

CHAPTER 2

P. E. 2.1

(a) At $P(1,3,5)$, $x = 1$, $y = 3$, $z = 5$,
 $\rho = \sqrt{x^2 + y^2} = \sqrt{10}$, $z = 5$, $\phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6^\circ$
 $P(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6^\circ, 5)}}$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$$

$$\theta = \tan^{-1} \sqrt{x^2 + y^2} / z = \tan^{-1} \sqrt{10} / 5 = \tan^{-1} 0.6325 = 32.31^\circ$$

$$P(r, \theta, \phi) = \underline{\underline{P(5.916, 32.31^\circ, 71.57^\circ)}}$$

At $T(0, -4, 3)$, $x = 0$, $y = -4$, $z = 3$;
 $\rho = \sqrt{x^2 + y^2} = 4$, $z = 3$, $\phi = \tan^{-1} y/x = \tan^{-1} -4/0 = 270^\circ$
 $T(\rho, \phi, z) = \underline{\underline{T(4, 270^\circ, 3)}}$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5$$

$$\theta = \tan^{-1} \rho / z = \tan^{-1} 4 / 3 = 53.13^\circ$$

$$T(r, \theta, \phi) = \underline{\underline{T(5, 53.13^\circ, 270^\circ)}}$$

At $S(-3, -4, -10)$, $x = -3$, $y = -4$, $z = -10$;
 $\rho = \sqrt{x^2 + y^2} = 5$, $\phi = \tan^{-1} \left(\frac{-4}{-3} \right) = 233.1^\circ$
 $S(\rho, \phi, z) = \underline{\underline{S(5, 233.1^\circ, -10)}}$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18$$

$$\theta = \tan^{-1} \rho / z = \tan^{-1} \frac{5}{-10} = 153.43^\circ$$

$$S(r, \theta, \phi) = \underline{\underline{S(11.18, 153.43^\circ, 233.1^\circ)}}$$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho \sin \phi$,
 $Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}$; $Q_y = 0$; $Q_z = -\frac{z\rho \sin \phi}{\sqrt{\rho^2 + z^2}}$;

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x \cos\phi = \frac{\rho \cos\phi}{\sqrt{\rho^2 + z^2}}, \quad Q_\phi = -Q_x \sin\phi = \frac{-\rho \sin\phi}{\sqrt{\rho^2 + z^2}}$$

Hence,

$$\underline{\underline{Q = \frac{\rho}{\sqrt{\rho^2 + z^2}} (\cos\phi \mathbf{a}_\rho - \sin\phi \mathbf{a}_\phi - z \sin\phi \mathbf{a}_z).}}$$

In Spherical coordinates:

$$Q_x = \frac{r \sin\theta}{r} = \sin\theta;$$

$$Q_z = -r \sin\phi \sin\theta r \cos\theta \frac{1}{r} = -r \sin\theta \cos\theta \sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x \sin\theta \cos\phi + Q_z \cos\theta = \sin^2\theta \cos\phi - r \sin\theta \cos^2\theta \sin\phi.$$

$$Q_\theta = Q_x \cos\theta \cos\phi - Q_z \sin\theta = \sin\theta \cos\theta \cos\phi + r \sin^2\theta \cos\theta \sin\phi.$$

$$Q_\phi = -Q_x \sin\phi = -\sin\theta \sin\phi.$$

$$\underline{\underline{\therefore Q = \sin\theta (\sin\theta \cos\phi - r \cos^2\theta \sin\phi) \mathbf{a}_r + \sin\theta \cos\theta (\cos\phi + r \sin\theta \sin\phi) \mathbf{a}_\theta - \sin\theta \sin\phi \mathbf{a}_\phi.}}$$

At T :

$$Q(x, y, z) = \frac{4}{5} \mathbf{a}_x + \frac{12}{5} \mathbf{a}_z = 0.8 \mathbf{a}_x + 2.4 \mathbf{a}_z;$$

$$\begin{aligned} Q(\rho, \phi, z) &= \frac{4}{5} (\cos 270^\circ \mathbf{a}_\rho - \sin 270^\circ \mathbf{a}_\phi - 3 \sin 270^\circ \mathbf{a}_z) \\ &= 0.8 \mathbf{a}_\phi + 2.4 \mathbf{a}_z; \end{aligned}$$

$$\begin{aligned} Q(r, \theta, \phi) &= \frac{4}{5} \left(0 - \frac{45}{25} (-1) \right) \mathbf{a}_r + \frac{4}{5} \left(\frac{3}{5} \right) \left(0 + \frac{20}{5} (-1) \right) \mathbf{a}_\theta - \frac{4}{5} (-1) \mathbf{a}_\phi \\ &= \frac{36}{25} \mathbf{a}_r - \frac{48}{25} \mathbf{a}_\theta + \frac{4}{5} \mathbf{a}_\phi = \underline{\underline{1.44 \mathbf{a}_r - 1.92 \mathbf{a}_\theta + 0.8 \mathbf{a}_\phi;}} \end{aligned}$$

Note, that the magnitude of vector Q = 2.53 in all 3 cases above.

P.E. 2.2 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z \sin \phi \\ 3\rho \cos \phi \\ \rho \cos \phi \sin \phi \end{bmatrix}$$

$$A = (\rho z \cos \phi \sin \phi - 3\rho \cos \phi \sin \phi) \mathbf{a}_x + (\rho z \sin^2 \phi + 3\rho \cos^2 \phi) \mathbf{a}_y + \rho \cos \phi \sin \phi \mathbf{a}_z.$$

$$\text{But } \rho = \sqrt{x^2 + y^2}, \quad \tan \phi = \frac{y}{x}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \phi = \frac{y}{\sqrt{x^2 + y^2}};$$

