STUDENT SOLUTIONS MANUAL

TO ACCOMPANY

Elementary Linear Algebra with Applications

NINTH EDITION

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EXERCISE SET 1.1

1. **(b)** Not linear because of the term x_1x_3.

(d) Not linear because of the term x_1^{-2}.

(e) Not linear because of the term $x_1^{3/5}$.

7. Since each of the three given points must satisfy the equation of the curve, we have the system of equations

$$ax_1^2 + bx_1 + c = y_1$$

$$ax_2^2 + bx_2 + c = y_2$$

$$ax_3^2 + bx_3 + c = y_3$$

If we consider this to be a system of equations in the three unknowns a, b, and c, the augmented matrix is clearly the one given in the exercise.

9. The solutions of $x_1 + kx_2 = c$ are $x_1 = c - kt$, $x_2 = t$ where t is any real number. If these satisfy $x_1 + \ell x_2 = d$, then $c - kt + \ell t = d$, or $(\ell - k)t = d - c$ for all real numbers t. In particular, if $t = 0$, then $d = c$, and if $t = 1$, then $\ell = k$.

11. If $x - y = 3$, then $2x - 2y = 6$. Therefore, the equations are consistent if and only if $k = 6$; that is, there are no solutions if $k \neq 6$. If $k = 6$, then the equations represent the same line, in which case, there are infinitely many solutions. Since this covers all of the possibilities, there is never a unique solution.

EXERCISE SET 1.2

1. **(e)** Not in reduced row-echelon form because Property 2 is not satisfied.

(f) Not in reduced row-echelon form because Property 3 is not satisfied.

(g) Not in reduced row-echelon form because Property 4 is not satisfied.

5. **(a)** The solution is

$$x_3 = 5$$

$$x_2 = 2 - 2\,x_3 = -8$$

$$x_1 = 7 - 4\,x_3 + 3x_2 = -37$$

(b) Let $x_4 = t$. Then $x_3 = 2 - \mathrm{t}$. Therefore

$$x_2 = 3 + 9t - 4x_3 = 3 + 9t - 4(2 - t) = -5 + 13t$$

$$x_1 = 6 + 5t - 8x_3 = 6 + 5t - 8(2 - t) = -10 + 13t$$

7. **(a)** In Problem 6(a), we reduced the augmented matrix to the following row-echelon matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

By Row 3, $x_3 = 2$. Thus by Row 2, $x_2 = 5x_3 - 9 = 1$. Finally, Row 1 implies that $x_1 = -x_2 - 2\,x_3 + 8 = 3$. Hence the solution is

$$x_1 = 3$$

$$x_2 = 1$$

$$x_3 = 2$$

(c) According to the solution to Problem 6(c), one row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row 2 implies that $y = 2z$. Thus if we let $z = s$, we have $y = 2s$. Row 1 implies that $x = -1 + y - 2z + w$. Thus if we let $w = t$, then $x = -1 + 2s - 2s + t$ or $x = -1 + t$. Hence the solution is

$$x = -1 + t$$

$$y = 2s$$

$$z = s$$

$$w = t$$

9. (a) In Problem 8(a), we reduced the augmented matrix of this system to row-echelon form, obtaining the matrix

$$\begin{bmatrix} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Row 3 again yields the equation $0 = 1$ and hence the system is inconsistent.

(c) In Problem 8(c), we found that one row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again if we let $x_2 = t$, then $x_1 = 3 + 2x_2 = 3 + 2t$.

11. **(a)** From Problem 10(a), a row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & -2/5 & 6/5 & 0 \\ 0 & 1 & 27 & 5 \end{bmatrix}$$

If we let $x_3 = t$, then Row 2 implies that $x_2 = 5 - 27t$. Row 1 then implies that $x_1 = (-6/5)x_3 + (2/5)x_2 = 2 - 12t$. Hence the solution is

$$x_1 = 2 - 12t$$

$$x_2 = 5 - 27t$$

$$x_3 = t$$

(c) From Problem 10(c), a row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1/2 & 7/2 & 0 & 7/2 \\ 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we let $y = t$, then Row 3 implies that $x = 3 + t$. Row 2 then implies that

$$w = 4 - 2x + t = -2 - t.$$

Now let $v = s$. By Row 1, $u = 7/2 - 2s - (1/2)w - (7/2)x = -6 - 2s - 3t$. Thus we have the same solution which we obtained in Problem 10(c).

