

Chapter 1: Solutions to Problems

Problem 1.1.

a. $\mathbf{A} = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) = \hat{\mathbf{x}}(6 - 1) + \hat{\mathbf{y}}(-3 - 0) + \hat{\mathbf{z}}(0 - 1) = \hat{\mathbf{x}}5 - \hat{\mathbf{y}}3 - \hat{\mathbf{z}}$
 Scalar components are: $A_x = 5$, $A_y = -3$, and $A_z = -1$

b. $\mathbf{B} = \hat{\mathbf{x}}(x_1 - x_0) + \hat{\mathbf{y}}(y_1 - y_0) + \hat{\mathbf{z}}(z_1 - z_0) = \hat{\mathbf{x}}(1 - 0) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(1 - 0) = \hat{\mathbf{x}} + \hat{\mathbf{z}}$
 Scalar components are: $A_x = 1$, $A_y = 0$, and $A_z = 1$

c. From (a), the vector is: $\mathbf{A} = \hat{\mathbf{x}}5 - \hat{\mathbf{y}}3 - \hat{\mathbf{z}}$
 Its magnitude is $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{5^2 + (-3)^2 + (-1)^2} = \sqrt{35}$

Problem 1.2.

Magnitude of velocity is $v = |\mathbf{v}| = 50 \text{ km/h} = 13.889 \text{ m/s}$

a. $\mathbf{v} = \hat{\mathbf{E}}v \cos 45^\circ + \hat{\mathbf{N}}v \sin 45^\circ = \hat{\mathbf{E}}9.821 + \hat{\mathbf{N}}9.821 \quad [\text{m/s}]$
 b. $t = 3,000,000 / 9.821 = 305,467.69 \text{ s} \rightarrow t = 84\text{hrs}, 51\text{min.}$
 c. $D = vt = 13.889 \times 305,467.69 = 4242640.7 \text{ m} = 4242.64 \text{ km}$

Problem 1.3.

a. First we convert speed to [m/s]:

$$800 [\text{km/h}] = 222.222 [\text{m/s}] \quad 100 [\text{km/h}] = 27.778 [\text{m/s}]$$

For the aircraft to fly west (in the negative x direction), with a wind pushing it south (negative y direction), it must fly in a northwesterly direction so that its north directed component (positive y direction) equals the magnitude of the wind velocity. **Figure A** shows this. From **Figure A**, the vertical and horizontal components of the aircraft speed are:

$$|v_x| = v \sin \theta \quad |v_y| = v \cos \theta$$

The y component of the aircraft velocity must equal (in magnitude) the wind velocity:

$$222.222 \cos \theta = 27.778 \rightarrow \cos \theta = \frac{27.778}{222.222} = 0.125 \rightarrow \theta = 82.82^\circ = 82^\circ 49'$$

The aircraft must fly at $82^\circ 49'$ from the north (y direction) or at $7^\circ 11'$ north relative to west (negative x direction).

b. The magnitudes of the horizontal and vertical components of the aircraft velocity are:

$$v_x = 222.222 \sin 82.82 = 220.48 \quad v_y = 27.778 \quad \left[\frac{\text{m}}{\text{s}} \right]$$

The velocity vector of the airplane is:

$$\mathbf{v} = -\hat{\mathbf{x}}220.48 + \hat{\mathbf{y}}27.778 \quad \left[\frac{\text{m}}{\text{s}} \right]$$

The speed in the London-NY direction is 220.48 m/s.

c. The total distance traveled in the westerly direction is 5,000 km at the speed above. The time required is:

$$T = \frac{5,000,000}{220.48} = 22,678 \quad [\text{s}]$$

This is 6 hours and 18 minutes.

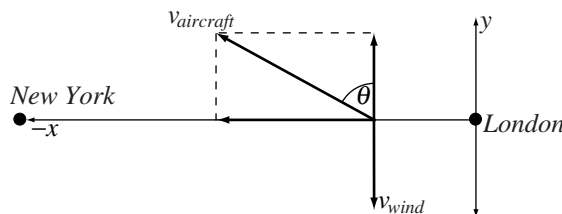


Figure A.

Problem 1.4.

a. $|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{5^2 + 3^2 + (-1)^2} = \sqrt{35} = 5.916$
 b. $\mathbf{A} + \mathbf{B} = \hat{\mathbf{x}}[5 + (-3)] + \hat{\mathbf{y}}(3 + 5) + \hat{\mathbf{z}}[(-1) + (-2)] = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}8 - \hat{\mathbf{z}}3$
 c. $\mathbf{A} - \mathbf{B} = \hat{\mathbf{x}}[5 - (-3)] + \hat{\mathbf{y}}(3 - 5) + \hat{\mathbf{z}}[(-1) - (-2)] = \hat{\mathbf{x}}8 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}$
 d. $\mathbf{B} - \mathbf{A} = \hat{\mathbf{x}}[(-3) - 5] + \hat{\mathbf{y}}(5 - 3) + \hat{\mathbf{z}}[(-2) - (-1)] = -\hat{\mathbf{x}}8 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}$
 e. $\hat{\mathbf{c}} = \frac{\mathbf{B} - \mathbf{A}}{|\mathbf{B} - \mathbf{A}|} = \frac{-\hat{\mathbf{x}}8 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}}{\sqrt{(-8)^2 + 2^2 + (-1)^2}} = -\hat{\mathbf{x}}\frac{8}{\sqrt{69}} + \hat{\mathbf{y}}\frac{2}{\sqrt{69}} - \hat{\mathbf{z}}\frac{1}{\sqrt{69}}$

Problem 1.5.

a. See Figures A through F

b.

$$\begin{aligned} \mathbf{A} + \mathbf{B} + \mathbf{C} &= \hat{x}(3 - 3 + 1) + \hat{y}(1 + 3 - 2) + \hat{z}(3 + 3 + 2) = \hat{x} + \hat{y}2 + \hat{z}8 \\ \mathbf{A} + \mathbf{B} - \mathbf{C} &= \hat{x}(3 - 3 - 1) + \hat{y}(1 + 3 + 2) + \hat{z}(3 + 3 - 2) = -\hat{x} + \hat{y}6 + \hat{z}4 \\ \mathbf{A} - \mathbf{B} - \mathbf{C} &= \hat{x}(3 + 3 - 1) + \hat{y}(1 - 3 + 2) + \hat{z}(3 - 3 - 2) = \hat{x}5 - \hat{z}2 \\ \mathbf{A} - \mathbf{B} + \mathbf{C} &= \hat{x}(3 + 3 + 1) + \hat{y}(1 - 3 - 2) + \hat{z}(3 - 3 + 2) = \hat{x}7 - \hat{y}4 + \hat{z}2 \\ \mathbf{A} + (\mathbf{B} - \mathbf{C}) &= \hat{x}[3 + (-3 - 1)] + \hat{y}[1 + (3 + 2)] + \hat{z}[3 + (3 - 2)] = -\hat{x} + \hat{y}6 + \hat{z}4 \\ (\mathbf{A} + \mathbf{B}) - \mathbf{C} &= \hat{x}[(3 - 3) - 1] + \hat{y}[(1 + 3) + 2] + \hat{z}[(3 + 3) - 2] = -\hat{x} + \hat{y}6 + \hat{z}4 \end{aligned}$$

c. The direct method is easier in computation but is not as expressive, especially when physical quantities are involved.

