Instructor's Resource Manual

Differential Equations with Boundary Value Problems

EIGHTH EDITION

and

A First Course in Differential Equations

TENTH EDITION

Dennis Zill

Warren S. Wright

Prepared by

Warren S. Wright

Carol D. Wright

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Brooks/Cole
20 Channel Center Street
Boston, MA 02210
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8. Second order; nonlinear because of $\dot{x}^2$

9. Writing the boundary-value problem in the form $x(dy/dx) + y^2 = 1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2 - 1)(dx/dy) + x = 0$, we see that it is linear in x.

10. Writing the differential equation in the form $u(dv/du) + (1 + u)v = ue^u$ we see that it is linear in v. However, writing it in the form $(v + uv - ue^u)(du/dv) + u = 0$, we see that it is nonlinear in u.

11. From $y = e^{-x/2}$ we obtain $y' = -\frac{1}{2}e^{-x/2}$. Then $2y' + y = -e^{-x/2} + e^{-x/2} = 0$.

12. From $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ we obtain $dy/dt = 24e^{-20t}$, so that

$$\frac{dy}{dt} + 20y = 24e^{-20t} + 20\left(\frac{6}{5} - \frac{6}{5}e^{-20t}\right) = 24.$$

13. From $y = e^{3x}\cos 2x$ we obtain $y' = 3e^{3x}\cos 2x - 2e^{3x}\sin 2x$ and $y'' = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$, so that $y'' - 6y' + 13y = 0$.

14. From $y = -\cos x \ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x \ln(\sec x + \tan x)$ and

$y'' = \tan x + \cos x \ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.

15. The domain of the function, found by solving $x+2 \geq 0$, is $[-2, \infty)$. From $y' = 1+2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+(2(x+2)^{-1/2}] \\ &= y - x + 2(y-x)(x+2)^{-1/2} \\ &= y - x + 2[x + 4(x+2)^{1/2} - x](x+2)^{-1/2} \\ &= y - x + 8(x+2)^{1/2}(x+2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x = -2$.

16. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5\tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$ or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

17. The domain of the function is $\{x \mid 4-x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ or } x \neq 2\}$. From $y' = 2x/(4-x^2)^2$ we have

$$y' = 2x\left(\frac{1}{4-x^2}\right)^2 = 2xy^2.$$

An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

18. The function is $y = 1/\sqrt{1-\sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1-\sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1-\sin x)^{-3/2}\cos x = [(1-\sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

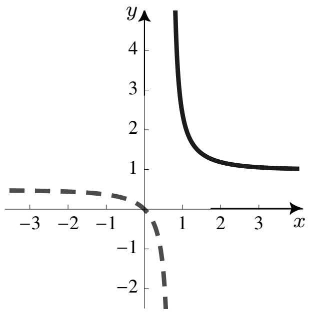
An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another interval is $(5\pi/2, 9\pi/2)$ and so on.

19. Writing $\ln(2X-1) - \ln(X-1) = t$ and differentiating implicitly we obtain

$$\begin{aligned}\frac{2}{2X-1}\frac{dX}{dt} - \frac{1}{X-1}\frac{dX}{dt} &= 1 \\ \left(\frac{2}{2X-1} - \frac{1}{X-1}\right)\frac{dX}{dt} &= 1 \\ \frac{2X-2-2X+1}{(2X-1)(X-1)}\frac{dX}{dt} &= 1 \\ \frac{dX}{dt} &= -(2X-1)(X-1) = (X-1)(1-2X).\end{aligned}$$

Exponentiating both sides of the implicit solution we obtain

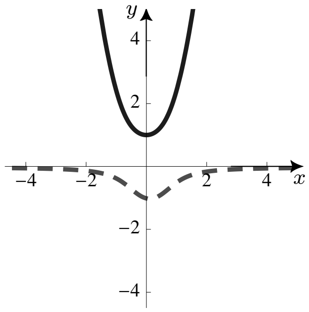
$$\frac{2X-1}{X-1} = e^t$$
$$2X - 1 = Xe^t - e^t$$
$$e^t - 1 = (e^t - 2)X$$
$$X = \frac{e^t - 1}{e^t - 2}.$$



Solving $e^t - 2 = 0$ we get $t = \ln 2$. Thus, the solution is defined on $(-\infty, \ln 2)$ or on $(\ln 2, \infty)$. The graph of the solution defined on $(-\infty, \ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2, \infty)$ is solid.

20. Implicitly differentiating the solution, we obtain

$$-2x^2\frac{dy}{dx} - 4xy + 2y\frac{dy}{dx} = 0$$
$$-x^2\,dy - 2xy\,dx + y\,dy = 0$$
$$2xy\,dx + (x^2 - y)dy = 0.$$



Using the quadratic formula to solve $y^2 - 2x^2y - 1 = 0$ for y, we get $y = \left(2x^2 \pm \sqrt{4x^4+4}\right)/2 = x^2 \pm \sqrt{x^4+1}$. Thus, two explicit solutions are $y_1 = x^2 + \sqrt{x^4+1}$ and $y_2 = x^2 - \sqrt{x^4+1}$. Both solutions are defined on $(-\infty, \infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.

21. Differentiating $P = c_1e^t/\left(1 + c_1e^t\right)$ we obtain

$$\frac{dP}{dt} = \frac{\left(1+c_1e^t\right)c_1e^t - c_1e^t\cdot c_1e^t}{\left(1+c_1e^t\right)^2} = \frac{c_1e^t}{1+c_1e^t}\,\frac{\left[\left(1+c_1e^t\right) - c_1e^t\right]}{1+c_1e^t}$$
$$= \frac{c_1e^t}{1+c_1e^t}\left[1 - \frac{c_1e^t}{1+c_1e^t}\right] = P(1-P).$$

22. Differentiating $y = e^{-x^2}\int_0^x e^{t^2}dt + c_1e^{-x^2}$ we obtain

$$y' = e^{-x^2}e^{x^2} - 2xe^{-x^2}\int_0^x e^{t^2}dt - 2c_1xe^{-x^2} = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt - 2c_1xe^{-x^2}.$$

Substituting into the differential equation, we have

$$y' + 2xy = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt - 2c_1xe^{-x^2} + 2xe^{-x^2}\int_0^x e^{t^2}dt + 2c_1xe^{-x^2} = 1.$$

23. From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\dfrac{dy}{dx} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2} = (4c_1 + 4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1)e^{2x} + (4c_2 - 8c_2 + 4c_2)xe^{2x} = 0.$$

24. From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$
$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2},$$

so that

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x \\ &\qquad + (-c_3 + c_3)x\ln x + (16 - 8 + 4)x^2 \\ &= 12x^2. \end{aligned}$$

25. From $y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ we obtain $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \ge 0 \end{cases}$ so that $xy' - 2y = 0$.

26. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x\to 0^-} y(x) = 5$ and $\lim_{x\to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

27. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $y' + 2y = 0$ implies

$$me^{mx} + 2e^{mx} = (m+2)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = -2$. Thus $y = e^{-2x}$ is a solution.

28. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $5y' = 2y$ implies

$$5me^{mx} = 2e^{mx} \quad \text{or} \quad m = \frac{2}{5}.$$

Thus $y = e^{2x/5} > 0$ is a solution.

29. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $y'' - 5y' + 6y = 0$ implies

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = (m-2)(m-3)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = 2$ and $m = 3$. Thus $y = e^{2x}$ and $y = e^{3x}$ are solutions.

30. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $2y'' + 7y' - 4y = 0$ implies

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = (2m-1)(m+4)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = \frac{1}{2}$ and $m = -4$. Thus $y = e^{x/2}$ and $y = e^{-4x}$ are solutions.

31. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $xy'' + 2y' = 0$ implies

$$\begin{aligned} xm(m-1)x^{m-2} + 2mx^{m-1} &= [m(m-1) + 2m]x^{m-1} = (m^2 + m)x^{m-1} \\ &= m(m+1)x^{m-1} = 0. \end{aligned}$$

Since $x^{m-1} > 0$ for $x > 0$, $m = 0$ and $m = -1$. Thus $y = 1$ and $y = x^{-1}$ are solutions.

32. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $x^2y'' - 7xy' + 15y = 0$ implies

$$\begin{aligned} x^2m(m-1)x^{m-2} - 7xmx^{m-1} + 15x^m &= [m(m-1) - 7m + 15]x^m \\ &= (m^2 - 8m + 15)x^m = (m-3)(m-5)x^m = 0. \end{aligned}$$

Since $x^m > 0$ for $x > 0$, $m = 3$ and $m = 5$. Thus $y = x^3$ and $y = x^5$ are solutions.

In Problems 33–36 we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$.

33. Solving $5c = 10$ we see that $y = 2$ is a constant solution.

34. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

35. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

36. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

37. From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}.$$

38. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \qquad \text{or} \qquad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \qquad \text{or} \qquad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}.$$

Discussion Problems

39. $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all functions $y = \phi(x)$.

40. The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since, if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

41. The first derivative of $f(x) = e^x$ is e^x. The first derivative of $f(x) = e^{kx}$ is $f'(x) = ke^{kx}$. The differential equations are $y' = y$ and $y' = ky$, respectively.

42. Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c\sin x$ or $y = c\cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.