13. **(b)** The augmented matrix of the homogeneous system is

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$$

This matrix may be reduced to

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 0 & 4 & 1 & 4 & 0 \end{bmatrix}$$

If we let $x_3 = 4s$ and $x_4 = t$, then Row 2 implies that

$$4x_2 = -4t - 4s \quad \text{or} \quad x_2 = -t - s$$

Now Row 1 implies that

$$3x_1 = -x_2 - 4s - t = t + s - 4s - t = -3s \quad \text{or} \quad x_1 = -s$$

Therefore the solution is

$$x_1 = -s$$

$$x_2 = -(t + s)$$

$$x_3 = 4s$$

$$x_4 = t$$

15. **(a)** The augmented matrix of this system is

$$\begin{bmatrix} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix}$$

Its reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence the solution is

$$I_1 = -1$$

$$I_2 = 0$$

$$I_3 = 1$$

$$I_4 = 2$$

(b) The reduced row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If we let $Z_2 = s$ and $Z_5 = t$, then we obtain the solution

$$Z_1 = -s - t$$

$$Z_2 = s$$

$$Z_3 = -t$$

$$Z_4 = 0$$

$$Z_5 = t$$

17. The Gauss-Jordan process will reduce this system to the equations

$$x + 2y - 3z = 4$$

$$y - 2z = 10/7$$

$$(a^2 - 16)z = a - 4$$

If $a = 4$, then the last equation becomes $0 = 0$, and hence there will be infinitely many solutions—for instance, $z = t$, $y = 2\,t + \frac{10}{7}$, $x = -2\,(2t + \frac{10}{7}) + 3t + 4$. If $a = -4$, then the last equation becomes $0 = -8$, and so the system will have no solutions. Any other value of a will yield a unique solution for z and hence also for y and x.

19. One possibility is

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Another possibility is

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \to \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 7/2 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 7/2 \\ 0 & 1 \end{bmatrix}$$

21. If we treat the given system as linear in the variables sin α, cos β, and tan γ, then the augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix}$$

This reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

so that the solution (for α, β, γ between 0 and 2 π) is

$$\sin \alpha = 0 \Rightarrow \alpha = 0, \pi, 2\pi$$

$$\cos \beta = 0 \Rightarrow \beta = \pi/2, 3\pi/2$$

$$\tan \gamma = 0 \Rightarrow \gamma = 0, \pi, 2\pi$$

That is, there are 3•2•3 = 18 possible triples α, β, γ which satisfy the system of equations.

23. If $\lambda = 2$, the system becomes

$$\begin{aligned} -x_2 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \\ -2x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

Thus $x_2 = 0$ and the third equation becomes –1 times the second. If we let $x_1 = t$, then $x_3 = -2t$.

25. Using the given points, we obtain the equations

$$\begin{aligned} d &= 10 \\ a + b + c + d &= 7 \\ 27a + 9b + 3c + d &= -11 \\ 64a + 16b + 4c + d &= -14 \end{aligned}$$

If we solve this system, we find that $a = 1$, $b = -6$, $c = 2$, and $d = 10$.

27. **(a)** If $a = 0$, then the reduction can be accomplished as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $a = 0$, then $b \neq 0$ and $c \neq 0$, so the reduction can be carried out as follows:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \to \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Where did you use the fact that $ad - bc \neq 0$? (This proof uses it twice.)

