

Chapter 1: Speaking Mathematically

Many college students appear to have difficulty using and interpreting language involving if-then statements and quantification. Section 1.1 is a gentle introduction to the relation between informal and formal ways of expressing important kinds of mathematical statements. Experience with the exercises in the section is meant as a warm-up to prepare students to master the linguistic aspects of learning mathematics to help them come to understand the meaning of mathematical statements and evaluate their truth or falsity in later chapters. Sections 1.2 and 1.3 form a brief introduction to the language of sets, relations, and functions. Covering them at the beginning of the course can help students relate discrete mathematics to the pre-calculus or calculus they have studied previously while broadening their perspective to include a larger proportion of discrete examples.

Proofs of set properties, such as the distributive laws, and proofs of properties of relations and functions, such as transitivity and surjectivity, are considerably more complex than the examples used in this book to introduce students to the idea of mathematical proof. Thus set theory as a theory is left to Chapter 6, properties of functions to Chapter 7, and properties of relations to Chapter 8. Instructors who wish to do so could cover Section 1.2 just before starting Chapter 6 and Section 1.3 just before starting Chapter 7.

An aspect of students' backgrounds that may surprise college and university mathematics instructors concerns their understanding of the meaning of "real number." When asked to evaluate the truth or falsity of a statement about real numbers, it is not unusual for students to think only of integers. Thus an informal description of the relationship between real numbers and points on a number line is given in Section 1.2 on page 8 to illustrate that there are many real numbers between any pair of consecutive integers, Examples 3.3.5 and 3.3.6 on page 121 show that while there is a smallest positive integer there is no smallest positive real number, and the discussion on pages 433 and 434 (preceding the proof of the uncountability of the real numbers between 0 and 1) describes a procedure for approximating the (possibly infinite) decimal expansion for an arbitrarily chosen point on a number line.

Section 1.1

2. *a.* a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6
b. an integer n ; n is divided by 6 the remainder is 3

4. *a.* a real number; greater than r *b.* real number r ; there is a real number s

6. *a.* s is negative *b.* negative; the cube root of s is negative (*Or:* $\sqrt[3]{s}$ is negative)
c. is negative; $\sqrt[3]{s}$ is negative (*Or:* the cube root of s is negative)

7. *b.* There is a real number whose square is less than itself.

True. For example, $(1/2)^2 = 1/4 < 1/2$.

- d.* The absolute value of the sum of any two numbers is less than or equal to the sum of their absolute values.

True. This is known as the triangle inequality. It is discussed in Section 4.4.

9. *a.* have at most two real solutions *b.* has at most two real solutions *c.* has at most two real solutions *d.* is a quadratic equation; has at most two real solutions *e.* E has at most two real solutions

11. *a.* have positive square roots *b.* a positive square root *c.* r is a square root for e

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13. *a.* real number; product with every real number equals zero
b. with every real number equals zero *c.* $ab = 0$

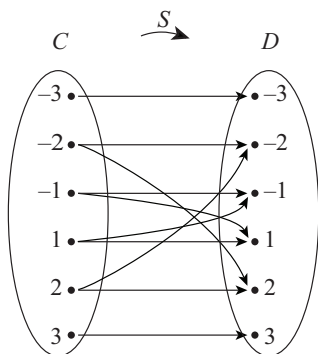
Section 1.2

2. *b.* The set of all real numbers x such that x is less than or equal to zero or x is greater than or equal to 1
d. The set of all positive integers n such that n is a factor of 6
4. *a.* Yes: $\{2\}$ is the set whose only element is 2. *b.* One: 2 is the only element in this set *c.* Two: The two elements are 0 and $\{0\}$ *d.* Yes: $\{0\}$ is one of the elements listed in the set.
e. No: The only elements listed in the set are $\{0\}$ and $\{1\}$, and 0 is not equal to either of these.
5. The only sets that are equal to each other are A and D .
 A contains the integers 0, 1, and 2 and nothing else.
 B contains all the real numbers that are greater than or equal to -1 and less than 3.
 C contains all the real numbers that are greater than -1 and less than 3. Thus -1 is in B but not in C .
 D contains all the integers greater than -1 and less than 3. Thus D contains the integers 0, 1, and 2 and nothing else, and so $D = \{0, 1, 2\} = A$.
 E contains all the positive integers greater than -1 and less than 3. Hence E contains the integers 1 and 2 and nothing else, that is, $E = \{1, 2\}$.
6. T_2 and T_{-3} each have two elements, and T_0 and T_1 each have one element.
Justification: $T_2 = \{2, 2^2\} = \{2, 4\}$, $T_{-3} = \{-3, (-3)^2\} = \{-3, 9\}$,
 $T_1 = \{1, 1^2\} = \{1, 1\} = \{1\}$, and $T_0 = \{0, 0^2\} = \{0, 0\} = \{0\}$.
7. *b.* $T = \{m \in \mathbf{Z} \mid m = 1 + (-1)^k \text{ for some integer } k\} = \{0, 2\}$. Exercises in Chapter 4 explore the fact that $(-1)^k = -1$ when k is odd and $(-1)^k = 1$ when k is even. So $1 + (-1)^k = 1 + (-1) = 0$ when k is odd, and $1 + (-1)^k = 1 + 1 = 2$ when k is even.
e. There are no elements in W because there are no integers that are both greater than 1 and less than -3 .
f. $X = \mathbf{Z}$ because every integer u satisfies at least one of the conditions $u \leq 4$ or $u \geq 1$.
8. *b.* Yes, because every element in C is in A . *c.* Yes, because every element in C is in C .
c.. Yes, because it is true that every element in C is in C .
9. *c.* No: The only elements in $\{1, 2\}$ are 1 and 2, and $\{2\}$ is not equal to either of these.
d. Yes: $\{3\}$ is one of the elements listed in $\{1, \{2\}, \{3\}\}$.
e. Yes: $\{1\}$ is the set whose only element is 1.
g. Yes: The only element in $\{1\}$ is 1, and 1 is an element in $\{1, 2\}$.
h. No: The only elements in $\{\{1\}, 2\}$ are $\{1\}$ and 2, and 1 is not equal to either of these.
j. Yes: The only element in $\{1\}$ is 1, which is an element in $\{1\}$. So every element in $\{1\}$ is in $\{1\}$.

10. *b.* No: For two ordered pairs to be equal, the elements in each pair must occur in the same order. In this case the first element of the first pair is 5, whereas the first element of the second pair is -5 , and the second element of the first pair is -5 whereas the second element of the second pair is 5.
- d.* Yes The first elements of both pairs equal $\frac{1}{2}$, and the second elements of both pairs equal -8 .
12. All four sets have nine elements.
- a.* $S \times T = \{(2, 1), (2, 3), (2, 5), (4, 1), (4, 3), (4, 5), (6, 1), (6, 3), (6, 5)\}$
- b.* $T \times S = \{(1, 2), (3, 2), (5, 2), (1, 4), (3, 4), (5, 4), (1, 6), (3, 6), (5, 6)\}$
- c.* $S \times S = \{(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)\}$
- d.* $T \times T = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

Section 1.3

2. *a.* $2 \ S \ 2$ because $\frac{1}{2} - \frac{1}{2} = 0$, which is an integer.
- $-1 \ S \ -1$ because $\frac{1}{-1} - \frac{1}{-1} = 0$, which is an integer.
- $(2, 2) \in S$ because $\frac{1}{2} - \frac{1}{2} = 0$, which is an integer.
- $3 \not\ S \ -3$ because $\frac{1}{3} - \frac{1}{-3} = \frac{2}{3}$, which is not an integer.
- b.* $S = \{(-3, -3), (-2, -2), (-1, -1), (1, 1), (2, 2), (3, 3), (1, -1), (-1, 1), (2, -2), (-2, 2)\}$
- c.* The domain and co-domain of S are both $\{-3, -2, -1, 1, 2, 3\}$.
- d.*



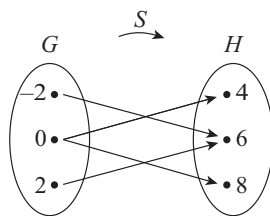
4. *a.* $2 \ V \ 6$ because $\frac{2-6}{4} = \frac{-4}{4} = -1$, which is an integer.
- $(-2) \not\ V \ 8$ because $\frac{(-2)-8}{4} = \frac{-6}{4}$, which is not an integer.
- $0 \not\ V \ 6$ because $\frac{0-6}{4} = \frac{-6}{4}$, which is not an integer.
- $2 \not\ V \ 4$ because $\frac{2-4}{4} = \frac{-2}{4}$, which is not an integer.

