Exercise 1.1. Many centuries ago, a mariner poured 100 cm^3 of water into the ocean. As time passed, the action of currents, tides, and weather mixed the liquid uniformly throughout the earth's oceans, lakes, and rivers. Ignoring salinity, estimate the probability that the next cup of water you drink will contain at least one water molecule that was dumped by the mariner. Assess your chances of ever drinking truly pristine water. [Some possibly useful facts: M_w for water is 18.0 *kg* per *kg-mole*, the radius of the earth is 6370 *km*, the mean depth of the oceans is approximately 3.8 *km* and they cover 71% of the surface of the earth. One cup is ~240 *ml*.]

Solution 1.1. To get started, first list or determine the volumes involved:

$$\upsilon_d = \text{volume of water dumped} = 100\ \text{cm}^3,\ \upsilon_c = \text{volume of a cup} \approx 240\ \text{cm}^3, \text{ and}$$

$$V = \text{volume of water in the oceans} = 4\pi R^2 D\gamma,$$

where, R is the radius of the earth, D is the mean depth of the oceans, and γ is the oceans' coverage fraction. Here we've ignored the ocean volume occupied by salt and have assumed that the oceans' depth is small compared to the earth's diameter. Putting in the numbers produces:

$$V = 4\pi(6.37\times10^6 m)^2(3.8\times10^3 m)(0.71) = 1.376\times10^{18} m^3.$$

For well-mixed oceans, the probability P_o that any water molecule in the ocean came from the dumped water is:

$$P_o = \frac{(100\ \text{cm}^3\ \text{of water})}{(\text{oceans' volume})} = \frac{\upsilon_d}{V} = \frac{1.0\times10^{-4} m^3}{1.376\times10^{18} m^3} = 7.27\times10^{-23},$$

Denote the probability that at least one molecule from the dumped water is part of your next cup as P_1 (this is the answer to the question). Without a lot of combinatorial analysis, P_1 is not easy to calculate directly. It is easier to proceed by determining the probability P_2 that all the molecules in your cup are <u>not</u> from the dumped water. With these definitions, P_1 can be determined from: $P_1 = 1 - P_2$. Here, we can calculate P_2 from:

$$P_2 = (\text{the probability that a molecule was not in the dumped water})^{[\text{number of molecules in a cup}]}.$$

The number of molecules, N_c, in one cup of water is

$$N_c = 240 cm^3 \times \frac{1.00 g}{cm^3} \times \frac{gmole}{18.0 g} \times 6.023\times10^{23} \frac{molecules}{gmole} = 8.03\times10^{24} molecules$$

Thus, $P_2 = (1-P_o)^{N_c} = (1-7.27\times10^{-23})^{8.03\times10^{24}}$. Unfortunately, electronic calculators and modern computer math programs cannot evaluate this expression, so analytical techniques are required. First, take the natural log of both sides, i.e.

$$\ln(P_2) = N_c \ln(1-P_o) = 8.03\times10^{24}\ln(1-7.27\times10^{-23})$$

then expand the natural logarithm using $\ln(1-\varepsilon) \approx -\varepsilon$ (the first term of a standard Taylor series for $\varepsilon \to 0$)

$$\ln(P_2) \cong -N_c \cdot P_o = -8.03\times10^{24}\cdot 7.27\times10^{-23} = -584,$$

and exponentiate to find:

$$P_2 \cong e^{-584} \cong 10^{-254}\ \ldots\ (!)$$

Therefore, $P_1 = 1 - P_2$ is very-very close to unity, so there is a virtual certainty that the next cup of water you drink will have at least one molecule in it from the 100 cm^3 of water dumped many years ago. So, if one considers the rate at which they themselves and everyone else on the planet uses water it is essentially impossible to get a truly fresh cup to drink.

Exercise 1.2. An adult human expels approximately 500 ml of air with each breath during ordinary breathing. Imagining that two people exchanged greetings (one breath each) many centuries ago, and that their breath subsequently has been mixed uniformly throughout the atmosphere, estimate the probability that the next breath you take will contain at least one air molecule from that age-old verbal exchange. Assess your chances of ever getting a truly fresh breath of air. For this problem, assume that air is composed of identical molecules having M_w = 29.0 *kg* per *kg-mole* and that the average atmospheric pressure on the surface of the earth is 100 kPa. Use 6370 *km* for the radius of the earth and 1.20 kg/m^3 for the density of air at room temperature and pressure.

Solution 1.2. To get started, first determine the masses involved.
m = mass of air in one breath = density x volume = $(1.20kg/m^3)(0.5\times10^{-3}m^3) = 0.60\times10^{-3}kg$

M = mass of air in the atmosphere = $4\pi R^2 \int_{z=0}^{\infty} \rho(z)dz$

Here, R is the radius of the earth, z is the elevation above the surface of the earth, and $\rho(z)$ is the air density as function of elevation. From the law for static pressure in a gravitational field, $dP/dz = -\rho g$, the surface pressure, P_s, on the earth is determined from $P_s - P_\infty = \int_{z=0}^{z=+\infty} \rho(z)gdz$ so that:

$$M = 4\pi R^2 \frac{P_s - P_\infty}{g} = 4\pi(6.37\times10^6 m)^2(10^5 Pa) = 5.2\times10^{18}kg.$$

where the pressure (vacuum) in outer space = P_∞ = 0, and g is assumed constant throughout the atmosphere. For a well-mixed atmosphere, the probability P_o that any molecule in the atmosphere came from the age-old verbal exchange is

$$P_o = \frac{2\times(\text{mass of one breath})}{(\text{mass of the whole atmosphere})} = \frac{2m}{M} = \frac{1.2\times10^{-3}kg}{5.2\times10^{18}kg} = 2.31\times10^{-22},$$

where the factor of two comes from one breath for each person. Denote the probability that at least one molecule from the age-old verbal exchange is part of your next breath as P_1 (this is the answer to the question). Without a lot of combinatorial analysis, P_1 is not easy to calculate directly. It is easier to proceed by determining the probability P_2 that all the molecules in your next breath are <u>not</u> from the age-old verbal exchange. With these definitions, P_1 can be determined from: $P_1 = 1 - P_2$. Here, we can calculate P_2 from:

P_2 = (the probability that a molecule was not in the verbal exchange)$^{[\text{number of molecules in a breath}]}$.

The number of molecules, N_b, involved in one breath is

$$N_b = \frac{0.6\times10^{-3}kg}{29.0g/gmole}\times\frac{10^3 g}{kg}\times6.023\times10^{23}\frac{molecules}{gmole} = 1.25\times10^{22}molecules$$

Thus, $P_2 = (1-P_o)^{N_b} = (1-2.31\times10^{-22})^{1.25\times10^{22}}$. Unfortunately, electronic calculators and modern computer math programs cannot evaluate this expression, so analytical techniques are required. First, take the natural log of both sides, i.e.

$$\ln(P_2) = N_b\ln(1-P_o) = 1.25\times10^{22}\ln(1-2.31\times10^{-22})$$

then expand the natural logarithm using $\ln(1-\varepsilon)\approx-\varepsilon$ (the first term of a standard Taylor series for $\varepsilon\to0$)

$$\ln(P_2) \cong -N_b\cdot P_o = -1.25\times10^{22}\cdot2.31\times10^{-22} = -2.89,$$

and exponentiate to find:

$$P_2 \cong e^{-2.89} = 0.056.$$

Therefore, $P_1 = 1 - P_2 = 0.944$ so there is a better than 94% chance that the next breath you take will have at least one molecule in it from the age-old verbal exchange. So, if one considers how often they themselves and everyone else breathes, it is essentially impossible to get a breath of truly fresh air.