Substituting all this yields:

$$A = \frac{1}{\sqrt{x^2 + y^2}} [(xyz - 3xy) \mathbf{a}_x + (zy^2 + 3x^2) \mathbf{a}_y + xy \mathbf{a}_z].$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin \theta \end{bmatrix}$$

$$\text{Since } r = \sqrt{x^2 + y^2 + z^2}, \quad \tan \theta = \frac{\sqrt{x^2 + y^2}}{z}, \quad \tan \phi = \frac{y}{x};$$

$$\text{and } \sin \theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}};$$

$$\text{and } \sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}};$$

$$B_x = r^2 \sin \theta \cos \phi - \sin \theta \sin \phi = rx - \frac{y}{r} = \frac{1}{r}(r^2 x - y).$$

$$B_y = r^2 \sin \theta \sin \phi + \sin \theta \cos \phi = ry + \frac{x}{r} = \frac{1}{r}(r^2 y + x).$$

$$B_z = r^2 \cos \theta = rz = \frac{1}{r}(r^2 z).$$

Hence,

$$\mathbf{B} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} [\{x(x^2 + y^2 + z^2) - y\} \mathbf{a}_x + \{y(x^2 + y^2 + z^2) + x\} \mathbf{a}_y + z(x^2 + y^2 + z^2) \mathbf{a}_z].$$

P.E.2.3 (a) At:

$$(1, \pi/3, 0), \quad \mathbf{H} = (0, 0.06767, 1)$$

$$\mathbf{a}_x = \cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi = \frac{1}{2}(\mathbf{a}_\rho - \sqrt{3} \mathbf{a}_\phi)$$

$$\mathbf{H} \bullet \mathbf{a}_x = \underline{\underline{-0.0586.}}$$

(b) At:

$$(1, \pi/3, 0), \quad \mathbf{a}_\theta = \cos \theta \mathbf{a}_\rho - \sin \theta \mathbf{a}_z = -\mathbf{a}_z.$$

$$\mathbf{H} \times \mathbf{a}_z = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{-0.06767 \mathbf{a}_\rho.}}$$

$$(c) \quad (\mathbf{H} \bullet \mathbf{a}_\rho) \mathbf{a}_\rho = \underline{\underline{0 \mathbf{a}_\rho.}}$$

$$(d) \quad \mathbf{H} \times \mathbf{a}_z = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.06767 \mathbf{a}_\rho.$$

$$|\mathbf{H} \times \mathbf{a}_z| = \underline{\underline{0.06767}}$$

P.E. 2.4

(a)

$$\mathbf{A} \bullet \mathbf{B} = (3, 2, -6) \bullet (4, 0, 3) = \underline{\underline{-6.}}$$

$$(b) \quad |\mathbf{A} \times \mathbf{B}| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = |6\mathbf{a}_r - 33\mathbf{a}_\theta - 8\mathbf{a}_\phi|.$$

$$\text{Thus the magnitude of } \mathbf{A} \times \mathbf{B} = \underline{\underline{34.48.}}$$

(c)

$$\text{At } (1, \pi/3, 5\pi/4), \quad \theta = \pi/3,$$

$$\mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta = \frac{1}{2} \mathbf{a}_r - \frac{\sqrt{3}}{2} \mathbf{a}_\theta.$$

$$(\mathbf{A} \bullet \mathbf{a}_z) \mathbf{a}_z = \left(\frac{3}{2} - \sqrt{3} \right) \left(\frac{1}{2} \mathbf{a}_r - \frac{\sqrt{3}}{2} \mathbf{a}_\theta \right) = \underline{\underline{-0.116 \mathbf{a}_r + 0.201 \mathbf{a}_\theta}}$$

Prob. 2.1

(a)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 25} = 5.3852, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} 2.5 = 68.2^\circ$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 25 + 1} = 5.477, \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{5.3852}{1} = 79.48^\circ$$

$$P(\rho, \phi, z) = \underline{\underline{P(5.3852, 68.2^\circ, 1)}}, \quad P(r, \theta, \phi) = \underline{\underline{P(5.477, 79.48^\circ, 68.2^\circ)}}$$

(b)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{9 + 16} = 5, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{4}{-3} = 360^\circ - 53.123^\circ = 306.88^\circ$$

$$r = \sqrt{x^2 + y^2 + z^2} = 5, \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \infty = 90^\circ$$

$$Q(\rho, \phi, z) = \underline{\underline{Q(5, 306.88^\circ, 0)}}, \quad P(r, \theta, \phi) = \underline{\underline{P(5, 90^\circ, 306.88^\circ)}}$$

(c)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{36 + 4} = 6.325, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{2}{6} = 18.43^\circ$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{36 + 4 + 16} = 7.483,$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{6.325}{-4} = 180^\circ - 57.69^\circ = 122.31^\circ$$

$$R(\rho, \phi, z) = \underline{\underline{R(6.325, 18.43^\circ, -4)}}, \quad R(r, \theta, \phi) = \underline{\underline{R(7.483, 122.31^\circ, 18.43^\circ)}}$$

Prob. 2.2

(a)

$$x = \rho \cos \phi = 2 \cos 30^\circ = 1.732;$$

$$y = \rho \sin \phi = 2 \sin 30^\circ = 1;$$

$$z = 5;$$

$$P_1(x, y, z) = \underline{\underline{P_1(1.732, 1, 5)}}.$$

(b)

$$x = 1 \cos 90^\circ = 0; \quad y = 1 \sin 90^\circ = 1; \quad z = -3.$$

$$P_2(x, y, z) = \underline{\underline{P_2(0, 1, -3)}}.$$

(c)

$$x = r \sin \theta \cos \phi = 10 \sin(\pi/4) \cos(\pi/3) = 3.535;$$

$$y = r \sin \theta \sin \phi = 10 \sin(\pi/4) \sin(\pi/3) = 6.124;$$

$$z = r \cos \theta = 10 \cos(\pi/4) = 7.0711$$

$$P_3(x, y, z) = \underline{\underline{P_3(3.535, 6.124, 7.0711)}}.$$

(d)

$$x = 4 \sin 30^\circ \cos 60^\circ = 1$$

$$y = 4 \sin 30^\circ \sin 60^\circ = 1.7321$$

$$z = r \cos \theta = 4 \cos 30^\circ = 3.464$$

$$P_4(x, y, z) = \underline{\underline{P_4(1, 1.7321, 3.464)}}.$$

Prob. 2.3

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.324$$

$$(a) \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{6}{2} = 71.56^\circ$$

$$P \text{ is } \underline{\underline{(6.324, 71.56^\circ, -4)}}$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 16} = 7.485$$