29. There are eight possibilities. They are

(a)

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&p\\0&1&q\\0&0&0\end{pmatrix},\ \begin{pmatrix}1&p&0\\0&0&1\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\ \begin{pmatrix}1&p&q\\0&0&0\\0&0&0\end{pmatrix},\ \begin{pmatrix}0&1&p\\0&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\ \text{where } p, q, \text{are any real numbers,}$$

and

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

(b)

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0&p\\0&1&0&q\\0&0&1&r\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}1&0&p&0\\0&1&q&0\\0&0&0&1\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&p&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}1&0&p&q\\0&1&r&s\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&p&0&q\\0&0&1&r\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}1&p&q&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&1&0&p\\0&0&1&q\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&p&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}1&p&q&r\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&p&q\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&1&p\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \text{and}\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

31. **(a)** False. The reduced row-echelon form of a matrix is unique, as stated in the remark in this section.

(b) True. The row-echelon form of a matrix is not unique, as shown in the following example:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(c) False. If the reduced row-echelon form of the augmented matrix for a system of 3 equations in 2 unknowns is

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

then the system has a unique solution. If the augmented matrix of a system of 3 equations in 3 unknowns reduces to

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then the system has no solutions.

(d) False. The system can have a solution only if the 3 lines meet in at least one point which is common to all 3.

EXERCISE SET 1.3

1. **(c)** The matrix AE is 4×4. Since B is 4×5, $AE + B$ is not defined.

(e) The matrix $A + B$ is 4×5. Since E is 5×4, $E\ (A + B)$is 5×5.

(h) Since A^T is 5×4 and E is 5×4, their sum is also 5×4. Thus $(A^T + E)D$ is 5×2.

3. **(e)** Since $2B$ is a 2×2 matrix and C is a 2×3 matrix, $2B - C$ is not defined.

(g) We have

$$-3(D + 2E) = -3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix}\right)$$

$$= -3\begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix} = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

(j) We have $tr(D - 3E) = (1 - 3(6)) + (0 - 3(1)) + (4 - 3(3)) = -25$.

5. **(b)** Since B is a 2×2 matrix and A is a 3×2 matrix, BA is not defined (although AB is).

(d) We have

$$AB = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

Hence

$$(AB)C = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(e) We have

$$A(BC) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$$

(f) We have

$$CC^T = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$$

(j) We have $tr(4E^T - D) = tr(4E - D) = (4(6) - 1) + (4(1) - 0) + (4(3) - 4) = 35$.

7. **(a)** The first row of A is

$$A_1 = [3 \quad -2 \quad 7]$$

Thus, the first row of AB is

$$A_1B = [\,3 \;\; -2 \;\; 7] \begin{vmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{vmatrix}$$

$$= [67 \;\; 41 \;\; 41]$$

(c) The second column of B is

$$B_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

Thus, the second column of AB is

$$AB_2 = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix}$$

(e) The third row of A is

$$A_3 = [0 \quad 4 \quad 9]$$

Thus, the third row of AA is

$$A_3A = [0 \quad 4 \quad 9] \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$
$$= [24 \quad 56 \quad 97]$$

9. (a) The product $\mathbf{y}A$ is the matrix

$$[y_1a_{11} + y_2a_{21} + \cdots + y_ma_{m1} \qquad y_1a_{12} + y_2a_{22} + \cdots + y_ma_{m2} \cdots$$
$$y_1a_{1n} + y_2a_{2n} + \cdots + y_ma_{mn}]$$

We can rewrite this matrix in the form

$$y_1\,[a_{11}\ a_{12} \cdots a_{1n}] + y_2\,[a_{21}\ a_{22} \cdots a_{2n}] + \cdots + y_m\,[a_{m1}\ \ a_{m2} \cdots a_{mn}]$$

which is, indeed, a linear combination of the row matrices of A with the scalar coefficients of $\mathbf{y}$.

(b) Let $y = [y_1, y_2, \cdots, y_m]$

and $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$ be the m rows of A.

by 9a, $yA = \begin{bmatrix} y_1 & A_1 \\ y_2 & A_2 \\ & \vdots & \\ y_m & A_m \end{bmatrix}$

Taking transposes of both sides, we have

$$(yA)^T = A^T y^T = (A_1 \mid A_2 \mid \cdots \mid A_m) \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & A_1 \\ y_2 & A_2 \\ \vdots & \\ y_m & A_m \end{bmatrix}^T = (y_1 A_1 \mid y_2 A_2 \mid \ldots \mid y_m A_m$$