Exercise 1.3. In Cartesian coordinates, the Maxwell probability distribution, $f(\mathbf{u}) = f(u_1,u_2,u_3)$, of molecular velocities in a gas flow with average velocity $\mathbf{U} = (U_1,U_2,U_3)$ is

$$f(\mathbf{u}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left\{-\frac{m}{2k_B T}|\mathbf{u}-\mathbf{U}|^2\right\}$$

where n is the number of gas molecules in volume V, m is the molecular mass, k_B is Boltzmann's constant and T is the absolute temperature.

a) Verify that $\mathbf{U}$ is the average molecular velocity, and determine the standard deviations (σ_1, σ_2, σ_3) of each component of $\mathbf{U}$ using $\sigma_i = \left[\iiint\limits_{all\,\mathbf{u}} (u_i - U_i)^2 f(\mathbf{u}) d^3u\right]^{1/2}$ for i = 1, 2, and 3.

b) Using the molecular version of perfect gas law (1.21), determine n/V at room temperature T = 295 K and atmospheric pressure p = 101.3 kPa.

c) Determine n for volumes $V = (10\ \mu m)^3$, $1\ \mu m^3$, and $(0.1\ \mu m)^3$.

d) For the i^{th} velocity component, the standard deviation of the average, $\sigma_{a,i}$, over n molecules is $\sigma_{a,i} = \sigma_i/\sqrt{n}$ when $n >> 1$. For an airflow at $\mathbf{U}$ = (1.0 ms^{-1}, 0, 0), compute the relative uncertainty, $2\sigma_{a,1}/U_1$, at the 95% confidence level for the average velocity for the three volumes listed in part c).

e) For the conditions specified in parts b) and d), what is the smallest volume of gas that ensures a relative uncertainty in U of less than one percent?

Solution 1.3. a) Use the given distribution, and the definition of an average:

$$\mathbf{u}_{ave} = \iiint\limits_{all\,\mathbf{u}} \mathbf{u} f(\mathbf{u}) d^3u = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{u} \exp\left\{-\frac{m}{2k_B T}|\mathbf{u}-\mathbf{U}|^2\right\} d^3u.$$

Consider the first component of $\mathbf{u}$, and separate out the integrations in the "2" and "3" directions.

$$(u_1)_{ave} = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1 \exp\left\{-\frac{m}{2k_B T}\left[(u_1-U_1)^2 + (u_2-U_2)^2 + (u_3-U_3)^2\right]\right\} du_1 du_2 du_3$$

$$= \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty} u_1 \exp\left\{-\frac{m(u_1-U_1)^2}{2k_B T}\right\} du_1 \int_{-\infty}^{+\infty} \exp\left\{-\frac{m(u_2-U_2)^2}{2k_B T}\right\} du_2 \int_{-\infty}^{+\infty} \exp\left\{-\frac{m(u_3-U_3)^2}{2k_B T}\right\} du_3$$

The integrations in the "2" and "3" directions are equal to: $(2\pi k_B T/m)^{1/2}$, so

$$(u_1)_{ave} = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \int_{-\infty}^{+\infty} u_1 \exp\left\{-\frac{m(u_1-U_1)^2}{2k_B T}\right\} du_1$$

The change of integration variable to $\beta = (u_1 - U_1)(m/2k_B T)^{1/2}$ changes this integral to:

$$(u_1)_{ave} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \left(\beta\left(\frac{2k_B T}{m}\right)^{1/2} + U_1\right) \exp\{-\beta^2\} d\beta = 0 + \frac{1}{\sqrt{\pi}} U_1 \sqrt{\pi} = U_1,$$

where the first term of the integrand is an odd function integrated on an even interval so its contribution is zero. This procedure is readily repeated for the other directions to find $(u_2)_{ave} = U_2$, and $(u_3)_{ave} = U_3$. Using the same simplifications and change of integration variables produces:

$$\sigma_1^2 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (u_1-U_1)^2 \exp\left\{-\frac{m}{2k_B T}\left[(u_1-U_1)^2 + (u_2-U_2)^2 + (u_3-U_3)^2\right]\right\} du_1 du_2 du_3$$

$$=\left(\frac{m}{2\pi k_B T}\right)^{1/2}\int_{-\infty}^{+\infty}(u_1-U_1)^2\exp\left\{-\frac{m(u_1-U_1)^2}{2k_B T}\right\}du_1=\frac{1}{\sqrt{\pi}}\left(\frac{2k_B T}{m}\right)\int_{-\infty}^{+\infty}\beta^2\exp\{-\beta^2\}d\beta.$$

The final integral over β is: $\sqrt{\pi}/2$, so the standard deviations of molecular speed are

$$\sigma_1=(k_B T/m)^{1/2}=\sigma_2=\sigma_3,$$

where the second two equalities follow from repeating this calculation for the second and third directions.

b) From (1.21), $n/V=p/k_B T=(101.2kPa)/[1.381\times10^{-23}J/K\cdot 295K]=2.487\times10^{25}m^{-3}$

c) From n/V from part b): $n=2.487\times10^{10}$ for $V = 10^3\ \mu m^3 = 10^{-15}\ m^3$
$n=2.487\times10^{7}$ for $V = 1.0\ \mu m^3 = 10^{-18}\ m^3$
$n=2.487\times10^{4}$ for $V = 0.001\ \mu m^3 = 10^{-21}\ m^3$

d) From (1.22), the gas constant is $R = (k_B/m)$, and $R = 287\ m^2/s^2K$ for air. Compute:
$2\sigma_{a,1}/U_1=2(k_B T/m\,n)^{1/2}/[1m/s]=2(RT/n)^{1/2}/1m/s=2(287\cdot 295/n)^{1/2}=582/\sqrt{n}$. Thus,
for $V = 10^{-15}\ m^3$: $2\sigma_{a,1}/U_1 = 0.00369$,
$V = 10^{-18}\ m^3$: $2\sigma_{a,1}/U_1 = 0.117$, and
$V = 10^{-21}\ m^3$: $2\sigma_{a,1}/U_1 = 3.69$.

e) To achieve a relative uncertainty of 1% we need $n \approx (582/0.01)^2 = 3.39\times10^9$, and this corresponds to a volume of $1.36\times10^{-16}\ m^3$ which is a cube with side dimension $\approx 5\ \mu m$.

Exercise 1.4. Using the Maxwell molecular velocity distribution given in Exercise 1.3 with **U** = 0, determine the average molecular speed = $\bar{v} = \left[\iiint_{all\,\mathbf{u}} |\mathbf{u}|^2 f(\mathbf{u}) d^3u \right]^{1/2}$ and compare it with c = speed of sound in a perfect gas under the same conditions.

Solution 1.4. Use the specified form for $\bar{v}$ and the Maxwell distribution

$$\bar{v}^2 = \iiint_{all\,\mathbf{u}} |\mathbf{u}|^2 f(\mathbf{u}) d^3u = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(u_1^2 + u_2^2 + u_3^2\right) \exp\left\{-\frac{m}{2k_B T}\left(u_1^2 + u_2^2 + u_3^2\right)\right\} du_1 du_2 du_3 .$$

This can be re-arranged and expanded into a total of nine one-variable integrations:

$$\bar{v}^2 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty} u_1^2 \exp\left\{-\frac{mu_1^2}{2k_B T}\right\} du_1 \int_{-\infty}^{+\infty} \exp\left\{-\frac{mu_2^2}{2k_B T}\right\} du_2 \int_{-\infty}^{+\infty} \exp\left\{-\frac{mu_3^2}{2k_B T}\right\} du_3$$
$$+ \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{mu_1^2}{2k_B T}\right\} du_1 \int_{-\infty}^{+\infty} u_2^2 \exp\left\{-\frac{mu_2^2}{2k_B T}\right\} du_2 \int_{-\infty}^{+\infty} \exp\left\{-\frac{mu_3^2}{2k_B T}\right\} du_3$$
$$+ \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{mu_1^2}{2k_B T}\right\} du_1 \int_{-\infty}^{+\infty} \exp\left\{-\frac{mu_2^2}{2k_B T}\right\} du_2 \int_{-\infty}^{+\infty} u_3^2 \exp\left\{-\frac{mu_3^2}{2k_B T}\right\} du_3 .$$

