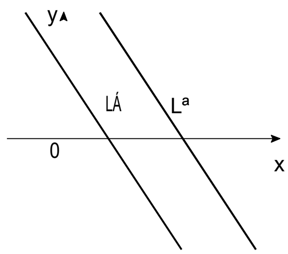
2 SYSTEMS OF LINEAR EQUATIONS AND MATRICES

2.1 Systems of Linear Equations: An Introduction

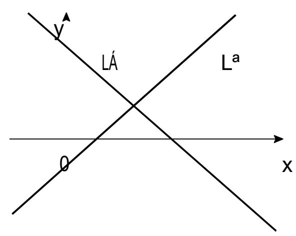
Concept Questions page 79

1. a. There may be no solution, a unique solution, or infinitely many solutions.

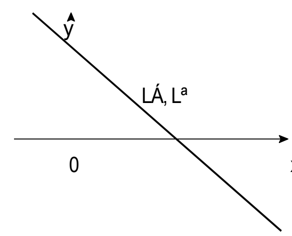
   b. There is no solution if the two lines represented by the given system of linear equations are parallel and distinct; there is a unique solution if the two lines intersect at precisely one point; there are infinitely many solutions if the two lines are parallel and coincident.



No solution



A unique solution

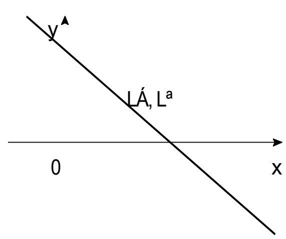


Infinitely many solutions

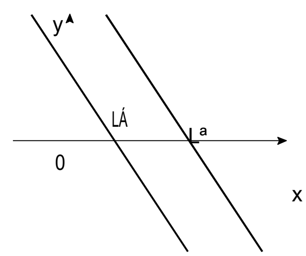
2. a. i. The system is dependent if the two equations in the system describe the same line.

   ii. The system is inconsistent if the two equations in the system describe two lines that are parallel and distinct.

   b.



Two (coincident) lines in a dependent system



Two lines in an inconsistent system

Exercises page 79

1. Solving the first equation for x, we find x □ $3y$ □ 1. Substituting this value of x into the second equation yields 4 □$3y$ □ 1□ □ $3y$ □ 11, so $12y$ □ 4 □ $3y$ □ 11 and y □ 1. Substituting this value of y into the first equation gives x □ 3□1□ □ 1 □ 2. Therefore, the unique solution of the system is □2□1□.

2. Solving the first equation for x, we have $2x$ □ $4y$ □ 10, so x □ $2y$ □ 5. Substituting this value of x into the second equation, we have 3 □$2y$ □ 5□ □ $2y$ □ 1, $6y$ □ 15 □ $2y$ □ 1, $8y$ □ 16, and y □ 2. Then x □ 2 □2□ □ 5 □ □1. Therefore, the solution is □□1□ 2□.

3. Solving the first equation for x, we have $x = 7 - 4y$. Substituting this value of x into the second equation, we have $\frac{1}{2}(7 - 4y) + 2y = 5$, so $7 - 4y + 4y = 10$, and $7 = 10$. Clearly, this is impossible and we conclude that the system of equations has no solution.

4. Solving the first equation for x, we obtain $3x = 7 + 4y$, so $x = \frac{7}{3} + \frac{4}{3}y$. Substituting this value of x into the second equation, we obtain $9\left(\frac{7}{3} + \frac{4}{3}y\right) - 12y = 14$, so $21 + 12y - 12y = 14$, or $21 = 14$. Since this is impossible, we conclude that the system of equations has no solution.

5. Solving the first equation for x, we obtain $x = 7 - 2y$. Substituting this value of x into the second equation, we have $2(7 - 2y) - y = 4$, so $14 - 4y - y = 4$, $-5y = -10$, and $y = 2$. Then $x = 7 - 2(2) = 7 - 4 = 3$. We conclude that the solution to the system is $(3, 2)$.