$$(b) \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{6.324}{-4} = 90^\circ + \tan^{-1} \frac{4}{6.324} = 122.3^\circ$$

$$P \text{ is } \underline{\underline{(7.483, 122.3^\circ, 71.56^\circ)}}$$

Prob. 2.4

(a)

$$x = \rho \cos \phi = 5 \cos 120^\circ = -2.5$$

$$y = \rho \sin \phi = 5 \sin 120^\circ = 4.33$$

$$z = 1$$

$$\text{Hence } Q = \underline{\underline{(-2.5, 4.33, 1)}}$$

(b)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = \sqrt{25 + 1} = 5.099$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{5}{1} = 78.69^\circ$$

$$\phi = 120^\circ$$

$$\text{Hence } Q = \underline{\underline{(5.099, 78.69^\circ, 120^\circ)}}$$

Prob. 2.5

$$T(r, \theta, \phi) \longrightarrow r = 10, \theta = 60^\circ, \phi = 30^\circ$$

$$x = r \sin \theta \cos \phi = 10 \sin 60^\circ \cos 30^\circ = 7.5$$

$$y = r \sin \theta \sin \phi = 10 \sin 60^\circ \sin 30^\circ = 4.33$$

$$z = r \cos \theta = 10 \cos 60^\circ = 5$$

$$T(x, y, z) = \underline{\underline{(7.5, 4.33, 5)}}$$

$$\rho = r \sin \theta = 10 \sin 60^\circ = 8.66$$

$$T(\rho, \phi, z) = \underline{\underline{(8.66, 30^\circ, 5)}}$$

Prob. 2.6

(a)

$$x = \rho \cos \phi, \quad y = \rho \sin \phi,$$

$$V = \underline{\underline{\rho z \cos \phi - \rho^2 \sin \phi \cos \phi + \rho z \sin \phi}}$$

(b)

$$\begin{aligned} U &= x^2 + y^2 + z^2 + y^2 + 2z^2 \\ &= r^2 + r^2 \sin^2 \theta \sin^2 \phi + 2r^2 \cos^2 \theta \\ &= \underline{\underline{r^2 [1 + \sin^2 \theta \sin^2 \phi + 2 \cos^2 \theta]}} \end{aligned}$$

Prob. 2.7

(a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{\rho^2 + z^2}} \\ \frac{y}{\sqrt{\rho^2 + z^2}} \\ \frac{4}{\sqrt{\rho^2 + z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2 + z^2}} [\rho \cos^2 \phi + \rho \sin^2 \phi] = \frac{\rho}{\sqrt{\rho^2 + z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2 + z^2}} [-\rho \cos \phi \sin \phi + \rho \cos \phi \sin \phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2 + z^2}};$$

$$\underline{\underline{\bar{F} = \frac{1}{\sqrt{\rho^2 + z^2}} (\rho \mathbf{a}_\rho + 4 \mathbf{a}_z)}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r} \sin^2 \theta \cos^2 \phi + \frac{r}{r} \sin^2 \theta \sin^2 \phi + \frac{4}{r} \cos \theta = \sin^2 \theta + \frac{4}{r} \cos \theta;$$

$$F_\theta = \sin \theta \cos \theta \cos^2 \phi + \sin \theta \cos \theta \sin^2 \phi - \frac{4}{r} \sin \theta = \sin \theta \cos \theta - \frac{4}{r} \sin \theta;$$

$$F_\phi = -\sin \theta \cos \phi \sin \phi + \sin \theta \sin \phi \cos \phi = 0;$$

$$\underline{\underline{\therefore \bar{F} = (\sin^2 \theta + \frac{4}{r} \cos \theta) \mathbf{a}_r + \sin \theta (\cos \theta - \frac{4}{r}) \mathbf{a}_\theta}}$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x\rho^2}{\sqrt{\rho^2 + z^2}} \\ \frac{y\rho^2}{\sqrt{\rho^2 + z^2}} \\ \frac{z\rho^2}{\sqrt{\rho^2 + z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2 + z^2}} [\rho \cos^2 \phi + \rho \sin^2 \phi] = \frac{\rho^3}{\sqrt{\rho^2 + z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2 + z^2}};$$

$$\underline{\underline{\mathbf{G} = \frac{\rho^2}{\sqrt{\rho^2 + z^2}} (\rho \mathbf{a}_\rho + z \mathbf{a}_z)}}$$

Spherical :

$$\mathbf{G} = \frac{\rho^2}{r} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) = \frac{r^2 \sin^2 \theta}{r} r\mathbf{a}_r = \underline{\underline{r^2 \sin^2 \theta \mathbf{a}_r}}$$

Prob. 2.8

$$\mathbf{B} = \rho \mathbf{a}_x + \frac{y}{\rho} \mathbf{a}_y + z \mathbf{a}_z$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ y/\rho \\ z \end{bmatrix}$$

$$B_\rho = \rho \cos \phi + \frac{y}{\rho} \sin \phi$$

$$B_\phi = -\rho \sin \phi + \frac{y}{\rho} \cos \phi$$

$$B_z = z$$

$$\text{But } y = \rho \sin \phi$$

$$B_\rho = \rho \cos \phi + \sin^2 \phi, B_\phi = -\rho \sin \phi + \sin \phi \cos \phi$$

Hence,

$$\underline{\underline{\mathbf{B} = (\rho \cos \phi + \sin^2 \phi) \mathbf{a}_\rho + \sin \phi (\cos \phi - \rho) \mathbf{a}_\phi + z \mathbf{a}_z}}$$

Prob. 2.9

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

At P, $\rho = 2$, $\phi = \pi/2$, $z = -1$

$$A_x = 2 \cos \phi - 3 \sin \phi = 2 \cos 90^\circ - 3 \sin 90^\circ = -3$$

$$A_y = 2 \sin \phi + 3 \cos \phi = 2 \sin 90^\circ + 3 \cos 90^\circ = 2$$

$$A_z = 4$$

Hence, $\underline{\underline{A = -3\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z}}$