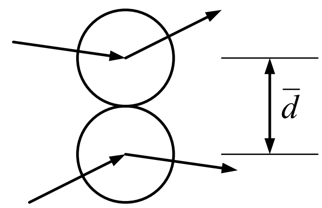
In this arrangement, the six off-diagonal integrals are equal to $\left(2\pi k_B T/m\right)^{1/2}$ and the three on-diagonal integrals are equal to $\left(2k_B T/m\right)^{3/2}\left(\sqrt{\pi}/2\right)$. Thus,

$$\bar{v}^2 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \left(\frac{2\pi k_B T}{m}\right) \left[\left(\frac{2k_B T}{m}\right)^{3/2} + \left(\frac{2k_B T}{m}\right)^{3/2} + \left(\frac{2k_B T}{m}\right)^{3/2}\right] \frac{\sqrt{\pi}}{2}, \text{ or } \bar{v}^2 = \frac{3k_B T}{m}.$$

From (1.22), $R = (k_B/m)$ so $\bar{v} = \sqrt{3RT}$ and this speed has the same temperature dependence but is a factor of $\sqrt{3/\gamma}$ larger than the speed of sound in a perfect gas: $c = \sqrt{\gamma RT}$.

Exercise 1.5. By considering the volume swept out by a moving molecule, estimate how the mean-free path, l, depends on the average molecular cross section dimension $\bar{d}$ and the molecular number density $\tilde{n}$ for nominally spherical molecules. Find a formula for $l\tilde{n}^{1/3}$ (= the ratio of the mean-free path to the mean intermolecular spacing) in terms of the *molecular volume* ($\bar{d}^3$) and the available *volume per molecule* ($1/\tilde{n}$). Is this ratio typically bigger or smaller than one?

Solution 1.5. The combined collision cross section for two spherical molecules having diameter $\bar{d}$ is $\pi\bar{d}^2$. The mean free path l is the average distance traveled by a molecule between collisions. Thus, the average molecule should experience one collision when sweeping a volume equal to $\pi\bar{d}^2 l$. If the molecular number density is $\tilde{n}$, then the volume per molecule is $\tilde{n}^{-1}$, and the mean intermolecular spacing is $\tilde{n}^{-1/3}$. Assuming that the swept volume necessary to produce one collision is proportional to the volume per molecule produces:



$$\pi\bar{d}^2 l = C/\tilde{n} \quad \text{or} \quad l = C\big/\left(\tilde{n}\pi\bar{d}^2\right),$$

where C is a dimensionless constant presumed to be of order unity. The dimensionless version of this equation is:

$$\frac{\text{mean free path}}{\text{mean intermolecular spacing}} = \frac{l}{\tilde{n}^{-1/3}} = l\tilde{n}^{1/3}$$

$$= \frac{C}{\tilde{n}^{2/3}\pi\bar{d}^2} = \frac{C}{\left(\tilde{n}\bar{d}^3\right)^{2/3}} = C\left(\frac{\tilde{n}^{-1}}{\bar{d}^3}\right)^{2/3} = C\left(\frac{\text{volume per molecule}}{\text{molecular volume}}\right)^{2/3},$$

where all numerical constants like π have been combined into C. Under ordinary conditions in gases, the molecules are not tightly packed so $l >> \tilde{n}^{-1/3}$. In liquids, the molecules are tightly packed so $l \sim \tilde{n}^{-1/3}$.

Exercise 1.6. In a gas, the molecular momentum flux (MF_{ij}) in the j-coordinate direction that crosses a flat surface of unit area with coordinate normal direction i is:
$MF_{ij} = \frac{n}{V}\iiint_{all\,\mathbf{u}} mu_iu_jf(\mathbf{u})d^3u$ where $f(\mathbf{u})$ is the Maxwell distribution given in Exercise 1.3. For a perfect gas that is not moving on average (i.e. $\mathbf{U} = 0$), show that $MF_{ij} = p$, the pressure, when $i = j$, and that $MF_{ij} = 0$, when $i \neq j$.

Solution 1.6. Start from the given equation using the Maxwell distribution:

$$MF_{ij} = \frac{n}{V}\iiint_{all\,\mathbf{u}} mu_iu_jf(\mathbf{u})d^3u = \frac{nm}{V}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_iu_j\exp\left\{-\frac{m}{2k_BT}\left(u_1^2+u_2^2+u_3^2\right)\right\}du_1du_2du_3$$

and first consider $i = j = 1$, and recognize $\rho = nm/V$ as the gas density (see (1.22)).

$$MF_{11} = \rho\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1^2\exp\left\{-\frac{m}{2k_BT}\left(u_1^2+u_2^2+u_3^2\right)\right\}du_1du_2du_3$$

$$= \rho\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_{-\infty}^{+\infty} u_1^2\exp\left\{-\frac{mu_1^2}{2k_BT}\right\}du_1\int_{-\infty}^{+\infty}\exp\left\{-\frac{mu_2^2}{2k_BT}\right\}du_2\int_{-\infty}^{+\infty}\exp\left\{-\frac{mu_3^2}{2k_BT}\right\}du_3$$

The first integral is equal to $\left(2k_BT/m\right)^{3/2}\left(\sqrt{\pi}/2\right)$ while the second two integrals are each equal to $\left(2\pi k_BT/m\right)^{1/2}$. Thus:

$$MF_{11} = \rho\left(\frac{m}{2\pi k_BT}\right)^{3/2}\left(\frac{2k_BT}{m}\right)^{3/2}\frac{\sqrt{\pi}}{2}\left(\frac{2\pi k_BT}{m}\right)^{1/2}\left(\frac{2\pi k_BT}{m}\right)^{1/2} = \rho\frac{k_BT}{m} = \rho RT = p$$

where $k_B/m = R$ from (1.22). This analysis may be repeated with $i = j = 2$, and $i = j = 3$ to find: $MF_{22} = MF_{33} = p$, as well.

Now consider the case $i \neq j$. First note that $MF_{ij} = MF_{ji}$ because the velocity product under the triple integral may be written in either order $u_iu_j = u_ju_i$, so there are only three cases of interest. Start with $i = 1$, and $j = 2$ to find:

$$MF_{12} = \rho\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1u_2\exp\left\{-\frac{m}{2k_BT}\left(u_1^2+u_2^2+u_3^2\right)\right\}du_1du_2du_3$$

$$= \rho\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_{-\infty}^{+\infty} u_1\exp\left\{-\frac{mu_1^2}{2k_BT}\right\}du_1\int_{-\infty}^{+\infty} u_2\exp\left\{-\frac{mu_2^2}{2k_BT}\right\}du_2\int_{-\infty}^{+\infty}\exp\left\{-\frac{mu_3^2}{2k_BT}\right\}du_3$$

Here we need only consider the first integral. The integrand of this integral is an odd function because it is product of an odd function, u_1, and an even function, $\exp\left\{-mu_1^2/2k_BT\right\}$. The integral of an odd function on an even interval $[-\infty,+\infty]$ is zero, so $MF_{12} = 0$. And, this analysis may be repeated for $i = 1$ and $j = 3$, and $i = 2$ and $j = 3$ to find $MF_{13} = MF_{23} = 0$.

Exercise 1.7. Consider the viscous flow in a channel of width $2b$. The channel is aligned in the x-direction, and the velocity u in the x-direction at a distance y from the channel centerline is given by the parabolic distribution $u(y) = U_0\left[1-(y/b)^2\right]$. Calculate the shear stress τ as a function y, μ, b, and U_o. What is the shear stress at $y = 0$?