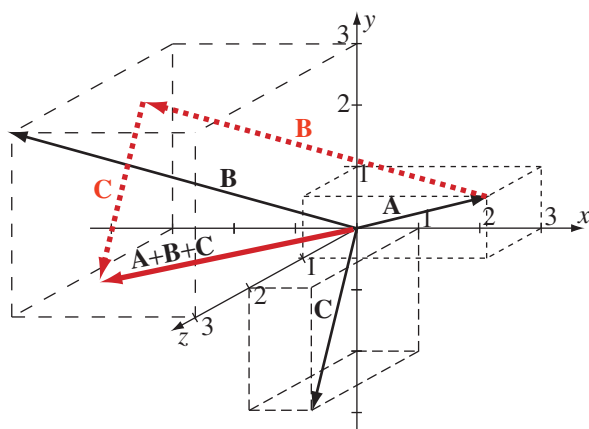


Figure A

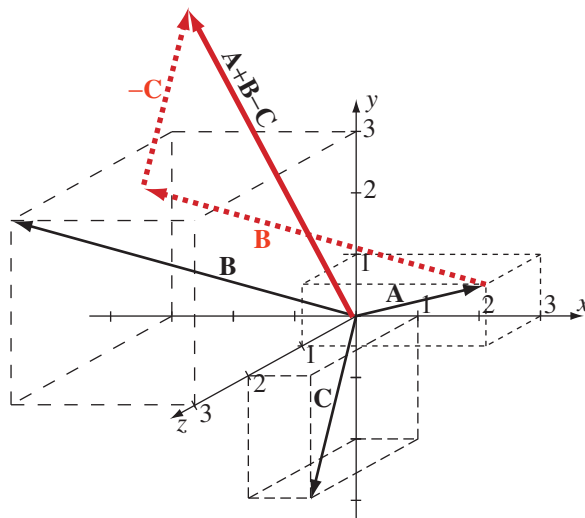


Figure B

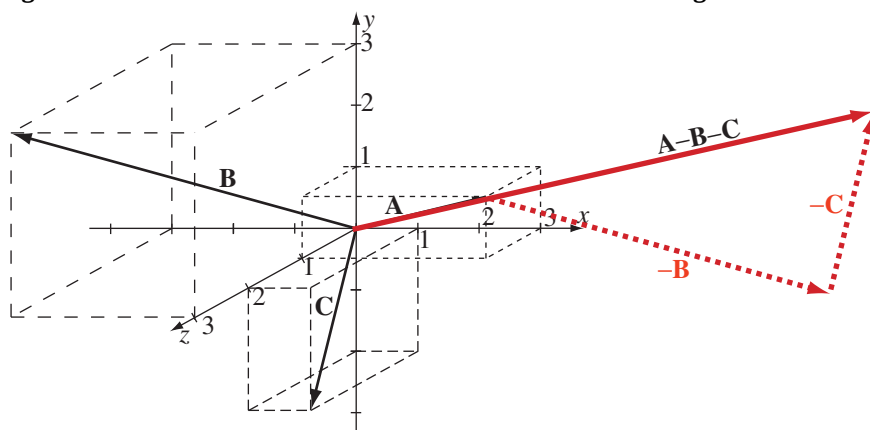


Figure C

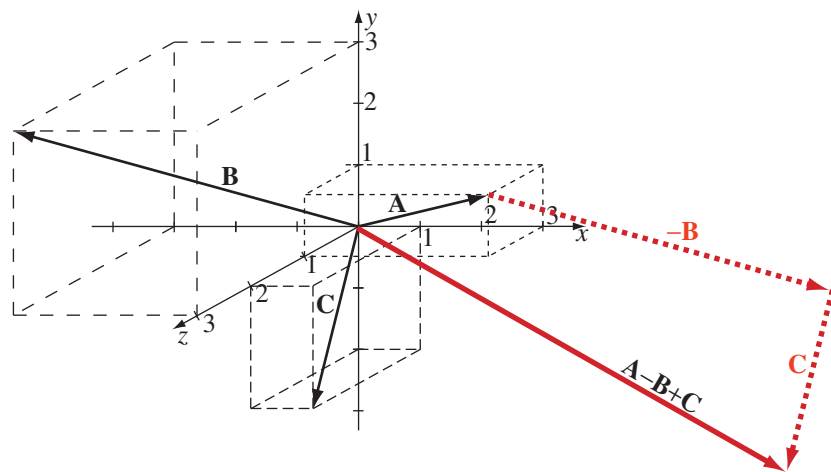


Figure D

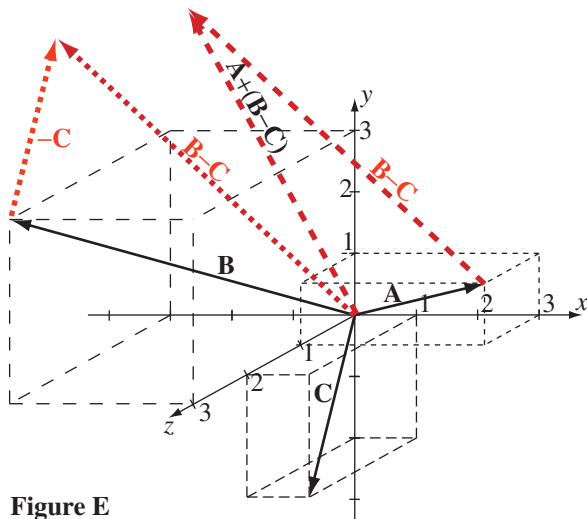


Figure E

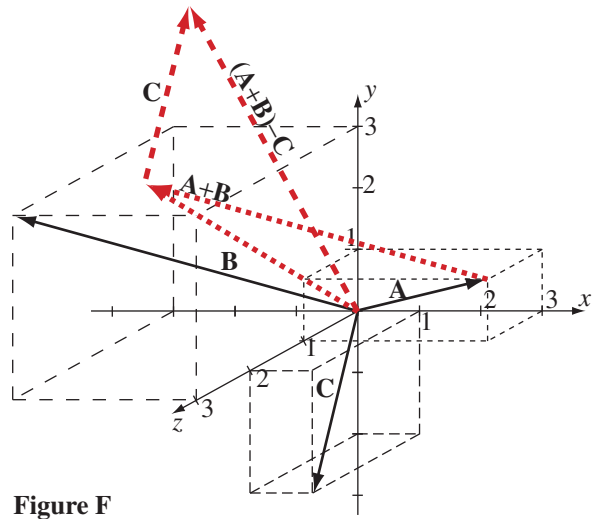


Figure F

Problem 1.6.

a. The velocity vector of the satellite before firing the rocket:

$$\mathbf{v}_1 = \hat{\phi} 16,000 \quad [\text{km/h}]$$

The velocity vector of the satellite after firing the rocket:

$$\mathbf{v}_2 = \hat{\phi}(16,000 - 1,000) = \hat{\phi} 15,000 \quad [\text{km/h}]$$

b. The scaling factor is the ratio between the final and initial speed:

$$k = \frac{v_2}{v_1} = \frac{15,000}{16,000} = 0.9375$$

Problem 1.7.

The vector is scaled by a factor of 2 without change in direction. Motion is in the direction of the unit vector.

a. The vector after scaling is:

$$\mathbf{v} = \hat{\mathbf{x}} \frac{v_x}{k} + \hat{\mathbf{y}} \frac{v_y}{k} + \hat{\mathbf{z}} \frac{v_z}{k} = \hat{\mathbf{x}} \frac{300}{2} + \hat{\mathbf{y}} \frac{50}{2} - \hat{\mathbf{z}} \frac{100}{2} = \hat{\mathbf{x}} 150 + \hat{\mathbf{y}} 25 - \hat{\mathbf{z}} 50 \quad \left[\frac{\text{m}}{\text{s}} \right]$$

The direction of motion is the unit vector in the direction of the vector:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\hat{\mathbf{x}} 150 + \hat{\mathbf{y}} 25 - \hat{\mathbf{z}} 50}{\sqrt{150^2 + 25^2 + (-50)^2}} = \hat{\mathbf{x}} \frac{6}{\sqrt{41}} + \hat{\mathbf{y}} \frac{1}{\sqrt{41}} - \hat{\mathbf{z}} \frac{2}{\sqrt{41}} = \hat{\mathbf{x}} \frac{150}{160.08} + \hat{\mathbf{y}} \frac{25}{160.08} - \hat{\mathbf{z}} \frac{50}{160.08} \quad \left[\frac{\text{m}}{\text{s}} \right]$$

b. The speed of the particle is the magnitude of its velocity: $v = |\mathbf{v}| = 160.08 \text{ m/s}$

Problem 1.8.

The scalar component of $\mathbf{F}$ in the direction of $\mathbf{A}$ is the projection of $\mathbf{F}$ on $\mathbf{A}$. The vector component is in the direction of the unit vector $\hat{\mathbf{A}}$. The unit vector in the direction of the $\mathbf{A}$ is:

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{x}} 3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}}{\sqrt{3^2 + 1^2 + (-1)^2}} = \hat{\mathbf{x}} \frac{3}{\sqrt{11}} + \hat{\mathbf{y}} \frac{1}{\sqrt{11}} - \hat{\mathbf{z}} \frac{1}{\sqrt{11}}$$

To find the projection of $\mathbf{F}$ onto $\mathbf{A}$ we use the scalar product:

$$\mathbf{F} \cdot \mathbf{A} = F A \cos \theta_{FA} \quad \Rightarrow \quad F \cos \theta_{FA} = \frac{\mathbf{F} \cdot \mathbf{A}}{|\mathbf{A}|}$$

$$F = |\mathbf{F}| = \frac{1}{r}, \quad A = |\mathbf{A}| = \sqrt{11} \quad \mathbf{F} \cdot \mathbf{A} = \frac{3}{r} \quad \rightarrow \quad F \cos \theta_{FA} = \frac{\mathbf{F} \cdot \mathbf{A}}{A} = \frac{(3/r)}{\sqrt{11}} = \frac{3}{r\sqrt{11}}$$

Thus, the vector component of $\mathbf{F}$ in the direction of $\mathbf{A}$ is $\hat{\mathbf{A}} F_A$:

$$\mathbf{F}_A = \hat{\mathbf{A}} F \cos \theta_{FA} = \left(\hat{\mathbf{x}} \frac{3}{\sqrt{11}} + \hat{\mathbf{y}} \frac{1}{\sqrt{11}} - \hat{\mathbf{z}} \frac{1}{\sqrt{11}} \right) \frac{1}{r} \frac{3}{\sqrt{11}} = \hat{\mathbf{x}} \frac{9}{11r} + \hat{\mathbf{y}} \frac{3}{11r} - \hat{\mathbf{z}} \frac{3}{11r}$$

Problem 1.9.