43. We first note that $\sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of x for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\frac{d}{dx}(\sin x)\right|_{x=\pi} = \cos x\big|_{x=\pi} = \cos\pi = -1,$$

but

$$\left.\sqrt{1-y^2}\,\right|_{x=\pi} = \sqrt{1-\sin^2\pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1-y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

44. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A\sin t + B\cos t$, could be a solution of the differential equation. Using $y' = A\cos t - B\sin t$ and $y'' = -A\sin t - B\cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 4A\sin t + 4B\cos t \\ &= (3A - 2B)\sin t + (2A + 3B)\cos t = 5\sin t. \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13}\sin t - \frac{10}{13}\cos t$.

45. One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.

46. One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

47. Differentiating $(x^3+y^3)/xy=3c$ we obtain

$$\begin{aligned}\frac{xy(3x^2+3y^2y')-(x^3+y^3)(xy'+y)}{x^2y^2}&=0\\ 3x^3y+3xy^3y'-x^4y'-x^3y-xy^3y'-y^4&=0\\ (3xy^3-x^4-xy^3)y'&=-3x^3y+x^3y+y^4\\ y'&=\frac{y^4-2x^3y}{2xy^3-x^4}=\frac{y(y^3-2x^3)}{x(2y^3-x^3)}\,.\end{aligned}$$

48. A tangent line will be vertical where y' is undefined, or in this case, where $x(2y^3-x^3)=0$. This gives $x=0$ and $2y^3=x^3$. Substituting $y^3=x^3/2$ into $x^3+y^3=3xy$ we get

$$\begin{aligned}x^3+\frac{1}{2}x^3&=3x\left(\frac{1}{2^{1/3}}\,x\right)\\ \frac{3}{2}x^3&=\frac{3}{2^{1/3}}x^2\\ x^3&=2^{2/3}x^2\\ x^2(x-2^{2/3})&=0.\end{aligned}$$

Thus, there are vertical tangent lines at $x=0$ and $x=2^{2/3}$, or at $(0,\,0)$ and $(2^{2/3},\,2^{1/3})$. Since $2^{2/3}\approx 1.59$, the estimates of the domains in Problem 46 were close.

49. The derivatives of the functions are $\phi_1'(x)=-x/\sqrt{25-x^2}$ and $\phi_2'(x)=x/\sqrt{25-x^2}$, neither of which is defined at $x=\pm 5$.

50. To determine if a solution curve passes through $(0,3)$ we let $t=0$ and $P=3$ in the equation $P=c_1e^t/(1+c_1e^t)$. This gives $3=c_1/(1+c_1)$ or $c_1=-\frac{3}{2}$. Thus, the solution curve

$$P=\frac{(-3/2)e^t}{1-(3/2)e^t}=\frac{-3e^t}{2-3e^t}$$

passes through the point $(0,3)$. Similarly, letting $t=0$ and $P=1$ in the equation for the one-parameter family of solutions gives $1=c_1/(1+c_1)$ or $c_1=1+c_1$. Since this equation has no solution, no solution curve passes through $(0,1)$.

51. For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y=\int(\int f(x)dx)dx+c_1x+c_2$.

52. Solving for y' using the quadratic formula we obtain the two differential equations

$$y'=\frac{1}{x}\left(2+2\sqrt{1+3x^6}\right)\quad\text{and}\quad y'=\frac{1}{x}\left(2-2\sqrt{1+3x^6}\right),$$

so the differential equation cannot be put in the form $dy/dx=f(x,y)$.

53. The differential equation $yy'-xy=0$ has normal form $dy/dx=x$. These are not equivalent because $y=0$ is a solution of the first differential equation but not a solution of the second.

54. Differentiating $y = c_1x + c_2x^2$ we get $y' = c_1 + 2c_2x$ and $y'' = 2c_2$. Then $c_2 = \frac{1}{2}y''$ and $c_1 = y' - xy''$, so

$$y = c_1x + c_2x^2 = (y' - xy'')x + \frac{1}{2}y''x^2 = xy' - \frac{1}{2}x^2y''.$$

The differential equation is $\frac{1}{2}x^2y'' - xy' + y = 0$ or $x^2y'' - 2xy' + 2y = 0$.

55. (a) Since e^{-x^2} is positive for all values of x, $dy/dx > 0$ for all x, and a solution, $y(x)$, of the differential equation must be increasing on any interval.

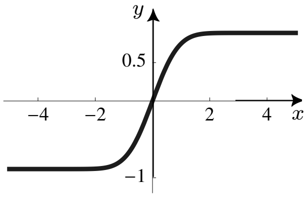
(b) $\displaystyle\lim_{x\to-\infty}\frac{dy}{dx} = \lim_{x\to-\infty} e^{-x^2} = 0$ and $\displaystyle\lim_{x\to\infty}\frac{dy}{dx} = \lim_{x\to\infty} e^{-x^2} = 0$. Since $\dfrac{dy}{dx}$ approaches 0 as x approaches $-\infty$ and ∞, the solution curve has horizontal asymptotes to the left and to the right.

(c) To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. x

(d)



56. (a) The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

(b) A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

57. (a) The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

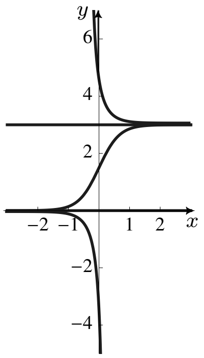
(b) A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

(c) Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Solving $d^2y/dx^2 = 0$ we obtain $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

(d)

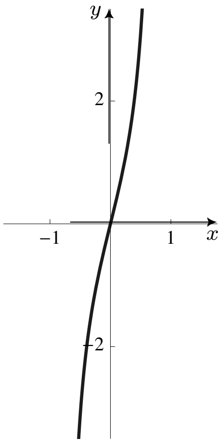


58. **(a)** If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of c.

(b) Since $y' = y^2 + 4 > 0$ for all x where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

(c) Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2+4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

(d)



Computer Lab Assignments

59. In *Mathematica* use

Clear[y]

y[x_]:= x **Exp**[5x] **Cos**[2x]

y[x]

y''''[x] - 20 y'''[x] + 158 y''[x] - 580 y'[x] + 841 y[x] // **Simplify**

The output will show $y(x) = e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

60. In *Mathematica* use

```
Clear[y]
y[x_]:= 20 Cos[5 Log[x]]/x - 3 Sin[5 Log[x]]/x
y[x]
x^3 y'''[x] + 2 x^2 y''[x] + 20 x y'[x] - 78 y[x] // Simplify
```

The output will show $y(x) = \dfrac{20\cos(5\ln x)}{x} - \dfrac{3\sin(5\ln x)}{x}$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

1.2 Initial-Value Problems

1. Solving $-1/3 = 1/(1+c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1-4e^{-x})$.

2. Solving $2 = 1/(1+c_1 e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2-e^{-(x+1)})$.

3. Letting $x = 2$ and solving $1/3 = 1/(4+c)$ we get $c = -1$. The solution is $y = 1/(x^2-1)$. This solution is defined on the interval $(1, \infty)$.

4. Letting $x = -2$ and solving $1/2 = 1/(4+c)$ we get $c = -2$. The solution is $y = 1/(x^2-2)$. This solution is defined on the interval $(-\infty, -\sqrt{2}\,)$.

5. Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2+1)$. This solution is defined on the interval $(-\infty, \infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4+c)$ we get $c = -1/2$. The solution is $y = 1/(x^2-1/2) = 2/(2x^2-1)$. This solution is defined on the interval $(-1/\sqrt{2}\,, 1/\sqrt{2}\,)$.

In Problems 7–10 we use $x = c_1\cos t + c_2\sin t$ and $x' = -c_1\sin t + c_2\cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

7. From the initial conditions we obtain the system

$$c_1 = -1$$
$$c_2 = 8.$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

8. From the initial conditions we obtain the system

$$c_2 = 0$$
$$-c_1 = 1.$$

The solution of the initial-value problem is $x = -\cos t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}\,c_1 + \frac{1}{2}\,c_2 = \frac{1}{2}$$
$$-\frac{1}{2}\,c_1 + \frac{\sqrt{3}}{2}\,c_2 = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

10. From the initial conditions we obtain

$$\frac{\sqrt{2}}{2}\,c_1 + \frac{\sqrt{2}}{2}\,c_2 = \sqrt{2}$$
$$-\frac{\sqrt{2}}{2}\,c_1 + \frac{\sqrt{2}}{2}\,c_2 = 2\sqrt{2}\,.$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is

$$x = -\cos t + 3\sin t.$$

In Problems 11–14 we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is

$$y = \frac{3}{2}\,e^x - \frac{1}{2}\,e^{-x}.$$

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}\,e^x - \frac{1}{2}\,e^2e^{-x} = \frac{1}{2}\,e^x - \frac{1}{2}\,e^{2-x}.$$

13. From the initial conditions we obtain

$$e^{-1}c_1 + ec_2 = 5$$
$$e^{-1}c_1 - ec_2 = -5.$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is

$$y = 5e^{-1}e^{-x} = 5e^{-1-x}.$$

14. From the initial conditions we obtain

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

15. Two solutions are $y = 0$ and $y = x^3$.

16. Two solutions are $y = 0$ and $y = x^2$. A lso, any constant multiple of x^2 is a solution.

17. For $f(x, y) = y^{2/3}$ we have Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

19. For $f(x, y) = \dfrac{y}{x}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ or where $x < 0$.

20. For $f(x, y) = x + y$ we have $\dfrac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

21. For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f/\partial y = 2x^2y/(4 - y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

22. For $f(x, y) = \dfrac{x^2}{1 + y^3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2}{(1 + y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

23. For $f(x, y) = \dfrac{y^2}{x^2 + y^2}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2x^2y}{(x^2 + y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0, 0)$.