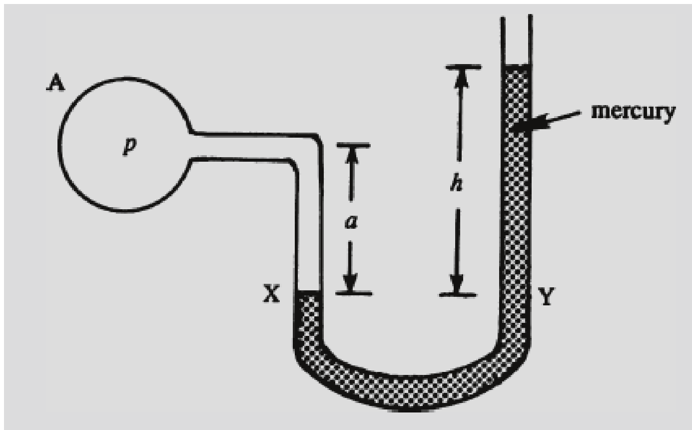
Solution 1.7. Start from (1.3): $\tau = \mu \dfrac{du}{dy} = \mu \dfrac{d}{dy} U_o \left[1-\left(\dfrac{y}{b}\right)^2\right] = -2\mu U_o \dfrac{y}{b^2}$. At $y = 0$ (the location of maximum velocity) $\tau = 0$. At At $y = \pm b$ (the locations of zero velocity), $\tau = \mp 2\mu U_o/b$.

Exercise 1.8. Estimate the height to which water at 20 °C will rise in a capillary glass tube 3 mm in diameter that is exposed to the atmosphere. For water in contact with glass the wetting angle is nearly 90°. At 20 °C, the surface tension of an water-air interface is σ = 0.073 N/m. (*Answer*: *h* = 0.99 cm.)

Solution 1.8. Start from the result of Example 1.1.

$$h = \frac{2\sigma \sin\alpha}{\rho g R} = \frac{2(0.073 N/m)\sin(90°)}{(10^3 kg/m^3)(9.81 m/s^2)(1.5 \times 10^{-3} m)} = 9.92 mm$$

Exercise 1.9. A *manometer* is a U-shaped tube containing mercury of density ρ_m. Manometers are used as pressure measuring devices. If the fluid in the tank A has a pressure p and density ρ, then show that the gauge pressure in the tank is: $p - p_{atm} = \rho_m gh - \rho ga$. Note that the last term on the right side is negligible if $\rho << \rho_m$. (*Hint*: Equate the pressures at X and Y.)



Solution 1.9. Start by equating the pressures at X and Y.

$$p_X = p + \rho ga = p_{atm} + \rho_m gh = p_Y.$$

Rearrange to find:

$$p - p_{atm} = \rho_m gh - \rho ga.$$

Exercise 1.10. Prove that if $e(T, \upsilon) = e(T)$ only and if $h(T, p) = h(T)$ only, then the (thermal) equation of state is (1.22) or $p\upsilon = kT$.

Solution 1.10. Start with the first member of (1.18): $de = Tds - pd\upsilon$, and rearrange it:

$$ds = \frac{1}{T}de + \frac{p}{T}d\upsilon = \left(\frac{\partial s}{\partial e}\right)_\upsilon de + \left(\frac{\partial s}{\partial \upsilon}\right)_e d\upsilon,$$

where the second equality holds assuming the entropy depends on e and υ. Here we see that:

$$\frac{1}{T} = \left(\frac{\partial s}{\partial e}\right)_\upsilon, \text{ and } \frac{p}{T} = \left(\frac{\partial s}{\partial \upsilon}\right)_e.$$

Equality of the crossed second derivatives of s, $\left(\frac{\partial}{\partial \upsilon}\left(\frac{\partial s}{\partial e}\right)_\upsilon\right)_e = \left(\frac{\partial}{\partial e}\left(\frac{\partial s}{\partial \upsilon}\right)_e\right)_\upsilon$, implies:

$$\left(\frac{\partial(1/T)}{\partial \upsilon}\right)_e = \left(\frac{\partial(p/T)}{\partial e}\right)_\upsilon.$$

However, if e depends only on T, then $(\partial/\partial\upsilon)_e = (\partial/\partial\upsilon)_T$, thus $\left(\frac{\partial(1/T)}{\partial \upsilon}\right)_e = \left(\frac{\partial(1/T)}{\partial \upsilon}\right)_T = 0$, so $\left(\frac{\partial(p/T)}{\partial e}\right)_\upsilon = 0$, and this can be integrated once to find: $p/T = f_1(\upsilon)$, where f_1 is an undetermined function.

Now repeat this procedure using the second member of (1.18), $dh = Tds + \upsilon dp$.

$$ds = \frac{1}{T}dh - \frac{\upsilon}{T}dp = \left(\frac{\partial s}{\partial h}\right)_p dh + \left(\frac{\partial s}{\partial p}\right)_h dp.$$

Here equality of the coefficients of the differentials implies: $\frac{1}{T} = \left(\frac{\partial s}{\partial h}\right)_p$, and $-\frac{\upsilon}{T} = \left(\frac{\partial s}{\partial p}\right)_h$.

So, equality of the crossed second derivatives implies: $\left(\frac{\partial(1/T)}{\partial p}\right)_h = -\left(\frac{\partial(\upsilon/T)}{\partial h}\right)_p$.

Yet, if h depends only on T, then $(\partial/\partial p)_h = (\partial/\partial p)_T$, thus $\left(\frac{\partial(1/T)}{\partial p}\right)_h = \left(\frac{\partial(1/T)}{\partial p}\right)_T = 0$, so $-\left(\frac{\partial(\upsilon/T)}{\partial h}\right)_p = 0$, and this can be integrated once to find: $\upsilon/T = f_2(p)$, where f_2 is an undetermined function.

Collecting the two results involving f_1 and f_2, and solving for T produces:

$$\frac{p}{f_1(\upsilon)} = T = \frac{\upsilon}{f_2(p)} \quad \text{or} \quad pf_2(p) = \upsilon f_1(\upsilon) = k,$$

where k must be is a constant since p and υ are independent thermodynamic variables. Eliminating f_1 or f_2 from either equation of the left, produces $p\upsilon = kT$.

And finally, using both versions of (1.18) we can write: $dh - de = \upsilon dp + pd\upsilon = d(p\upsilon)$. When e and h only depend on T, then $dh = C_p dT$ and $de = C_v dT$, so

$$dh - de = (C_p - C_v)dT = d(p\upsilon) = kdT, \text{ thus } k = C_p - C_v = R,$$

where R is the gas constant. Thus, the final result is the perfect gas law: $p = kT/\upsilon = \rho RT$.

Exercise 1.11. Starting from the property relationships (1.18) prove (1.25) and (1.26) for a reversible adiabatic process when the specific heats C_p and C_v are constant.

Solution 1.11. For an isentropic process: $de = Tds - pd\upsilon = -pd\upsilon$, and $dh = Tds + \upsilon dp = +\upsilon dp$. Equations (1.25) and (1.26) apply to a perfect gas so the definition of the specific heat capacities (1.14), and (1.15) for a perfect gas, $dh = C_p dT$, and $de = C_v dT$, can be used to form the ratio dh/de:

$$\frac{dh}{de} = \frac{C_p dT}{C_v dT} = \frac{C_p}{C_v} = \gamma = -\frac{\upsilon dp}{pd\upsilon} \quad \text{or} \quad -\gamma\frac{d\upsilon}{\upsilon} = \gamma\frac{d\rho}{\rho} = \frac{dp}{p}.$$

The final equality integrates to: $\ln(p) = \gamma\ln(\rho) + const$ which can be exponentiated to find:

$$p = const.\rho^{\gamma},$$

which is (1.25). The constant may be evaluated at a reference condition p_o and ρ_o to find: $p/p_o = (\rho/\rho_o)^{\gamma}$ and this may be inverted to put the density ratio on the left

$$\rho/\rho_o = (p/p_o)^{1/\gamma},$$

which is the second member of (1.26). The remaining relationship involving the temperature is found by using the perfect gas law, $p = \rho RT$, to eliminate $\rho = p/RT$:

$$\frac{\rho}{\rho_o} = \frac{p/RT}{p_o/RT_o} = \frac{pT_o}{p_o T} = \left(\frac{p}{p_o}\right)^{1/\gamma} \quad \text{or} \quad \frac{T}{T_o} = \frac{p}{p_o}\left(\frac{p}{p_o}\right)^{-1/\gamma} = \left(\frac{p}{p_o}\right)^{(\gamma-1)/\gamma},$$

which is the first member of (1.26).