The unit vector is found through the cross product. Two vectors in each of the planes are first found. Then their cross-product gives a normal vector. Division of this vector by the magnitude of the vector gives the normal unit vector. The plane is given as: $3x + 4y + z = 0$. Find three points on that plane so that two vectors may be identified. The points are arbitrary as long as they are distinct and on the plane. The following three points satisfy these conditions:

$$P_0(0, 0, 0), \quad P_1(0, 1, -4), \quad \text{and} \quad P_2(1, 1, -7)$$

Let P_0 and P_1 form a vector $\mathbf{A}$; P_0 and P_2 form a vector $\mathbf{B}$:

$$\mathbf{A} = \hat{\mathbf{y}} - \hat{\mathbf{z}}4 \quad \mathbf{B} = \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}7$$

The cross-product is:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & -4 \\ 1 & 1 & -7 \end{vmatrix} = \hat{\mathbf{x}}(-7 + 4) + \hat{\mathbf{y}}(-4 - 0) + \hat{\mathbf{z}}(0 - 1) = -\hat{\mathbf{x}}3 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}$$

A unit vector normal to this plane is

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = -\hat{\mathbf{x}}\frac{3}{\sqrt{26}} - \hat{\mathbf{y}}\frac{4}{\sqrt{26}} - \hat{\mathbf{z}}\frac{1}{\sqrt{26}}$$

Problem 1.10.

The unit vector is found through the cross product. Two vectors in each of the planes are first found. Then their cross-product gives a normal vector. Division of this vector by the magnitude of the vector gives the normal unit vector.

a. The surface is described as $z = -x - y$. Find three points on that plane so that two vectors may be identified. The points are arbitrary as long as they are distinct and on the plane. The following three points satisfy these conditions:

$$P_0(0, 0, 0), \quad P_1(-1, -1, 2), \quad \text{and} \quad P_2(0, 1, -1)$$

Let P_0 and P_1 form a vector $\mathbf{A}$; P_0 and P_2 form a vector $\mathbf{B}$:

$$\mathbf{A} = -\hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}2 \quad \mathbf{B} = \hat{\mathbf{y}} - \hat{\mathbf{z}}$$

To evaluate the vector product we use the determinant method:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\mathbf{x}}(1 - 2) + \hat{\mathbf{y}}(0 - 1) + \hat{\mathbf{z}}(-1 - 0) = -\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$$

A unit vector normal to the surface is :

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{-\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}}{\sqrt{(-1)^2 + (-1)^2 + (-1)^2}} = -\hat{\mathbf{x}}\frac{1}{\sqrt{3}} - \hat{\mathbf{y}}\frac{1}{\sqrt{3}} - \hat{\mathbf{z}}\frac{1}{\sqrt{3}}$$

Note: The solution is not unique. It depends on the choice of the vectors $\mathbf{A}$ and $\mathbf{B}$.

b. To find two vectors we again specify three points in the plane as follows:

$$\begin{aligned} 4x - 3y + z + 5 = 0 &\Rightarrow x = 0, y = 0 \Rightarrow 0 - 0 + z + 5 = 0 &\rightarrow P_1(0, 0, -5) \\ x = 1, y = 1 &\Rightarrow 4 - 3 + z + 5 = 0 \Rightarrow z = -6 &\rightarrow P_2(1, 1, -6) \\ x = -1, y = +1 &\Rightarrow -4 - 3 + z + 5 = 0 \Rightarrow z = -5 + 7 = 2 &\rightarrow P_3(-1, 1, 2) \end{aligned}$$

The vectors are from P_1 to P_2 and from P_1 to P_3 are:

$$\begin{aligned} \mathbf{A} = \overrightarrow{P_1 P_2} &= \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}} \quad \mathbf{B} = \overrightarrow{P_1 P_3} = -\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}7 \\ \mathbf{A} \times \mathbf{B} &= (\hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}) \times (-\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}7) = \hat{\mathbf{z}} - \hat{\mathbf{y}}7 + \hat{\mathbf{z}} + \hat{\mathbf{x}}7 + \hat{\mathbf{y}} + \hat{\mathbf{x}} = \hat{\mathbf{x}}8 - \hat{\mathbf{y}}6 + \hat{\mathbf{z}}2 \\ |\mathbf{A} \times \mathbf{B}| &= \sqrt{64 + 36 + 4} = \sqrt{104} \end{aligned}$$

therefore:

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \hat{\mathbf{x}}\frac{8}{\sqrt{104}} - \hat{\mathbf{y}}\frac{6}{\sqrt{104}} + \hat{\mathbf{z}}\frac{2}{\sqrt{104}} = \hat{\mathbf{x}}0.784 - \hat{\mathbf{y}}0.588 + \hat{\mathbf{z}}0.196$$

See note in (a)

c. The surface is given as: $z = ax + by$. Three points on the surface are:

$$P_0(0, 0, 0), \quad P_1(0, 1, b), \quad \text{and} \quad P_2(1, 1, a+b)$$

The vectors from P_0 to P_1 and from P_1 and P_2 are:

$$\mathbf{A} = \hat{\mathbf{y}} + \hat{\mathbf{z}}b \quad \mathbf{B} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}(a+b)$$

The vector product is:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & b \\ 1 & 1 & a+b \end{vmatrix} = \hat{\mathbf{x}}(a+b-b) + \hat{\mathbf{y}}(b-0) + \hat{\mathbf{z}}(0-1) = \hat{\mathbf{x}}a + \hat{\mathbf{y}}b - \hat{\mathbf{z}}$$

A normal unit vector is therefore:

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{\hat{\mathbf{x}}a + \hat{\mathbf{y}}b - \hat{\mathbf{z}}}{\sqrt{a^2 + b^2 + 1}}$$

See note in (a)

Problem 1.11.

The area of the triangle may be found from the fact that the magnitude of the cross product of two vectors equals the area of the parallelogram defined by the two vectors. Taking two vectors to be two sides of the triangle, the area calculated equals twice the area of the triangle (see **Figure A**). The area of the triangle is then given as:

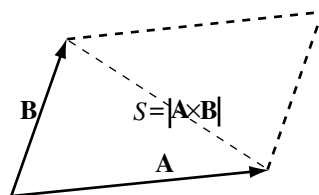


Figure A.

$$S = \frac{|\mathbf{A} \times \mathbf{B}|}{2}$$

The three vertex points define two vectors as follows:

$$\mathbf{A} = \hat{\mathbf{x}}(a' - a) + \hat{\mathbf{y}}(b' - b) + \hat{\mathbf{z}}(c' - c) \quad \mathbf{B} = \hat{\mathbf{x}}(a'' - a) + \hat{\mathbf{y}}(b'' - b) + \hat{\mathbf{z}}(c'' - c)$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a' - a & b' - b & c' - c \\ a'' - a & b'' - b & c'' - c \end{vmatrix} = \hat{\mathbf{x}}[(b' - b)(c'' - c) - (b'' - b)(c' - c)] + \hat{\mathbf{y}}[(a'' - a)(c' - c) - (a' - a)(c'' - c)] + \hat{\mathbf{z}}[(a' - a)(b'' - b) - (a'' - a)(b' - b)]$$

The area of the triangle is therefore:

$$S = \frac{1}{2} \left\{ [(b' - b)(c'' - c) - (b'' - b)(c' - c)]^2 + [(a'' - a)(c' - c) - (a' - a)(c'' - c)]^2 + [(a' - a)(b'' - b) - (a'' - a)(b' - b)]^2 \right\}^{1/2}$$

Problem 1.12.

Since we need to express the relations in terms of sines, the vector product may be used, again by defining the sides of the triangle as vectors as shown in **Figure A**. The magnitude of the vector product of each of the two vectors equals twice the area of the triangle. Thus:

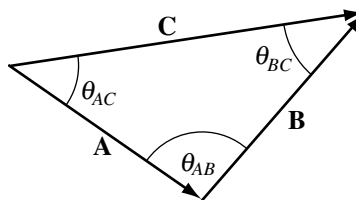


Figure A.