In the first printing of the book, the second question in part (a) was “Is $(-2) \ V \ (-6)$?”. You might accept both of the following as acceptable answers::

- (1) $-2 \not\ V \ -6$ because $-6 \notin B$.
- (2) $(-2) \ V \ (-6)$ because $\frac{-2-(-6)}{4} = \frac{4}{4} = 1$, which is an integer.
- b.* $V = \{(-2, 6), (0, 4), (0, 8), (2, 6)\}$
- c.* Domain of $V = \{-2, 0, 2\}$, co-domain of $V = \{4, 6, 8\}$

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d.



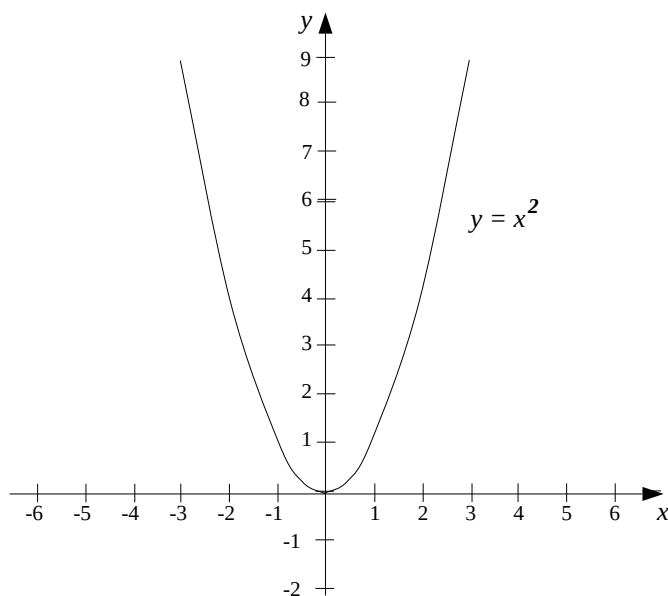
6. a. $(2, 4) \in R$ because $4 = 2^2$.

$(4, 2) \notin R$ because $2 \neq 4^2$.

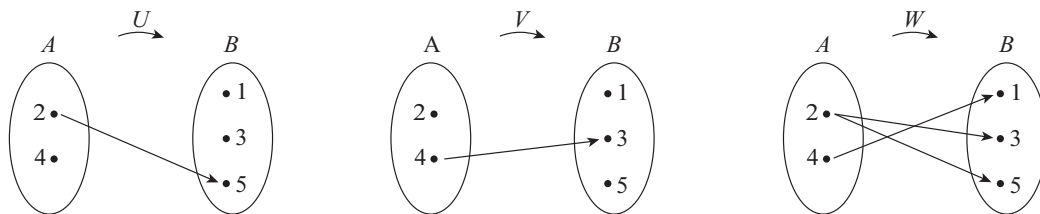
$(-3, 9) \in R$ because $9 = (-3)^2$.

$(9, -3) \notin R$ because $-3 \neq 9^2$.

b.



8. a.



b. None of U , V , or W are functions.

U is not a function because $(4, y)$ is not in U for any y in B , and so U does not satisfy property (1) of the definition of function.

V is not a function because $(2, y)$ is not in V for any y in B , and so V does not satisfy property (1) of the definition of function.

W is not a function because both $(2, 3)$ and $(2, 5)$ are in W and $3 \neq 5$, and so W does not satisfy property (2) of the definition of function.

10. The following sets are relations from $\{a, b\}$ to $\{x, y\}$ that are not functions:
 $\{(a, x)\}$, $\{(a, y)\}$, $\{(b, x)\}$, $\{(b, y)\}$, $\{(a, x), (a, y)\}$, $\{(b, x), (b, y)\}$, $\{(a, x), (a, y), (b, x)\}$,
 $\{(a, x), (a, y), (b, y)\}$, $\{(b, x), (b, y), (a, x)\}$, $\{(b, x), (b, y), (a, y)\}$, $\{(a, x), (a, y), (b, x), (b, y)\}$.
12. T is not a function because, for example, both $(0, 1)$ and $(0, -1)$ are in T but $1 \neq -1$. Many other examples could be given showing that T does not satisfy property (2) of the definition of function.
14. *a.* Domain of $G = \{1, 2, 3, 4\}$, co-domain of $G = \{a, b, c, d\}$
b. $G(1) = G(2) = G(3) = G(4) = c$
15. *c.* This diagram does not determine a function because 4 is related to both 1 and 2, which violates property (2) of the definition of function.
d. This diagram defines a function; both properties (1) and (2) are satisfied.
e. This diagram does not determine a function because 2 is in the domain but it is not related to any element in the co-domain.
17. $g(-1000) = -999$, $g(0) = 1$, $g(999) = 1000$
18. $h(-\frac{12}{5}) = h(\frac{0}{1}) = h(\frac{9}{17}) = 2$
20. For all $x \in \mathbf{R}$, $K(x) = (x-1)(x-3) + 1 = (x^2 - 4x + 3) + 1 = x^2 + 4x + 4 = (x-2)^2 = H(x)$.
 Therefore, by definition of equality of functions, $H = K$.

Chapter 2: The Logic of Compound Statements

The ability to reason using the principles of logic is essential for solving problems in abstract mathematics and computer science and for understanding the reasoning used in mathematical proof and disproof. Because a significant number of students who come to college have had limited opportunity to develop this ability, a primary aim of Chapters 2 and 3 is to help students develop an inner voice that speaks with logical precision. Consequently, the various rules used in logical reasoning are developed both symbolically and in the context of their somewhat limited but very important use in everyday language. Exercise sets for Sections 2.1–2.3 and 3.1–3.4 contain sentences for students to negate, write the contrapositive for, and so forth. Virtually all students benefit from doing these exercises. Another aim of Chapters 2 and 3 is to teach students the rudiments of symbolic logic as a foundation for a variety of upper-division courses. Symbolic logic is used in, among others, the study of digital logic circuits, relational databases, artificial intelligence, and program verification.

Suggestions

1. In Section 2.1 a surprising number of students apply De Morgan's law to write the negation of, for example, " $1 < x \leq 3$ " as " $1 \geq x > 3$." You may find that it takes some effort to teach them to avoid making this mistake.

2. In Sections 2.1 and 2.4, students have more difficulty than you might expect simplifying statement forms and circuits. Only through trial and error can you learn the extent to which this is the case at your institution. If it is, you might either assign only the easier exercises or build in extra time to teach students how to do the more complicated ones. Discussion of simplification techniques occurs again in Chapter 6 in the context of set theory. At this later point in the course most students are able to deal with it successfully.

3. In ordinary English, the phrase "only if" is often used as a synonym for "if and only if." But it is possible to find informal sentences in which the intuitive interpretation is the same as the logical definition, and it is helpful to give examples of such statements when you introduce the logical definition. For instance, it is not hard to get students to agree that "The team will win the championship only if it wins the semifinal game" means the same as "If the team does not win the semifinal game then it will not win the championship." Once students see this, you can suggest that they remember this translation when they encounter more abstract statements of the form " A only if B ."

Through guided discussion, students will also come to agree that the statement "Winning the semifinal game is a necessary condition for winning the championship" translates to "If the team does not win the semifinal game then it will not win the championship." They can be encouraged to use this (or a similar statement) as a reference to help develop intuition for general statements of the form " A is a necessary condition for B ."

With students who have weaker backgrounds, you may find yourself tying up excessive amounts of class time discussing "only if" and "necessary and sufficient conditions." You might just assign the easier exercises, or you might assign exercises on these topics to be done for extra credit (putting corresponding extra credit problems on exams) and use the results to help distinguish A from B students. It is probably best not to omit these topics altogether, though, because the language of "only if" and "necessary and sufficient conditions" is a standard part of the technical vocabulary of textbooks used in upper-division courses, as well as occurring regularly in non-mathematical writing.

4. In Section 2.3, many students mistakenly conclude that an argument is valid if, when they compute the truth table, they find a single row in which both the premises and the conclusion are true. The source of students' difficulty appears to be their tendency to ignore quantification and to misinterpret if-then statements as "and" statements. Since the definition of validity includes both a universal quantifier and if-then, it is helpful to go back over the definition and the procedures for

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testing for validity and invalidity after discussing the general topic of universal conditional statements in Section 3.1. As a practical measure to help students assess validity and invalidity correctly, the first example in Section 2.3 is of an invalid argument whose truth table has eight rows, several of which have true premises and a true conclusion. In addition, to further focus students' attention on the situations where all the premises are true, the truth values for the conclusions of arguments are simply omitted when at least one premise is false.

5. In Section 2.3, you might suggest that students just familiarize themselves with, but not memorize, the various forms of valid arguments covered in Section 2.3. It is wise, however, to have them learn the terms modus ponens and modus tollens (because these are used in some upper-division computer science courses) and converse and inverse errors (because these errors are so common).

Section 2.1

2. common form: If p then q .

$$\sim q$$

Therefore, $\sim p$.