6. Solving the second equation for x, we obtain $x = -\frac{1}{3}y + 2$. Substituting this value of x into the first equation gives $\frac{3}{2}\left(-\frac{1}{3}y + 2\right) - 2y = 4$, $-\frac{1}{2}y - 2y = 4 - 3$, $-\frac{5}{2}y = 1$, and $y = -\frac{2}{5}$. Then $x = 2 - \frac{1}{3}\left(-\frac{2}{5}\right) = \frac{32}{15}$. Therefore, the solution of the system is $\left(\frac{32}{15}, -\frac{2}{5}\right)$.

7. Solving the first equation for x, we have $2x = 5y + 10$, so $x = \frac{5}{2}y + 5$. Substituting this value of x into the second equation, we have $6\left(\frac{5}{2}y + 5\right) - 15y = 30$, $15y + 30 - 15y = 30$, and $0 = 0$. This result tells us that the second equation is equivalent to the first. Thus, any ordered pair of numbers (x, y) satisfying the equation $2x - 5y = 10$ (or $6x - 15y = 30$) is a solution to the system. In particular, by assigning the value t to x, where t is any real number, we find that $y = -2 + \frac{2}{5}t$ so the ordered pair, $\left(t, \frac{2}{5}t - 2\right)$ is a solution to the system, and we conclude that the system has infinitely many solutions.

8. Solving the first equation for x, we have $5x = 6y + 8$, so $x = \frac{6}{5}y + \frac{8}{5}$. Substituting this value of x into the second equation gives $10\left(\frac{6}{5}y + \frac{8}{5}\right) - 12y = 16$, $12y + 16 - 12y = 16$, and $0 = 0$. This result tells us that the second equation is equivalent to the first. Thus, any ordered pair of numbers (x, y) satisfying the equation $5x - 6y = 8$ (or $10x - 12y = 16$) is a solution to the system. In particular, by assigning the value t to x, where t is any real number, we find that $y = \frac{5}{6}t - \frac{4}{3}$. So the ordered pair, $\left(t, \frac{5}{6}t - \frac{4}{3}\right)$ is a solution to the system, and we conclude that the system has infinitely many solutions.

9. Solving the first equation for x, we obtain $4x - 5y = 14$, so $4x = 14 + 5y$, and $x = \frac{14}{4} + \frac{5}{4}y$. Substituting this value of x into the second equation gives $2\left(\frac{7}{2} + \frac{5}{4}y\right) + 3y = -4$, so $7 + \frac{5}{2}y + 3y = -4$, $\frac{11}{2}y = -11$, and $y = -2$. Thus, $x = \frac{7}{2} + \frac{5}{4}(-2) = 1$. We conclude that the ordered pair $(1, -2)$ satisfies the given system of equations.

10. Solving the first equation for x, we have $\frac{5}{4}x - \frac{2}{3}y = 3$, so $\frac{5}{4}x = \frac{2}{3}y + 3$ and $x = \frac{4}{5}\left(\frac{2}{3}y + 3\right) = \frac{8}{15}y + \frac{12}{5}$. Substituting this value of x into the second equation yields $\frac{1}{4}\left(\frac{8}{15}y + \frac{12}{5}\right) + \frac{5}{3}y = 6$, so $\frac{2}{15}y + \frac{3}{5} + \frac{5}{3}y = 6$, $\frac{27}{15}y = \frac{27}{5}$, and $y = 3$. Then $x = \frac{8}{15}(3) + \frac{12}{5} = \frac{20}{5} = 4$. Thus, the ordered pair $(4, 3)$ satisfies the given equation.

11. Solving the first equation for x, we obtain $2x = 3y + 6$, so $x = \frac{3}{2}y + 3$. Substituting this value of x into the second equation gives $6\left(\frac{3}{2}y + 3\right) - 9y = 12$, so $9y + 18 - 9y = 12$ and $18 = 12$, which is impossible. We conclude that the system of equations has no solution.

12. Solving the first equation for y, we obtain $\frac{2}{3}x - y = 5$, so $y = \frac{2}{3}x - 5$. Substituting this value of y into the second equation yields $\frac{1}{2}x - \frac{3}{4}\left(\frac{2}{3}x - 5\right) = \frac{15}{4}$, so $\frac{1}{2}x - \frac{1}{2}x + \frac{15}{4} = \frac{15}{4}$ and $\frac{15}{4} = \frac{15}{4}$. We conclude that the system of equations has infinitely many solutions of the form $\left(t, \frac{2}{3}t - 5\right)$.