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}AB \sin \theta_{AB}, \quad \mathbf{A} \times \mathbf{C} = \hat{\mathbf{n}}AC \sin \theta_{AC}, \quad \mathbf{B} \times \mathbf{C} = \hat{\mathbf{n}}BC \sin \theta_{BC}$$

Taking the magnitudes:

$$\frac{AB \sin \theta_{AB}}{2} = \frac{AC \sin \theta_{AC}}{2} = \frac{BC \sin \theta_{BC}}{2}$$

Dividing each product by ABC gives:

$$\frac{\sin \theta_{AB}}{C} = \frac{\sin \theta_{AC}}{B} = \frac{\sin \theta_{BC}}{A} \quad \rightarrow \quad \frac{A}{\sin \theta_{BC}} = \frac{B}{\sin \theta_{AC}} = \frac{C}{\sin \theta_{AB}}$$

Problem 1.13.

The z axis is described by a unit vector in the z direction. The angle between **A** and the z axis is then found from the scalar product. In (b), the cross product between the vectors is found.

a. Let vector **B** be along the z axis, $\mathbf{B} = \hat{\mathbf{z}}$. The angle between **A** and the z axis is:

$$\theta_{Az} = \cos^{-1} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} \right) = \cos^{-1} \left(\frac{(\hat{\mathbf{x}}3 + \hat{\mathbf{y}}1 - \hat{\mathbf{z}}2) \cdot \hat{\mathbf{z}}}{\sqrt{(3^2 + 1 + 2^2)}(1)} \right) = \cos^{-1} \left(\frac{-2}{\sqrt{14}} \right) = 180^\circ \pm 57.688^\circ$$

b. A vector **C** perpendicular to **A** and the z axis may be defined as follows:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = \hat{\mathbf{x}} - \hat{\mathbf{y}}3$$

Problem 1.14.

The magnitude of a vector in the direction of another vector is the magnitude of its projection. This may be calculated by first finding the unit vector in the direction of **A** and then the scalar product between **F** and $\hat{\mathbf{A}}$ (see **Figure A**).

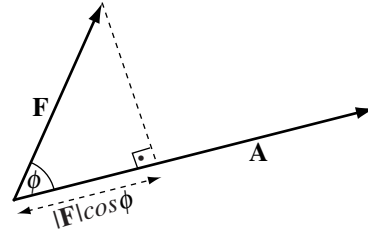


Figure A

$$\hat{\mathbf{A}} = \frac{-\hat{\mathbf{x}}3 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2}{\sqrt{3^2 + 2^2 + 2^2}} = -\hat{\mathbf{x}} \frac{3}{\sqrt{17}} + \hat{\mathbf{y}} \frac{2}{\sqrt{17}} - \hat{\mathbf{z}} \frac{2}{\sqrt{17}}$$

Now:

$$\mathbf{F} \cdot \hat{\mathbf{A}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}) \cdot \left(-\hat{\mathbf{x}} \frac{3}{\sqrt{17}} + \hat{\mathbf{y}} \frac{2}{\sqrt{17}} - \hat{\mathbf{z}} \frac{2}{\sqrt{17}} \right) = -\frac{3}{\sqrt{17}} + \frac{10}{\sqrt{17}} - \frac{2}{\sqrt{17}} = \frac{9}{\sqrt{17}}$$

The scalar product of **F** in the direction of **A** is $9/\sqrt{17}$.

An alternative way is to calculate the product:

$$\frac{\mathbf{F} \cdot \mathbf{A}}{|\mathbf{A}|} = \frac{(\hat{\mathbf{x}} + \hat{\mathbf{y}}5 - \hat{\mathbf{z}}) \cdot (-\hat{\mathbf{x}}3 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}2)}{\sqrt{3^2 + 2^2 + 2^2}} = \frac{-3 + 10 - 2}{\sqrt{3^2 + 2^2 + 2^2}} = \frac{9}{\sqrt{17}}$$

which gives the same result.

Problem 1.15.

The area of a parallelogram equals the magnitude of the vector product between any two vectors that make its sides, provided both vectors emanate from the same vertex. While the choice of vectors is not unique, the area remains the same regardless of the pair of vectors chosen.

a. For the given vertices, we define two vectors emanating from P_1 . One ends in P_2 , the other in P_3 . The two vectors are:

$$\begin{aligned} \mathbf{A} &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) = \hat{\mathbf{x}}(2 - 7) + \hat{\mathbf{y}}(1 - 3) + \hat{\mathbf{z}}(0 - 1) = -\hat{\mathbf{x}}5 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}} \\ \mathbf{B} &= \hat{\mathbf{x}}(x_3 - x_1) + \hat{\mathbf{y}}(y_3 - y_1) + \hat{\mathbf{z}}(z_3 - z_1) = \hat{\mathbf{x}}(2 - 7) + \hat{\mathbf{y}}(2 - 3) + \hat{\mathbf{z}}(5 - 1) = -\hat{\mathbf{x}}5 - \hat{\mathbf{y}} + \hat{\mathbf{z}}4 \end{aligned}$$

The vector product $\mathbf{A} \times \mathbf{B}$ is:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -5 & -2 & -1 \\ -5 & -1 & 4 \end{vmatrix} = \hat{\mathbf{x}}(-8 - 1) + \hat{\mathbf{y}}(5 + 20) + \hat{\mathbf{z}}(5 - 10) = -\hat{\mathbf{x}}9 + \hat{\mathbf{y}}25 - \hat{\mathbf{z}}5$$

The area of the parallelogram is:

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(-9)^2 + (25)^2 + (-5)^2} = \sqrt{731} = 27.04$$

b. The solution is not unique. There are a total of three different parallelograms (one given above in **Figure A**, the other two shown in **Figures B** and **C**) but the area of each parallelogram is the same.

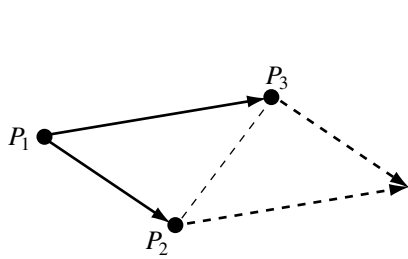


Figure A

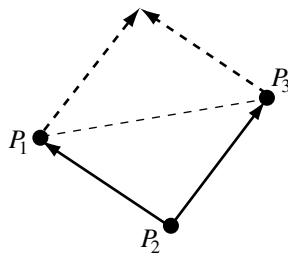


Figure B

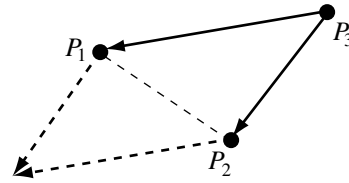


Figure C

Problem 1.16.

a. The four vertices of the parallelepiped form six vectors:

1. Vector $\mathbf{A}_1$ from P_1 to P_2
2. Vector $\mathbf{A}_2$ from P_1 to P_3
3. Vector $\mathbf{A}_3$ from P_1 to P_4
4. Vector $\mathbf{A}_4$ from P_3 to P_2
5. Vector $\mathbf{A}_5$ from P_4 to P_2
6. Vector $\mathbf{A}_6$ from P_4 to P_3 .

Of course, any one of these vectors may be reversed.

b. There are four vertices, each emanating three vectors. Thus, there are four parallelepipeds:

1. With P_1 as the root vertex.
2. With P_2 as the root vertex.
3. With P_3 as the root vertex.
4. With P_4 as the root vertex.

c. There are four distinct parallelepiped that can be defined that satisfy the requirement of three vectors emanating from a node. These are defined by taking each vertex as a root node.

1. P_1 as the root node:

$$\begin{aligned}\mathbf{A}_1 &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) = \hat{\mathbf{x}}(a - 0) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(1 - 0) = \hat{\mathbf{x}}a + \hat{\mathbf{z}} \\ \mathbf{A}_2 &= \hat{\mathbf{x}}(x_3 - x_1) + \hat{\mathbf{y}}(y_3 - y_1) + \hat{\mathbf{z}}(z_3 - z_1) = \hat{\mathbf{x}}a + \hat{\mathbf{y}}2 + \hat{\mathbf{z}}c \\ \mathbf{A}_3 &= \hat{\mathbf{x}}(x_4 - x_1) + \hat{\mathbf{y}}(y_4 - y_1) + \hat{\mathbf{z}}(z_4 - z_1) = \hat{\mathbf{x}} + \hat{\mathbf{y}}b + \hat{\mathbf{z}}\end{aligned}$$

The volume of the parallelepiped is defined by $\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3)$. We calculate the vector product first:

$$\mathbf{A}_2 \times \mathbf{A}_3 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a & 2 & c \\ 1 & b & 1 \end{vmatrix} = \hat{\mathbf{x}}(2 - bc) + \hat{\mathbf{y}}(c - a) + \hat{\mathbf{z}}(ab - 2)$$