- b. all prime numbers are odd; 2 is odd

4. common form: If p then q .

If q then r .

Therefore, if p then r .

- b. a polynomial; differentiable; is continuous

5. b. The truth or falsity of this sentence depends on the reference for the pronoun "she." Considered on its own, the sentence cannot be said to be either true or false, and so it is not a statement.

- c. This sentence is false; hence it is a statement.

- d. This is not a statement because its truth or falsity depends on the value of x .

7. $m \wedge \sim c$

8. b. $\sim w \wedge (h \wedge s)$

- c. $\sim w \wedge \sim h \wedge \sim s$

- e. $w \wedge \sim (h \wedge s)$ ($w \wedge (\sim h \vee \sim s)$ is also acceptable)

9. $(n \vee k) \wedge \sim (n \wedge k)$

10. b. $p \wedge \sim q$ d. $(\sim p \wedge q) \wedge \sim r$ e. $\sim p \vee (q \wedge r)$

- 13.

p	q	$p \wedge q$	$p \vee q$	$\sim (p \wedge q)$	$\sim (p \wedge q) \vee (p \vee q)$
T	T	T	T	F	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	F	T	T

15.

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

17.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim (p \wedge q)$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	T

←
←

different truth values in rows 2 and 3

The truth table shows that $\sim (p \wedge q)$ and $\sim p \wedge \sim q$ do not always have the same truth values. Therefore they are not logically equivalent.

19.

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p
T	T	T	T
F	T	F	F

same truth values

The truth table shows that $p \wedge \mathbf{t}$ and p always have the same truth values. Thus they are logically equivalent. This proves the identity law for $\wedge$.

20.

p	$\mathbf{c}$	$p \wedge \mathbf{c}$	$p \vee \mathbf{c}$
T	F	F	T
F	F	F	F

←

different truth values in row 1

The truth table shows that $p \wedge \mathbf{c}$ and $p \vee \mathbf{c}$ do not always have the same truth values. Thus they are not logically equivalent.

22.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	T

same truth values

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The truth table shows that $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ always have the same truth values. Therefore they are logically equivalent. This proves the distributive law for $\wedge$ over $\vee$.

24.

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \vee (p \wedge r)$	$(p \vee q) \wedge r$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	F	T	T
F	T	F	T	F	T	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

different truth values

The truth table shows that $(p \vee q) \vee (p \wedge r)$ and $(p \vee q) \wedge r$ have different truth values in rows 2, 3, and 6. Hence they are not logically equivalent.

26. Sam is not an orange belt or Kate is not a red belt.
28. The units digit of 4^{67} is not 4 and it is not 6.
29. This computer program does not have a logical error in the first ten lines and it is not being run with an incomplete data set.
30. The dollar is not at an all-time high or the stock market is not at a record low.
31. The train is not late and my watch is not fast.
33. $-10 \geq x$ or $x \geq 2$
35. $x > -1$ and $x \leq 1$
37. $0 \leq x$ or $x < -7$
39. The statement's logical form is $(p \wedge q) \vee ((r \wedge s) \wedge t)$, so its negation has the form

$$\begin{aligned}
 \sim((p \wedge q) \vee ((r \wedge s) \wedge t)) &\equiv \sim(p \wedge q) \wedge \sim((r \wedge s) \wedge t) \\
 &\equiv (\sim p \vee \sim q) \wedge (\sim(r \wedge s) \vee \sim t) \\
 &\equiv (\sim p \vee \sim q) \wedge ((\sim r \vee \sim s) \vee \sim t).
 \end{aligned}$$

Thus a negation is $(num_orders \geq 50 \text{ or } num_instock \leq 300)$ and $((50 > num_orders \text{ or } num_orders \geq 75) \text{ or } num_instock \leq 500)$.

42.

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge q$	$q \wedge r$	$((\sim p \wedge q) \wedge (q \wedge r))$	$((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	F	F	F	F
T	F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F	F
F	T	T	T	F	T	T	T	F
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F
F	F	F	T	T	F	F	F	F

all F 's

Since all the truth values of $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$ are F , $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$ is a contradiction.

43.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \wedge \sim q$	$(\sim p \vee q) \vee (p \wedge \sim q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

all T 's

Since all the truth values of $(\sim p \vee q) \vee (p \wedge \sim q)$ are T , $(\sim p \vee q) \vee (p \wedge \sim q)$ is a tautology.

45. Let b be “Bob is majoring in both math and computer science,” m be “Ann is majoring in math,” and a be “Ann is not majoring in both math and computer science.” Then the two statements can be symbolized as follows: a. $(b \wedge m) \wedge \sim a$ and b. $\sim (b \wedge a) \wedge (m \wedge b)$.

b	m	a	$\sim a$	$b \wedge m$	$m \wedge b$	$b \wedge a$	$\sim (b \wedge a)$	$(b \wedge m) \wedge \sim a$	$\sim (b \wedge a) \wedge (m \wedge b)$
T	T	T	F	T	T	T	F	F	F
T	T	F	T	F	T	F	T	T	T
T	F	T	F	T	F	T	F	F	F
T	F	F	T	F	F	F	T	F	F
F	T	T	F	F	F	F	T	F	F
F	T	F	T	F	F	F	T	F	F
F	F	T	F	F	F	F	T	F	F
F	F	F	T	F	F	F	T	F	F

same truth values

The truth table shows that $(b \wedge m) \wedge \sim a$ and $\sim (b \wedge a) \wedge (m \wedge b)$ always have the same truth values. Hence they are logically equivalent.

46. b. Yes.

p	q	r	$p \oplus q$	$q \oplus r$	$(p \oplus q) \oplus r$	$p \oplus (q \oplus r)$
T	T	T	F	F	T	T
T	T	F	F	T	F	F
T	F	T	T	T	F	F
T	F	F	T	F	T	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

same truth values

The truth table shows that $(p \oplus q) \oplus r$ and $p \oplus (q \oplus r)$ always have the same truth values. So they are logically equivalent.

- c. Yes.

p	q	r	$p \oplus q$	$p \wedge r$	$q \wedge r$	$(p \oplus q) \wedge r$	$(p \wedge r) \oplus (q \wedge r)$
T	T	T	F	T	T	F	F
T	T	F	F	F	F	F	F
T	F	T	T	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	T	F	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

same truth values

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The truth table shows that $(p \oplus q) \wedge r$ and $(p \wedge r) \oplus (q \wedge r)$ always have the same truth values. So they are logically equivalent.

Section 2.2

2. If I catch the 8:05 bus, then I am on time for work.
4. If you don't fix my ceiling, then I won't pay my rent.

6.

p	q	$\sim p$	$\sim p \wedge q$	$p \vee q$	$(p \vee q) \vee (\sim p \wedge q)$	$(p \vee q) \vee (\sim p \wedge q) \rightarrow q$
T	T	F	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	F	T	F	F	F	T

8.

p	q	r	$\sim p$	$\sim p \vee q$	$\sim p \vee q \rightarrow r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	F

10.

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \leftrightarrow (q \rightarrow r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

11.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \wedge q$	$p \wedge q \rightarrow r$	$(p \rightarrow (q \rightarrow r)) \leftrightarrow (p \wedge q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

13. b.

p	q	$\sim q$	$p \rightarrow q$	$\sim (p \rightarrow q)$	$p \wedge \sim q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

same truth values

The truth table shows that $\sim(p \rightarrow q)$ and $p \wedge \sim q$ always have the same truth values. Hence they are logically equivalent.

14. a.

p	q	r	$\sim q$	$\sim r$	$q \vee r$	$p \wedge \sim q$	$p \wedge \sim r$	$p \rightarrow q \vee r$	$p \wedge \sim q \rightarrow r$	$p \wedge \sim r \rightarrow q$
T	T	T	F	F	T	F	F	T	T	T
T	T	F	F	T	T	F	T	T	T	T
T	F	T	T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	T	F	F	F
F	T	T	F	F	T	F	F	T	T	T
F	T	F	F	T	T	F	F	T	T	T
F	F	T	T	F	T	F	F	T	T	T
F	F	F	T	T	F	F	F	T	T	T

same truth values

The truth table shows that the three statement forms $p \rightarrow q \vee r$, $p \wedge \sim q \rightarrow r$, and $p \wedge \sim r \rightarrow q$ always have the same truth values. Thus they are all logically equivalent.

b. If n is prime and n is not odd, then n is 2.

And: If n is prime and n is not 2, then n is odd.

15.

p	q	r	$q \rightarrow r$	$p \rightarrow q$	$p \rightarrow (q \rightarrow r)$	$(p \rightarrow q) \rightarrow r$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	F
F	F	F	T	T	T	F

different truth values

The truth table shows that $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$ do not always have the same truth values. (They differ for the combinations of truth values for p , q , and r shown in rows 6, 7, and 8.) Therefore they are not logically equivalent.