24. For $f(x, y) = (y + x)/(y - x)$ we have $\partial f/\partial y = -2x/(y - x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28 we identify $f(x, y) = \sqrt{y^2 - 9}$ and $\partial f/\partial y = y/\sqrt{y^2 - 9}$. We see that f and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on x.

25. Since $4 > 3$, $(1, 4)$ is in the region defined by $y > 3$ and the differential equation has a unique solution through $(1, 4)$.

26. Since $(5, 3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(5, 3)$.

27. Since $(2, -3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2, -3)$.

28. Since $(-1, 1)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(-1, 1)$.

29. **(a)** A one-parameter family of solutions is $y = cx$. Since $y' = c$, $xy' = xc = y$ and $y(0) = c \cdot 0 = 0$.

(b) Writing the equation in the form $y' = y/x$, we see that R cannot contain any point on the y-axis. Thus, any rectangular region disjoint from the y-axis and containing (x_0, y_0) will determine an interval around x_0 and a unique solution through (x_0, y_0). Since $x_0 = 0$ in part (a), we are not guaranteed a unique solution through $(0, 0)$.

(c) The piecewise-defined function which satisfies $y(0) = 0$ is not a solution since it is not differentiable at $x = 0$.

30. **(a)** Since $\dfrac{d}{dx}\tan(x + c) = \sec^2(x + c) = 1 + \tan^2(x + c)$, we see that $y = \tan(x + c)$ satisfies the differential equation.

(b) Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2, 2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

31. **(a)** Since $\dfrac{d}{dx}\left(-\dfrac{1}{x + c}\right) = \dfrac{1}{(x + c)^2} = y^2$, we see that $y = -\dfrac{1}{x + c}$ is a solution of the differential equation.

(b) Solving $y(0) = -1/c = 1$ we obtain $c = -1$ and $y = 1/(1 - x)$. Solving $y(0) = -1/c = -1$ we obtain $c = 1$ and $y = -1/(1 + x)$. Being sure to include $x = 0$, we see that the interval of existence of $y = 1/(1 - x)$ is $(-\infty, 1)$, while the interval of existence of $y = -1/(1 + x)$ is $(-1, \infty)$.

(c) By inspection we see that $y = 0$ is a solution on $(-\infty, \infty)$.

32. **(a)** Applying $y(1) = 1$ to $y = -1/(x+c)$ gives

$$1 = -\frac{1}{1+c} \quad \text{or} \quad 1 + c = -1.$$

Thus $c = -2$ and

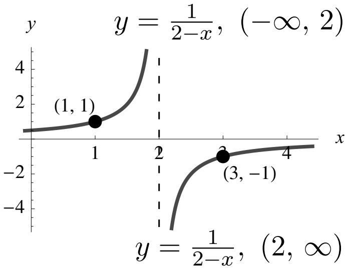
$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

(b) Applying $y(3) = -1$ to $y = -1/(x+c)$ gives

$$-1 = -\frac{1}{3+c} \quad \text{or} \quad 3 + c = 1.$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

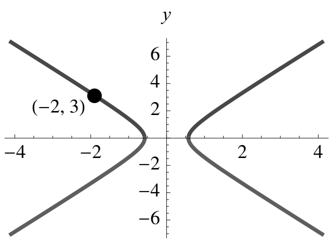


(c) No, they are not the same solution. The interval I of definition for the solution in part (a) is $(-\infty, 2)$; whereas the interval I of definition for the solution in part (b) is $(2, \infty)$. See the figure.

33. **(a)** Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

(b) Solving $3x^2 - y^2 = 3$ for y we get

$$\begin{aligned}
y &= \phi_1(x) = \sqrt{3(x^2-1)}\,, && 1 < x < \infty,\\
y &= \phi_2(x) = -\sqrt{3(x^2-1)}\,, && 1 < x < \infty,\\
y &= \phi_3(x) = \sqrt{3(x^2-1)}\,, && -\infty < x < -1,\\
y &= \phi_4(x) = -\sqrt{3(x^2-1)}\,, && -\infty < x < -1.
\end{aligned}$$

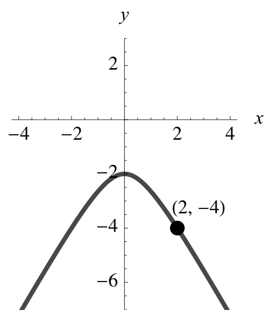


(c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

34. **(a)** Setting $x = 2$ and $y = -4$ in $3x^2 - y^2 = c$ we get $12 - 16 = -4 = c$, so the explicit solution is

$$y = -\sqrt{3x^2 + 4}\,, \quad -\infty < x < \infty.$$

(b) Setting $c = 0$ we have $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, both defined on $(-\infty, \infty)$ and both passing through the origin.



In Problems 35–38 we consider the points on the graphs with x-coordinates $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).

35. The graph satisfies the conditions in (b) and (f).

36. The graph satisfies the conditions in (e).

37. The graph satisfies the conditions in (c) and (d).

38. The graph satisfies the conditions in (a).

In Problems 39-44 $y = c_1 \cos 2x + c_2 \sin 2x$ *is a two parameter family of solutions of the second-order differential equation* $y'' + 4y = 0$. *In some of the problems we will use the fact that* $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$.

39. From the boundary conditions $y(0) = 0$ and $y\left(\frac{\pi}{4}\right) = 3$ we obtain

$$y(0) = c_1 = 0$$

$$y\left(\frac{\pi}{4}\right) = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = c_2 = 3.$$

Thus, $c_1 = 0$, $c_2 = 3$, and the solution of the boundary-value problem is $y = 3\sin 2x$.

40. From the boundary conditions $y(0) = 0$ and $y(\pi) = 0$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 0.$$

Thus, $c_1 = 0$, c_2 is unrestricted, and the solution of the boundary-value problem is $y = c_2 \sin 2x$, where c_2 is any real number.

41. From the boundary conditions $y'(0) = 0$ and $y'\left(\frac{\pi}{6}\right) = 0$ we obtain

$$y'(0) = 2c_2 = 0$$

$$y'\left(\frac{\pi}{6}\right) = -2c_1 \sin\left(\frac{\pi}{3}\right) = -\sqrt{3}\, c_1 = 0.$$

Thus, $c_2 = 0$, $c_1 = 0$, and the solution of the boundary-value problem is $y = 0$.

42. From the boundary conditions $y(0) = 1$ and $y'(\pi) = 5$ we obtain

$$y(0) = c_1 = 1$$

$$y'(\pi) = 2c_2 = 5.$$

Thus, $c_1 = 1$, $c_2 = \frac{5}{2}$, and the solution of the boundary-value problem is $y = \cos 2x + \frac{5}{2}\sin 2x$.

43. From the boundary conditions $y(0) = 0$ and $y(\pi) = 2$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 2.$$

Since $0 \neq 2$, this is not possible and there is no solution.

44. From the boundary conditions $y' = \left(\frac{\pi}{2}\right) = 1$ and $y'(\pi) = 0$ we obtain

$$y'\left(\frac{\pi}{2}\right) = -2c_2 = 1$$

$$y'(\pi) = 2c_2 = 0.$$

Since $0 \neq -1$, this is not possible and there is no solution.

Discussion Problems

45. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

46. Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2)dx = 6x^2 - 2x + c_1.$$

Then, integrating y' we obtain

$$y = \int (6x^2 - 2x + c_1)dx = 2x^3 - x^2 + c_1x + c_2.$$

At $x = 1$ the y-coordinate of the point of tangency is $y = -1 + 5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \qquad \text{or} \qquad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \qquad \text{or} \qquad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

47. When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the red curve.

48. If the solution is tangent to the x-axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.

49. The theorem guarantees a unique (meaning single) solution through any point. Thus, there cannot be two distinct solutions through any point.

50. When $y = \frac{1}{16}x^4$, $y' = \frac{1}{4}x^3 = x(\frac{1}{4}x^2) = xy^{1/2}$, and $y(2) = \frac{1}{16}(16) = 1$. When

$$y = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \ge 0 \end{cases}$$

we have

$$y' = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^3, & x \ge 0 \end{cases} = x\begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & x \ge 0 \end{cases} = xy^{1/2},$$

and $y(2) = \frac{1}{16}(16) = 1$. The two different solutions are the same on the interval $(0, \infty)$, which is all that is required by Theorem 1.2.1.

51. At $t = 0$, $dP/dt = 0.15P(0) + 20 = 0.15(100) + 20 = 35$. Thus, the population is increasing at a rate of 3,500 individuals per year. If the population is 500 at time $t = T$ then

$$\left.\frac{dP}{dt}\right|_{t=T} = 0.15P(T) + 20 = 0.15(500) + 20 = 95.$$

Thus, at this time, the population is increasing at a rate of 9,500 individuals per year.

1.3 Differential Equations as Mathematical Models

Population Dynamics

1. $\dfrac{dP}{dt} = kP + r; \qquad \dfrac{dP}{dt} = kP - r$

2. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P$.

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P^2$.