11. Let f_{ij} denote the entry in the i^{th} row and j^{th} column of $C(DE)$. We are asked to find f_{23}. In order to compute f_{23}, we must calculate the elements in the second row of C and the third column of DE. According to Equation (3), we can find the elements in the third column of DE by computing DE_3 where E_3 is the third column of E. That is,

$$\begin{aligned} f_{23} &= [3\ \ 1\ \ 5] \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right) \\ &= [3\ \ 1\ \ 5] \begin{bmatrix} 19 \\ 0 \\ 25 \end{bmatrix} = 182 \end{aligned}$$

15. **(a)** By block multiplication,

$$AB = \left[\begin{array}{c|c} \begin{bmatrix}-1 & 2\\0 & -3\end{bmatrix}\begin{bmatrix}2 & 1\\-3 & 5\end{bmatrix}+\begin{bmatrix}1 & 5\\4 & 2\end{bmatrix}\begin{bmatrix}7 & -1\\0 & 3\end{bmatrix} & \begin{bmatrix}-1 & 2\\0 & -3\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}1 & 5\\4 & 2\end{bmatrix}\begin{bmatrix}5\\-3\end{bmatrix} \\ \hline \begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}2 & 1\\-3 & 5\end{bmatrix}+\begin{bmatrix}6 & 1\end{bmatrix}\begin{bmatrix}7 & -1\\0 & 3\end{bmatrix} & \begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}6 & 1\end{bmatrix}\begin{bmatrix}5\\-3\end{bmatrix}\end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix}-8 & 9\\9 & -15\end{bmatrix}+\begin{bmatrix}7 & 14\\28 & 2\end{bmatrix} & \begin{bmatrix}0\\-6\end{bmatrix}+\begin{bmatrix}-10\\14\end{bmatrix} \\ \hline \begin{bmatrix}-13 & 26\end{bmatrix}+\begin{bmatrix}42 & -3\end{bmatrix} & \begin{bmatrix}14\end{bmatrix}+\begin{bmatrix}27\end{bmatrix}\end{array}\right]$$

$$= \left[\begin{array}{cc|c} -1 & 23 & -10\\ 37 & -13 & 8\\ \hline 29 & 23 & 41\end{array}\right]$$

17. **(a)** The partitioning of A and B makes them each effectively 2×2 matrices, so block multiplication might be possible. However, if

$$A = \begin{bmatrix}A_{11} & A_{12}\\A_{21} & A_{22}\end{bmatrix} \text{ and } B = \begin{bmatrix}B_{11} & B_{12}\\B_{21} & B_{22}\end{bmatrix}$$

then the products $A_{11}B_{11}$, $A_{12}B_{21}$, $A_{11}B_{12}$, $A_{12}B_{22}$, $A_{21}B_{11}$, $A_{22}B_{21}$, $A_{21}B_{12}$, and $A_{22}B_{22}$ are *all* undefined. If even *one* of these is undefined, block multiplication is impossible.

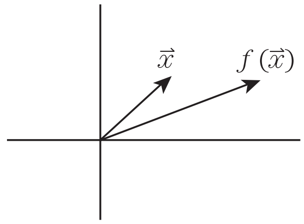
21. **(b)** If $i > j$, then the entry a_{ij} has row number larger than column number; that is, it lies below the matrix diagonal. Thus $[a_{ij}]$ has all zero elements below the diagonal.

(d) If $|i - j| > 1$, then either $i - j > 1$ or $i - j < -1$; that is, either $i > j + 1$ or $j > i + 1$. The first of these inequalities says that the entry a_{ij} lies below the diagonal and also below the "subdiagonal" consisting of all entries immediately below the diagonal ones. The second inequality says that the entry a_{ij} lies above the diagonal and also above the entries immediately above the diagonal ones. Thus we have

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0\\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0\\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0\\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0\\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56}\\ 0 & 0 & 0 & 0 & a_{65} & a_{66}\end{bmatrix}$$

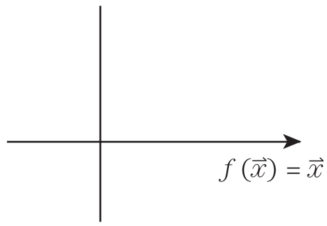
23.