17. Let p represent “2 is a factor of n ,” q represent “3 is a factor of n ,” and r represent “6 is a factor of n .” The statement “If 2 is a factor of n and 3 is a factor of n , then 6 is a factor of n ” has the form $p \wedge q \rightarrow r$. And the statement “2 is not a factor of n or 3 is not a factor of n or 6 is a factor of n ” has the form $\sim p \vee \sim q \vee r$.

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge q \rightarrow r$	$\sim p \vee \sim q \vee r$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	T
F	T	T	T	F	F	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

same truth values

The truth table shows that $p \wedge q \rightarrow r$ and $\sim p \vee \sim q \vee r$ always have the same truth values. Therefore they are logically equivalent.

18. *Part 1:* Let p represent “It walks like a duck,” q represent “It talks like a duck,” and r represent “It is a duck.” The statement “If it walks like a duck and it talks like a duck, then it is a duck” has the form $p \wedge q \rightarrow r$. And the statement “Either it does not walk like a duck or it does not talk like a duck or it is a duck” has the form $\sim p \vee \sim q \vee r$.

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$p \wedge q \rightarrow r$	$(\sim p \vee \sim q) \vee r$
T	T	T	F	F	T	F	T	T
T	T	F	F	F	T	F	F	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

same truth values

The truth table shows that $p \wedge q \rightarrow r$ and $(\sim p \vee \sim q) \vee r$ always have the same truth values. Thus the following statements are logically equivalent: “If it walks like a duck and it talks like a duck, then it is a duck” and “Either it does not walk like a duck or it does not talk like a duck or it is a duck.”

Part 2: The statement “If it does not walk like a duck and it does not talk like a duck then it is not a duck” has the form $\sim p \wedge \sim q \rightarrow \sim r$.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \wedge q$	$\sim p \wedge \sim q$	$p \wedge q \rightarrow r$	$(\sim p \wedge \sim q) \rightarrow \sim r$
T	T	T	F	F	F	T	F	T	T
T	T	F	F	F	T	T	F	F	T
T	F	T	F	T	F	F	F	T	T
T	F	F	F	T	T	F	F	T	T
F	T	T	T	F	F	F	F	T	T
F	T	F	T	F	T	F	F	T	T
F	F	T	T	T	F	F	T	T	F
F	F	F	T	T	T	F	T	T	T

different truth values

The truth table shows that $p \wedge q \rightarrow r$ and $(\sim p \wedge \sim q) \rightarrow \sim r$ do not always have the same truth values. (They differ for the combinations of truth values of p , q , and r shown in rows 2 and 7.) Thus they are not logically equivalent, and so the statement “If it walks like a duck and it talks like a duck, then it is a duck” is not logically equivalent to the statement “If it does not walk like a duck and it does not talk like a duck then it is not a duck.” In addition, because of the logical equivalence shown in Part 1, we can also conclude that the following two statements are not logically equivalent: “Either it does not walk like a duck or it does not talk like a duck or it is a duck” and “If it does not walk like a duck and it does not talk like a duck then it is not a duck.”

20. *b. Negation:* Today is New Year’s Eve and tomorrow is not January.

c. Negation: The decimal expansion of r is terminating and r is not rational.

e. Negation: x is nonnegative and x is not positive and x is not 0.

Or: x is nonnegative but x is not positive and x is not 0.

Or: x is nonnegative and x is neither positive nor 0.

g. Negation: n is divisible by 6 and either n is not divisible by 2 or n is not divisible by 3.

21. By the truth table for $\rightarrow$, $p \rightarrow q$ is false if, and only if, p is true and q is false. Under these circumstances, (b) $p \vee q$ is true and (c) $q \rightarrow p$ is also true.
22. b. *Contrapositive*: If tomorrow is not January, then today is not New Year's Eve.
 c. *Contrapositive*: If r is not rational, then the decimal expansion of r is not terminating.
 e. *Contrapositive*: If x is not positive and x is not 0, then x is not nonnegative.
Or: If x is neither positive nor 0, then x is negative.
 g. *Contrapositive*: If n is not divisible by 2 or n is not divisible by 3, then n is not divisible by 6.
23. b. *Converse*: If tomorrow is January, then today is New Year's Eve.
Inverse: If today is not New Year's Eve, then tomorrow is not January.
 c. *Converse*: If r is rational then the decimal expansion of r is terminating.
Inverse: If the decimal expansion of r is not terminating, then r is not rational.
 e. *Converse*: If x is positive or x is 0, then x is nonnegative.
Inverse: If x is not nonnegative, then both x is not positive and x is not 0.
Or: If x is negative, then x is neither positive nor 0.

25.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

←
←

different truth values

The truth table shows that $p \rightarrow q$ and $\sim p \rightarrow \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent. Thus a conditional statement is not logically equivalent to its inverse.

27.

p	q	$\sim p$	$\sim q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

same truth values

The truth table shows that $q \rightarrow p$ and $\sim p \rightarrow \sim q$ always have the same truth values, so they are logically equivalent. Thus the converse and inverse of a conditional statement are logically equivalent to each other.

28. The if-then form of "I say what I mean" is "If I mean something, then I say it."
 The if-then form of "I mean what I say" is "If I say something, then I mean it."
 Thus "I mean what I say" is the converse of "I say what I mean." The two statements are not logically equivalent.
30. The corresponding tautology is $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	F	T	T	T	T
T	F	F	F	F	F	F	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	F	F	F	T
F	F	T	T	F	F	F	F	T
F	F	F	F	F	F	F	F	T

all T 's

The truth table shows that $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$ is always true. Hence it is a tautology.

31. The corresponding tautology is $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$.

p	q	r	$q \rightarrow r$	$p \wedge q$	$p \rightarrow (q \rightarrow r)$	$(p \wedge q) \rightarrow r$	$p \rightarrow (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	T	F	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

all T 's

The truth table shows that $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$ is always true. Hence it is a tautology.

33. If this integer is even, then it equals twice some integer, and if this integer equals twice some integer, then it is even.
35. If Sam is not an expert sailor, then he will not be allowed on Signe's racing boat.
If Sam is allowed on Signe's racing boat, then he is an expert sailor.
36. The Personnel Director did not lie. By using the phrase "only if," the Personnel Director set forth conditions that were necessary but not sufficient for being hired: if you did not satisfy those conditions then you would not be hired. The Personnel Director's statement said nothing about what would happen if you did satisfy those conditions.
38. If it doesn't rain, then Ann will go.
39. *b.* If a security code is not entered, then the door will not open.
41. If this triangle has two 45° angles, then it is a right triangle.
43. If Jim does not do his homework regularly, then Jim will not pass the course.
If Jim passes the course, then he will have done his homework regularly.
45. If this computer program produces error messages during translation, then it is not correct.
If this computer program is correct, then it does not produce error messages during translation.
46. *c.* must be true *d.* not necessarily true *e.* must be true *f.* not necessarily true

Note: To solve this problem, it may be helpful to imagine a compound whose boiling point is greater than 150° C. For concreteness, suppose it is 200° C. Then the given statement would be true for this compound, but statements a , d , and f would be false.

Section 2.3

2. $1 - 0.99999\dots$ is less than every positive real number.

4. This figure is not a quadrilateral.

5. They did not telephone.

9.

<i>premises</i>						<i>conclusion</i>			
p	q	r	$\sim q$	$\sim r$	$p \wedge q$	$p \wedge q \rightarrow \sim r$	$p \vee \sim q$	$\sim q \rightarrow p$	$\sim r$
T	T	T	F	F	T	F	T	T	
T	T	F	F	T	T	T	T	T	$T \leftarrow$ critical row
T	F	T	T	F	F	T	T	T	$F \leftarrow$ critical row
T	F	F	T	T	F	T	T	T	$T \leftarrow$ critical row
F	T	T	F	F	F	T	F	T	
F	T	F	F	T	F	T	F	T	
F	F	T	T	F	F	T	T	F	
F	F	F	T	T	F	T	T	F	

Rows 2, 3, and 4 of the truth table are the critical rows in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

10.

<i>premises</i>				<i>conclusion</i>		
p	q	r	$p \vee \sim q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$
T	T	T	T	T	T	$T \leftarrow$ critical row
T	T	F	T	F	F	
T	F	T	T	T	T	$T \leftarrow$ critical row
T	F	F	T	F	T	
F	T	T	T	T	T	$T \leftarrow$ critical row
F	T	F	T	T	F	
F	F	T	F	T	T	$T \leftarrow$ critical row
F	F	F	F	T	T	$T \leftarrow$ critical row

Rows 1, 3, 5, 7, and 8 of the truth table represent the situations in which all the premises are true, and in all of these rows the conclusion is also true. Therefore, the argument form is valid.