Prob. 2.10

(a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \sin \phi \\ \rho \cos \phi \\ -2z \end{bmatrix}$$

$$A_x = \rho \sin \phi \cos \phi - \rho \cos \phi \sin \phi = 0$$

$$A_y = \rho \sin^2 \phi + \rho \cos^2 \phi = \rho = \sqrt{x^2 + y^2}$$

$$A_z = -2z$$

Hence,

$$\underline{\underline{A = \sqrt{x^2 + y^2} \mathbf{a}_y - 2z \mathbf{a}_z}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} 4r \cos \phi \\ r \\ 0 \end{bmatrix}$$

$$B_x = 4r \sin \theta \cos^2 \phi + r \cos \theta \cos \phi$$

$$B_y = 4r \sin \theta \sin \phi \cos \phi + r \cos \theta \sin \phi$$

$$B_z = 4r \cos \theta \cos \phi - r \sin \theta$$

But $r = \sqrt{x^2 + y^2 + z^2}$, $\sin \theta = \frac{\sqrt{x^2 + y^2}}{r}$, $\cos \theta = \frac{z}{r}$

$$\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$B_x = 4\sqrt{x^2 + y^2} \frac{x^2}{x^2 + y^2} + \frac{zx}{\sqrt{x^2 + y^2}}$$

$$B_y = 4\sqrt{x^2 + y^2} \frac{xy}{x^2 + y^2} + \frac{zy}{\sqrt{x^2 + y^2}}$$

$$B_z = 4z \frac{x}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2}$$

$$\mathbf{B} = \frac{1}{\sqrt{x^2 + y^2}} \left[x(4x + z)\mathbf{a}_x + y(4x + z)\mathbf{a}_y + (4xz - x^2 - y^2)\mathbf{a}_z \right]$$

Prob. 2.11

Method 1:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} 4/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$F_x = \frac{4}{r^2} \sin \theta \cos \phi, \quad F_y = \frac{4}{r^2} \sin \theta \sin \phi, \quad F_z = \frac{4}{r^2} \cos \theta$$

$$r^2 = x^2 + y^2 + z^2, \quad \sin \theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$F_x = \frac{4}{x^2 + y^2 + z^2} \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{x}{\sqrt{x^2 + y^2}} = \frac{4x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_y = \frac{4}{x^2 + y^2 + z^2} \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \frac{y}{\sqrt{x^2 + y^2}} = \frac{4y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_z = \frac{4}{x^2 + y^2 + z^2} \frac{z}{(x^2 + y^2 + z^2)} = \frac{4z}{(x^2 + y^2 + z^2)^{3/2}}$$

Thus,

$$\mathbf{F} = \frac{4}{(x^2 + y^2 + z^2)^{3/2}} \left[x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z \right]$$

Method 2:

$$\mathbf{F} = \frac{4\mathbf{a}_r}{r^2} \cdot \frac{\mathbf{r}}{r} = \frac{4r\mathbf{a}_r}{r^3}$$

$$\mathbf{F} = \frac{4}{(x^2 + y^2 + z^2)^{3/2}} [x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z]$$

Prob. 2.12

$$r = 2, \quad \theta = \pi/2, \quad \phi = 3\pi/2$$

$$(a) \quad \mathbf{B} = 2 \sin(\pi/2)\mathbf{a}_r - 4 \cos(3\pi/2)\mathbf{a}_\phi = \underline{\underline{2\mathbf{a}_r}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \cos \theta \cos \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} r \sin \theta \\ 0 \\ -r^2 \cos \phi \end{bmatrix}$$

$$B_x = r \sin^2 \theta \cos \phi - r^2 \sin \phi \cos \phi, \quad B_y = r \sin \theta \cos \theta \cos \phi - r^2 \cos^2 \phi$$

$$B_z = r \sin \theta \cos \theta$$

$$\text{But } r = \sqrt{x^2 + y^2 + z^2}, \cos \theta = \frac{z}{r}, \sin \theta = \frac{\rho}{r} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos \phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\begin{aligned} B_x &= \sqrt{x^2 + y^2 + z^2} \frac{x^2 + y^2}{x^2 + y^2 + z^2} \frac{x}{\sqrt{x^2 + y^2}} - (x^2 + y^2 + z^2) \frac{xy}{x^2 + y^2} \\ &= \frac{x\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} - \frac{xy(x^2 + y^2 + z^2)}{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} B_y &= \sqrt{x^2 + y^2 + z^2} \frac{z\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} \frac{x}{\sqrt{x^2 + y^2}} - (x^2 + y^2 + z^2) \frac{x^2}{x^2 + y^2} \\ &= \frac{xz}{\sqrt{x^2 + y^2 + z^2}} - \frac{x^2(x^2 + y^2 + z^2)}{x^2 + y^2} \end{aligned}$$

$$B_z = \sqrt{x^2 + y^2 + z^2} \frac{z\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} = \frac{z\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

Prob. 2.13

$$(a) \quad x = \rho \cos \phi$$

$$\underline{\underline{\mathbf{B} = \rho \cos \phi \mathbf{a}_z}}$$

$$x = r \sin \theta \cos \phi$$

$$(b) \quad \mathbf{B} = r \sin \theta \cos \phi \mathbf{a}_z, \quad B_x = 0 = B_y, B_z = r \sin \theta \cos \phi$$

$$\begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \sin \theta \cos \phi \end{bmatrix}$$

$$B_r = r \sin \theta \cos \theta \cos \phi = 0.5r \sin(2\theta) \cos \phi$$

$$B_\theta = -r \sin^2 \theta \cos \phi, \quad B_\phi = 0$$

$$\underline{\underline{\mathbf{B} = 0.5r \sin(2\theta) \cos \phi \mathbf{a}_r - r \sin^2 \theta \cos \phi \mathbf{a}_\theta}}$$

Prob. 2.14

(a)

$$\mathbf{a}_x \times \mathbf{a}_\rho = (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi) \times \mathbf{a}_\rho = \cos \phi$$

$$\mathbf{a}_x \times \mathbf{a}_\phi = (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi) \times \mathbf{a}_\phi = -\sin \phi$$

$$\mathbf{a}_y \times \mathbf{a}_\rho = (\sin \phi \mathbf{a}_\rho + \cos \phi \mathbf{a}_\phi) \times \mathbf{a}_\rho = \sin \phi$$

$$\bar{\mathbf{a}}_y \times \bar{\mathbf{a}}_\phi = (\sin \phi \mathbf{a}_\rho + \sin \phi \mathbf{a}_\phi) \times \mathbf{a}_\phi = \cos \phi$$