$$f(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



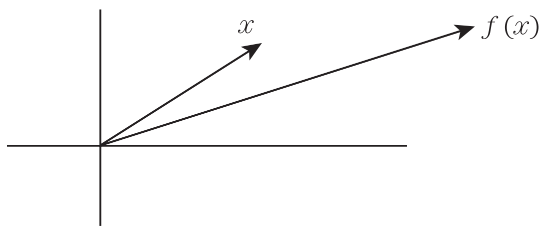
$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f(x) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$



$$x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

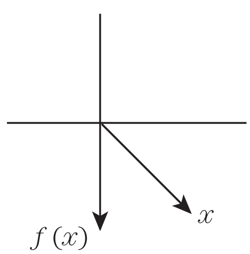
$$f(x) = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$



$$x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

(a)

$$f(x) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$



$$x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

(b)

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$$

(c)

27. The only solution to this system of equations is, by inspection,

(d)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

29. **(a)** Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $B^2 = A$ implies that

$$(*) \qquad \begin{aligned} a^2 + bc &= 2 \qquad & ab + bd &= 2 \\ ac + cd &= 2 \qquad & bc + d^2 &= 2 \end{aligned}$$

One might note that $a = b = c = d = 1$ and $a = b = c = d = -1$ satisfy $(*)$. Solving the first and last of the above equations simultaneously yields $a^2 = d^2$. Thus $a = \pm d$. Solving the remaining 2 equations yields $c(a + d) = b(a + d) = 2$. Therefore $a \neq -d$ and a and d cannot both be zero. Hence we have $a = d \neq 0$, so that $ac = ab = 1$, or $b = c = 1/a$. The first equation in $(*)$ then becomes $a^2 + 1/a^2 = 2$ or $a^4 - 2a^2 + 1 = 0$. Thus $a = \pm 1$. That is,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

are the only square roots of A.

(b) Using the reasoning and the notation of Part (a), show that either $a = -d$ or $b = c = 0$. If $a = -d$, then $a^2 + bc = 5$ and $bc + a^2 = 9$. This is impossible, so we have $b = c = 0$. This implies that $a^2 = 5$ and $d^2 = 9$. Thus

$$\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$$

are the 4 square roots of A.

Note that if A were $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, say, then $B = \begin{bmatrix} 1 & r \\ 4/r & -1 \end{bmatrix}$ would be a square root of A for every nonzero real number r and there would be infinitely many other square roots as well.

(c) By an argument similar to the above, show that if, for instance,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $BB = A$, then either $a = -d$ or $b = c = 0$. Each of these alternatives leads to a contradiction. Why?

31. **(a)** True. If A is an $m \times n$ matrix, then A^{T} is $n \times m$. Thus AA^T is $m \times m$ and A^TA is $n \times n$. Since the trace is defined for every square matrix, the result follows.

(b) True. Partition A into its row matrices, so that

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \text{ and } A^T = \begin{bmatrix} r_1^T & r_2^T & \cdots & r_m^T \end{bmatrix}$$

Then

$$AA^T = \begin{bmatrix} r_1r_1^T & r_1r_2^T & \cdots & r_1r_m^T \\ r_2r_1^T & r_2r_2^T & \cdots & r_2r_m^T \\ \vdots & \vdots & & \vdots \\ r_mr_1^T & r_mr_2^T & \cdots & r_mr_m^T \end{bmatrix}$$

Since each of the rows r_i is a $1 \times n$ matrix, each r_i^T is an $n \times 1$ matrix, and therefore each matrix $r_i\, r_j^T$ is a 1×1 matrix. Hence

$$\operatorname{tr}(AA^T) = r_1\, r_1^T + r_2\, r_2^T + \cdots + r_m\, r_m^T$$

Note that since $r_i\, r_i^T$ is just the sum of the squares of the entries in the i^{th} row of A, $r_1\, r_1^T + r_2\, r_2^T + \cdots + r_m\, r_m^T$ is the sum of the squares of *all* of the entries of A.