13. Solving the first equation for x, we obtain $3x = 5y - 1$, so $x = \frac{5}{3}y - \frac{1}{3}$. Substituting this value of y into the second equation yields $2\left(\frac{5}{3}y - \frac{1}{3}\right) - 4y = -1$, $\frac{10}{3}y - \frac{2}{3} - 4y = -1$, $-\frac{2}{3}y = -\frac{1}{3}$, and $y = \frac{1}{2}$. Thus, $x = \frac{5}{3}\left(\frac{1}{2}\right) - \frac{1}{3} = \frac{1}{2}$, and the system has the unique solution $\left(\frac{1}{2}, \frac{1}{2}\right)$.

14. Solving the first equation for x, we obtain $10x = 15y - 3$, so $x = \frac{3}{2}y - \frac{3}{10}$. Substituting this value of y into the second equation yields $4\left(\frac{3}{2}y - \frac{3}{10}\right) - 6y = -3$, $6y - \frac{6}{5} - 6y = -3$, and $-\frac{6}{5} = -3$, which is impossible. We conclude that the system of equations has no solution.

15. Solving the first equation for x, we obtain $3x = 6y + 2$, so $x = 2y + \frac{2}{3}$. Substituting this value of y into the second equation yields $-\frac{3}{2}\left(2y + \frac{2}{3}\right) + 3y = -1$, $-3y - 1 + 3y = -1$, and $0 = 0$. We conclude that the system of equations has infinitely many solutions of the form $\left(2t + \frac{2}{3}, t\right)$, where t is a parameter.

16. Solving the first equation for x, we obtain $\frac{3}{2}x - \frac{1}{2}y = 1$, so $x = \frac{1}{3}y + \frac{2}{3}$. Substituting this value of y into the second equation yields $-\left(\frac{1}{3}y + \frac{2}{3}\right) + \frac{1}{3}y = -\frac{2}{3}$ and $0 = 0$. We conclude that the system of equations has infinitely many solutions of the form $\left(\frac{1}{3}t + \frac{2}{3}, t\right)$, where t is a parameter.

17. Solving the first equation for y, we obtain $y = -0.2x + 1.8$. Substituting this value of y into the second equation gives $0.4x + 0.3(-0.2x + 1.8) = 0.2$, $0.34x = -0.34$, and $x = -1$. Substituting this value of x into the first equation, we have $y = -0.2(-1) + 1.8 = 2$. Therefore, the solution is $(-1, 2)$.

18. Solving the first equation for x, we find $0.3x = 0.4y + 0.2$, $3x = 4y + 2$, and $x = \frac{4}{3}y + \frac{2}{3}$. Substituting this value of x into the second equation, which we rewrite as $-2x + 5y = 1$, we have $-2\left(\frac{4}{3}y + \frac{2}{3}\right) + 5y = 1$, $-\frac{8}{3}y - \frac{4}{3} + 5y = 1$, $\frac{7}{3}y = \frac{7}{3}$, and $y = 1$. Thus, $x = \frac{4}{3}(1) + \frac{2}{3} = 2$ and the unique solution is $(2, 1)$.

19. Solving the first equation for y, we obtain $y = 2x - 3$. Substituting this value of y into the second equation yields $4x + k(2x - 3) = 4$, so $4x + 2xk - 3k = 4$, $2x(2 + k) = 4 + 3k$, and $x = \frac{4 + 3k}{2(2 + k)}$. Since x is not defined when the denominator of this last expression is zero, we conclude that the system has no solution when $k = -2$.

20. Solving the second equation for x, we have $x = 4 - ky$. Substituting this value of x into the first equation gives $3(4 - ky) + 4y = 12$, so $12 - 3ky + 4y = 12$ and $y(-3k + 4) = 0$. Since this last equation is always true when $k = \frac{4}{3}$, we see that the system has infinitely many solutions when $k = \frac{4}{3}$. When $k = \frac{4}{3}$, $x = 4 - ky = 4 - \frac{4}{3}y$, so the solutions are the set of all ordered pairs $\left(4 - \frac{4}{3}t, t\right)$, where t is a parameter.