4. $\dfrac{dP}{dt} = k_1 P - k_2 P^2 - h,\ h > 0$

Newton's Law of cooling/Warming

5. From the graph in the text we estimate $T_0 = 180°$ and $T_m = 75°$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30\cos \pi t/12$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\frac{\pi}{12}t\right)\right], \quad t > 0.$$

Spread of a Disease/Technology

7. The number of students with the flu is x and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.

8. By analogy, with the differential equation modeling the spread of a disease, we assume that the rate at which the technological innovation is adopted is proportional to the number of people who have adopted the innovation and also to the number of people, $y(t)$, who have not yet adopted it. Then $x + y = n$, and assuming that initially one person has adopted the innovation, we have

$$\frac{dx}{dt} = kx(n - x), \quad x(0) = 1.$$

Mixtures

9. The rate at which salt is leaving the tank is

$$R_{out}\ (3\ \text{gal/min})\left(\frac{A}{300}\ \text{lb/gal}\right) = \frac{A}{100}\ \text{lb/min}.$$

Thus $dA/dt = -A/100$ (where the minus sign is used since the amount of salt is decreasing). The initial amount is $A(0) = 50$.

10. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3-2)$gal/min $= 1$ gal/min. After t minutes there are $300+t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300+t} \text{ lb/gal}\right) = \frac{2A}{300+t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}.$$

11. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min})(2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the tank loses liquid at the net rate of

$$3 \text{ gal/min} - 3.5 \text{ gal/min} = -0.5 \text{ gal/min},$$

after t minutes the number of gallons of brine in the tank is $300 - \frac{1}{2}t$ gallons. Thus the rate at which salt is leaving is

$$R_{out} = \left(\frac{A}{300-t/2} \text{ lb/gal}\right)(3.5 \text{ gal/min}) = \frac{3.5A}{300-t/2} \text{ lb/min} = \frac{7A}{600-t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{7A}{600-t} \quad \text{or} \quad \frac{dA}{dt} + \frac{7}{600-t}A = 6.$$

12. The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ lb/gal})(r_{in} \text{ gal/min}) = c_{in}r_{in} \text{ lb/min}.$$

Now let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time t. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/gal}\right)(r_{out} \text{ gal/min}) = r_{out}\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in}r_{in} - r_{out}\frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t}A = c_{in}r_{in}.$$

Draining a Tank

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}\,.$$

Using $A_h = \pi\left(\dfrac{2}{12}\right)^2 = \dfrac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100}\sqrt{64h} = -\frac{c\pi}{450}\sqrt{h}\,.$$

14. The volume of water in the tank at time t is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the tank at height h. From the figure in the text we see that $r/h = 8/20$ so that $r = \frac{2}{5}h$ and $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{75}\pi h^3$. Differentiating with respect to t we have $dV/dt = \frac{4}{25}\pi h^2\,dh/dt$ or

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}\,.$$

From Problem 13 we have $dV/dt = -cA_h\sqrt{2gh}$ where $c = 0.6$, $A_h = \pi\left(\frac{2}{12}\right)^2$, and $g = 32$. Thus $dV/dt = -2\pi\sqrt{h}/15$ and

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\left(-\frac{2\pi\sqrt{h}}{15}\right) = -\frac{5}{6h^{3/2}}\,.$$

Series Circuits

15. Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

16. By Kirchhoff's second law we obtain $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$.

Falling Bodies and Air Resistance

17. From Newton's second law we obtain $m\dfrac{dv}{dt} = -kv^2 + mg$.

Newton's Second Law and Archimedes' Principle

18. Since the barrel in Figure 1.3.17(b) in the text is submerged an additional y feet below its equilibrium position the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi\times(\text{radius})^2\times\text{height}$ or $\pi(s/2)^2y$. Then we have from Archimedes' principle

$$\begin{aligned}\text{upward force of water on barrel} &= \text{weight of water displaced}\\ &= (62.4)\times(\text{volume of water displaced})\\ &= (62.4)\pi(s/2)^2y = 15.6\pi s^2 y.\end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2} = -15.6\pi s^2 y \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \frac{15.6\pi s^2 g}{w}\,y = 0,$$

where $g = 32$ and w is the weight of the barrel in pounds.

Newton's Second Law and Hooke's Law

19. The net force acting on the mass is

$$F = ma = m\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m\frac{d^2x}{dt^2} = -kx.$$

20. From Problem 19, without a damping force, the differential equation is $m\,d^2x/dt^2 = -kx$. With a damping force proportional to velocity, the differential equation becomes

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

Newton's Second Law and Rocket Motion

21. Since the positive direction is taken to be upward, and the acceleration due to gravity g is positive, (14) in Section 1.3 becomes

$$m\frac{dv}{dt} = -mg - kv + R.$$

This equation, however, only applies if m is constant. Since in this case m includes the variable amount of fuel we must use (17) in Exercises 1.3:

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}.$$

Thus, replacing $m\,dv/dt$ with $m\,dv/dt + v\,dm/dt$, we have

$$m\frac{dv}{dt} + v\frac{dm}{dt} = -mg - kv + R \qquad \text{or} \qquad m\frac{dv}{dt} + v\frac{dm}{dt} + kv = -mg + R.$$

22. Here we are given that the variable mass of the rocket is $m(t) = m_p + m_\nu + m_f(t)$, where m_p and m_ν are the constant masses of the payload and vehicle, respectively, and $m_f(t)$ is the variable mass of the fuel.

(a) Since

$$\frac{d}{dt}m(t) = \frac{d}{dt}\big(m_p + m_\nu + m_f(t)\big) = \frac{d}{dt}m_f(t),$$

the rates at which the mass of the rocket and the mass of the fuel change are the same.

(b) If the rocket loses fuel at a constant rate λ then we take $dm/dt = -\lambda$. We use $-\lambda$ instead of λ because the fuel is decreasing over time. We next divide the resulting differential equation in Problem 21 by m, obtaining

$$\frac{dv}{dt} + \frac{v}{m}(-\lambda) + \frac{kv}{m} = -g + \frac{R}{m} \qquad \text{or} \qquad \frac{dv}{dt} + \frac{k-\lambda}{m}v = -g + \frac{R}{m}.$$

Integrating $dm/dt = -\lambda$ with respect to t we have $m(t) = -\lambda + C$. Since $m(0) = m_0$, $C = m_0$ and $m(t) = -\lambda t + m_0$. The differential equation then may be written as

$$\frac{dv}{dt} + \frac{k-\lambda}{m_0 - \lambda t}v = -g + \frac{R}{m_0 - \lambda t}.$$

(c) We integrate $dm_f/dt = -\lambda$ to obtain $m_f(t) = -\lambda t + C$. Since $m_f(0) = C$ we have $m_f(t - \lambda t + m_f(0)$. At burnout $m_f(t_b) = -\lambda t_b + m_f(0) = 0$, so $t_b = m_f(0)/\lambda$.

Newton's Second Law and the Law of Universal Gravitation

23. From $g = k/R^2$ we find $k = gR^2$. Using $a = d^2r/dt^2$ and the fact that the positive direction is upward we get

$$\frac{d^2r}{dt^2} = -a = -\frac{k}{r^2} = -\frac{gR^2}{r^2} \qquad \text{or} \qquad \frac{d^2r}{dt^2} + \frac{gR^2}{r^2} = 0.$$

24. The gravitational force on m is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$ we have $M_r = r^3M/R^3$ and

$$F = -k\frac{M_r m}{r^2} = -k\frac{r^3Mm/R^3}{r^2} = -k\frac{mM}{R^3}r.$$

Now from $F = ma = d^2r/dt^2$ we have

$$m\frac{d^2r}{dt^2} = -k\frac{mM}{R^3}r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}r.$$

Additional Mathematical Models

25. The differential equation is $\dfrac{dA}{dt} = k(M - A)$ where $k > 0$.

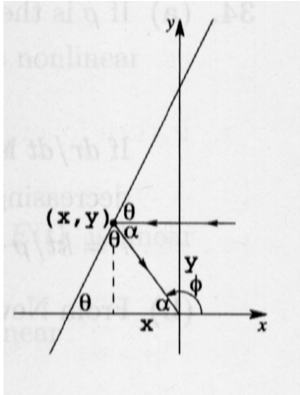
26. The differential equation is $\dfrac{dA}{dt} = k_1(M - A) - k_2A$.

27. The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.

28. By the Pythagorean Theorem the slope of the tangent line is $y' = \dfrac{-y}{\sqrt{s^2 - y^2}}$.

29. We see from the figure that $2\theta + \alpha = \pi$. Thus

$$\frac{y}{-x} = \tan\alpha = \tan(\pi - 2\theta) = -\tan 2\theta = -\frac{2\tan\theta}{1 - \tan^2\theta}\,.$$



Since the slope of the tangent line is $y' = \tan\theta$ we have $y/x = 2y'/[1 - (y')^2]$ or $y - y(y')^2 = 2xy'$, which is the quadratic equation $y(y')^2 + 2xy' - y = 0$ in y'. Using the quadratic formula, we get

$$y' = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}\,.$$

Since $dy/dx > 0$, the differential equation is

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y} \qquad \text{or} \qquad y\,\frac{dy}{dx} - \sqrt{x^2 + y^2} + x = 0.$$

Discussion Problems

30. The differential equation is $dP/dt = kP$, so from Problem 41 in Exercises 1.1, $P = e^{kt}$, and a one-parameter family of solutions is $P = ce^{kt}$.

31. The differential equation in (3) is $dT/dt = k(T - T_m)$. When the body is cooling, $T > T_m$, so $T - T_m > 0$. Since T is decreasing, $dT/dt < 0$ and $k < 0$. When the body is warming, $T < T_m$, so $T - T_m < 0$. Since T is increasing, $dT/dt > 0$ and $k < 0$.

32. The differential equation in (8) is $dA/dt = 6 - A/100$. If $A(t)$ attains a maximum, then $dA/dt = 0$ at this time and $A = 600$. If $A(t)$ continues to increase without reaching a maximum, then $A'(t) > 0$ for $t > 0$ and A cannot exceed 600. In this case, if $A'(t)$ approaches 0 as t increases to infinity, we see that $A(t)$ approaches 600 as t increases to infinity.