11.

<i>premises</i>							<i>conclusion</i>		
p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \vee r$	$p \rightarrow q \vee r$	$\sim q \vee \sim r$	$\sim p \vee \sim r$
T	T	T	F	F	F	T	T	F	
T	T	F	F	F	T	T	T	T	$T \leftarrow$ critical row
T	F	T	F	T	F	T	T	T	$F \leftarrow$ critical row
T	F	F	F	T	T	F	F	T	
F	T	T	T	F	F	T	T	F	
F	T	F	T	F	T	T	T	T	$T \leftarrow$ critical row
F	F	T	T	T	F	T	T	T	$T \leftarrow$ critical row
F	F	F	T	T	T	F	T	T	$T \leftarrow$ critical row

Rows 2, 3, 6, 7, and 8 of the truth table represent the situations in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

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12. b. premises conclusion

p	q	$p \rightarrow q$		$\sim p$	$\sim q$
T	T	T	F		
T	F	F	F		
F	T	T	T	F	critical row
F	F	T	T	T	critical row

Rows 3, and 4 of the truth table represent the situations in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

13. premises conclusion

p	q	$p \rightarrow q$		$\sim q$	$\sim p$
T	T	T	F		
T	F	F	T		
F	T	T	F		
F	F	T	T	T	critical row

Row 4 of the truth table represents the only situation in which all the premises are true, and in this row the conclusion is also true. Therefore, the argument form (modus tollens) is valid.

15. premise conclusion

p	q	q	$p \vee q$
T	T	T	T
T	F	F	
F	T	T	T
F	F	F	

The truth table shows that in the two situations (represented by rows 1 and 3) in which the premise is true, the conclusion is also true. Therefore, the the second version of generalization is valid.

16. premise conclusion

p	q	$p \wedge q$	p
T	T	T	T
T	F	F	
F	T	F	
F	F	F	

The truth table shows that in the only situation (represented by row 1) in which both premises are true, the conclusion is also true. Therefore, the the first version of specialization is valid.

17. premise conclusion

p	q	$p \wedge q$	q
T	T	T	T
T	F	F	
F	T	F	
F	F	F	

The truth table shows that in the only situation (represented by row 1) in which both premises are true, the conclusion is also true. Therefore, the second version of specialization is valid.

19.

		<i>premises</i>		<i>conclusion</i>
p	q	$p \vee q$	$\sim p$	q
T	T	T	F	$T \leftarrow$ critical row
T	F	T	F	
F	T	T	T	
F	F	F	T	

The truth table shows that in the only situation (represented by row 3) in which both premises are true, the conclusion is also true. Therefore, the the second version of elimination is valid.

20.

		<i>premises</i>			<i>conclusion</i>
p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$
T	T	T	T	T	$T \leftarrow$ critical row
T	T	F	T	F	
T	F	T	F	T	
T	F	F	F	T	
F	T	T	T	T	$T \leftarrow$ critical row
F	T	F	T	F	
F	F	T	T	T	$T \leftarrow$ critical row
F	F	F	T	T	$T \leftarrow$ critical row

The truth table shows that in the four situations (represented by rows 1, 5, 7, and 8) in which both premises are true, the conclusion is also true. Therefore, the argument form (transitivity) is valid.

21.

		<i>premises</i>				<i>conclusion</i>
p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	r
T	T	T	T	T	T	$T \leftarrow$ critical row
T	T	F	T	F	F	
T	F	T	T	T	T	$T \leftarrow$ critical row
T	F	F	T	F	T	
F	T	T	T	T	T	$T \leftarrow$ critical row
F	T	F	T	T	F	
F	F	T	F	T	T	
F	F	F	F	T	T	

The truth table shows that in the three situations (represented by rows 1, 3, 5) in which all three premises are true, the conclusion is also true. Therefore, proof by division into cases is valid.

23. form: $p \vee q$
 $p \rightarrow r$
 $\therefore q \vee \sim r$

				<i>premises</i>		<i>conclusion</i>	
p	q	r	$\sim r$	$p \vee q$	$p \rightarrow r$	$q \vee \sim r$	
T	T	T	F	T	T	T	← <i>critical row</i>
T	T	F	T	T	F		
T	F	T	F	T	T	F	← <i>critical row</i>
T	F	F	T	T	F		
F	T	T	F	T	T	T	← <i>critical row</i>
F	T	F	T	T	T	T	← <i>critical row</i>
F	F	T	F	F	T		
F	F	F	T	F	T		

Rows 1, 3, 5, and 6 represent the situations in which both premises are true, but in row 3 the conclusion is false. Hence, it is possible for an argument of this form to have true premises and a false conclusion, and so the given argument is invalid.

28. form: $p \rightarrow q$ invalid, converse error
 q
 $\therefore p$

29. form: $p \rightarrow q$ invalid, inverse error
 $\sim p$
 $\therefore \sim q$

30. form: $p \rightarrow q$ invalid, converse error
 q
 $\therefore p$

31. form: $p \wedge q$ valid, generalization
 $\therefore q$

32. form: $p \rightarrow r$ valid, proof by division into cases
 $q \rightarrow r$
 $\therefore p \vee q \rightarrow r$

33. A valid argument with a false conclusion must have at least one false premise. In the following example, the second premise is false. (The first premise is true because its hypothesis is false.)

If the square of every real number is positive, then no real number is negative.

The square of every real number is positive.

Therefore, no real number is negative.

34. An invalid argument with a true conclusion can have premises that are either true or false. In the following example the first premise is true for either one of following two reasons: its hypothesis is false and its conclusion is true.

If the square of every real number is positive, then some real numbers are positive.

Some real numbers are positive.

Therefore, the square of every real number is positive.

35. A correct answer should indicate that for a valid argument, any argument of the same form that has true premises has a true conclusion, whereas for an invalid argument, it is possible to find an argument of the same form that has true premises and a false conclusion. The validity of an argument does not depend on whether the conclusion is true or not. The validity of an argument only depends on the formal relationship between its premises and its conclusion.
38. b. 1. Suppose C is a knight.
 2. $\therefore C$ is a knave (because what C said was true).
 3. $\therefore C$ is both a knight and a knave (by (1) and (2)), which is a contradiction.
 4. $\therefore C$ is not a knight (because by the contradiction rule the supposition is false).
 5. $\therefore$ What C says is false (because since C is not a knight he is a knave and knaves always speak falsely).
 6. $\therefore$ At least one of C or D is a knight (by De Morgan's law).
 7. $\therefore D$ is a knight (by (4) and (6) and elimination).
 8. $\therefore C$ is a knave and D is a knight (by (4) and (7)).
 To check that the problem situation is not inherently contradictory, note that if C is a knave and D is a knight, then each could have spoken as reported.
- c. There is one knave. E and F cannot both be knights because then both would also be knaves (since each would have spoken the truth), which is a contradiction. Nor can E and F both be knaves because then both would be telling the truth which is impossible for knaves. Hence, the only possible answer is that one is a knight and the other is a knave. But in this case both E and F could have spoken as reported, without contradiction.
- d. The following is one of many solutions.
1. The statement made by U must be false because if it were true then U would not be a knight (since none would be a knight), but since he spoke the truth he would be a knight and this would be a contradiction.
 2. $\therefore$ there is at least one knight, and U is a knave (since his statement that there are no knights is false).
 3. Suppose Z spoke the truth. Then so did W (since if there is exactly one knight then it is also true that there are at most three knights). But this implies that there are at least two knights, which contradicts Z 's statement. Hence Z cannot have spoken the truth.
 4. $\therefore$ there are at least two knights, and Z is a knave (since his statement that there is exactly one knight is false). Also X 's statement is false because since both U and Z are knaves it is impossible for there to be exactly five knights. Hence X also is a knave.
 5. $\therefore$ there are at least three knaves (U , Z , and X), and so there are at most three knights.
 6. $\therefore W$'s statement is true, and so W is a knight.
 7. Suppose V spoke the truth. Then V , W , and Y are all knights (otherwise there would not be at least three knights because U , Z , and X are known to be knaves). It follows that Y spoke the truth. But Y said that exactly two were knights. This contradicts the result that V , W , and Y are all knights.
 8. $\therefore V$ cannot have spoken the truth, and so V is a knave.
 9. $\therefore U$, Z , X , and V are all knaves, and so there are at most two knights.
 10. Suppose that Y is a knave. Then the only knight is W , which means that Z spoke the truth. But we have already seen that this is impossible. Hence Y is a knight.
 11. By 6, 9, and 10, the only possible solution is that U , Z , X , and V are knaves and W and Y are knights. Examination of the statements shows that this solution is consistent: in this case, the statements of U , Z , X , and V are false and those of W and Y are true.
40. Suppose Socko is telling the truth. Then Fats is also telling the truth because if Lefty killed Sharky then Muscles didn't kill Sharky. Consequently, two of the men were telling the truth, which contradicts the fact that all were lying except one. Therefore, Socko is not telling the truth: Lefty did not kill Sharky. Hence Muscles is telling the truth and all the others are lying. It follows that Fats is lying, and so Muscles killed Sharky.