(b) and (c)

In spherical system :

$$\mathbf{a}_x = \sin \theta \cos \phi \mathbf{a}_r + \cos \theta \cos \phi \mathbf{a}_\theta - \sin \phi \mathbf{a}_\phi.$$

$$\mathbf{a}_y = \sin \theta \sin \phi \mathbf{a}_r + \cos \theta \sin \phi \mathbf{a}_\theta - \cos \phi \mathbf{a}_\phi.$$

$$\mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta.$$

Hence,

$$\mathbf{a}_x \times \mathbf{a}_r = \sin \theta \cos \phi;$$

$$\mathbf{a}_x \times \mathbf{a}_\theta = \cos \theta \cos \phi;$$

$$\mathbf{a}_y \times \mathbf{a}_r = \sin \theta \sin \phi;$$

$$\mathbf{a}_y \times \mathbf{a}_\theta = \cos \theta \sin \phi;$$

$$\bar{\mathbf{a}}_z \times \bar{\mathbf{a}}_r = \cos \theta;$$

$$\bar{\mathbf{a}}_z \times \bar{\mathbf{a}}_\theta = -\sin \theta;$$

Prob. 2.15

(a)

$$\mathbf{a}_\rho = \cos \phi \mathbf{a}_x + \sin \phi \mathbf{a}_y, \quad \mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \begin{vmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 \end{vmatrix} = (\cos^2 \phi + \sin^2 \phi) \mathbf{a}_z = \mathbf{a}_z$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \begin{vmatrix} 0 & 0 & 1 \\ \cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 \end{vmatrix} = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y = \mathbf{a}_\phi$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \begin{vmatrix} -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \cos \phi \mathbf{a}_x + \sin \phi \mathbf{a}_y = \mathbf{a}_\rho$$

(b)

$$\mathbf{a}_r = \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos \theta \cos \phi \mathbf{a}_x + \cos \theta \sin \phi \mathbf{a}_y - \sin \theta \mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \begin{vmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \end{vmatrix}$$

$$= (-\sin^2 \theta \sin \phi - \cos^2 \theta \sin \phi) \mathbf{a}_x + (\cos^2 \theta \cos \phi + \sin^2 \theta \cos \phi) \mathbf{a}_y$$

$$+ (\sin \theta \cos \theta \sin \phi \cos \phi - \sin \theta \cos \theta \sin \phi \cos \phi) \mathbf{a}_z$$

$$= -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y = \mathbf{a}_\phi$$

$$\begin{aligned}
\mathbf{a}_\phi \times \mathbf{a}_r &= \begin{vmatrix} -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{vmatrix} \\
&= \cos \theta \cos \phi \mathbf{a}_x + \cos \theta \sin \phi \mathbf{a}_y + (-\sin \theta \sin^2 \phi - \sin \theta \cos^2 \phi) \mathbf{a}_z \\
&= \cos \theta \cos \phi \mathbf{a}_x + \cos \theta \sin \phi \mathbf{a}_y - \sin \theta \mathbf{a}_z = \mathbf{a}_\theta \\
\mathbf{a}_\theta \times \mathbf{a}_\phi &= \begin{vmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{vmatrix} \\
&= \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + (\cos \theta \cos^2 \phi + \cos \theta \sin^2 \phi) \mathbf{a}_z \\
&= \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z = \mathbf{a}_r
\end{aligned}$$

Prob. 2.16

(a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1} \frac{\rho}{z}; \quad \phi = \phi.$$

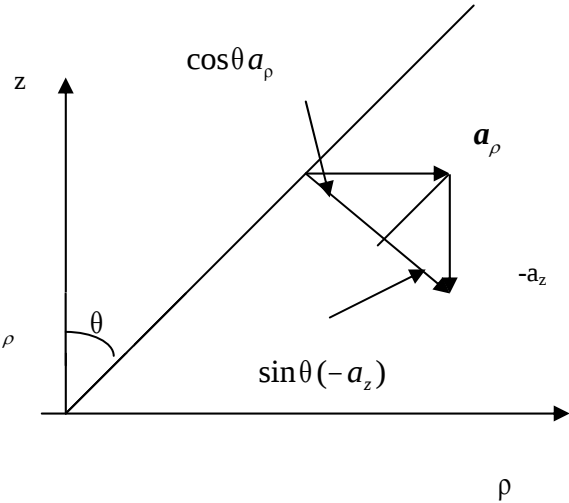
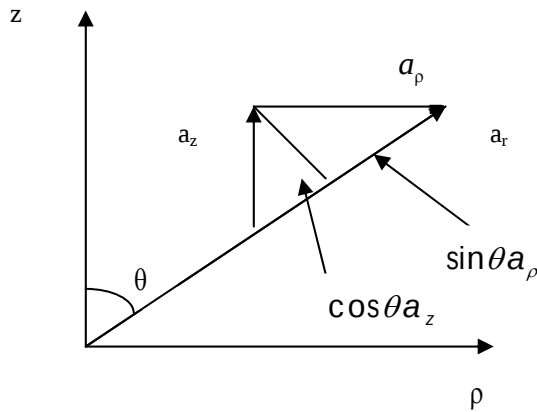
or