33. This differential equation could describe a population that undergoes periodic fluctuations.

34. **(a)** As shown in Figure 1.3.24(b) in the text, the resultant of the reaction force of magnitude F and the weight of magnitude mg of the particle is the centripetal force of magnitude $m\omega^2 x$. The centripetal force points to the center of the circle of radius x on which the particle rotates about the y-axis. Comparing parts of similar triangles gives

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = m\omega^2 x.$$

(b) Using the equations in part (a) we find

$$\tan\theta = \frac{F\sin\theta}{F\cos\theta} = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{\omega^2 x}{g}\,.$$

35. From Problem 23, $d^2r/dt^2 = -gR^2/r^2$. Since R is a constant, if $r = R + s$, then $d^2r/dt^2 = d^2s/dt^2$ and, using a Taylor series, we get

$$\frac{d^2s}{dt^2} = -g\frac{R^2}{(R+s)^2} = -gR^2(R+s)^{-2} \approx -gR^2\left[R^{-2} - 2sR^{-3} + \cdots\right] = -g + \frac{2gs}{R^3} + \cdots.$$

Thus, for R much larger than s, the differential equation is approximated by $d^2s/dt^2 = -g$.

36. (a) If ρ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = \rho\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \rho\left(4\pi r^2\frac{dr}{dt}\right) = \rho S\frac{dr}{dt}.$$

If dr/dt is a constant, then $dm/dt = kS$ where $\rho\, dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

(b) From Newton's second law, $\dfrac{d}{dt}[mv] = mg$, where v is the velocity of the raindrop. Then

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg \qquad \text{or} \qquad \rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v(k4\pi r^2) = \rho\left(\frac{4}{3}\pi r^3\right)g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r}v = g \qquad \text{or} \qquad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0}v = g, \quad k < 0.$$

37. We assume that the plow clears snow at a constant rate of k cubic miles per hour. Let t be the time in hours after noon, $x(t)$ the depth in miles of the snow at time t, and $y(t)$ the distance the plow has moved in t hours. Then dy/dt is the velocity of the plow and the assumption gives

$$wx\frac{dy}{dt} = k,$$

where w is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour. Now let t_0 be the number of hours before noon when it started snowing and let s be the constant rate in miles per hour at which x increases. Then for $t > -t_0$, $x = s(t + t_0)$. The differential equation then becomes

$$\frac{dy}{dt} = \frac{k}{ws}\frac{1}{t + t_0}.$$

Integrating, we obtain

$$y = \frac{k}{ws}\left[\ln(t + t_0) + c\right],$$

where c is a constant. Now when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws}\ln\left(1 + \frac{t}{t_0}\right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$, we obtain

$$\left(1 + \frac{2}{t_0}\right)^2 = \left(1 + \frac{1}{t_0}\right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. Since $t_0 > 0$, we find $t_0 \approx 0.618$ hours ≈ 37 minutes. Thus it started snowing at about 11:23 in the morning.

38. (1): $\dfrac{dP}{dt} = kP$ is linear

(2): $\dfrac{dA}{dt} = kA$ is linear

(3): $\dfrac{dT}{dt} = k(T - T_m)$ is linear

(5): $\dfrac{dx}{dt} = kx(n + 1 - x)$ is nonlinear

(6): $\dfrac{dX}{dt} = k(\alpha - X)(\beta - X)$ is nonlinear

(8): $\dfrac{dA}{dt} = 6 - \dfrac{A}{100}$ is linear

(10): $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$ is nonlinear

(11): $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$ is linear

(12): $\dfrac{d^2s}{dt^2} = -g$ is linear

(14): $m\dfrac{dv}{dt} = mg - kv$ is linear

(15): $m\dfrac{d^2s}{dt^2} + k\dfrac{ds}{dt} = mg$ is linear

(16): $\dfrac{dy}{dx} = \dfrac{W}{T_1}$ linearity or nonlinearity is determined by the manner in which W and T_1 involve x.

1.R Chapter 1 in Review

1. $\dfrac{d}{dx}c_1e^{10x} = 10c_1e^{10x}$; $\quad \dfrac{dy}{dx} = 10y$

2. $\dfrac{d}{dx}(5 + c_1e^{-2x}) = -2c_1e^{-2x} = -2(5 + c_1e^{-2x} - 5)$; $\quad \dfrac{dy}{dx} = -2(y - 5)$ or $\dfrac{dy}{dx} = -2y + 10$

3. $\dfrac{d}{dx}(c_1\cos kx + c_2\sin kx) = -kc_1\sin kx + kc_2\cos kx$;

$$\frac{d^2}{dx^2}(c_1\cos kx + c_2\sin kx) = -k^2c_1\cos kx - k^2c_2\sin kx = -k^2(c_1\cos kx + c_2\sin kx);$$

$$\frac{d^2y}{dx^2} = -k^2y \quad \text{or} \quad \frac{d^2y}{dx^2} + k^2y = 0$$

4. $\dfrac{d}{dx}(c_1\cosh kx + c_2\sinh kx) = kc_1\sinh kx + kc_2\cosh kx$;

$$\frac{d^2}{dx^2}(c_1\cosh kx + c_2\sinh kx) = k^2c_1\cosh kx + k^2c_2\sinh kx = k^2(c_1\cosh kx + c_2\sinh kx);$$

$$\frac{d^2y}{dx^2} = k^2y \quad \text{or} \quad \frac{d^2y}{dx^2} - k^2y = 0$$

5. $y = c_1e^x + c_2xe^x$; $\quad y' = c_1e^x + c_2xe^x + c_2e^x$; $\quad y'' = c_1e^x + c_2xe^x + 2c_2e^x$;
$y'' + y = 2(c_1e^x + c_2xe^x) + 2c_2e^x = 2(c_1e^x + c_2xe^x + c_2e^x) = 2y'$; $\quad y'' - 2y' + y = 0$

6. $y' = -c_1e^x \sin x + c_1e^x \cos x + c_2e^x \cos x + c_2e^x \sin x$;

$$y'' = -c_1e^x \cos x - c_1e^x \sin x - c_1e^x \sin x + c_1e^x \cos x - c_2e^x \sin x + c_2e^x \cos x + c_2e^x \cos x + c_2e^x \sin x$$
$$= -2c_1e^x \sin x + 2c_2e^x \cos x;$$

$y'' - 2y' = -2c_1e^x \cos x - 2c_2e^x \sin x = -2y$; $\quad y'' - 2y' + 2y = 0$

7. a, d $\quad$ **(8.)** c $\quad$ **(9.)** b $\quad$ **(10.)** a, c $\quad$ **(11.)** b $\quad$ **(12.)** a, b, d

13. A few solutions are $y = 0$, $y = c$, and $y = e^x$. In general, $y = c_1 + c_2e^x$ is a solution for any constants c_1 and c_2.

14. When y is a constant, then $y' = 0$. Thus, easy solutions to see are $y = 0$ and $y = 3$.

15. The slope of the tangent line at (x, y) is y', so the differential equation is $y' = x^2 + y^2$.

16. The rate at which the slope changes is $dy'/dx = y''$, so the differential equation is $y'' = -y'$ or $y'' + y' = 0$.

17. **(a)** The domain is all real numbers.

(b) Since $y' = 2/3x^{1/3}$, the solution $y = x^{2/3}$ is undefined at $x = 0$. This function is a solution of the differential equation on $(-\infty, 0)$ and also on $(0, \infty)$.

18. **(a)** Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get

$$y = \frac{2 \pm \sqrt{4 + 4(x^2 - x - 1)}}{2} = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}\,.$$

Since $x(x-1) \ge 0$ for $x \le 0$ or $x \ge 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

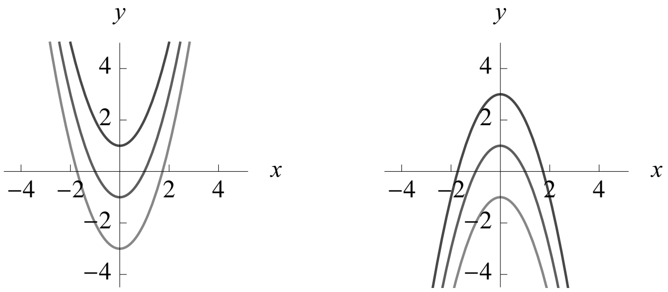
19. Setting $x = x_0$ and $y = 1$ in $y = -2/x + x$, we get

$$1 = -\frac{2}{x_0} + x_0 \qquad \text{or} \qquad x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) = 0.$$

Thus, $x_0 = 2$ or $x_0 = -1$. Since $x \ne 0$ in $y = -2/x + x$, we see that $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(-1) = 1$ on the interval $(-\infty, 0)$ $(-1 < 0)$, and $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(2) = 1$, on the interval $(0, \infty)$ $(2 > 0)$.

20. From the differential equation, $y'(1) = 1^2 + [y(1)]^2 = 1 + (-1)^2 = 2 > 0$, so $y(x)$ is increasing in some neighborhood of $x = 1$. From $y'' = 2x + 2yy'$ we have $y''(1) = 2(1) + 2(-1)(2) = -2 < 0$, so $y(x)$ is concave down in some neighborhood of $x = 1$.