16 Solutions for Exercises: The Logic of Compound Statements

42. (1) $q \rightarrow r$ premise b
 $\sim r$ premise d
 $\therefore \sim q$ by modus tollens
- (2) $p \vee q$ premise a
 $\sim q$ by (1)
 $\therefore p$ by elimination
- (3) $\sim q \rightarrow u \wedge s$ premise e
 $\sim q$ by (1)
 $\therefore u \wedge s$ by modus ponens
- (4) $u \wedge s$ by (3)
 $\therefore s$ by specialization
- (5) p by (2)
 s by (4)
 $\therefore p \wedge s$ by conjunction
- (6) $p \wedge s \rightarrow t$ premise c
 $p \wedge s$ by (5)
 $\therefore t$ by modus ponens
44. (1) $\sim q \vee s$ premise d
 $\sim s$ premise e
 $\therefore \sim q$ by elimination
- (2) $p \rightarrow q$ premise a
 $\sim q$ by (1)
 $\therefore \sim p$ by modus tollens
- (3) $r \vee s$ premise b
 $\sim s$ premise e
 $\therefore r$ by elimination
- (4) $\sim p$ by (2)
 r by (3)
 $\therefore \sim p \wedge r$ by conjunction
- (5) $\sim p \wedge r \rightarrow u$ premise f
 $\sim p \wedge r$ by (4)
 $\therefore u$ by modus ponens
- (6) $\sim s \rightarrow \sim t$ premise c
 $\sim s$ premise e
 $\therefore \sim t$ by modus ponens

(7) $w \vee t$ premise g
 $\sim t$ by (6)
 $\therefore w$ by elimination

(8) u by (5)
 w by (7)
 $\therefore u \wedge w$ by conjunction

Chapter 3: The Logic of Quantified Statements

Ability to use the logic of quantified statements correctly is necessary for doing mathematics because mathematics is, in a very broad sense, about quantity. The main purpose of this chapter is to familiarize students with the language of universal and existential statements. The various facts about quantified statements developed in this chapter are used extensively in Chapter 4 and are referred to throughout the rest of the book. Experience with the formalism of quantification is especially useful to students planning to study LISP or Prolog, program verification, or relational databases.

Many students come to college with inconsistent interpretations of quantified statements. In tests made at DePaul University, over 60% of students chose the statement “No fire trucks are red” as the negation of “All fire trucks are red.” Yet, through guided discussion, these same students came fairly quickly to accept that “Some fire trucks are not red” conveys the negation more accurately, and most learned to take negations of general statements of the form “ $\forall x$, if $P(x)$ then $Q(x)$,” “ $\forall x$, $\exists y$ such that $P(x, y)$,” and so forth with reliable accuracy.

One thing to keep in mind is the tolerance for potential ambiguity in ordinary language, which is typically resolved through context or inflection. For instance, as the “Caution” on page 111 of the text indicates, the sentence “All mathematicians do not wear glasses” is one way to phrase a negation to “All mathematicians wear glasses.” (To see this, say it out loud, stressing the word “not.”) Some grammarians ask us to avoid such phrasing because of its potentially ambiguity, but the usage is widespread even in formal writing in high-level publications (“All juvenile offenders are not alike,” Anthony Lewis, *The New York Times*, 19 May 1997, Op-Ed page) and in literary works (“All that glitters is not gold,” William Shakespeare, *The Merchant of Venice*, Act 2, Scene 7, 1596-1597).

Even rather complex sentences can be negated in this way. For instance, when asked to write a negation for “The sum of any two irrational numbers is irrational,” a student wrote “The sum of any two irrational numbers is not irrational,” which is an acceptable informal negation (again, say it out loud, stressing the word “not”). To avoid such responses, it may be necessary to specify to students that simply inserting the word “not” is not an acceptable answer to a problem that asks for a negation.

The modified formal language of the text includes the words “such that” in statements containing an existential quantifier because when students write multiply-quantified statements “formally,” they often insert the words “such that” in the wrong place. That is, they insert it in a place that changes the meaning of the statement they were given. If they were not required to include the words “such that,” an opportunity to correct their misunderstanding would be missed. It can also be helpful to alternate between writing out the words “if-then” (to encourage students to take the word “if” more seriously than they may be inclined to do¹) and using an arrow to denote the conditional (to communicate the dynamic nature of deductive reasoning).

The discussion about the geometry of the real numbers begun in Section 1.2 (page 8) is continued in this chapter. Examples 3.3.5 and 3.3.6 on page 121 show that while there is a smallest positive integer there is no smallest positive real number. The topic is further explored in Section 7.4, where a proof of the uncountability of the real numbers between 0 and 1 is given.

Suggestions

1. The exercises in Sections 3.1–3.3 are designed to try to imprint new language patterns on students’ minds. Because it takes time to develop new habits, it is helpful to continue assigning exercises from these sections for several days after covering them in class. To prepare for Chapter 4, universal

¹Many students (and other people) mistakenly interpret *if-then* statements as *and* statements. For instance, when students are asked to state what it means for a relation R to be symmetric, a significant fraction write “ aRb and bRa .”

2 Solutions for Exercises: The Logic of Quantified Statements

conditional statements should especially be emphasized. As in Section 2.2, care may need to be taken not to spend excessive class time going over the more difficult exercises, such as those on “necessary” and “sufficient” conditions.

2. The most important idea of Section 3.4 is also the simplest: the rule of universal instantiation. Yet this inference rule drives an enormous amount of mathematical reasoning. If you wish to move rapidly through Chapter 3, you could focus on this rule and its immediate consequences in Section 3.4 and omit the discussion of how to use diagrams to check validity of arguments.

Section 3.1

1. *c.* False *d.* True *e.* False *f.* True

4. *a.* $Q(2)$: $2^2 \leq 30$ - true because $2^2 = 4$ and $4 \leq 30$,

$Q(-2)$: $(-2)^2 \leq 30$ - true because $(-2)^2 = 4$ and $4 \leq 30$

$Q(7)$: $7^2 \leq 30$ - false because $7^2 = 49$ and $49 \not\leq 30$,

$Q(-7)$: $(-7)^2 \leq 30$ - false because $(-7)^2 = 49$ and $49 \not\leq 30$

c. truth set = $\{n \in \mathbf{Z}^+ \mid n^2 \leq 30\} = \{1, 2, 3, 4, 5\}$

5. *b.* Let $x = -1$ and $y = 0$. Then $x < y$ because $-1 < 0$ but $x^2 < y^2$ because $(-1)^2 = 1 \not< 0^2 = 0$. Thus the hypothesis $x < y$ is true and the conclusion $x^2 < y^2$ is false, so the statement as a whole is false.

d. Here are examples of three kinds of correct answers:

(1) Let $x = 2$ and $y = 3$. Then $x < y$ because $2 < 3$ and $x^2 < y^2$ because $2^2 = 4 < 3^2 = 9$. Thus both the hypothesis and the conclusion are true, so the statement as a whole is true.

(2) Let $x = 3$ and $y = 2$. Then $x \not< y$ because $3 \not< 2$ and $x^2 \not< y^2$ because $3^2 = 9 \not< 2^2 = 4$. Thus both the hypothesis and the conclusion are false, so the statement as a whole is true.

(3) Let $x = 2$ and $y = -3$. Then $x \not< y$ because $2 \not< -3$ and $x^2 < y^2$ because $2^2 = 4 < (-3)^2 = 9$. Thus the hypothesis is false and the conclusion is true, so the statement as a whole is true.

6. *a.* When $m = 25$ and $n = 10$, the statement “ m is a factor of n^2 ” is true because $n^2 = 100$ and $100 = 4 \cdot 25$. But the statement “ m is a factor of n ” is false because 10 is not a product of 25 times any integer. Thus the hypothesis is true and the conclusion is false, so the statement as a whole is false.

b. $R(m, n)$ is also false when $m = 8$ and $n = 4$ because 8 is a factor of $4^2 = 16$, but 8 is not a factor of 4.

c. When $m = 5$ and $n = 10$, both statements “ m is a factor of n^2 ” and “ m is a factor of n ” are true because $n = 10 = 5 \cdot 2 = m \cdot 2$ and $n^2 = 100 = 5 \cdot 20 = m \cdot 20$. Thus both the hypothesis and conclusion of $R(m, n)$ are true, and so the statement as a whole is true.

d. Here are examples of two kinds of correct answers:

(1) Let $m = 2$ and $n = 6$. Then both statements “ m is a factor of n^2 ” and “ m is a factor of n ” are true because $n = 6 = 2 \cdot 3 = m \cdot 3$ and $n^2 = 36 = 2 \cdot 18 = m \cdot 18$. Thus both the hypothesis and conclusion of $R(m, n)$ are true, and so the statement as a whole is true.