$_{3}t^{\square t}$

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21. Solving the first equation for x in terms of y, we have $ax = by + c$ or $x = \frac{b}{a}y + \frac{c}{a}$ (provided $a \neq 0$). Substituting this value of x into the second equation gives $a\left(\frac{b}{a}y + \frac{c}{a}\right) + by = d$, $by + c + by = d$, $2by = d - c$, and $y = \frac{d-c}{2b}$ (provided $b \neq 0$). Substituting this into the expression for x gives $x = \frac{b}{a}\left(\frac{d-c}{2b}\right) + \frac{c}{a} = \frac{d-c}{2a} + \frac{c}{a} = \frac{c+d}{2a}$. Thus, the system has the unique solution $\left(\frac{c+d}{2a}, \frac{d-c}{2b}\right)$ if $a \neq 0$ and $b \neq 0$.

22. Solving the first equation for x in terms of y, we have $ax = -by + e$ or $x = -\frac{b}{a}y + \frac{e}{a}$ (provided $a \neq 0$). Substituting this value of x into the second equation gives $c\left(-\frac{b}{a}y + \frac{e}{a}\right) + dy = f$, $\left(\frac{ad - bc}{a}\right)y = f - \frac{ce}{a}$, and $y = \frac{a}{ad - bc}\left(\frac{af - ce}{a}\right) = \frac{af - ce}{ad - bc}$ (provided $ad - bc \neq 0$). Substituting this into the expression for x gives $x = -\frac{b}{a}\left(\frac{af - ce}{ad - bc}\right) + \frac{e}{a} = \frac{-b(af - ce) + e(ad - bc)}{a(ad - bc)} = \frac{ed - bf}{ad - bc}$. If $a = 0$, the system reduces to

$$\begin{aligned} by &= e \\ cx + dy &= f \end{aligned}$$

and so $y = \frac{e}{b}$ and $x = \frac{bf - ed}{bc}$, provided $b \neq 0$ and $c \neq 0$. Thus, if $a \neq 0$, $b \neq 0$, $c \neq 0$, and $ad - bc \neq 0$, the system has the unique solution $\left(\frac{ed - bf}{ad - bc}, \frac{af - ce}{ad - bc}\right)$.

23. Let x and y denote the number of acres of corn and wheat planted, respectively. Then $x + y = 500$. Since the cost of cultivating corn is \$42/acre and that of wheat \$30/acre and Mr. Johnson has \$18,600 available for cultivation, we have $42x + 30y = 18{,}600$. Thus, the solution is found by solving the system of equations

$$\begin{aligned} x + y &= 500 \\ 42x + 30y &= 18{,}600 \end{aligned}$$

24. Let x be the amount of money Michael invests in the institution that pays interest at the rate of 3% per year and y the amount of money invested in the institution paying 4% per year. Since his total investment is \$2000, we have $x + y = 2000$. Next, since the interest earned during a one-year period was \$72, we have $0.03x + 0.04y = 72$. Thus, the solution is found by solving the system of equations

$$\begin{aligned} x + y &= 2000 \\ 0.03x + 0.04y &= 72 \end{aligned}$$

25. Let x denote the number of pounds of the \$8.00/lb coffee and y denote the number of pounds of the \$9/lb coffee. Then $x + y = 100$. Since the blended coffee sells for \$8.60/lb, we know that the blended mixture is worth $8.60(100) = \$860$. Therefore, $8x + 9y = 860$. Thus, the solution is found by solving the system of equations

$$\begin{aligned} x + y &= 100 \\ 8x + 9y &= 860 \end{aligned}$$

26. Let the amount of money invested in the bonds yielding 4% be x dollars and the amount of money invested in the bonds yielding 5% be y dollars. Then $x + y = 30{,}000$. Also, since the yield from both investments totals \$1320, we have $0.04x + 0.05y = 1320$. Thus, the solution to the problem can be found by solving the system of equations

$$\begin{aligned} x + \quad y &= 30{,}000 \\ 0.04x + 0.05y &= \quad 1320 \end{aligned}$$