21. (a)



$y = x^2 + c_1$ $\qquad$ $y = -x^2 + c_2$

(b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

(c) Pasting together x^2, $x \geq 0$, and $-x^2$, $x \leq 0$, we get

$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0. \end{cases}$$

22. The slope of the tangent line is $y'\Big|_{(-1,4)} = 6\sqrt{4} + 5(-1)^3 = 7$.

23. Differentiating $y = x \sin x + x \cos x$ we get

$$y' = x \cos x + \sin x - x \sin x + \cos x$$

and

$$\begin{aligned} y'' &= -x \sin x + \cos x + \cos x - x \cos x - \sin x - \sin x \\ &= -x \sin x - x \cos x + 2 \cos x - 2 \sin x. \end{aligned}$$

Thus

$$y'' + y = -x \sin x - x \cos x + 2 \cos x - 2 \sin x + x \sin x + x \cos x = 2 \cos x - 2 \sin x.$$

An interval of definition for the solution is $(-\infty, \infty)$.

24. Differentiating $y = x \sin x + (\cos x) \ln(\cos x)$ we get

$$\begin{aligned} y' &= x \cos x + \sin x + \cos x \left(\frac{-\sin x}{\cos x} \right) - (\sin x) \ln(\cos x) \\ &= x \cos x + \sin x - \sin x - (\sin x) \ln(\cos x) \\ &= x \cos x - (\sin x) \ln(\cos x) \end{aligned}$$

and,
$$\begin{aligned} y'' &= -x\sin x + \cos x - \sin x\left(\frac{-\sin x}{\cos x}\right) - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{1-\cos^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \sec x - \cos x - (\cos x)\ln(\cos x) \\ &= -x\sin x + \sec x - (\cos x)\ln(\cos x). \end{aligned}$$

Thus
$$y'' + y = -x\sin x + \sec x - (\cos x)\ln(\cos x) + x\sin x + (\cos x)\ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0,\infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

25. Differentiating $y = \sin(\ln x)$ we obtain $y' = \cos(\ln x)/x$ and $y'' = -[\sin(\ln x) + \cos(\ln x)]/x^2$. Then
$$x^2y'' + xy' + y = x^2\left(-\frac{\sin(\ln x) + \cos(\ln x)}{x^2}\right) + x\frac{\cos(\ln x)}{x} + \sin(\ln x) = 0.$$
An interval of definition for the solution is $(0,\infty)$.

26. Differentiating $y = \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$ we obtain
$$\begin{aligned} y' &= \cos(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right) + \ln(\cos(\ln x))\left(-\frac{\sin(\ln x)}{x}\right) + \ln x\frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x} \\ &= -\frac{\ln(\cos(\ln x))\sin(\ln x)}{x} + \frac{(\ln x)\cos(\ln x)}{x} \end{aligned}$$
and
$$\begin{aligned} y'' &= -x\left[\ln(\cos(\ln x))\frac{\cos(\ln x)}{x} + \sin(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right)\right]\frac{1}{x^2} \\ &\quad + \ln(\cos(\ln x))\sin(\ln x)\frac{1}{x^2} + x\left[(\ln x)\left(-\frac{\sin(\ln x)}{x}\right) + \frac{\cos(\ln x)}{x}\right]\frac{1}{x^2} - (\ln x)\cos(\ln x)\frac{1}{x^2} \\ &= \frac{1}{x^2}\left[-\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x)\right. \\ &\qquad\left. - (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x)\right]. \end{aligned}$$

Then
$$\begin{aligned} x^2y'' + xy' + y &= -\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x) \\ &\quad - (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x) - \ln(\cos(\ln x))\sin(\ln x) \\ &\quad + (\ln x)\cos(\ln x) + \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x) \\ &= \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) = \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x). \end{aligned}$$

To obtain an interval of definition, we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos(\ln x) > 0$. Since $\cos x > 0$ when $-\pi/2 < x < \pi/2$, we require $-\pi/2 < \ln x < \pi/2$. Since e^x is an increasing function, this is equivalent to $e^{-\pi/2} < x < e^{\pi/2}$. Thus, an interval of definition is $(e^{-\pi/2}, e^{\pi/2})$. Much of this problem is more easily done using a computer algebra system such as *Mathematica* or *Maple*.

27. Using implicit differentiation on $x^3y^3 = x^3 + 1$ we have

$$\begin{aligned} 3x^3y^2y' + 3x^2y^3 &= 3x^2 \\ xy^2y' + y^3 &= 1 \\ xy' + y &= \frac{1}{y^2}. \end{aligned}$$

28. Using implicit differentiation on $(x-5)^2 + y^2 = 1$ we have

$$\begin{aligned} 2(x-5) + 2yy' &= 0 \\ x - 5 + yy' &= 0 \\ y' &= -\frac{x-5}{y} \\ \left(y'\right)^2 &= \frac{(x-5)^2}{y^2} = \frac{1-y^2}{y^2} = \frac{1}{y^2} - 1 \\ \left(y'\right)^2 + 1 &= \frac{1}{y^2}. \end{aligned}$$

29. Using implicit differentiation on $y^3 + 3y = 1 - 3x$ we have

$$\begin{aligned} 3y^2y' + 3y' &= -3 \\ y^2y' + y' &= -1 \\ y' &= -\frac{1}{y^2+1}. \end{aligned}$$

Again, using implicit differentiation, we have

$$y'' = -\frac{-2yy'}{(y^2+1)^2} = 2yy'\left(\frac{1}{y^2+1}\right)^2 = 2yy'\left(-\frac{1}{y^2+1}\right)^2 = 2yy'\left(-y'\right)^2 = 2y(y')^3.$$

30. Using implicit differentiation on $y = e^{xy}$ we have

$$\begin{aligned} y' &= e^{xy}\left(xy' + y\right) \\ \left(1 - xe^{x}y\right)y' &= ye^{xy}. \end{aligned}$$

Since $y = e^{xy}$ we have

$$(1 - xy)y' = y \cdot y \qquad \text{or} \qquad (1 - xy)y' = y^2.$$

In Problems 31–34 we have $y' = 3c_1e^{3x} - c_2e^x - 2$.

31. The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 - c_2 - 2 &= 0, \end{aligned}$$

so $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus $y = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-x} - 2x$.

32. The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - c_2 - 2 &= -3, \end{aligned}$$

so $c_1 = 0$ and $c_2 = 1$. Thus $y = e^{-x} - 2x$.

33. The initial conditions imply

$$\begin{aligned} c_1e^3 + c_2e^{-1} - 2 &= 4 \\ 3c_1e^3 - c_2e^{-1} - 2 &= -2, \end{aligned}$$

so $c_1 = \frac{3}{2}e^{-3}$ and $c_2 = \frac{9}{2}e$. Thus $y = \frac{3}{2}e^{3x-3} + \frac{9}{2}e^{-x+1} - 2x$.

34. The initial conditions imply

$$\begin{aligned} c_1e^{-3} + c_2e + 2 &= 0 \\ 3c_1e^{-3} - c_2e - 2 &= 1, \end{aligned}$$

so $c_1 = \frac{1}{4}e^3$ and $c_2 = -\frac{9}{4}e^{-1}$. Thus $y = \frac{1}{4}e^{3x+3} - \frac{9}{4}e^{-x-1} - 2x$.

35. From the graph we see that estimates for y_0 and y_1 are $y_0 = -3$ and $y_1 = 0$.

36. Figure 1.3.3 in the text can be used for reference in this problem. The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w}\sqrt{2gh}\,.$$

Using $A_0 = \pi(1/24)^2 = \pi/576$, $A_w = \pi(2)^2 = 4\pi$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/576}{4\pi}\sqrt{64h} = \frac{c}{288}\sqrt{h}\,.$$

37. Let $P(t)$ be the number of owls present at time t. Then $dP/dt = k(P - 200 + 10t)$.

38. Setting $A'(t) = -0.002$ and solving $A'(t) = -0.0004332A(t)$ for $A(t)$, we obtain

$$A(t) = \frac{A'(t)}{-0.0004332} = \frac{-0.002}{-0.0004332} \approx 4.6 \text{ grams.}$$

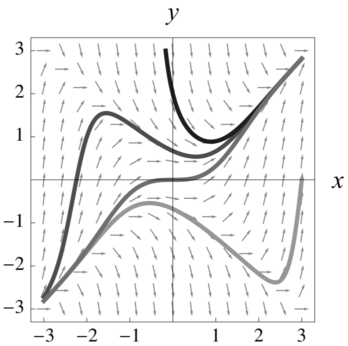
2 FIRST-ORDER DIFFERENTIAL EQUATIONS

2.1 Solution Curves Without a Solution

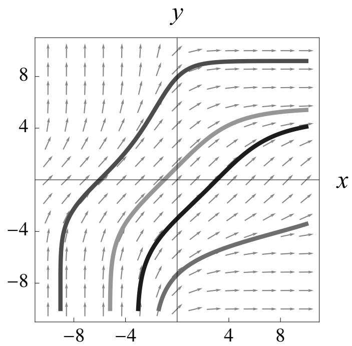
2.1.1 DIRECTION FIELDS

In Problems 1–4 *the graph corresponding to the initial condition in* Part (a) *is red,* Part (b) *is green,* Part (c) *is blue, and* Part (d) *is brown. The pictures are obtain using Mathematica with* **VectorPlot[{1, f[x, y]}, {x, lhs, rhs}, {y, down, up}, . . .].**

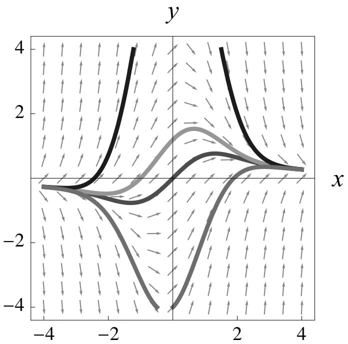
1.