$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi}.$$

$$= r \sin \theta;$$

$$z = r \cos \theta; \quad \phi = \phi.$$

(b) From the figures below,



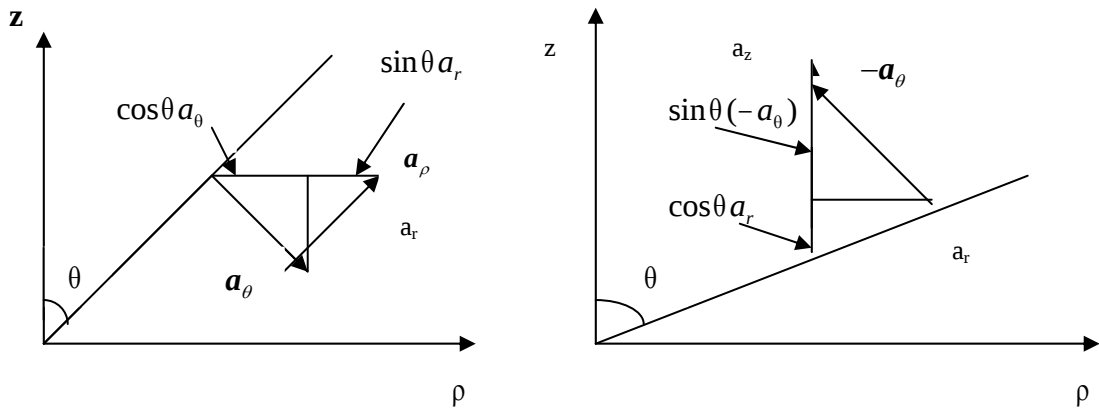
$$\mathbf{a}_r = \sin\theta \mathbf{a}_\rho + \cos\theta \mathbf{a}_z; \quad \mathbf{a}_\theta = \cos\theta \mathbf{a}_\rho - \sin\theta \mathbf{a}_z; \quad \mathbf{a}_\phi = \mathbf{a}_\phi.$$

Hence,

$$\begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix}$$

From the figures below,

$$\mathbf{a}_\rho = \cos\theta \mathbf{a}_\theta + \sin\theta \mathbf{a}_r; \quad \mathbf{a}_z = \cos\theta \mathbf{a}_r - \sin\theta \mathbf{a}_\theta; \quad \mathbf{a}_\phi = \mathbf{a}_\phi.$$



$$\begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_z \end{bmatrix}$$

Prob. 2.17

At $P(2, 0, -1)$, $\phi = 0$, $\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) = 116.56^\circ$

- (a) $\mathbf{a}_\rho \bullet \mathbf{a}_x = \cos\phi = \underline{\underline{1}}$
- (b) $\mathbf{a}_\phi \bullet \mathbf{a}_y = \cos\phi = \underline{\underline{1}}$
- (c) $\mathbf{a}_r \bullet \mathbf{a}_z = \cos\theta = \underline{\underline{-0.4472}}$

Prob. 2.18

If **A** and **B** are perpendicular to each other, $\mathbf{A} \cdot \mathbf{B} = 0$

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= \rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi - \rho^2 \\ &= \rho^2 (\sin^2 \phi + \cos^2 \phi) - \rho^2 \\ &= \rho^2 - \rho^2 \\ &= 0\end{aligned}$$

As expected.

Prob. 2.19

$$(a) \mathbf{A} + \mathbf{B} = \underline{\underline{8\mathbf{a}_\rho + 2\mathbf{a}_\phi - 7\mathbf{a}_z}}$$

$$(b) \mathbf{A} \cdot \mathbf{B} = 15 + 0 - 8 = \underline{\underline{7}}$$

$$\begin{aligned}(c) \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & -8 \end{vmatrix} \\ &= -16\mathbf{a}_\rho + (5 + 24)\mathbf{a}_\phi - 10\mathbf{a}_z \\ &= \underline{\underline{-16\mathbf{a}_\rho + 29\mathbf{a}_\phi - 10\mathbf{a}_z}}\end{aligned}$$

$$\begin{aligned}(d) \cos \theta_{AB} &= \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{7}{\sqrt{9+4+1}\sqrt{25+64}} = \frac{7}{\sqrt{14}\sqrt{89}} \\ &= 0.19831\end{aligned}$$

$$\underline{\underline{\theta_{AB} = 78.56^\circ}}$$

Prob. 2.20

$$\begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix}$$

$$\begin{aligned}G_x &= G_\rho \cos \phi - G_\phi \sin \phi = 3\rho \cos \phi - \rho \cos \phi \sin \phi \\ &= 3x - x \sin \phi = 3(3) - (3)\sin(306.87^\circ) = 11.4\end{aligned}$$

$$\mathbf{G}_x = G_x \mathbf{a}_x = \underline{\underline{11.4\mathbf{a}_x}}$$

Prob. 2.21

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

$$G_\rho = yz \cos \phi + xz \sin \phi$$

$$x = \rho \cos \phi, y = \rho \sin \phi, \quad yz = \rho z \sin \phi, xz = \rho z \cos \phi$$

$$G_\rho = \rho z \sin \phi \cos \phi + \rho z \cos \phi \sin \phi = 2\rho z \sin \phi \cos \phi = \rho z \sin 2\phi$$

$$G_\phi = -yz \sin \phi + xz \cos \phi = \rho z (\cos^2 \phi - \sin^2 \phi) = \rho z \cos 2\phi$$

$$G_z = xy = \rho^2 \cos \phi \sin \phi = 0.5\rho^2 \sin 2\phi$$

$$\underline{\underline{G = \rho z \sin 2\phi \mathbf{a}_y + \rho z \cos 2\phi \mathbf{a}_\phi + 0.5\rho^2 \sin 2\phi \mathbf{a}_z}}$$

Prob. 2.22

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & -\frac{y}{\sqrt{x^2 + y^2}} & 0 \\ \frac{y}{\sqrt{x^2 + y^2}} & \frac{x}{\sqrt{x^2 + y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2 + z^2}} & \frac{xz}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2 + z^2}} & \frac{-y}{\sqrt{x^2 + y^2}} \\ \frac{y}{\sqrt{x^2 + y^2 + z^2}} & \frac{yz}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2 + z^2}} & \frac{x}{\sqrt{x^2 + y^2}} \\ \frac{z}{\sqrt{x^2 + y^2 + z^2}} & -\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Prob. 2.23 (a) Using the results in Prob.2.14,

$$A_\rho = \rho z \sin \phi = r^2 \sin \theta \cos \theta \sin \phi$$

$$A_\phi = 3\rho \cos \phi = 3r \sin \theta \cos \phi$$

$$A_z = \rho \cos \phi \sin \phi = r \sin \theta \cos \phi \sin \phi$$

Hence,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2 \sin \theta \cos \theta \sin \phi \\ 3r \sin \theta \cos \phi \\ r \sin \theta \cos \phi \sin \phi \end{bmatrix}$$

$$\underline{\underline{\mathbf{A}(r, \theta, \phi) = r \sin \theta \left[\sin \phi \cos \theta (r \sin \theta + \cos \phi) \mathbf{a}_r + \sin \phi (r \cos^2 \theta - \sin \theta \cos \phi) \mathbf{a}_\theta + 3 \cos \phi \mathbf{a}_\phi \right]}}$$