Exercise 1.12. A cylinder contains 2 kg of air at 50 °C and a pressure of 3 bars. The air is compressed until its pressure rises to 8 bars. What is the initial volume? Find the final volume for both isothermal compression and isentropic compression.

Solution 1.12. Use the perfect gas law but explicitly separate the mass M of the air and the volume V it occupies via the substitution $\rho = M/V$:

$$p = \rho RT = (M/V)RT.$$

Solve for V at the initial time:

$$V_i = \text{initial volume} = MRT/p_i = (2\ kg)(287\ m^2/s^2K)(273 + 50°)/(300\ kPa) = 0.618\ m^3.$$

For an isothermal process:

$$V_f = \text{final volume} = MRT/p_f = (2\ kg)(287\ m^2/s^2K)(273 + 50°)/(800\ kPa) = 0.232\ m^3.$$

For an isentropic process:

$$V_f = V_i\left(p_i/p_f\right)^{1/\gamma} 0.618 m^3 \left(300 kPa/800 kPa\right)^{1/1.4} = 0.307 m^3.$$

Exercise 1.13. Derive (1.29) starting from the arguments provided at the beginning of Section 1.10 and Figure 1.8.

Solution 1.13. Take the z axis vertical, and consider a small fluid element δm of fluid having volume δV that starts at height z_0 in a stratified fluid medium having a vertical density profile = $\rho(z)$, and a vertical pressure profile $p(z)$. Without any vertical displacement, the small mass and its volume are related by $\delta m = \rho(z_0)\delta V$. If the small mass is displaced vertically a small distance ζ via an isentropic process, its density will change isentropically according to:

$$\rho_a(z_0+\zeta) = \rho(z_0) + (d\rho_a/dz)\zeta + \ldots$$

where $d\rho_a/dz$ is the isentropic density at z_0. For a constant δm, the volume of the fluid element will be:

$$\delta V = \frac{\delta m}{\rho_a} = \frac{\delta m}{\rho(z_0) + (d\rho_a/dz)\zeta + \ldots} = \frac{\delta m}{\rho(z_0)}\left(1 - \frac{1}{\rho(z_0)}\frac{d\rho_a}{dz}\zeta + \ldots\right)$$

The background density at $z_0 + \zeta$ is:

$$\rho(z_0+\zeta) = \rho(z_0) + (d\rho/dz)\zeta + \ldots$$

If g is the acceleration of gravity, the (upward) buoyant force on the element at the vertically displaced location will be $g\rho(z_0+\zeta)\delta V$, while the (downward) weight of the fluid element at any vertical location is $g\delta m$. Thus, a vertical application Newton's second law implies:

$$\delta m\frac{d^2\zeta}{dt^2} = +g\rho(z_0+\zeta)\delta V - g\delta m = g\big(\rho(z_0) + (d\rho/dz)\zeta + \ldots\big)\frac{\delta m}{\rho(z_0)}\left(1 - \frac{1}{\rho(z_0)}\frac{d\rho_a}{dz}\zeta + \ldots\right) - g\delta m,$$

where the second equality follows from substituting for $\rho(z_0+\zeta)$ and δV from the above equations. Multiplying out the terms in (,)-parentheses and dropping second order terms produces:

$$\delta m\frac{d^2\zeta}{dt^2} = g\delta m + \frac{g\delta m}{\rho(z_0)}\frac{d\rho}{dz}\zeta - \frac{g\delta m}{\rho(z_0)}\frac{d\rho_a}{dz}\zeta + \ldots - g\delta m \cong \frac{g\delta m}{\rho(z_0)}\left(\frac{d\rho}{dz} - \frac{d\rho_a}{dz}\right)\zeta$$

Dividing by δm and moving all the terms to the right side of the equation produces:

$$\frac{d^2\zeta}{dt^2} - \frac{g}{\rho(z_0)}\left(\frac{d\rho}{dz} - \frac{d\rho_a}{dz}\right)\zeta = 0$$

Thus, for oscillatory motion at frequency N, we must have

$$N^2 = -\frac{g}{\rho(z_0)}\left(\frac{d\rho}{dz} - \frac{d\rho_a}{dz}\right),$$

which is (1.29).

Exercise 1.14. Starting with the hydrostatic pressure law (1.8), prove (1.30) without using perfect gas relationships.

Solution 1.14. The adiabatic temperature gradient dT_a/dz, can be written terms of the pressure gradient:

$$\frac{dT_a}{dz} = \left(\frac{\partial T}{\partial p}\right)_s \frac{dp}{dz} = -g\rho\left(\frac{\partial T}{\partial p}\right)_s$$

where the hydrostatic law $dp/dz = -\rho g$ has been used to reach the second equality. Here, the final partial derivative can be exchanged for one involving $\upsilon = 1/\rho$ and s, by considering:

$$dh = \left(\frac{\partial h}{\partial s}\right)_p ds + \left(\frac{\partial h}{\partial p}\right)_s dp = Tds + \upsilon dp.$$

Equality of the crossed second derivatives of h, $\left(\frac{\partial}{\partial p}\left(\frac{\partial h}{\partial s}\right)_p\right)_s = \left(\frac{\partial}{\partial s}\left(\frac{\partial h}{\partial p}\right)_s\right)_p$, implies:

$$\left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial \upsilon}{\partial s}\right)_p = \left(\frac{\partial \upsilon}{\partial T}\right)_p \left(\frac{\partial T}{\partial s}\right)_p = \left(\frac{\partial \upsilon}{\partial T}\right)_p \Big/ \left(\frac{\partial s}{\partial T}\right)_p,$$

where the second two equalities are mathematical manipulations that allow the introduction of

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p = \rho\left(\frac{\partial \upsilon}{\partial T}\right)_p, \text{ and } C_{\mathrm{p}} = \left(\frac{\partial h}{\partial T}\right)_p = T\left(\frac{\partial s}{\partial T}\right)_p.$$

Thus,

$$\frac{dT_a}{dz} = -g\rho\left(\frac{\partial T}{\partial p}\right)_s = -g\rho\left(\frac{\partial \upsilon}{\partial T}\right)_p \Big/ \left(\frac{\partial s}{\partial T}\right)_p = -g\alpha \Big/ \left(\frac{C_{\mathrm{p}}}{T}\right) = -\frac{g\alpha T}{C_{\mathrm{p}}}.$$

Exercise 1.15. Assume that the temperature of the atmosphere varies with height z as $T = T_0 + Kz$ where K is a constant. Show that the pressure varies with height as $p = p_0\left[\dfrac{T_0}{T_0 + Kz}\right]^{g/KR}$, where g is the acceleration of gravity and R is the gas constant for the atmospheric gas.

Solution 1.15. Start with the hydrostatic and perfect gas laws, $dp/dz = -\rho g$, and $p = \rho RT$, eliminate the density, and substitute in the given temperature profile to find:

$$\frac{dp}{dz} = -\rho g = -\frac{p}{RT}g = -\frac{p}{R(T_0 + Kz)}g \quad \text{or} \quad \frac{dp}{p} = -\frac{g}{R}\frac{dz}{(T_0 + Kz)}.$$

The final form may be integrated to find:

$$\ln p = -\frac{g}{RK}\ln(T_0 + Kz) + const.$$

At $z = 0$, the pressure must be p_0, therefore:

$$\ln p_0 = -\frac{g}{RK}\ln(T_0) + const.$$

Subtracting this from the equation above and invoking the properties of logarithms produces:

$$\ln\left(\frac{p}{p_0}\right) = -\frac{g}{RK}\ln\left(\frac{T_0 + Kz}{T_0}\right)$$

Exponentiating produces:

$$\frac{p}{p_0} = \left[\frac{T_0 + Kz}{T_0}\right]^{-g/KR}, \text{ which is the same as: } p = p_0\left[\frac{T_0}{T_0 + Kz}\right]^{g/KR}.$$

Exercise 1.16. Suppose the atmospheric temperature varies according to: $T = 15 - 0.001z$, where T is in degrees Celsius and height z is in meters. Is this atmosphere stable?