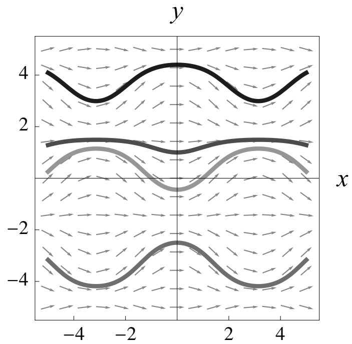
2.



3.

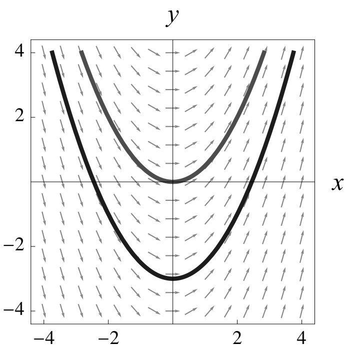


4.

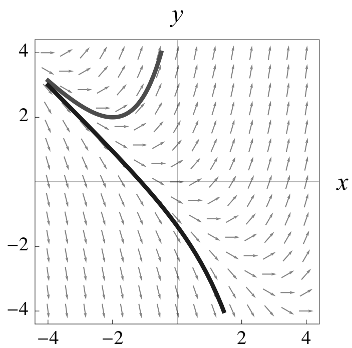


In Problems 5–12 *the graph corresponding to the initial condition in* Part (a) *is red, and* Part (b) *is blue. The pictures are obtain using Mathematica, as mentioned before* Problem 1.

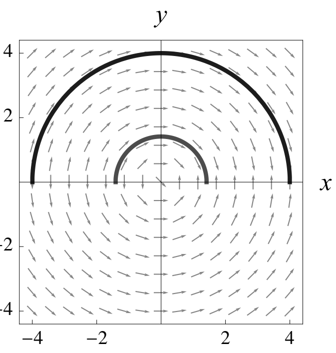
5.



6.



7.



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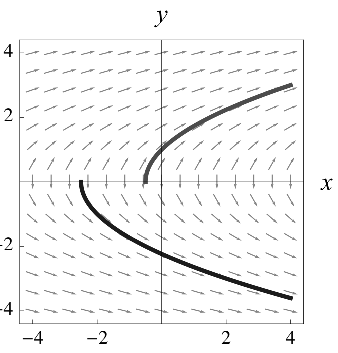
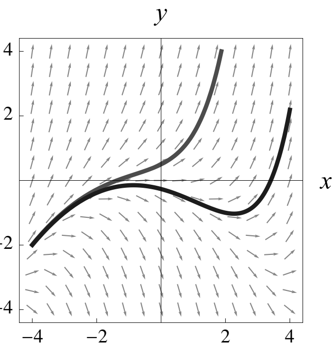
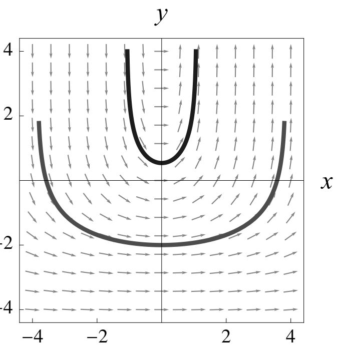


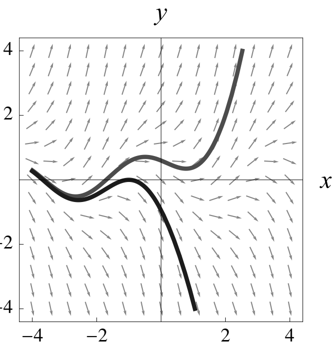
9.



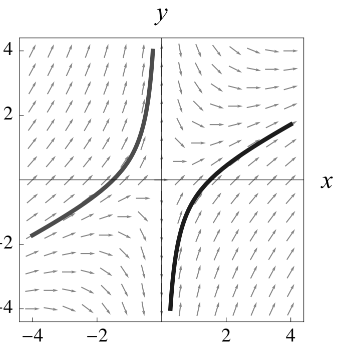
10.



11.

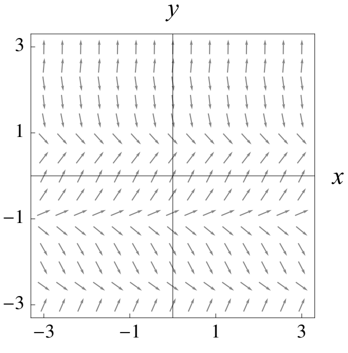


12.

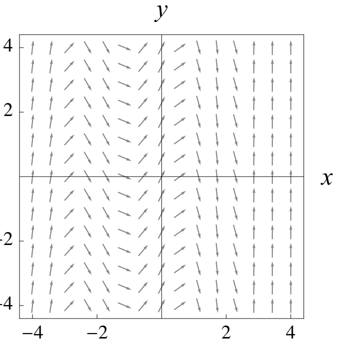


In Problems 13 *and* 14 *Mathematica was used, as mentioned before Problem* 1.

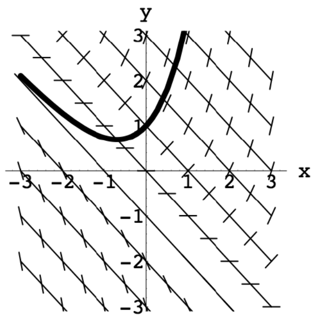
13.



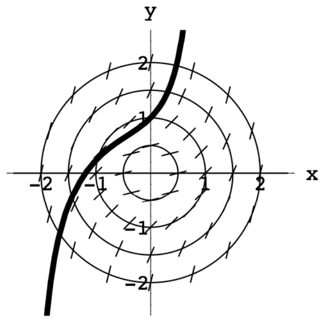
14.



15. (a) The isoclines have the form $y = -x + c$, which are straight lines with slope -1.



(b) The isoclines have the form $x^2 + y^2 = c$, which are circles centered at the origin.

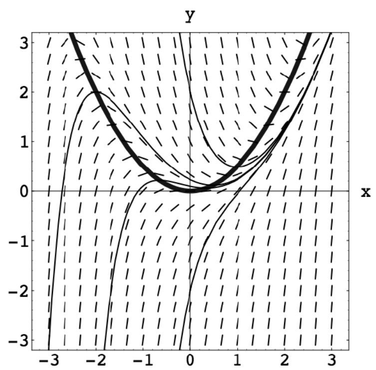


Discussion Problems

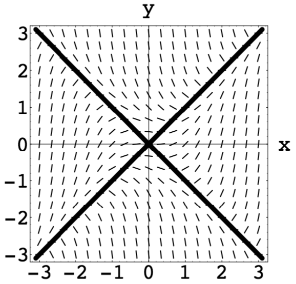
16. (a) When $x = 0$ or $y = 4$, $dy/dx = -2$ so the lineal elements have slope -2. When $y = 3$ or $y = 5$, $dy/dx = x - 2$, so the lineal elements at $(x, 3)$ and $(x, 5)$ have slopes $x - 2$.

(b) At $(0, y_0)$ the solution curve is headed down. If $y \to \infty$ as x increases, the graph must eventually turn around and head up, but while heading up it can never cross $y = 4$ where a tangent line to a solution curve must have slope -2. Thus, y cannot approach ∞ as x approaches ∞.

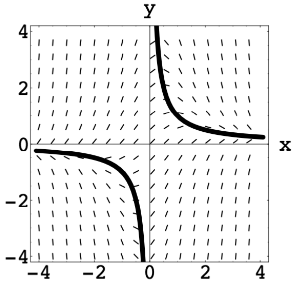
17. When $y < \frac{1}{2}x^2$, $y' = x^2 - 2y$ is positive and the portions of solution curves "outside" the nullcline parabola are increasing. When $y > \frac{1}{2}x^2$, $y' = x^2 - 2y$ is negative and the portions of the solution curves "inside" the nullcline parabola are decreasing.



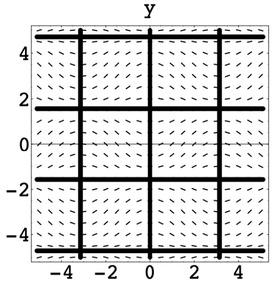
18. (a) Any horizontal lineal element should be at a point on a nullcline. In Problem 1 the nullclines are $x^2 - y^2 = 0$ or $y = \pm x$. In Problem 3 the nullclines are $1 - xy = 0$ or $y = 1/x$. In Problem 4 the nullclines are $(\sin x)\cos y = 0$ or $x = n\pi$ and $y = \pi/2 + n\pi$, where n is an integer. The graphs on the next page show the nullclines for the differential equations in Problems 1, 3, and 4 superimposed on the corresponding direction field.



Problem 1



Problem 3

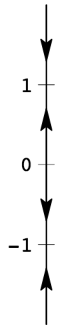


Problem 4

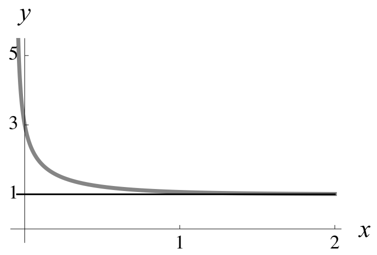
(b) An autonomous first-order differential equation has the form $y' = f(y)$. Nullclines have the form $y = c$ where $f(c) = 0$. These are the equilibrium solutions of the differential equation.