27. Let x denote the number of children who ride the bus during the morning shift and y the number of adults who ride the bus during the morning shift. Then $x + y = 1000$. Since the total fare collected is \$1300, we have $0.5x + 1.5y = 1300$. Thus, the solution to the problem can be found by solving the system of equations

$$\begin{aligned} x + \quad y &= 1000 \\ 0.5x + 1.5y &= 1300 \end{aligned}$$

28. Let x, y, and z denote the number of one-bedroom units, two-bedroom townhouses, and three-bedroom townhouses, respectively. Since the total number of units is 192, we have $x + y + z = 192$. Next, the number of family units is equal to the number of one-bedroom units, and this implies that $y + z = x$, or $x - y - z = 0$. Finally, the number of one-bedroom units is three times the number of three-bedroom units, and this implies that $x = 3z$, or $x - 3z = 0$. Summarizing, we have the system

$$\begin{aligned} x + y + z &= 192 \\ x - y - z &= \quad 0 \\ x \qquad - 3z &= \quad 0 \end{aligned}$$

29. Let x and y denote the costs of the ball and the bat, respectively. Then

$$\begin{aligned} x + y &= 110 \\ y - x &= 100 \end{aligned} \quad \text{or} \quad \begin{aligned} x + y &= 110 \\ -x + y &= 100 \end{aligned}$$

30. Let x and y denote the amounts of money invested in projects A and B, respectively. Then

$$\begin{aligned} x + y &= 70{,}000 \\ x - y &= 20{,}000 \end{aligned}$$

31. Let x be the amount of money invested at 3% in a savings account, y the amount of money invested at 4% in mutual funds, and z the amount of money invested at 6% in bonds. Since the total interest was \$10,800, we have $0.03x + 0.04y + 0.06z = 10{,}800$. Also, since the amount of Sid's investment in bonds is twice the amount of the investment in the savings account, we have $z = 2x$. Finally, the interest earned from his investment in bonds was equal to the dividends earned from his money mutual funds, so $0.04y = 0.06z$. Thus, the solution to the problem can be found by solving the system of equations

$$\begin{aligned} 0.03x + 0.04y + 0.06z &= 10{,}800 \\ 2x \qquad\qquad - z &= \qquad 0 \\ 0.04y - 0.06z &= \qquad 0 \end{aligned}$$

32. Let x, y, and z denote the amount to be invested in high-risk, medium-risk, and low-risk stocks, respectively. Since all of the \$400,000 is to be invested, we have $x + y + z = 400{,}000$. The investment goal of a return of \$40,000/year leads to $0.15x + 0.10y + 0.06z = 40{,}000$. Finally, the decision that the investment in low-risk stocks be equal to the sum of the investments in the stocks of the other two categories leads to $z = x + y$. So, we are led to the problem of solving the system

$$\begin{aligned} x + y + z &= 400{,}000 \\ 0.15x + 0.1y + 0.06z &= 40{,}000 \\ x + y - z &= 0 \end{aligned}$$

33. The percentages must add up to 100%, so

$$\begin{aligned} x + y + z &= 100 \\ x + y \phantom{{}+z} &= 67 \\ x \phantom{{}+y} + z &= 17 \end{aligned}$$

34. Let x, y, and z denote the numbers of respondents who answered "yes," "no," and "not sure," respectively. Then we have

$$\begin{aligned} x + y + z &= 1000 \\ y - z &= 370 \\ x - y \phantom{{}-z} &= 340 \end{aligned}$$

35. Let x, y, and z denote the number of 100-lb. bags of grade A, grade B, and grade C fertilizers to be produced. The amount of nitrogen required is $18x + 20y + 24z$, and this must be equal to 26,400, so we have $18x + 20y + 24z = 26{,}400$. Similarly, the constraints on the use of phosphate and potassium lead to the equations $4x + 4y + 3z = 4900$ and $5x + 4y + 6z = 6200$, respectively. Thus we have the problem of finding the solution to the system

$$\begin{aligned} 18x + 20y + 24z &= 26{,}400 && \text{(nitrogen)} \\ 4x + 4y + 3z &= 4900 && \text{(phosphate)} \\ 5x + 4y + 6z &= 6200 && \text{(potassium).} \end{aligned}$$