Solution 1.16. Compute the temperature gradient:

$$\frac{dT}{dz} = \frac{d}{dz}(15 - 0.001z) = -0.001\frac{°C}{m} = -1.0\frac{°C}{km}.$$

For air in the earth's gravitational field, the adiabatic temperature gradient is:

$$\frac{dT_a}{dz} = -\frac{g\alpha T}{C_{\mathrm{p}}} = \frac{(9.81 m/s^2)(1/T)T}{1004 m^2/s^2 °C} = -9.8\frac{°C}{km}.$$

Thus, the given temperature profile is *stable* because the magnitude of its gradient is less than the magnitude of the adiabatic temperature gradient.

Exercise 1.17. Consider the case of a pure gas planet where the hydrostatic law is: $dp/dz = -\rho(z)Gm(z)/z^2$. Here G is the gravitational constant, and $m(z) = 4\pi\int_o^z \rho(\zeta)\zeta^2 d\zeta$ is the planetary mass up to distance z from the center of the planet. If the planetary gas is perfect with gas constant R, determine $\rho(z)$ and $p(z)$ if this atmosphere is isothermal at temperature T. Are these vertical profiles of ρ and p valid as z increases without bound?

Solution 1.17. Start with the given relationship for $m(z)$, differentiate it with respect to z, and use the perfect gas law, $p = \rho RT$ to replace the ρ with p.

$$\frac{dm}{dz} = \frac{d}{dz}\left(4\pi\int_0^z \rho(\zeta)\zeta^2 d\zeta\right) = 4\pi z^2\rho(z) = 4\pi z^2\frac{p(z)}{RT}.$$

Now use this and the hydrostatic law to obtain a differential equation for $m(z)$,

$$\frac{dp}{dz} = -\rho(z)\frac{Gm(z)}{z^2} \rightarrow \frac{d}{dz}\left(\frac{RT}{4\pi z^2}\frac{dm}{dz}\right) = -\left(\frac{1}{4\pi z^2}\frac{dm}{dz}\right)\frac{Gm(z)}{z^2}.$$

After recognizing T as a constant, the nonlinear second-order differential equation for $m(z)$ simplifies to:

$$\frac{RT}{G}\frac{d}{dz}\left(\frac{1}{z^2}\frac{dm}{dz}\right) = -\frac{1}{z^4}m\frac{dm}{dz}.$$

This equation can be solved by assuming a power law: $m(z) = Az^n$. When substituted in, this trial solution produces:

$$\frac{RT}{G}\frac{d}{dz}\left(z^{-2}Anz^{n-1}\right) = \frac{RT}{G}(n-3)Anz^{n-4} = -z^{-4}A^2nz^{2n-1}.$$

Matching exponents of z across the last equality produces: $n - 4 = 2n - 5$, and this requires $n = 1$. For this value of n, the remainder of the equation is:

$$\frac{RT}{G}(-2)Az^{-3} = -z^{-4}A^2z^1\text{, which reduces to: } A = 2\frac{RT}{G}.$$

Thus, we have $m(z) = 2RTz/G$, and this leads to:

$$\rho(z) = \frac{2RT}{G}\frac{1}{4\pi z^2}\text{ , and } p(z) = \frac{2R^2T^2}{G}\frac{1}{4\pi z^2}.$$

Unfortunately, these profiles are *not* valid as z increases without bound, because this leads to an unbounded planetary mass.

Exercise 1.18. Consider a heat-insulated enclosure that is separated into two compartments of volumes V_1 and V_2, containing perfect gases with pressures and temperatures of p_1, p_2, and T_1, T_2, respectively. The compartments are separated by an impermeable membrane that conducts heat (but not mass). Calculate the final steady-state temperature assuming each gas has constant specific heats.

Solution 1.18. Since no work is done and no heat is transferred out of the enclosure, the final energy E_f is the sum of the energies, E_1 and E_2, in the two compartments.

$$E_1 + E_2 = E_f \quad \text{implies} \quad \rho_1 V_1 C_{v1} T_1 + \rho_2 V_2 C_{v2} T_2 = (\rho_1 V_1 C_{v1} + \rho_2 V_2 C_{v2}) T_f,$$

where the C_v's are the specific heats at constant volume for the two gases. The perfect gas law can be used to find the densities: $\rho_1 = p_1/R_1T_1$ and $\rho_2 = p_2/R_2T_2$, so

$$p_1 V_1 C_{v1}/R_1 + p_2 V_2 C_{v2}/R_2 = (p_1 V_1 C_{v1}/R_1 T_1 + p_2 V_2 C_{v2}/R_2 T_2) T_f.$$

A little more simplification is possible, $C_{v1}/R_1 = 1/(\gamma_1 - 1)$ and $C_{v2}/R_1 = 1/(\gamma_2 - 1)$. Thus, the final temperature is:

$$T_f = \frac{p_1 V_1/(\gamma_1 - 1) + p_2 V_2/(\gamma_2 - 1)}{p_1 V_1/\left[(\gamma_1 - 1)T_1\right] + p_2 V_2/\left[(\gamma_2 - 1)T_2\right]}.$$

Exercise 1.19. Consider the initial state of an enclosure with two compartments as described in Exercise 1.18. At $t = 0$, the membrane is broken and the gases are mixed. Calculate the final temperature.

Solution 1.19. No heat is transferred out of the enclosure and the work done by either gas is delivered to the other so the total energy is unchanged. First consider the energy of either gas at temperature T, and pressure P in a container of volume V. The energy E of this gas will be:

$$E = \rho V C_v T = (p/RT) V C_v T = pV(C_v/R) = pV/(\gamma - 1).$$

where γ is the ratio of specific heats. For the problem at hand the final energy E_f will be the sum of the gas energies, E_1 and E_2, in the two compartments. Using the above formula:

$$E_f = p_1 V_1/(\gamma_1 - 1) + p_2 V_2/(\gamma_2 - 1).$$

Now consider the mixture. The final volume and temperature for both gases is V_1+V_2, and T_f. However, from Dalton's law of partial pressures, the final pressure of the mixture p_f can be considered a sum of the final partial pressures of gases "1" and "2", p_{1f} and p_{2f}:

$$p_f = p_{1f} + p_{2f}.$$

Thus, the final energy of the mixture is a sum involving each gases partial pressure and the total volume:

$$E_f = p_{1f}(V_1 + V_2)/(\gamma_1 - 1) + p_{2f}(V_1 + V_2)/(\gamma_2 - 1).$$

However, the perfect gas law implies: $p_{1f}(V_1+V_2) = n_1 R_u T_f$, and $p_{2f}(V_1+V_2) = n_2 R_u T_f$ where n_1 and n_2 are the mole numbers of gases "1" and "2", and R_u is the universal gas constant. The mole numbers are obtained from:

$$n_1 = p_1 V_1/R_u T_1, \text{ and } n_2 = p_2 V_2/R_u T_2,$$

Thus, final energy determined from the mixture is:

$$E_f = \frac{n_1 R_u T_f}{\gamma_1 - 1} + \frac{n_2 R_u T_f}{\gamma_1 - 1} = \left(\frac{p_1 V_1}{R_u T_1}\right)\frac{R_u T_f}{\gamma_1 - 1} + \left(\frac{p_2 V_2}{R_u T_2}\right)\frac{R_u T_f}{\gamma_2 - 1} = \left(\frac{p_1 V_1}{T_1}\right)\frac{T_f}{\gamma_1 - 1} + \left(\frac{p_2 V_2}{T_2}\right)\frac{T_f}{\gamma_2 - 1}.$$

Equating this and the first relationship for E_f above then produces:

$$T_f = \frac{p_1 V_1/(\gamma_1 - 1) + p_2 V_2/(\gamma_2 - 1)}{p_1 V_1/[(\gamma_1 - 1)T_1] + p_2 V_2/[(\gamma_2 - 1)T_2]}.$$

Exercise 1.20. A heavy piston of weight W is dropped onto a thermally insulated cylinder of cross-sectional area A containing a perfect gas of constant specific heats, and initially having the external pressure p_1, temperature T_1, and volume V_1. After some oscillations, the piston reaches an equilibrium position L meters below the equilibrium position of a weightless piston. Find L. Is there an entropy increase?