The volume of the parallelepiped is:

$$\mathbf{A}_1 \cdot (\mathbf{A}_2 \times \mathbf{A}_3) = a(2 - bc) + (ab - 2) = a(2 - bc + b) - 2$$

2. P_2 as the root node.

$$\begin{aligned}\mathbf{B}_1 &= \hat{\mathbf{x}}(x_1 - x_2) + \hat{\mathbf{y}}(y_1 - y_2) + \hat{\mathbf{z}}(z_1 - z_2) = -\hat{\mathbf{x}}a - \hat{\mathbf{z}} \\ \mathbf{B}_2 &= \hat{\mathbf{x}}(x_3 - x_2) + \hat{\mathbf{y}}(y_3 - y_2) + \hat{\mathbf{z}}(z_3 - z_2) = \hat{\mathbf{y}}2 + \hat{\mathbf{z}}(c - 1) \\ \mathbf{B}_3 &= \hat{\mathbf{x}}(x_4 - x_2) + \hat{\mathbf{y}}(y_4 - y_2) + \hat{\mathbf{z}}(z_4 - z_2) = \hat{\mathbf{x}}(1 - a) + \hat{\mathbf{y}}b \\ \mathbf{B}_2 \times \mathbf{B}_3 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 2 & c-1 \\ 1-a & b & 0 \end{vmatrix} = \hat{\mathbf{x}}[-b(c-1)] + \hat{\mathbf{y}}[(c-1)(1-a)] + \hat{\mathbf{z}}[-2(1-a)] \\ \mathbf{B}_1 \cdot (\mathbf{B}_2 \times \mathbf{B}_3) &= ab(c-1) + 2(1-a) = a(bc - b - 2) + 2\end{aligned}$$

3. P_3 as the root node.

$$\begin{aligned}\mathbf{C}_1 &= \hat{\mathbf{x}}(x_1 - x_3) + \hat{\mathbf{y}}(y_1 - y_3) + \hat{\mathbf{z}}(z_1 - z_3) = -\hat{\mathbf{x}}a - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}c \\ \mathbf{C}_2 &= \hat{\mathbf{x}}(x_2 - x_3) + \hat{\mathbf{y}}(y_2 - y_3) + \hat{\mathbf{z}}(z_2 - z_3) = -\hat{\mathbf{y}}2 + \hat{\mathbf{z}}(1 - c) \\ \mathbf{C}_3 &= \hat{\mathbf{x}}(x_4 - x_3) + \hat{\mathbf{y}}(y_4 - y_3) + \hat{\mathbf{z}}(z_4 - z_3) = \hat{\mathbf{x}}(1 - a) + \hat{\mathbf{y}}(b - 2) + \hat{\mathbf{z}}(1 - c)\end{aligned}$$

$$\mathbf{C}_2 \times \mathbf{C}_3 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & -2 & 1-c \\ 1-a & b-2 & c-1 \end{vmatrix} = \hat{\mathbf{x}}[-2(1-c) - (b-2)(1-c)] + \hat{\mathbf{y}}[(1-c)(1-a)] + \hat{\mathbf{z}}[-2(1-a)]$$

$$\mathbf{C}_1 \cdot (\mathbf{C}_2 \times \mathbf{C}_3) = 2a(1-c) + a(b-2)(1-c) - 2(1-a)(1-c) - 2c(1-a) = a(b-bc+2) - 2$$

4. P_4 as the root node.

$$\mathbf{D}_1 = \hat{\mathbf{x}}(x_1 - x_4) + \hat{\mathbf{y}}(y_1 - y_4) + \hat{\mathbf{z}}(z_1 - z_4) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}b - \hat{\mathbf{z}}$$

$$\mathbf{D}_2 = \hat{\mathbf{x}}(x_2 - x_4) + \hat{\mathbf{y}}(y_2 - y_4) + \hat{\mathbf{z}}(z_2 - z_4) = \hat{\mathbf{x}}(a-1) - \hat{\mathbf{y}}b$$

$$\mathbf{D}_3 = \hat{\mathbf{x}}(x_3 - x_4) + \hat{\mathbf{y}}(y_3 - y_4) + \hat{\mathbf{z}}(z_3 - z_4) = \hat{\mathbf{x}}(a-1) + \hat{\mathbf{y}}(2-b) + \hat{\mathbf{z}}(c-1)$$

$$\mathbf{D}_2 \times \mathbf{D}_3 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a-1 & -b & 0 \\ a-1 & 2-b & c-1 \end{vmatrix} = \hat{\mathbf{x}}[-b(c-1)] + \hat{\mathbf{y}}[-(a-1)(c-1)] + \hat{\mathbf{z}}[(a-1)(2-b) + b(a-1)]$$

$$\mathbf{D}_1 \cdot (\mathbf{D}_2 \times \mathbf{D}_3) = b(c-1) + b(a-1)(c-1) - (a-1)(2-b) - b(a-1) = a(bc-b-2) + 2$$

Note: the four parallelepipeds have the same volume. In the calculations above, volumes (2) and (4) are the negatives of volumes (1) and (3). This merely means that the sequence chosen for the vector products in (2) and (4) needs to be reversed or, alternatively, that the magnitude of the scalar triple product needs to be used.

Problem 1.17.

The volume of the parallelepiped equals the triple scalar product $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$. The area of the parallelogram forming the basis of the parallelepiped equals $|\mathbf{A} \times \mathbf{B}|$. Thus, the height equals:

$$h_1 = \frac{\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}}{|\mathbf{A} \times \mathbf{B}|}$$

Similar calculation provides the height of the second and the third parallelepiped.

$$\mathbf{a.} \quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \hat{\mathbf{x}}(2-2) + \hat{\mathbf{y}}(4-2) + \hat{\mathbf{z}}(1-2) = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = -4 - 3 = -7 \quad \frac{(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}}{|\mathbf{A} \times \mathbf{B}|} = \frac{-7}{\sqrt{5}} = -3.13 \quad \rightarrow \quad h = 3.13$$

Note: The negative sign simply means that we took the vector product $\mathbf{A} \times \mathbf{B}$ rather than $\mathbf{B} \times \mathbf{A}$, hence we need to take the absolute value of the projection.

$$\mathbf{b.} \quad \mathbf{A} \times \mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \hat{\mathbf{x}}(3+4) + \hat{\mathbf{y}}(2-3) + \hat{\mathbf{z}}(-2-1) = \hat{\mathbf{x}}7 - \hat{\mathbf{y}} - \hat{\mathbf{z}}3$$

$$(\mathbf{A} \times \mathbf{C}) \cdot \mathbf{B} = 14 - 1 - 6 = 7 \quad \frac{(\mathbf{A} \times \mathbf{C}) \cdot \mathbf{B}}{|\mathbf{A} \times \mathbf{C}|} = \frac{7}{\sqrt{59}} = 0.911 \quad \rightarrow \quad h = 0.911$$

$$\mathbf{c.} \quad \mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \hat{\mathbf{x}}(3+4) + \hat{\mathbf{y}}(2-6) + \hat{\mathbf{z}}(-4-1) = \hat{\mathbf{x}}7 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}5$$

$$(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = 7 - 4 - 10 = -7 \quad \frac{(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}}{|\mathbf{B} \times \mathbf{C}|} = \frac{-7}{\sqrt{90}} = -0.738 \quad \rightarrow \quad h = 0.738$$

The height of the parallelepipeds is different (but their volumes is the same)

Problem 1.18.

a. See **Figure A**.

b. To find the zero slope points we calculate the derivative with respect to x and y and set to zero. The scalar function may be written as:

$$P = x^2(y - 2) + x(y - 2) = x^2y - 2x^2 - xy + 2x + 1$$

$$\frac{\partial P}{\partial y} = x^2 - x = x(x - 1) = 0 \quad \rightarrow \quad x = 0, x = 1$$

$$\frac{\partial P}{\partial x} = 2x(y - 2) - (y - 2) = 0 \quad \rightarrow \quad y = 2 \text{ for } x = 0 \text{ or } x = 1$$

At point $x = 1, y = 2$ and $x = 0, y = 2$, the slope of the field is zero.