36. Let x be the number of tickets sold to children, y the number of tickets sold to students, and z the number of tickets sold to adults at that particular screening. Since there was a full house at that screening, we have $x + y + z = 900$. Next, since the number of adults present was equal to one-half the number of students and children present, we have $z = \frac{1}{2}(x + y)$. Finally, the receipts totaled \$5600, and this implies that $4x + 6y + 8z = 5600$. Summarizing, we have the system

$$\begin{aligned} x + y + z &= 900 \\ x + y - 2z &= 0 \\ 4x + 6y + 8z &= 5600 \end{aligned}$$

37. Let x, y, and z denote the number of compact, intermediate, and full-size cars to be purchased, respectively. The cost incurred in buying the specified number of cars is $18{,}000x + 27{,}000y + 36{,}000z$. Since the budget is \$2.25 million, we have the system

$$\begin{aligned} 18{,}000x + 27{,}000y + 36{,}000z &= 2{,}250{,}000 \\ x - 2y \phantom{{}+36{,}000z} &= 0 \\ x + y + z &= 100 \end{aligned}$$

38. Let x be the amount of money invested in high-risk stocks, y the amount of money invested in medium-risk stocks, and z the amount of money invested in low-risk stocks. Since a total of \$200,000 is to be invested, we have $x + y + z = 200{,}000$. Next, since the investment in low-risk stocks is to be twice the sum of the investments in high- and medium-risk stocks, we have $z = 2(x + y)$. Finally, the expected return of the three investments is given by $0.15x + 0.10y + 0.06z$ and the goal of the investment club is that an average return of 9% be realized on the total investment. If this goal is realized, then $0.15x + 0.10y + 0.06z = 0.09(x + y + z)$. Summarizing, we have the system of equations

$$\begin{aligned} x + y + z &= 200{,}000 \\ 2x + 2y - z &= 0 \\ 6x + y - 3z &= 0 \end{aligned}$$

39. Let x be the number of ounces of Food I used in the meal, y the number of ounces of Food II used in the meal, and z the number of ounces of Food III used in the meal. Since 100% of the daily requirement of proteins, carbohydrates, and iron is to be met by this meal, we have the system of linear equations

$$\begin{aligned} 10x + 6y + 8z &= 100 \\ 10x + 12y + 6z &= 100 \\ 5x + 4y + 12z &= 100 \end{aligned}$$

40. Let x, y, and z denote the amounts of money invested in stocks, bonds, and the money market, respectively. Then we have

$$\begin{aligned} x + y + z &= 100{,}000 && \text{(the investments total \$100,000)} \\ 0.12x + 0.08y + 0.04z &= 10{,}000 && \text{(the annual income is \$10,000)} \\ z &= 0.20x + 0.10y && \text{(the investment mix)} \end{aligned}$$

Equivalently,

$$\begin{aligned} x + y + z &= 100{,}000 \\ 12x + 8y + 4z &= 1{,}000{,}000 \\ 20x + 10y - 100z &= 0 \end{aligned}$$

41. Let x, y, and z denote the numbers of front orchestra, rear orchestra, and front balcony seats sold for this performance, respectively. Then we have

$$\begin{aligned} x + y + z &= 1000 && \text{(tickets sold total 1000)} \\ 80x + 60y + 50z &= 62{,}800 && \text{(total revenue)} \\ x + y - 2z &= 400 && \text{(relationship among different types of tickets)} \end{aligned}$$

42. Let x, y, and z denote the numbers of dozens of sleeveless, short-sleeve, and long-sleeve blouses produced per day, respectively. Then we have

$$\begin{aligned} 9x + 12y + 15z &= 4800 \\ 22x + 24y + 28z &= 9600 \\ 6x + 8y + 8z &= 2880 \end{aligned}$$

43. Let x, y, and z denote the numbers of days spent in London, Paris, and Rome, respectively. Then we have

$$\begin{aligned} 280x + 330y + 260z &= 4060 && \text{(hotel bills)} \\ 130x + 140y + 110z &= 1800 && \text{(meals)} \\ x - y - z &= 0 && \text{(since } x = y + z\text{)} \end{aligned}$$