(2) Let $m = 6$ and $n = 2$. Then both statements “ m is a factor of n^2 ” and “ m is a factor of n ” are false because $n = 2 \neq 6 \cdot k$, for any integer k , and $n^2 = 4 \neq 6 \cdot j$, for any integer j . Thus both the hypothesis and conclusion of $R(m, n)$ are false, and so the statement as a whole is true.

7. *b.* Truth set = $\{1, 2, 3, 6\}$

d. Truth set = $\{-2, -1, 1, 2\}$

8. *b.* Truth set = $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
d. Truth set = $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$
10. Counterexample 1: Let $a = 1$, and note that $(a - 1)/a = (1 - 1)/1 = 0$ is an integer.
Counterexample 2: Let $a = -1$, and note that $(a - 1)/a = (-1 - 1)/(-1) = 2$ is an integer.
12. Counterexample: Let $x = 1$ and $y = 1$, and note that

$$\sqrt{x + y} = \sqrt{1 + 1} = \sqrt{2}$$

whereas

$$\sqrt{x} + \sqrt{y} = \sqrt{1} + \sqrt{1} = 1 + 1 = 2,$$

and

$$2 \neq \sqrt{2}.$$

(This is one counterexample among many. Any real numbers x and y with $xy \neq 0$ will produce a counterexample.)

15. *a. Some acceptable answers*: All rectangles are quadrilaterals. If a figure is a rectangle then that figure is a quadrilateral. Every rectangle is a quadrilateral. All figures that are rectangles are quadrilaterals. Any figure that is a rectangle is a quadrilateral.
b. Some acceptable answers: There is a set with sixteen subsets. Some set has sixteen subsets. Some sets have sixteen subsets. There is at least one set that has sixteen subsets.
16. *b.* $\forall$ real numbers x , x is positive, negative, or zero.
d. $\forall$ logicians x , x is not lazy.
f. $\forall$ real numbers x , $x^2 \neq -1$.
17. *b.* $\exists$ a real number x such that x is rational.
18. *c.* $\forall s$, if $C(s)$ then $\sim E(s)$.
d. $\exists x$ such that $C(s) \wedge M(s)$.
20. *Some acceptable answers*: If a real number is positive, then its square root is positive. All positive real numbers have positive square roots. Every real number that is positive has a positive square root. Each real number that is positive has a positive square root. Given any positive real number, that number has a positive square root.
21. *b.* The base angles of T are equal, for any isosceles triangle T .
d. f is not differentiable, for some continuous function f .
22. *b.* $\forall x$, if x is a valid argument with true premises, then x has a true conclusion.
Or: $\forall$ arguments x , if x is valid and x has true premises then x has a true conclusion.
Or: $\forall$ valid arguments x , if x has true premises then x has a true conclusion.
23. *b.* $\forall x$, if x is a computer science student then x needs to take data structures.
 $\forall$ computer science students x , x needs to take data structures.
24. *b.* $\exists$ a question x such that x is easy.
 $\exists x$ such that x is a question and x is easy.

25. *b.* $\forall$ polynomial functions x , the derivative of x is a polynomial function.
 $\forall x$, if x is a polynomial function, then the derivative of x is a polynomial function.
d. $\forall$ triangles x , the sum of the angles of x is 180° .
 $\forall x$, if x is a triangle then the sum of the angles of x is 180°
f. $\forall$ fractions x and y , the product of x and y is a fraction.
 $\forall x$ and y , if x and y are fractions then the product of x and y is a fraction.
26. *a.* $\forall x$, if x is an integer then x is a rational number, but $\exists x$ such that x is a rational number and x is not an integer.
27. *c.* This statement translates as “There is a square that is above d .” This is false because the only objects above d are a (a triangle) and b (a circle).
d. This statement translates as “There is a triangle that has f above it,” or, “ f is above some triangle.” This is true because g is a triangle and f is above g .
28. *a* This statement translates as “0 is a positive real number.” This is false: 0 is neither positive nor negative. (See also the order axiom Ord3 on page A-2 of Appendix A: Properties of the Real Numbers.)
c. This statement translates as “All integers are real numbers.” This is true; each integer corresponds to a position along the number line.
29. *a.* This statement translates as “There is a geometric figure that is both a rectangle and a square.” This is true. As an example take any square; it is a rectangle whose sides all have the same length.
b. This statement translates as “There is a geometric figure that is a rectangle but is not a square.” This is true. Any rectangle whose sides are not all of the same length is a rectangle that is not a square. For example, one pair of parallel sides could be twice as long as the other pair of parallel sides.
c. This statement translates as “Every square is a rectangle.” This is true. A square is a rectangle satisfying the additional condition that all its sides have the same length.
30. *a.* This statement translates as “There is a prime number that is not odd.” This is true. The number 2 is prime and it is not odd.
c. This statement translates as “There is a number that is both an odd number and a perfect square.” This is true. For example, the number 9 is odd and it is also a perfect square (because $9 = 3^2$).
32. *b.* This statement translates as “For all real numbers x , if $x > 2$ then $x^2 > 4$,” which is true.
d. This statement translates as “For all real numbers x , $x^2 > 4$ if, and only if, $|x| > 2$.” This is true because $x^2 > 4$ if, and only if, $x > 2$ or $x < -2$, and $|x| > 2$ means that either $x > 2$ or $x < -2$.
33. *c.* This statement translates as “For all real numbers a and b , if $ab = 0$ then $a = 0$ or $b = 0$,” which is true.
d. This statement translates as “For all real numbers a , b , c , and d , if $a < b$ and $c < d$ then $ac < bd$,” which is false.
Counterexample: Let $a = -2$, $b = 1$, $c = -3$, and $d = 0$. Then $a < b$ because $-2 < 1$ and $c < d$ because $-3 < 0$, but $ac \not< bd$ because $ac = (-2)(-3) = 6$ and $bd = 1 \cdot 0 = 0$ and $6 \not< 0$.

Section 3.2

2. Statements c and f are negations for the given statement.
3. b . $\exists$ a computer C such that C does not have a CPU.
 d . $\forall$ bands b , b has won fewer than 10 Grammy awards.
4. b . Some people are unhappy.
 d . All estimates are inaccurate. *Or*: No estimates are accurate.
5. b . *Formal negation*: $\exists$ a real number x such that x is not positive and x is not negative and x is not zero.
Some acceptable informal negations: There is a real number that is not positive, negative, or zero. There is a real number that is neither positive, negative, nor zero.
6. b . *Formal negation*: $\exists$ a road r on the map such that r connects towns P and Q .
Some acceptable informal negations: There is a road on the map that connects towns P and Q . Some road on the map connects towns P and Q . Towns P and Q are connected by a road on the map.
8. *Informal negation of the statement*: “Some of life’s problems have simple solutions,” or “At least one of life’s problems has a simple solution.”
Formal version of the statement: “ $\forall$ of life’s problems x , x does not have a simple solution” or “ $\forall x$, if x is one of life’s problems, then x does not have a simple solution.”
Informal version of the statement: “None of life’s problems has a simple solution.”
10. $\exists$ a computer program P such that P compiles without error messages but P is not correct.
12. The proposed negation is not correct. *Correct negation*: There are an irrational number x and a rational number y such that xy is rational. *Or*: There are an irrational number and a rational number whose product is rational.
14. The proposed negation is not correct. There are two mistakes: The negation of a “for all” statement is not a “for all” statement, and the negation of an “if-then” statement is not an “if-then” statement. *Correct negation*: $\exists$ real numbers x_1 and x_2 such that $x_1^2 = x_2^2$ and $x_1 \neq x_2$.
15. b . True d . True
 e . False: $x = 36$ is a counterexample because the ones digit of x is 6 and the tens digit is neither 1 nor 2.
17. $\exists$ an integer d such that $6/d$ is an integer and $d \neq 3$.
19. $\exists n \in \mathbf{Z}$ such that n is prime and n is not odd and $n \neq 2$.
Or: $\exists$ an integer n such that n is prime and both n is not odd and $n \neq 2$.
Or: $\exists$ an integer n such that n is prime and n is neither odd nor equal to 2.
21. $\exists$ an integer n such that n is divisible by 6 and n is not divisible by 2 and n is not divisible by 3.
23. There is a function that is differentiable but not continuous.
Or: $\exists$ a function g such that g is differentiable and g is not continuous.

24. *b.* If an integer greater than 5 ends in 1, 3, 7, or 9, then the integer is prime.

If an integer greater than 5 is prime, then the integer ends in 1, 3, 7, or 9.