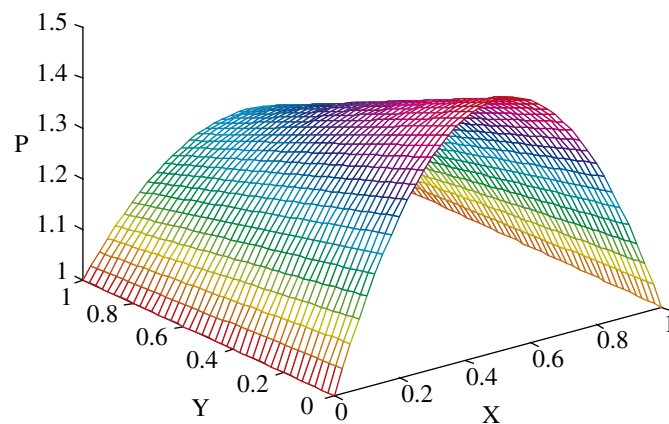


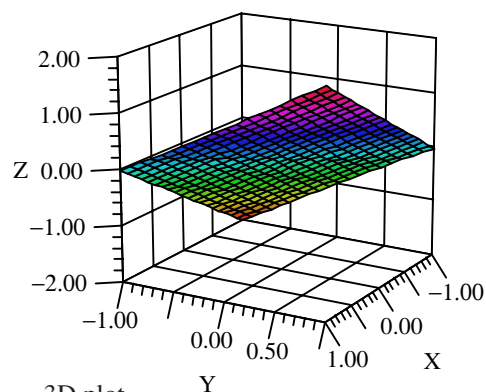
Figure A

Problem 1.19.

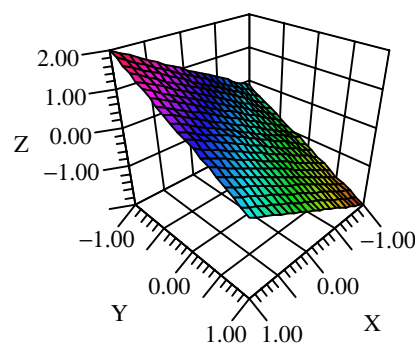
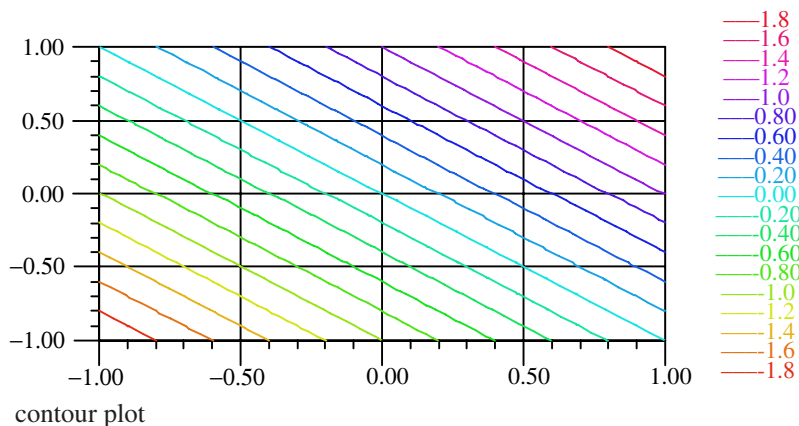
$A = x + y$. See **Figure A** for a 3D plot and a contour plot.

$B = x - y$. See **Figure B** for a 3D plot and a contour plot.

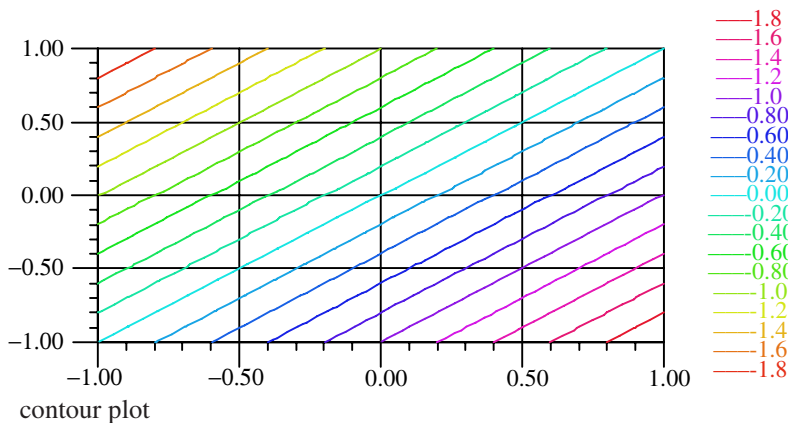
$C = \frac{x + y}{\sqrt{x^2 + y^2}}$. See **Figure C** for a 3D plot and a contour plot.



3D plot
Figure A



3D plot
Figure B



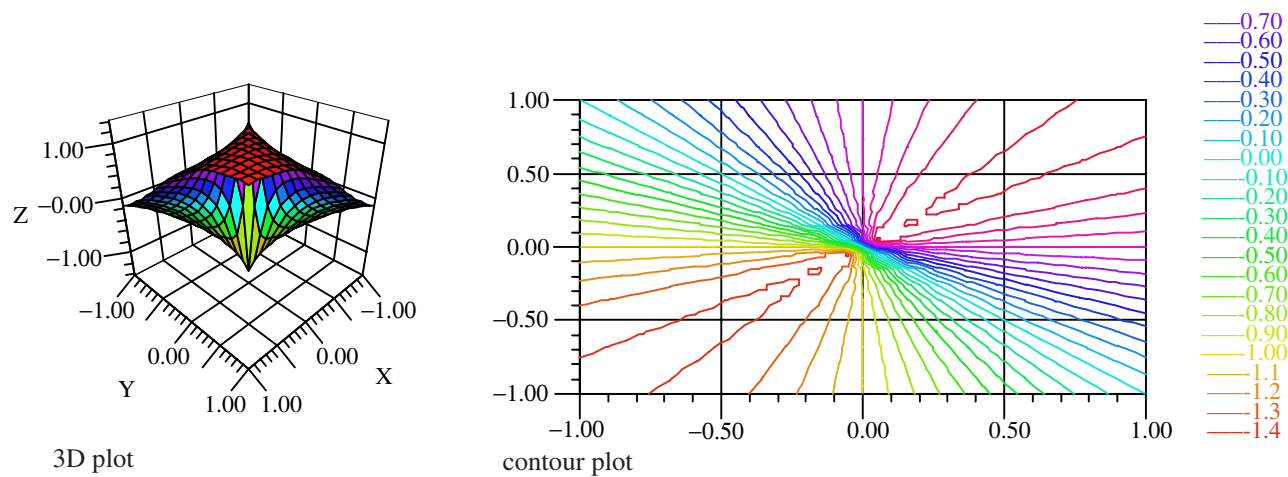


Figure C

Problem 1.20.

$\mathbf{A} = \hat{x}\hat{y} + \hat{y}\hat{x}$. See Figure A.

$\mathbf{B} = \hat{x}\hat{y} - \hat{y}\hat{x}$. See Figure B.

$\mathbf{C} = \frac{\hat{x}\hat{x} + \hat{y}\hat{y}}{\sqrt{x^2 + y^2}}$. See Figure C.

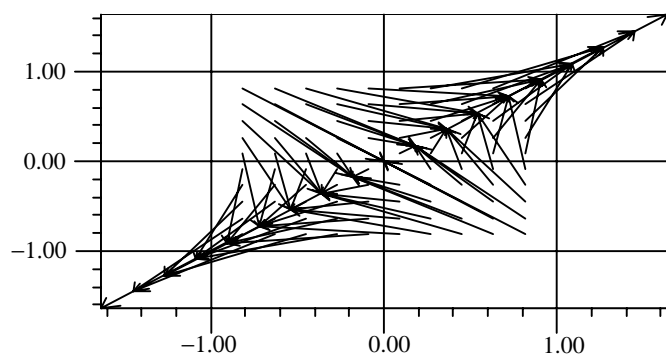


Figure A

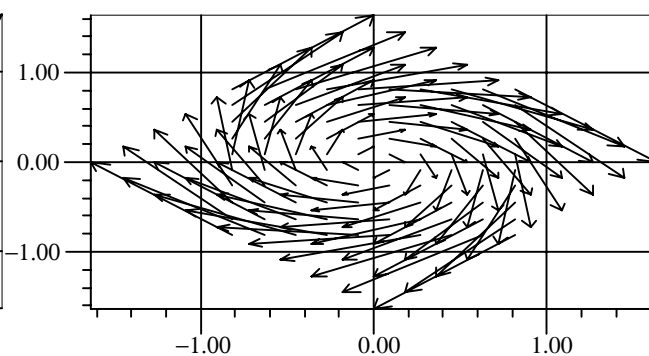


Figure B

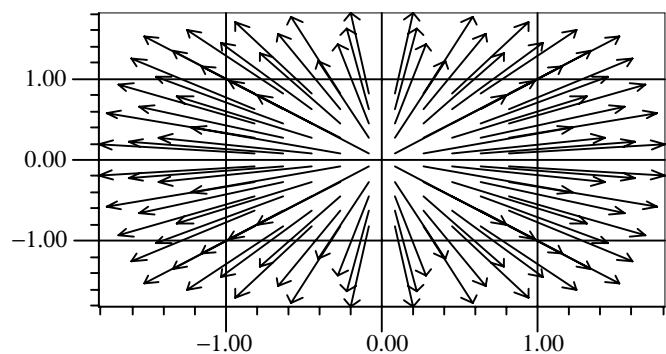


Figure C

Problem 1.21.