Solution 1.20. From the first law of thermodynamics, with $Q = 0$, ΔE = Work = WL. For a perfect gas with constant specific heats, $E = C_vT$, so $\Delta E = E_2 - E_1 = C_v(T_2 - T_1) = WL$. Then $T_2 = T_1 + WL/C_v$. Also, for a perfect gas, PV/T = constant so $p_1V_1/T_1 = p_2V_2/T_2$. For the cylinder, $V_2 = V_1 - AL$, and $p_2 = p_1 + W/A$. Therefore:

$$\frac{p_1V_1}{T_1} = \frac{(p_1 + W/A)(V_1 - AL)}{T_1 + WL/C_v}.$$

Solve for L.

$$\frac{p_1V_1}{T_1}\left(T_1 + \frac{WL}{C_v}\right) = \left(p + \frac{W}{A}\right)(V_1 - AL),$$

$$L\left[\frac{p_1V_1}{T_1}\frac{W}{C_v} + \left(p + \frac{W}{A}\right)A\right] = \left(p + \frac{W}{A}\right)V_1 - \frac{p_1V_1}{T_1}T_1 = \frac{W}{A}V_1,$$

$$L = \frac{WV_1/A}{(pV_1/T_1)(W/C_v) + pA + W}.$$

Exercise 1.21. A gas of non-interacting particles of mass m at temperature T has density ρ, and internal energy per unit volume ε.
a) Using dimensional analysis, determine how ε must depend on ρ, T, and m. In your formulation use k_B = Boltzmann's constant, h = Plank's constant, and c = speed of light to include possible quantum and relativistic effects.
b) Consider the limit of slow moving particles without quantum effects by requiring c and h to drop out of your dimensionless formulation. How does ε depend on ρ and T? What type of gas follows this thermodynamic law?
c) Consider the limit of massless particles (*i.e.* photons) by requiring m and ρ to drop out of your dimensionless formulation of part a). How does ε depend on T in this case? What is the name of this radiation law?

Solution 1.21. a) Construct the parameter & units matrix noting that k_B and T must go together since they are the only parameters that involve temperature units.

	ε	ρ	k_BT	m	h	c
M	1	1	1	1	1	0
L	-1	-3	2	0	2	1
T	-2	0	-2	0	-1	-1

This rank of this matrix is three. There are 6 parameters and 3 independent units, so there will be 3 dimensionless groups. Two of the dimensionless groups are energy ratios that are easy spot: $\Pi_1 = \varepsilon/\rho c^2$ and $\Pi_2 = k_BT/mc^2$. There is one dimensionless group left that must contain h. A bit of work produces: $\Pi_3 = \dfrac{\rho h^3}{m^4c^3}$, so $\dfrac{\varepsilon}{\rho c^2} = \varphi_1\left(\dfrac{k_BT}{mc^2}, \dfrac{\rho h^3}{m^4c^3}\right)$.
b) Dropping h means dropping Π_3. Eliminating c means combining Π_1 and Π_2 to create a new dimensionless group that lacks c: $\dfrac{\Pi_1}{\Pi_2} = \dfrac{\varepsilon/\rho c^2}{k_BT/mc^2} = \dfrac{\varepsilon m}{\rho k_BT}$. However, now there is only one dimensionless group so it must be a constant. This implies: $\varepsilon = const\cdot\left(\dfrac{\rho k_BT}{m}\right)$ which is the caloric equation of state for a perfect gas.
c) Eliminating ρ means combining Π_1 and Π_3 to create a new dimensionless group that lacks ρ: $\Pi_1\cdot\Pi_3 = \dfrac{\varepsilon}{\rho c^2}\cdot\dfrac{\rho h^3}{m^4c^3} = \dfrac{\varepsilon h^3}{m^4c^5}$. Now combine this new dimensionless group with Π_2 to eliminate m: $\dfrac{\varepsilon h^3}{m^4c^5}\cdot\dfrac{1}{\Pi_2^4} = \dfrac{\varepsilon h^3}{m^4c^5}\cdot\left(\dfrac{mc^2}{k_BT}\right)^4 = \dfrac{\varepsilon h^3c^3}{(k_BT)^4}$. Again there is only a single dimensionless group so it must equal a constant; therefore $\varepsilon = \dfrac{const}{h^3c^3}\cdot(k_BT)^4$. This is the Stephan-Boltzmann radiation law.

Exercise 1.22. Many flying and swimming animals – as well as human engineered vehicles – rely on some type of repetitive motion for propulsion through air or water. For this problem, assume the average travel speed U, depends on the repetition frequency f, the characteristic length scale of the animal or vehicle L, the acceleration of gravity g, the density of the animal or vehicle ρ_o, the density of the fluid ρ, and the viscosity of the fluid μ.
a) Formulate a dimensionless scaling law for U involving all the other parameters.
b) Simplify your answer for a) for turbulent flow where μ is no longer a parameter.
c) Fish and animals that swim at or near a water surface generate waves that move and propagate because of gravity, so g clearly plays a role in determining U. However, if fluctuations in the propulsive thrust are small, then f may not be important. Thus, eliminate f from your answer for b) while retaining L, and determine how U depends on L. Are successful competitive human swimmers likely to be shorter or taller than the average person?
d) When the propulsive fluctuations of a surface swimmer are large, the characteristic length scale may be U/f instead of L. Therefore, drop L from your answer for b). In this case, will higher speeds be achieved at lower or higher frequencies?
e) While traveling submerged, fish, marine mammals, and submarines are usually neutrally buoyant ($\rho_a \approx \rho$) or very nearly so. Thus, simplify your answer for b) so that g drops out. For this situation, how does the speed U depends on the repetition frequency f?
f) Although fully submerged, aircraft and birds are far from neutrally buoyant in air, so their travel speed is predominately set by balancing lift and weight. Ignoring frequency and viscosity, use the remaining parameters to construct dimensionally-accurate surrogates for lift & weight to determine how U depends on ρ_o/ρ, L, and g.

Solution 1.22. a) Construct the parameter & units matrix

	U	f	L	g	ρ_a	ρ	μ
M	0	0	0	0	1	1	1
L	1	0	1	1	-3	-3	-1
T	-1	-1	0	-2	0	0	-1

This rank of this matrix is three. There are 7 parameters and 3 independent units, so there will be 4 dimensionless groups. First try to assemble traditional dimensionless groups, but its best to use the solution parameter U only once. Here U is used in the Froude number, so its dimensional counter part, $\sqrt{gL}$, is used in place of U in the Reynolds number.