At $(10, \pi/2, 3\pi/4)$, $r = 10, \theta = \pi/2, \phi = 3\pi/4$

$$\underline{\underline{\mathbf{A} = 10(0\mathbf{a}_r + 0.5\mathbf{a}_\theta - \frac{3}{\sqrt{2}}\mathbf{a}_\phi) = 5\mathbf{a}_\theta - 21.21\mathbf{a}_\phi}}$$

$$(b) \quad B_r = r^2 = (\rho^2 + z^2), \quad B_\theta = 0, \quad B_\phi = \sin \theta = \frac{\rho}{\sqrt{\rho^2 + z^2}}$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}(\rho, \phi, z) = \sqrt{\rho^2 + z^2} \left(\rho \mathbf{a}_\rho + \frac{\rho}{\rho^2 + z^2} \mathbf{a}_\phi + z \mathbf{a}_z \right)}}$$

At $(2, \pi/6, 1)$, $\rho = 2, \phi = \pi/6, z = 1$

$$\underline{\underline{\mathbf{B} = \sqrt{5}(2\mathbf{a}_\rho + 0.4\mathbf{a}_\phi + \mathbf{a}_z) = 4.472\mathbf{a}_\rho + 0.8944\mathbf{a}_\phi + 2.236\mathbf{a}_z}}$$

Prob. 2.24

$$(a) \quad d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$$

$$(b) \quad d^2 = 3^2 + 5^2 - 2(3)(5) \cos \pi + (-1-5)^2 = 100$$

$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)

$$\begin{aligned}
 d^2 &= 10^2 + 5^2 - 2(10)(5) \cos \frac{\pi}{4} \cos \frac{\pi}{6} - 2(10)(5) \sin \frac{\pi}{4} \sin \frac{\pi}{6} \cos \left(7\frac{\pi}{4} - \frac{3\pi}{4} \right) \\
 &= 125 - 100(0.7071)(0.866) - 100(0.7071)(0.5)(-0.2334) \\
 &= 125 - 61.23 + 35.33 = 99.118 \\
 d &= \sqrt{99.118} = \underline{\underline{9.956}}.
 \end{aligned}$$

Prob. 2.25

Using eq. (2.33),

$$\begin{aligned}
 d^2 &= r_1^2 + r_2^2 - 2r_1r_2 \cos \theta_1 \cos \theta_2 - 2r_1r_2 \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1) \\
 &= 16 + 36 - 2(4)(6) \cos 30^\circ \cos 90^\circ - 2(4)(6) \sin 30^\circ \sin 90^\circ \cos(180^\circ) \\
 &= 16 + 36 - 0 - 48(0.5)(1)(-1) = 52 + 24 = 76 \\
 d &= 8.718
 \end{aligned}$$

Prob. 2.26

$$\begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix}$$

$$\mathbf{a}_\rho = \cos \phi \mathbf{a}_x + \sin \phi \mathbf{a}_y, \quad \mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$$

$$\text{At } (0, 4, -1), \quad \phi = 90^\circ$$

$$\mathbf{a}_\rho = \sin 90^\circ \mathbf{a}_y = \underline{\underline{\mathbf{a}_y}}$$

$$\mathbf{a}_\phi = -\sin 90^\circ \mathbf{a}_x = \underline{\underline{-\mathbf{a}_x}}$$

Prob. 2.27

$$\text{At } (1, 60^\circ, -1), \quad \rho = 1, \phi = 60^\circ, z = -1,$$

$$\begin{aligned}
 \text{(a) } \mathbf{A} &= (-2 - \sin 60^\circ) \mathbf{a}_\rho + (4 + 2 \cos 60^\circ) \mathbf{a}_\phi - 3(1)(-1) \mathbf{a}_z \\
 &= -2.866 \mathbf{a}_\rho + 5 \mathbf{a}_\phi + 3 \mathbf{a}_z
 \end{aligned}$$

$$\mathbf{B} = 1 \cos 60^\circ \mathbf{a}_\rho + \sin 60^\circ \mathbf{a}_\phi + \mathbf{a}_z = 0.5 \mathbf{a}_\rho + 0.866 \mathbf{a}_\phi + \mathbf{a}_z$$

$$\mathbf{A} \cdot \mathbf{B} = -1.433 + 4.33 + 3 = 5.897$$

$$AB = \sqrt{2.866^2 + 26 + 9} \sqrt{0.25 + 1 + 0.866^2} = 9.1885$$

$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{5.897}{9.1885} = 0.6419 \quad \longrightarrow \quad \theta_{AB} = \underline{\underline{50.07^\circ}}$$

Let $\mathbf{D} = \mathbf{A} \times \mathbf{B}$. At $(1, 90^\circ, 0)$, $\rho = 1, \phi = 90^\circ, z = 0$

$$(b) \mathbf{A} = -\sin 90^\circ \mathbf{a}_\rho + 4\mathbf{a}_\phi = -\mathbf{a}_\rho + 4\mathbf{a}_\phi$$

$$\mathbf{B} = 1 \cos 90^\circ \mathbf{a}_\rho + \sin 90^\circ \mathbf{a}_\phi + \mathbf{a}_z = \mathbf{a}_\phi + \mathbf{a}_z$$

$$\mathbf{D} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 4\mathbf{a}_\rho + \mathbf{a}_\phi - \mathbf{a}_z$$

$$\mathbf{a}_D = \frac{\mathbf{D}}{D} = \frac{(4, 1, -1)}{\sqrt{16+1+1}} = \underline{\underline{0.9428\mathbf{a}_\rho + 0.2357\mathbf{a}_\phi - 0.2357\mathbf{a}_z}}$$