A similar argument works for A^TA, and since the sum of the squares of the entries of A^T is the same as the sum of the squares of the entries of A, the result follows.

31. **(c)** False. For instance, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(d) True. Every entry in the first row of AB is the matrix product of the first row of A with a column of B. If the first row of A has all zeros, then this product is zero.

EXERCISE SET 1.4

1. (a) We have

$$A + B = \begin{bmatrix} 10 & -4 & -2 \\ 0 & 5 & 7 \\ 2 & -6 & 10 \end{bmatrix}$$

Hence,

$$(A + B) + C = \begin{bmatrix} 10 & -4 & -2 \\ 0 & 5 & 7 \\ 2 & -6 & 10 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -6 & 1 \\ 1 & 12 & 11 \\ 5 & -1 & 19 \end{bmatrix}$$

On the other hand,

$$B + C = \begin{bmatrix} 8 & -5 & -2 \\ 1 & 8 & 6 \\ 7 & -2 & 15 \end{bmatrix}$$

Hence,

$$A + (B + C) = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 8 & -5 & -2 \\ 1 & 8 & 6 \\ 7 & -2 & 15 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 1 \\ 1 & 12 & 11 \\ 5 & -1 & 19 \end{bmatrix}$$

1. **(c)** Since $a + b = -3$, we have

$$(a+b)C = (-3)\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -9 \\ -3 & -21 & -12 \\ -9 & -15 & -27 \end{bmatrix}$$

Also

$$aC + bC = \begin{bmatrix} 0 & -8 & 12 \\ 4 & 28 & 16 \\ 12 & 20 & 36 \end{bmatrix} + \begin{bmatrix} 0 & 14 & -21 \\ -7 & -49 & -28 \\ -21 & -35 & -63 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -9 \\ -3 & -21 & -12 \\ -9 & -15 & -27 \end{bmatrix}$$

3. **(b)** Since

$$(A+B)^T = \begin{bmatrix} 10 & -4 & -2 \\ 0 & 5 & 7 \\ 2 & -6 & 10 \end{bmatrix}^T = \begin{bmatrix} 10 & 0 & 2 \\ -4 & 5 & -6 \\ -2 & 7 & 10 \end{bmatrix}$$

and

$$A^T + B^T = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 3 & 5 & -4 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 4 \\ -3 & 1 & -7 \\ -5 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ -4 & 5 & -6 \\ -2 & 7 & 10 \end{bmatrix}$$

the two matrices are equal.

3. **(d)** Since

$$(AB)^T = \begin{bmatrix} 28 & -28 & 6 \\ 20 & -31 & 38 \\ 0 & -21 & 36 \end{bmatrix}^T = \begin{bmatrix} 28 & 20 & 0 \\ -28 & -31 & -21 \\ 6 & 38 & 36 \end{bmatrix}$$

and

$$B^T A^T = \begin{bmatrix} 8 & 0 & 4 \\ -3 & 1 & -7 \\ -5 & 2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 28 & 20 & 0 \\ -28 & -31 & -21 \\ 6 & 38 & 36 \end{bmatrix}$$

the two matrices are equal.

5. **(b)**

$$(B^T)^{-1} = \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -4 \\ 3 & 2 \end{bmatrix}$$

$$(B^{-1})^T = \left[\frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}\right]^T = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}^T = \frac{1}{20}\begin{bmatrix} 4 & -4 \\ 3 & 2 \end{bmatrix}$$

7. **(b)** We are given that $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$. Therefore

$$7A = ((7A)^{-1})^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$$

7. **(d)** If $(I+2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$, then $I + 2A = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix}$. Hence

$$2A = \begin{bmatrix} -\frac{5}{13} & \frac{2}{13} \\ \frac{4}{13} & \frac{1}{13} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{18}{13} & \frac{2}{13} \\ \frac{4}{13} & -\frac{12}{13} \end{bmatrix}, \text{ so that } A = \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}.$$

9. **(b)** We have

$$p(A) = 2\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^2 - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$$

11. Call the matrix A. By Theorem 1.4.5,

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

since $\cos^2\theta + \sin^2\theta = 1$.