a. The transformation from Cartesian to cylindrical coordinates is (Eq. (1.63)):

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z \quad (1)$$

Substitution of the Cartesian coordinates of the three points in Eq. (1) gives:

$$\begin{aligned} r_1 &= \sqrt{1^2 + 1^2} = \sqrt{2}, & \phi_1 &= \tan^{-1} \left(\frac{1}{1} \right) = 45^\circ, & z_1 &= 1 \\ r_2 &= \sqrt{1^2 + 1^2} = \sqrt{2}, & \phi_2 &= \tan^{-1} \left(\frac{1}{1} \right) = 45^\circ, & z_2 &= 0 \\ r_3 &= \sqrt{0^2 + 1^2} = 1, & \phi_3 &= \tan^{-1} \left(\frac{1}{0} \right) = 90^\circ, & z_3 &= 1 \end{aligned}$$

$$P_1(\sqrt{2}, 45^\circ, 1), \quad P_2(\sqrt{2}, 45^\circ, 0), \quad \text{and} \quad P_3(1, 90^\circ, 1) \quad (2)$$

The transformation to spherical coordinates is:

$$R = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right), \quad \phi = \tan^{-1} \left(\frac{y}{x} \right) \quad (3)$$

Substitution of the Cartesian coordinates of the three points in Eq. (3) gives:

$$\begin{aligned} R_1 &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}, & \theta_1 &= \tan^{-1} \left(\frac{\sqrt{1^2 + 1^2}}{1} \right) = 54.736^\circ, & \phi_1 &= \tan^{-1} \left(\frac{1}{1} \right) = 45^\circ \\ R_2 &= \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}, & \theta_2 &= \tan^{-1} \left(\frac{\sqrt{1^2 + 1^2}}{0} \right) = 90^\circ, & \phi_2 &= \tan^{-1} \left(\frac{1}{1} \right) = 45^\circ \\ R_3 &= \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}, & \theta_3 &= \tan^{-1} \left(\frac{\sqrt{0^2 + 1^2}}{1} \right) = 45^\circ, & \phi_3 &= \tan^{-1} \left(\frac{1}{0} \right) = 90^\circ \end{aligned}$$

$$P_1(\sqrt{3}, 54.736^\circ, 45^\circ), \quad P_2(\sqrt{2}, 90^\circ, 45^\circ), \quad \text{and} \quad P_3(\sqrt{2}, 45^\circ, 90^\circ)$$

b. To find the equation of the plane we define two vectors connecting the three points so that they emanate from P_1 . A third vector also emanating from P_1 to some arbitrary point $P(x, y, z)$ is also defined. Now we calculate the volume of the parallelepiped defined by the three vectors. Setting this volume to zero forces the three vectors to be in a plane. The resulting equation is the equation of the plane.

$$\begin{aligned} \mathbf{A} &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1) = -\hat{\mathbf{z}} \\ \mathbf{B} &= \hat{\mathbf{x}}(x_3 - x_1) + \hat{\mathbf{y}}(y_3 - y_1) + \hat{\mathbf{z}}(z_3 - z_1) = -\hat{\mathbf{x}} \\ \mathbf{C} &= \hat{\mathbf{x}}(x - x_1) + \hat{\mathbf{y}}(y - y_1) + \hat{\mathbf{z}}(z - z_1) = \hat{\mathbf{x}}(x - 1) + \hat{\mathbf{y}}(y - 1) + \hat{\mathbf{z}}(z - 1) \end{aligned}$$

The equation of a plane through three points in Cartesian coordinates is:

$$\begin{aligned} \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) &= (\hat{\mathbf{x}}(x - 1) + \hat{\mathbf{y}}(y - 1) + \hat{\mathbf{z}}(z - 1)) \cdot ((-\hat{\mathbf{z}}) \times (-\hat{\mathbf{x}})) = \\ &= (\hat{\mathbf{x}}(x - 1) + \hat{\mathbf{y}}(y - 1) + \hat{\mathbf{z}}(z - 1)) \cdot \hat{\mathbf{y}} = (y - 1) = 0 \end{aligned}$$

or:

$$y = 1$$

c. In the cylindrical coordinate system, we use the coordinate transformation (Eq. (1.62)):

$$x = r \cos \phi \quad y = r \sin \phi \quad z = z$$

Since we only need the transformation for y, we write:

$$y = r \sin \phi \quad \rightarrow \quad r \sin \phi = 1$$

Note that this describes a plane as ϕ varies from zero to 2π for any value of r .

d. In the spherical coordinate system, we use again the transformation from spherical to Cartesian coordinates (Eq. (1.82)):

$$x = R \sin(\theta) \cos(\phi) \quad y = R \sin(\theta) \sin(\phi) \quad z = R \cos(\theta)$$

Again we only need the transformation for y:

$$y = R \sin(\theta) \sin(\phi) \quad \rightarrow \quad R \sin(\theta) \sin(\phi) = 1$$

and again, this describes a plane for values of θ between zero and π , and for values of ϕ varying between zero and 2π . For these values, R varies from 1 to ∞ over the plane.

Problem 1.22.

a. In Cartesian coordinates:

$$x^2 + y^2 + z^2 = a^2$$

b. In cylindrical coordinates, since $r = \sqrt{x^2 + y^2}$:

$$r^2 + z^2 = a^2$$

c. In spherical coordinates:

$$R = a$$

Problem 1.23.

We choose a general point on the sphere and transform it into Cartesian coordinates.

a. We choose a point:

$$P(a, \theta, \phi)$$

In Cartesian coordinates:

$$x = a \sin(\theta) \cos(\phi), \quad y = a \sin(\theta) \sin(\phi), \quad z = a \cos(\theta)$$

or:

$$P(a \sin(\theta) \cos(\phi), a \sin(\theta) \sin(\phi), a \cos(\theta))$$

b. In Spherical coordinates, the point is simply described as:

$$P(a, \theta, \phi)$$

Problem 1.24.

The transformation from Cartesian to spherical coordinates for the scalar components is (Eq. (1.89)):

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The vector has scalar components $A_x=2, A_y=-5, A_z=3$. Substituting these gives:

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Expanding, this gives:

$$\begin{aligned} A_R &= 2 \sin \theta \cos \phi - 5 \sin \theta \sin \phi + 3 \cos \theta \\ A_\theta &= 2 \cos \theta \cos \phi - 5 \cos \theta \sin \phi - 3 \sin \theta \\ A_\phi &= -2 \sin \phi - 5 \cos \phi \end{aligned}$$

The vector **A** in spherical coordinates is:

$$\mathbf{A} = \hat{\mathbf{R}}(2 \sin \theta \cos \phi - 5 \sin \theta \sin \phi + 3 \cos \theta) + \hat{\boldsymbol{\theta}}(2 \cos \theta \cos \phi - 5 \cos \theta \sin \phi - 3 \sin \theta) + \hat{\boldsymbol{\phi}}(-2 \sin \phi - 5 \cos \phi) \quad (1)$$

Now we must transform the coordinates given from Cartesian to spherical coordinates (Eq. (1.81)):

$$R = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right), \quad \phi = \tan^{-1} \left(\frac{y}{x} \right)$$

for $P(-2, 3, 1)$, these give:

$$\begin{aligned} R &= \sqrt{x^2 + y^2 + z^2} = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14} \\ \theta &= \tan^{-1} \left(\frac{\sqrt{(-2)^2 + 3^2}}{1} \right) = \tan^{-1}(3.60555) = 74.5^\circ \\ \phi &= \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(\frac{3}{-2} \right) = \tan^{-1}(-1.5) = -56.31^\circ \quad \text{or:} \quad \phi = 123.69^\circ \end{aligned}$$

The second solution is valid since $x < 0, y > 0$ (second quadrant). Thus, $\phi = 123.69^\circ$. Substituting these values into the vector **A** in Eq. (1) gives:

$$\begin{aligned} \mathbf{A} &= \hat{\mathbf{R}}(2 \sin(74.5^\circ) \cos(123.69^\circ) - 5 \sin(74.5^\circ) \sin(123.69^\circ) + 3 \cos(74.5^\circ)) + \\ &\quad \hat{\boldsymbol{\theta}}(2 \cos(74.5^\circ) \cos(123.69^\circ) - 5 \cos(74.5^\circ) \sin(123.69^\circ) - 3 \sin(74.5^\circ)) + \\ &\quad \hat{\boldsymbol{\phi}}(-2 \sin(123.69^\circ) - 5 \cos(123.69^\circ)) \end{aligned}$$

or, after evaluating the various $\sin$ and $\cos$ terms:

$$\mathbf{A} = -\hat{\mathbf{R}} 4.276 - \hat{\boldsymbol{\theta}} 4.299 + \hat{\boldsymbol{\phi}} 1.1094$$

