CHAPTER 1

Introduction

1.4 The general way to do this is to write a procedure with heading

```
void processFile( String fileName );
```

which opens *fileName,* does whatever processing is needed, and then closes it. If a line of the form

```
#include SomeFile
```

is detected, then the call

```
processFile( SomeFile );
```

is made recursively. Self-referential includes can be detected by keeping a list of files for which a call to *processFile* has not yet terminated, and checking this list before making a new call to *processFile.*

1.5

```
public static int ones( int n )
{
    if( n < 2 )
        return n;
    return n % 2 + ones( n / 2 );
}
```

1.7 **(a)** The proof is by induction. The theorem is clearly true for $0 < X$ □ 1, since it is true for $X = 1$, and for $X < 1$, $\log X$ is negative. It is also easy to see that the theorem holds for $1 < X$ □ 2, since it is true for $X = 2$, and for $X < 2$, $\log X$ is at most 1. Suppose the theorem is true for $p < X$ □ $2p$ (where p is a positive integer), and consider any $2p < Y$ □ $4p$ (p □ 1). Then $\log Y = 1 + \log(Y/2) < 1 + Y/2 < Y/2 + Y/2$ □ Y, where the first inequality follows by the inductive hypothesis.

(b) Let $2^X = A$. Then $A^B = (2^X)^B = 2^{XB}$. Thus $\log A^B = XB$. Since $X = \log A$, the theorem is proved.

1.8 **(a)** The sum is 4/3 and follows directly from the formula.

(b) $S = \frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \cdots$. $4S = 1 + \frac{2}{4} + \frac{3}{4^2} + \cdots$. Subtracting the first equation from the second gives $3S = 1 + \frac{1}{4} + \frac{2}{4^2} + \cdots$. By part (a), $3S = 4/3$ so $S = 4/9$.

(c) $S = \frac{1}{4} + \frac{4}{4^2} + \frac{9}{4^3} + \cdots$. $4S = 1 + \frac{4}{4} + \frac{9}{4^2} + \frac{16}{4^3} + \cdots$. Subtracting the first equation from the second gives

$3S = 1 + \frac{3}{4} + \frac{5}{4^2} + \frac{7}{4^3} + \cdots$. Rewriting, we get $3S = 2\sum_{i=0}^{\infty} \frac{i}{4^i} + \sum_{i=0}^{\infty} \frac{1}{4^i}$. Thus $3S = 2(4/9) + 4/3 = 20/9$. Thus $S = 20/27$.

(d) Let $S_N = \sum_{i=0}^{\infty} \frac{i^N}{4^i}$. Follow the same method as in parts (a) – (c) to obtain a formula for S_N in terms of S_{N-1}, $S_{N-2}, \ldots, S_0$ and solve the recurrence. Solving the recurrence is very difficult.

1.9 $\sum_{i=\lfloor N/2 \rfloor}^{N} \frac{1}{i} = \sum_{i=1}^{N} \frac{1}{i} - \sum_{i=1}^{\lfloor N/2-1 \rfloor} \frac{1}{i} \approx \ln N - \ln N/2 \approx \ln 2.$

1.10 $2^4 = 16 \equiv 1 \pmod 5$. $(2^4)^{25} \equiv 1^{25} \pmod 5$. Thus $2^{100} \equiv 1 \pmod 5$.

1.11 **(a)** Proof is by induction. The statement is clearly true for $N = 1$ and $N = 2$. Assume true for $N = 1, 2, \ldots, k$. Then $\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1}$. By the induction hypothesis, the value of the sum on the right is $F_{k+2} - 2 + F_{k+1} = F_{k+3} - 2$, where the latter equality follows from the definition of the Fibonacci numbers. This proves the claim for $N = k + 1$, and hence for all N.

(b) As in the text, the proof is by induction. Observe that □ + 1 = □^{2}. This implies that □ $^{-1}$ + □ $^{-2}$ = 1. For $N = 1$ and $N = 2$, the statement is true. Assume the claim is true for $N = 1, 2, \ldots, k$.

$$F_{k+1} = F_k + F_{k-1}$$

by the definition, and we can use the inductive hypothesis on the right-hand side, obtaining

$$\begin{aligned} F_{k+1} &< \phi^k + \phi^{k-1} \\ &< \phi^{-1}\phi^{k+1} + \phi^{-2}\phi^{k+1} \\ F_{k+1} &< (\phi^{-1} + \phi^{-2})\phi^{k+1} < \phi^{k+1} \end{aligned}$$

and proving the theorem.

(c) See any of the advanced math references at the end of the chapter. The derivation involves the use of generating functions.

1.12 **(a)** $\sum_{i=1}^{N}(2i-1) = 2\sum_{i=1}^{N} i - \sum_{i=1}^{N} 1 = N(N+1) - N = N^2$.

(b) The easiest way to prove this is by induction. The case $N = 1$ is trivial. Otherwise,

$$
\begin{aligned}
\sum_{i=1}^{N+1} i^3 &= (N+1)^3 + \sum_{i=1}^{N} i^3 \\
&= (N+1)^3 + \frac{N^2(N+1)^2}{4} \\
&= (N+1)^2 \left[\frac{N^2}{4} + (N+1) \right] \\
&= (N+1)^2 \left[\frac{N^2 + 4N + 4}{4} \right] \\
&= \frac{(N+1)^2 (N+2)^2}{2^2} \\
&= \left[\frac{(N+1)(N+2)}{2} \right]^2 \\
&= \left[\sum_{i=1}^{N+1} i \right]^2
\end{aligned}
$$