The second statement is the converse of the first.

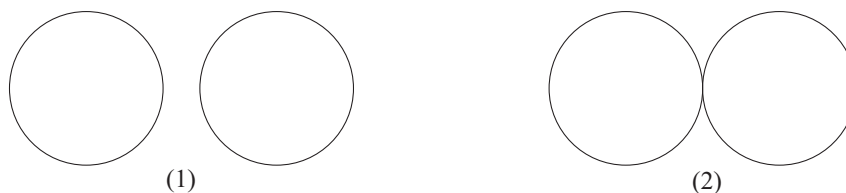
25. *b. Converse:* If $2m$ is even, then m is an odd integer.

Counterexample: Let $m = 2$. Then $2m = 4$, which is even, but m is not even.

- c. Converse:* If two circles do not have a common center, then they intersect in exactly two points.

Counterexample 1: The two circles in diagram (1) do not have a common center and they do not intersect in exactly two points. In fact, they do not intersect at all.

Counterexample 2: The two circles in diagram (2) illustrate the situation of two circles that do not have a common center and do not intersect in exactly two points because they are tangent to each other, that is, they intersect in exactly one point.



27. *Converse:* $\forall$ integers d , if $d = 3$ then $6/d$ is an integer.

Inverse: $\forall$ integers d , if $6/d$ is not an integer, then $d \neq 3$.

Contrapositive: $\forall$ integers d , if $d \neq 3$ then $6/d$ is not an integer.

The converse and inverse of the statement are both true, but both the statement and its contrapositive are false. For example, when $d = 2$, then $d \neq 3$ but $6/d = 3$ is an integer.

29. *Converse:* $\forall n \in \mathbf{Z}$, if n is odd or $n = 2$, then n is prime.

Inverse: $\forall n \in \mathbf{Z}$, if n is not prime, then both n is not odd and $n \neq 2$.

Or: $\forall n \in \mathbf{Z}$, if n is not prime, then neither is n odd nor is n equal to 2.

Contrapositive: $\forall n \in \mathbf{Z}$, if n is not odd and $n \neq 2$ then n is not prime.

The statement and its contrapositive are both true, but both the converse and the inverse are false. For example, when $n = 2$, then n is prime.

31. *Converse:* $\forall$ integers n , if n is divisible by 2 and n is divisible by 3, then n is divisible by 6.

Inverse: $\forall$ integers d , if n is not divisible by 6., then n is not divisible by 2 or n is not divisible by 3.

Contrapositive: $\forall$ integers n , if n is not divisible by 2 or n is not divisible by 3, then n is not divisible by 6.

The statement, its contrapositive, the converse, and the inverse are all true.

33. *Converse:* If a function is continuous, then it is differentiable.

Inverse: If a function is not differentiable, then it is not continuous.

Contrapositive: If a function is not continuous, then it is not differentiable.

The statement and its contrapositive are true, but both the converse and inverse are false. For example, take the function f defined by $f(x) = |x|$ for all real numbers x . This function is continuous for all real numbers, but it is not differentiable at $x = 0$.

34. *b.* If A and B are not disjoint, then they have at least one element in common.
Or: If A and B are not disjoint, then there is some element that they have in common.
35. *Possible example 1:* Consider the statement: $\forall$ real numbers x , if $x > 0$ then $x^2 > 0$. This statement is true. But its inverse is “ $\forall$ real numbers x , if $x \not> 0$ then $x^2 \not> 0$,” which is false. (One counterexample is $x = -1$ because $-1 \not> 0$ but $(-1)^2 > 0$.)
Possible example 2: Consider the statement: $\forall$ integers n that are greater than 2, if n is prime, then n is odd. This statement is true. But its inverse is “ $\forall$ integers n that are greater than 2, if n is not prime, then n is not odd, which is false. (One counterexample is $x = 15$ because 15 is not prime but it is odd.)
36. *b. One possible answer:* Let $P(x)$ be “ $x^2 \neq 2$.” The statements “ $\forall x \in \mathbf{Z}, x^2 \neq 2$ ” and “ $\forall x \in \mathbf{Q}, x^2 \neq 2$ ” are true, but the statement “ $\forall x \in \mathbf{R}, x^2 \neq 2$ ” is false.
38. The given statement cannot be false because its negation is “There exists an occurrence of the letter u in *Discrete Mathematics* that is not lowercase,” which is not true because there are no occurrences of the letter u in *Discrete Mathematics*. Hence the given statement is true because it is not false. Recall that in a situation such as this we call the statement “true by default” or “vacuously true.”
40. If an integer is divisible by 8, then it is divisible by 4.
42. If a person does not pass a comprehensive exam, then that person cannot obtain a master’s degree. *Or:* If a person obtains a master’s degree then that person passed a comprehensive exam.
44. There is a person who does not have a large income and is happy.
46. There is a function that is a polynomial but does not have a real root.
47. *Formal Versions:* $\forall$ computer programs P , if P is correct then P translates without error messages. However, $\exists$ a computer program P such that P translates without error messages and P is not correct.
Informal Versions: If a computer program is correct, then it translates without error messages. But some incorrect computer programs also do not produce error messages during translation.

Section 3.3

1. *c.* True: Paris is the capital of France.
d. False: Miami is not the capital of Brazil.
2. *c.* False: $\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not> \frac{1}{2}$ *d.* True: $(-2)^2 = 4 > 2$
3. *c.* Let $y = \frac{4}{3}$. Then $xy = \left(\frac{3}{4}\right)\left(\frac{4}{3}\right) = 1$.
4. *b. One possible answer:* Let $n = 10^8 + 1$ *c. One possible answer:* Let $n = 10^{10^{10}} + 1$.
6. True.

Given $x =$	Choose $y =$	Is y a circle above x , with a different color from x ?
e	a, b , or c	yes ✓
g	a or c	yes ✓
h	a or c	yes ✓
j	b	yes ✓

8 Solutions for Exercises: The Logic of Quantified Statements

8. True. Let $x = f$ or $x = i$. The statement “ $\forall$ circles y , y is above x ” is true for either choice of x because all the circles are above both of these triangles.

9. b. True. *Solution 1:* Let $x = 0$. Then for any real number r , $x + r = r + x = r$ because 0 is an identity for addition of real numbers. Thus, because every element in E is a real number, $\forall y \in E, x + y = y$.

Solution 2: Let $x = 0$. Then $x + y = y$ is true for each individual element y of E :

Choose $x = 0$	Given $y =$	Is $x + y = y$?
	-2	yes: $0 + (-2) = -2$ ✓
	-1	yes: $0 + (-1) = -1$ ✓
	0	yes: $0 + 0 = 0$ ✓
	1	yes: $0 + 1 = 1$ ✓
	2	yes: $0 + 2 = 2$ ✓

10. b. This statement says that every student chose a salad. This is false: Yuen did not choose a salad.

d. This statement says that some particular beverage was chosen by every student. This is false: There is no beverage that was chosen by every student.

e. This statement says that some particular item was not chosen by any student. This is false: every item was chosen by at least one student.

f. This statement says that there is a station from which every student made a selection. This is true. In fact, there are three such stations: every student chose a main course, every student chose a dessert, and every student chose a beverage.

11. b. Every student has seen Star Wars.

e. There are two different students who have both seen the same movie.

f. There are two different students, one of whom has seen all the movies that the other has seen.

12. c. *first version of negation:* $\exists x$ in D such that $\sim (\exists y$ in E such that $xy \geq y)$.

final version of negation: $\exists x$ in D such that $\forall y$ in E , $xy \not\geq y$. (Or: $\exists x$ in D such that $\forall y$ in E , $xy < y$.)

The statement is true. For each number x in D , you can find a y in D so that $xy \geq y$. Here is a table showing one way to do this: how all possible choices for x could be matched with a y so that $xy \geq y$:

Given $x =$	you could take $y =$	Is $xy \geq y$?
-2	-2	$(-2) \cdot (-2) = 4 \geq -2$ ✓
-1	0	$(-1) \cdot 0 = 0 \geq -1$ ✓
0	1	$0 \cdot 1 = 0 \geq 0$ ✓
1	1	$1 \cdot 1 = 1 \geq 1$ ✓
2	2	$2 \cdot 2 = 4 \geq 2$ ✓

d. *first version of negation:* $\forall x$ in D , $\sim (\forall y$ in E , $x \leq y)$.

final version of negation: $\forall x$ in D , $\exists y$ in E such that $x \not\leq y$. (Or: $\forall x$ in D , $\exists y$ in E such that $x > y$.)

The statement is true. It says that there is a number in D that is less than or equal to every number in D . In fact, -2 is in D and -2 is less than or equal to every number in D (-2, -1, 0, 1, and 2).

In 16, 17, and 19 there are other correct answers besides those shown.