Problem 1.25.

a. To transform the vector to Cartesian coordinates we use the general transformation from cylindrical to Cartesian coordinates (Eq. (1.70)):

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\cos\phi \\ -2\sqrt{r} \\ r\phi \end{bmatrix}$$

Expanding, we obtain:

$$\begin{aligned} A_x &= A_r \cos\phi - A_\phi \sin\phi = 3\cos^2\phi + 2\sqrt{r}\sin\phi \\ A_y &= A_r \sin\phi + A_\phi \cos\phi = 3\cos\phi \sin\phi - 2\sqrt{r}\cos\phi \\ A_z &= r\phi \end{aligned}$$

Thus, the vector is:

$$\mathbf{A} = \hat{\mathbf{x}}(3\cos^2\phi + 2\sqrt{r}\sin\phi) + \hat{\mathbf{y}}(3\cos\phi \sin\phi - 2\sqrt{r}\cos\phi) + \hat{\mathbf{z}}r\phi$$

b. In spherical coordinates we use the transformation from cylindrical to spherical coordinates in Eq. (1.92):

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\cos\phi \\ -2\sqrt{r} \\ r\phi \end{bmatrix}$$

Expanding gives the scalar components:

$$A_R = 3\cos\phi \sin\theta + r\phi \cos\theta, \quad A_\theta = 3\cos\phi \cos\theta - r\phi \sin\theta, \quad A_\phi = -2\sqrt{r}$$

Problem 1.26.

Position vectors are calculated as any vector but the root of the vector is at (0,0,0). Thus:

$$\begin{aligned} \text{a.} \quad \mathbf{r}_1 &= \hat{\mathbf{x}}(a - 0) + \hat{\mathbf{y}}(b - 0) + \hat{\mathbf{z}}(c - 0) = \hat{\mathbf{x}}a + \hat{\mathbf{y}}b + \hat{\mathbf{z}}c \\ \text{b.} \quad \mathbf{r}_2 &= \hat{\mathbf{x}}(a' - 0) + \hat{\mathbf{y}}(b' - 0) + \hat{\mathbf{z}}(c' - 0) = \hat{\mathbf{x}}a' + \hat{\mathbf{y}}b' + \hat{\mathbf{z}}c' \\ \text{c.} \quad \mathbf{R} &= \hat{\mathbf{x}}(a' - a) + \hat{\mathbf{y}}(b' - b) + \hat{\mathbf{z}}(c' - c) \\ \text{d.} \quad \mathbf{r}_2 - \mathbf{r}_1 &= (\hat{\mathbf{x}}a' + \hat{\mathbf{y}}b' + \hat{\mathbf{z}}c') - (\hat{\mathbf{x}}a + \hat{\mathbf{y}}b + \hat{\mathbf{z}}c) = \\ &\quad \hat{\mathbf{x}}(a' - a) + \hat{\mathbf{y}}(b' - b) + \hat{\mathbf{z}}(c' - c) = \mathbf{R} \end{aligned}$$

Problem 1.27.

It is easiest to first transform the coordinates into Cartesian coordinates. Then we can define vectors, position vectors and the like and, if necessary, transform them back into any other system of coordinates. This is not the only way but the most straightforward way.

Transformation of coordinates from spherical to Cartesian coordinates (Eq. (1.82)):

$$x = R\sin(\theta)\cos(\phi) \quad y = R\sin(\theta)\sin(\phi) \quad z = R\cos(\theta)$$

Substituting the coordinates for point P_1 and P_2 we get:

$$\begin{aligned} x_1 &= 3\sin(0^\circ)\cos(30^\circ) = 0 & y_1 &= 3\sin(0^\circ)\sin(30^\circ) = 0 & z_1 &= 3\cos(0^\circ) = 3 \\ x_2 &= 3\sin(45^\circ)\cos(45^\circ) = 1.5 & y_2 &= 3\sin(45^\circ)\sin(45^\circ) = 1.5 & z_2 &= 3\cos(45^\circ) = \frac{3\sqrt{2}}{2} \end{aligned}$$

The two points are:

$$P_1(0,0,3) \quad P_2(1.5,1.5,\frac{3\sqrt{2}}{2})$$

a. The position vectors for P_1 and P_2 , denoted as $\mathbf{r}_1$ and $\mathbf{r}_2$ are:

$$\mathbf{r}_1 = \hat{\mathbf{z}}3 \quad \mathbf{r}_2 = \hat{\mathbf{x}}1.5 + \hat{\mathbf{y}}1.5 + \hat{\mathbf{z}}\frac{3\sqrt{2}}{2}$$

b. The vector $\mathbf{R}$ from P_1 to P_2 is:

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1 = \hat{\mathbf{x}}1.5 + \hat{\mathbf{y}}1.5 + \hat{\mathbf{z}}\frac{3\sqrt{2}}{2} - \hat{\mathbf{z}}3 = \hat{\mathbf{x}}1.5 + \hat{\mathbf{y}}1.5 + \hat{\mathbf{z}}(\frac{3\sqrt{2}}{2} - 3)$$

c. The position vectors as well as the vectors are now transformed into Cylindrical coordinates and spherical coordinates using the appropriate transformation formulas.

Cylindrical coordinates:

For $\mathbf{r}_1$

$$\begin{bmatrix} r_{1r} \\ r_{1\phi} \\ r_{1z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{1x} \\ r_{1y} \\ r_{1z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Expanding gives:

$$r_{1r} = \hat{\mathbf{z}}3 \rightarrow \mathbf{r}_1 = \hat{\mathbf{z}}3$$

For $\mathbf{r}_2$:

$$\begin{bmatrix} r_{2r} \\ r_{2\phi} \\ r_{2z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1.5 \\ 3\sqrt{2}/2 \end{bmatrix}$$

or:

$$r_{2r} = 1.5\cos\phi + 1.5\sin\phi, \quad r_{2\phi} = -1.5\sin\phi + 1.5\cos\phi, \quad r_{2z} = 3\sqrt{2}/2$$

the vector is:

$$\mathbf{r}_2 = \hat{\mathbf{r}}(1.5\cos\phi + 1.5\sin\phi) + \hat{\boldsymbol{\phi}}(-1.5\sin\phi + 1.5\cos\phi) + \hat{\mathbf{z}}3\sqrt{2}/2$$

For $\mathbf{R}$:

$$\begin{bmatrix} R_r \\ R_\phi \\ R_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1.5 \\ \frac{3\sqrt{2}-6}{2} \end{bmatrix}$$

or:

$$\mathbf{R} = \hat{\mathbf{r}}(1.5\cos\phi + 1.5\sin\phi) + \hat{\boldsymbol{\phi}}(-1.5\sin\phi + 1.5\cos\phi) + \hat{\mathbf{z}}\frac{3\sqrt{2}-6}{2}$$

Spherical coordinates:

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

or:

$$r_{1R} = 3\cos\theta, \quad r_{1\theta} = -3\sin\theta$$

The vector $\mathbf{r}_1$ is:

$$\mathbf{r}_1 = \hat{\mathbf{R}}3\cos\theta - \hat{\boldsymbol{\theta}}3\sin\theta$$

For $\mathbf{r}_2$:

$$\begin{bmatrix} r_{2R} \\ r_{2\theta} \\ r_{2\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1.5 \\ 3\sqrt{2}/2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{r}_2 = & \hat{\mathbf{R}}\left(1.5\sin\theta\cos\phi + 1.5\sin\theta\sin\phi + \frac{3\sqrt{2}}{2}\cos\theta\right) \\ & + \hat{\boldsymbol{\theta}}\left(1.5\cos\theta\cos\phi + 1.5\cos\theta\sin\phi - \frac{3\sqrt{2}}{2}\sin\theta\right) + \hat{\boldsymbol{\phi}}(-1.5\sin\phi + 1.5\cos\phi) \end{aligned}$$

For $\mathbf{R}$:

$$\begin{bmatrix} r_{2R} \\ r_{2\theta} \\ r_{2\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1.5 \\ \frac{3\sqrt{2}-6}{2} \end{bmatrix}$$

and:

$$\begin{aligned} \mathbf{R} = & \hat{\mathbf{R}}\left(1.5\sin\theta\cos\phi + 1.5\sin\theta\sin\phi + \frac{3\sqrt{2}-6}{2}\cos\theta\right) \\ & + \hat{\boldsymbol{\theta}}\left(1.5\cos\theta\cos\phi + 1.5\cos\theta\sin\phi - \frac{3\sqrt{2}-6}{2}\sin\theta\right) + \hat{\boldsymbol{\phi}}(-1.5\sin\phi + 1.5\cos\phi) \end{aligned}$$