2.1.2 AUTONOMOUS FIRST-ORDER DEs

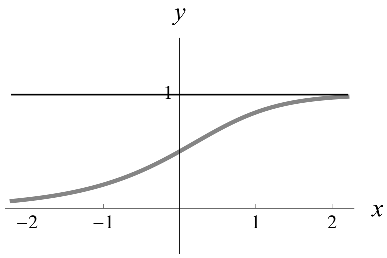
19. Writing the differential equation in the form $dy/dx = y(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



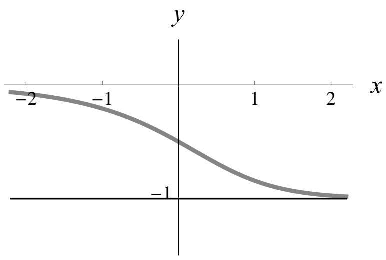
(a)



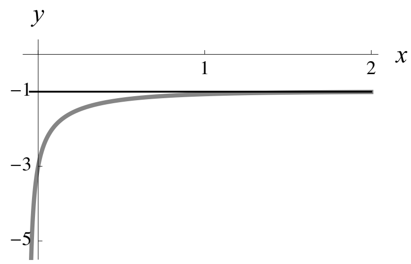
(b)



(c)

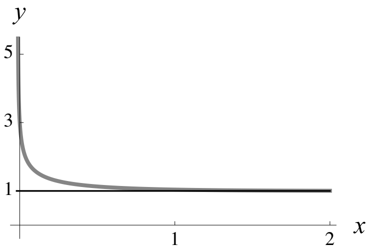


(d)

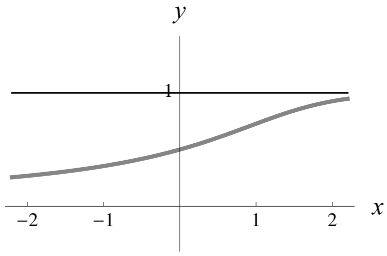


20. Writing the differential equation in the form $dy/dx = y^2(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right, and the graphs of the typical solutions are shown on the next page.

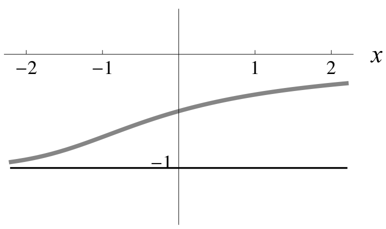
(a)



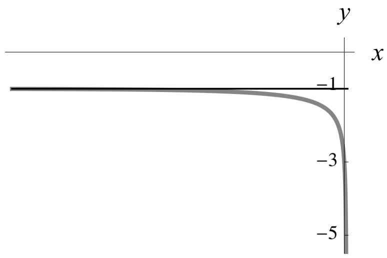
(b)



(c)

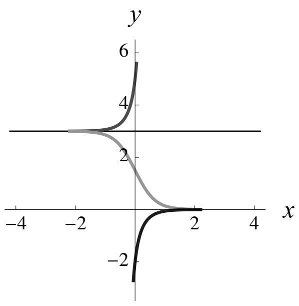


(d)

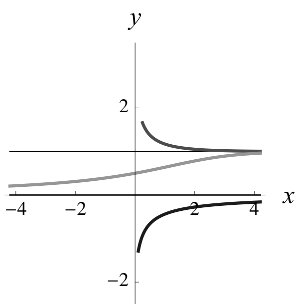


In Problems 21–28 graphs of typical solutions are shown. However, in some of the solutions, even though the upper and lower graphs either actually bend up or down, they display as straight line segments. This is a peculiarity of the Mathematica graphing routine and may be due to the fact that the **NDSolve** *function was used rather than* **DSolve**. **NDSolve** *uses a numerical routine (see* Section 2.6 *in the text), and involves sampling x-coordinates where the corresponding y-coordinates are approximated. It may be that the routine involved breaks down as the graph becomes nearly vertical, forcing the x-coordinates on the graph to becomes closer and closer together.*

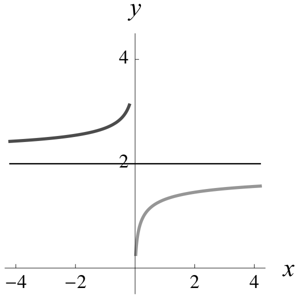
21. Solving $y^2 - 3y = y(y-3) = 0$ we obtain the critical points 0 and 3. From the phase portrait we see that 0 is asymptotically stable (attractor) and 3 is unstable (repeller).

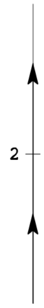


22. Solving $y^2 - y^3 = y^2(1-y) = 0$ we obtain the critical points 0 and 1. From the phase portrait we see that 1 is asymptotically stable (attractor) and 0 is semi-stable.

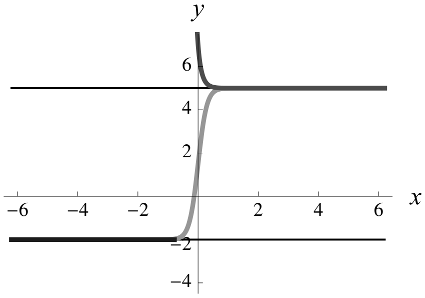


23. Solving $(y-2)^4 = 0$ we obtain the critical point 2. From the phase portrait we see that 2 is semi-stable.

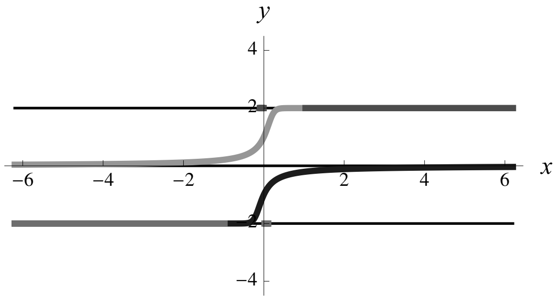


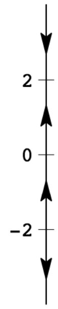


24. Solving $10 + 3y - y^2 = (5-y)(2+y) = 0$ we obtain the critical points -2 and 5. From the phase portrait we see that 5 is asymptotically stable (attractor) and -2 is unstable (repeller).

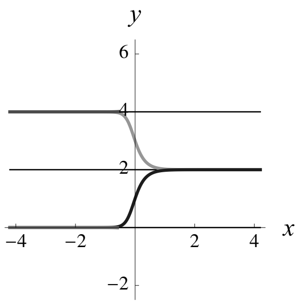


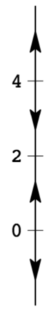
25. Solving $y^2(4-y^2) = y^2(2-y)(2+y) = 0$ we obtain the critical points -2, 0, and 2. From the phase portrait we see that 2 is asymptotically stable (attractor), 0 is semi-stable, and -2 is unstable (repeller).



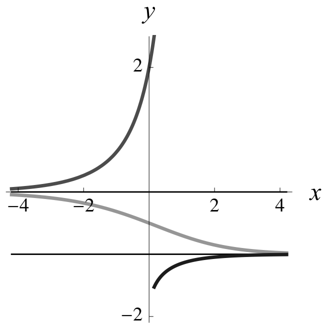


26. Solving $y(2-y)(4-y) = 0$ we obtain the critical points 0, 2, and 4. From the phase portrait we see that 2 is asymptotically stable (attractor) and 0 and 4 are unstable (repellers).

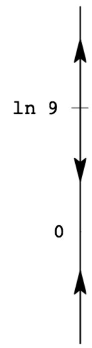


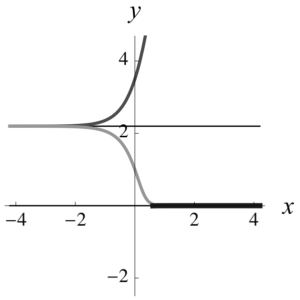


27. Solving $y\ln(y+2) = 0$ we obtain the critical points -1 and 0. From the phase portrait we see that -1 is asymptotically stable (attractor) and 0 is unstable (repeller).

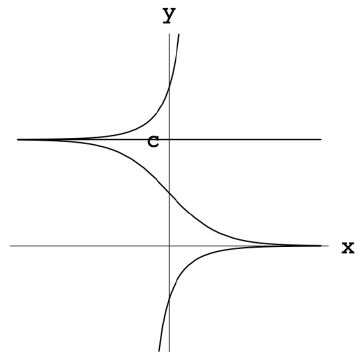


28. Solving $ye^y - 9y = y(e^y - 9) = 0$ (since e^y is always positive) we obtain the critical points 0 and $\ln 9$. From the phase portrait we see that 0 is asymptotically stable (attractor) and $\ln 9$ is unstable (repeller).





29. The critical points are 0 and c because the graph of $f(y)$ is 0 at these points. Since $f(y) > 0$ for $y < 0$ and $y > c$, the graph of the solution is increasing on $(-\infty, 0)$ and (c, ∞). Since $f(y) < 0$ for $0 < y < c$, the graph of the solution is decreasing on $(0, c)$.



30. The critical points are approximately at -2, 2, 0.5, and 1.7. Since $f(y) > 0$ for $y < -2.2$ and $0.5 < y < 1.7$, the graph of the solution is increasing on $(-\infty, -2.2)$ and $(0.5, 1.7)$. Since $f(y) < 0$ for $-2.2 < y < 0.5$ and $y > 1.7$, the graph is decreasing on $(-2.2, 0.5)$ and $(1.7, \infty)$.