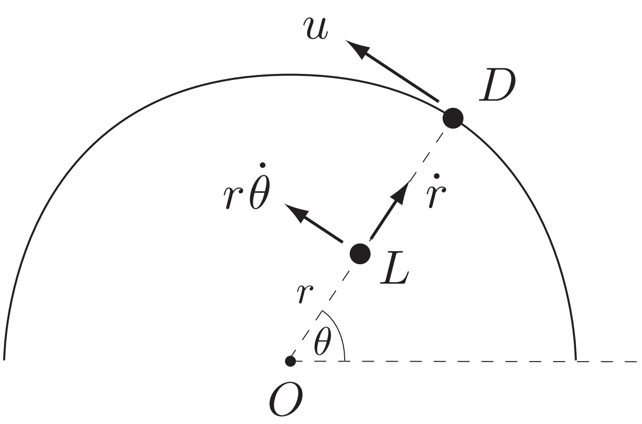

FIGURE 2.2 Daniel D is pursued by the lion L. The lion remains on the rotating radius OD.

Solution

Let the lion have polar coordinates r, θ as shown in Figure 2.2. Then the **velocity** vector of the lion is

$$\begin{aligned}\boldsymbol{v} &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(r\dot{\theta}\right)\widehat{\boldsymbol{\theta}}\\ &= \dot{r}\,\widehat{\boldsymbol{r}} + \left(\frac{ur}{a}\right)\widehat{\boldsymbol{\theta}},\end{aligned}$$