Prob.2.28

At $P(0, 2, -5)$, $\phi = 90^\circ$;

$$\begin{aligned} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} &= \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix} \end{aligned}$$

$$\mathbf{B} = -\mathbf{a}_x - 5\mathbf{a}_y - 3\mathbf{a}_z$$

$$(a) \mathbf{A} + \mathbf{B} = (2, 4, 10) + (-1, -5, -3)$$

$$= \underline{\underline{\mathbf{a}_x - \mathbf{a}_y + 7\mathbf{a}_z}}$$

$$(b) \cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{AB} = \frac{-52}{\sqrt{4200}}$$

$$\theta_{AB} = \cos^{-1}\left(\frac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36^\circ}}$$

$$(c) A_B = \mathbf{A} \cdot \mathbf{a}_B = \frac{\mathbf{A} \cdot \mathbf{B}}{B} = -\frac{52}{\sqrt{35}} = \underline{\underline{-8.789}}$$

Prob. 2.29

$$\mathbf{B} \cdot \mathbf{a}_x = B_x$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$\begin{aligned} B_x &= B_\rho \cos \phi - B_\phi \sin \phi = \rho^2 \sin \phi \cos \phi - (z-1) \cos \phi \sin \phi \\ &= 16(0.5) - (-2)(0.5) = 8 + 1 = \underline{\underline{9}} \end{aligned}$$

Prob. 2.30

$$\bar{\mathbf{G}} = \cos^2 \phi \mathbf{a}_x + \frac{2r \cos \theta \sin \phi}{r \sin \theta} \bar{\mathbf{a}}_y + (1 - \cos^2 \phi) \bar{\mathbf{a}}_z$$

$$= \cos^2 \phi \bar{\mathbf{a}}_x + 2 \cot \theta \sin \phi \bar{\mathbf{a}}_y + \sin^2 \phi \bar{\mathbf{a}}_z$$

$$\begin{pmatrix} G_r \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2 \phi \\ 2 \cot \theta \sin \phi \\ \sin^2 \phi \end{bmatrix}$$

$$\begin{aligned} G_r &= \sin \theta \cos^3 \phi + 2 \cos \theta \sin^2 \phi + \cos \theta \sin^2 \phi \\ &= \sin \theta \cos^3 \phi + 3 \cos \theta \sin^2 \phi \end{aligned}$$

$$G_\theta = \cos \theta \cos^3 \phi + 2 \cot \theta \cos \theta \sin^2 \phi - \sin \theta \sin^2 \phi$$

$$G_\phi = -\sin \phi \cos^2 \phi + 2 \cot \theta \sin \phi \cos \phi$$

$$\begin{aligned} \bar{\mathbf{G}} &= [\sin \theta \cos^3 \phi + 3 \cos \theta \sin^2 \phi] \mathbf{a}_r \\ &\quad + [\cos \theta \cos^3 \phi + 2 \cot \theta \cos \theta \sin^2 \phi - \sin \theta \sin^2 \phi] \mathbf{a}_\theta \\ &\quad + \underline{\underline{\sin \phi \cos \phi (2 \cot \theta - \cos \phi) \mathbf{a}_\phi}} \end{aligned}$$

Prob. 2.31

- (a) An infinite line parallel to the z-axis.
- (b) Point (2,-1,10).
- (c) A circle of radius $r \sin \theta = 5$, i.e. the intersection of a cone and a sphere.
- (d) An infinite line parallel to the z-axis.
- (e) A semi-infinite line parallel to the x-y plane.
- (f) A semi-circle of radius 5 in the y-z plane.

Prob. 2.32

$$(a) \mathbf{J}_z = (\mathbf{J} \bullet \mathbf{a}_z) \mathbf{a}_z.$$

$$\text{At } (2, \pi/2, 3\pi/2), \mathbf{a}_z = \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta = -\mathbf{a}_\theta.$$

$$\mathbf{J}_z = -\cos 2\theta \sin \phi \mathbf{a}_\theta = -\cos \pi \sin(3\pi/2) \mathbf{a}_\theta = -\mathbf{a}_\theta.$$

$$(b) \mathbf{J}_\phi = \tan \frac{\theta}{2} \ln r \mathbf{a}_\phi = \tan \frac{\pi}{4} \ln 2 \mathbf{a}_\phi = \ln 2 \mathbf{a}_\phi = 0.6931 \mathbf{a}_\phi.$$

$$(c) \mathbf{J}_t = \mathbf{J} - \mathbf{J}_n = \mathbf{J} - \mathbf{J}_r = -\mathbf{a}_\theta + \ln 2 \mathbf{a}_\phi = \underline{\underline{-\mathbf{a}_\theta + 0.6931 \mathbf{a}_\phi}}$$

$$(d) \mathbf{J}_p = (\mathbf{J} \bullet \mathbf{a}_x) \mathbf{a}_x$$

$$\mathbf{a}_x = \sin \theta \cos \phi \mathbf{a}_r + \cos \theta \cos \phi \mathbf{a}_\theta - \sin \phi \mathbf{a}_\phi = \mathbf{a}_\phi.$$

$$\text{At } (2, \pi/2, 3\pi/2),$$

$$\mathbf{J}_p = \underline{\underline{\ln 2 \mathbf{a}_\phi.}}$$

Prob. 2.33

$$\mathbf{H} \bullet \mathbf{a}_x = H_x$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho^2 \cos \phi \\ -\rho \sin \phi \\ 0 \end{bmatrix}$$

$$H_x = \rho^2 \cos^2 \phi + \rho \sin^2 \phi$$

$$\text{At P, } \rho = 2, \phi = 60^\circ, z = -1$$

$$H_x = 4(1/4) + 2(3/4) = 1 + 1.5 = \underline{\underline{2.5}}$$

Prob. 2.34

$$(a) 5 = \mathbf{r} \bullet \mathbf{a}_x + \mathbf{r} \bullet \mathbf{a}_y = x + y \quad \underline{\underline{\text{a plane}}}$$

$$(b) 10 = |\mathbf{r} \times \mathbf{a}_z| = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \end{vmatrix} = |y \mathbf{a}_x - x \mathbf{a}_y| = \sqrt{x^2 + y^2} = \rho$$

a cylinder of infinite length