$\Pi_1 = \dfrac{U}{\sqrt{gL}}$ = Froude number, $\Pi_2 = \dfrac{\rho\sqrt{gL^3}}{\mu}$ = a Reynolds number

The next two groups can be found by inspection:

$\Pi_3 = \dfrac{\rho_o}{\rho}$ = a density ratio , and the final group must include f: $\Pi_4 = \dfrac{f}{\sqrt{g/L}}$, and is a frequency ratio between f and that of simple pendulum with length L. Putting these together produces:

$\dfrac{U}{\sqrt{gL}} = \psi_1\left(\dfrac{\rho\sqrt{gL^3}}{\mu}, \dfrac{\rho_o}{\rho}, \dfrac{f}{\sqrt{g/L}}\right)$ where, throughout this problem solution, ψ_i, i = 1, 2, 3, … are unknown functions.

b) When μ is no longer a parameter, the Reynolds number drops out: $\frac{U}{\sqrt{gL}} = \psi_2\left(\frac{\rho_o}{\rho}, \frac{f}{\sqrt{g/L}}\right)$.

c) When f is no longer a parameter, then $U = \sqrt{gL} \cdot \psi_3(\rho_o/\rho)$, so that U is proportional to $\sqrt{L}$. This scaling suggests that taller swimmers have an advantage over shorter ones. [Human swimmers best approach the necessary conditions for this part of this problem while doing freestyle (crawl) or backstroke where the arms (and legs) are used for propulsion in an alternating (instead of simultaneous) fashion. Interestingly, this length advantage also applies to ships and sailboats. Aircraft carriers are the longest and fastest (non-planing) ships in any Navy, and historically the longer boat typically won the America's Cup races under the 12-meter rule. Thus, if you bet on a swimming or sailing race where the competitors aren't known to you but appear to be evenly matched, choose the taller swimmer or the longer boat.]

d) Dropping L from the answer for b) requires the creation of a new dimensionless group from f, g, and U to replace Π_1 and Π_4. The new group can be obtained via a product of original dimensionless groups: $\Pi_1\Pi_4 = \frac{U}{\sqrt{gL}}\frac{f}{\sqrt{g/L}} = \frac{Uf}{g}$. Thus, $\frac{Uf}{g} = \psi_4\left(\frac{\rho_o}{\rho}\right)$, or $U = \frac{g}{f}\psi_4\left(\frac{\rho_o}{\rho}\right)$. Here, U is inversely proportional to f which suggests that higher speeds should be obtained at lower frequencies. [Human swimmers of butterfly (and breaststroke to a lesser degree) approach the conditions required for this part of this problem. Fewer longer strokes are typically preferred over many short ones. Of course, the trick for reaching top speed is to properly lengthen each stroke without losing propulsive force].

e) When g is no longer a parameter, a new dimensionless group that lacks g must be made to replace Π_1 and Π_5. This new dimensionless group is $\frac{\Pi_1}{\Pi_5} = \frac{U/\sqrt{gL}}{f/\sqrt{g/L}} = \frac{U}{fL}$, so the overall scaling law must be: $U = fL \cdot \psi_5\left(\frac{\rho_o}{\rho}\right)$. Thus, U will be directly proportional to f. Simple observations of swimming fish, dolphins, whales, etc. verify that their tail oscillation frequency increases at higher swimming speeds, as does the rotation speed of a submarine or torpedo's propeller.

f) Dimensionally-accurate surrogates for weight and lift are: $\rho_o L^3 g$ and $\rho U^2 L^2$, respectively. Set these proportional to each other, $\rho_o L^3 g \propto \rho U^2 L^2$, to find $U \propto \sqrt{\rho_o gL/\rho}$, which implies that larger denser flying objects must fly faster. This result is certainly reasonable when comparing similarly shaped aircraft (or birds) of different sizes.

Exercise 1.23. The acoustic power W generated by a large industrial blower depends on its volume flow rate Q, the pressure rise ΔP it works against, the air density ρ, and the speed of sound c. If hired as an acoustic consultant to quiet this blower by changing its operating conditions, what is your first suggestion?

Solution 1.23. The boundary condition and material parameters are: Q, ρ, ΔP, and c. The solution parameter is W. Create the parameter matrix:

	Q	ΔP	ρ	c	W
Mass:	0	1	1	0	1
Length:	3	-1	-3	1	2
Time:	-1	-2	0	-1	-3

This rank of this matrix is three. Next, determine the number of dimensionless groups: 5 parameters - 3 dimensions = 2 groups. Construct the dimensionless groups: $\prod_1 = W/Q\Delta P$, $\prod_2 = \Delta P/\rho c^2$. Now write the dimensionless law: $W = Q\Delta P\Phi(\Delta P/\rho c^2)$, where Φ is an unknown function. Since the sound power W must be proportional to volume flow rate Q, you can immediately suggest a decrease in Q as means of lowering W. At this point you do not know if Q must be maintained at high level, so this solution may be viable even though it may oppose many of the usual reasons for using a blower. Note that since Φ is unknown the dependence of W on ΔP cannot be determined from dimensional analysis alone.

Exercise 1.24. A machine that fills peanut-butter jars must be reset to accommodate larger jars. The new jars are twice as large as the old ones but they must be filled in the same amount of time by the same machine. Fortunately, the viscosity of peanut butter decreases with increasing temperature, and this property of peanut butter can be exploited to achieve the desired results since the existing machine allows for temperature control.
a) Write a dimensionless law for the jar-filling time t_f based on: the density of peanut butter ρ, the jar volume V, the viscosity of peanut butter μ, the driving pressure that forces peanut butter out of the machine P, and the diameter of the peanut butter delivery tube d.
b) Assuming that the peanut butter flow is dominated by viscous forces, modify the relationship you have written for part a) to eliminate the effects of fluid inertia.
c) Make a reasonable assumption concerning the relationship between t_f and V when the other variables are fixed, so that you can determine the viscosity ratio μ_{new}/μ_{old} necessary for proper operation of the old machine with the new jars.

Solution 1.24. a) The boundary condition and material parameters are: V, ρ, P, μ, and d. The solution parameter is t_f. First create the parameter matrix:

	V	P	ρ	d	μ	t_f
Mass:	0	1	1	0	1	0
Length:	3	-1	-3	1	-1	0
Time:	0	-2	0	0	-1	1

This rank of this matrix is three. Next determine the number of dimensionless groups: 6 parameters - 3 dimensions = 3 groups. Construct the dimensionless groups: $\Pi_1 = Pt_f/\mu$, $\Pi_2 = \mu^2/\rho d^2 P$, $\Pi_3 = V/d^3$, and write a dimensionless law: $t_f = (\mu/P)\Phi(\mu^2/\rho d^2 P, V/d^3)$, where Φ is an unknown function.
b) When fluid inertia is not important the fluid's density is not a parameter. Therefore, drop Π_2 from the dimensional analysis formula: $t_f = (\mu/P)\Psi(V/d^3)$, where Ψ is yet another unknown function.
c) One might reasonably expect that $t_f \propto V$ (these are the two extensive variables). Therefore, we end up with $t_f =$ const$\cdot\mu V/Pd^3$. Now form a ratio between the old and new conditions and cancel common terms:

$$\frac{(t_f)_{new}}{(t_f)_{old}} = 1 = \frac{(\mu V/Pd^3)_{new}}{(\mu V/Pd^3)_{old}} = \frac{(\mu V)_{new}}{(\mu V)_{old}}, \text{ so } \frac{V_{new}}{V_{old}} = 2 \rightarrow \frac{\mu_{new}}{\mu_{old}} = \frac{1}{2}$$

Exercise 1.25. As an idealization of fuel injection in a Diesel engine, consider a stream of high-speed fluid (called a *jet*) that emerges into a quiescent air reservoir at $t = 0$ from a small hole in an infinite plate to form a *plume* where the fuel and air mix.

a) Develop a scaling law via dimensional analysis for the penetration distance D of the plume as a function of: Δp the pressure difference across the orifice that drives the jet, d_o the diameter of the jet orifice, ρ_o the density of the fuel, μ_∞ and ρ_∞ the viscosity and density of the air, and t the time since the jet was turned on.

b) Simplify this scaling law for turbulent flow where air viscosity is no longer a parameter.

c) For turbulent flow and $D \ll d_o$, d_o and ρ_∞ are not parameters. Recreate the dimensionless law for D.

d) For turbulent flow and $D \gg d_o$, only the momentum flux of the jet matters, so Δp and d_o are replaced by the single parameter J_o = jet momentum flux (J_o has the units of force and is approximately equal to $\Delta p d_o^2$). Recreate the dimensionless law for D using the new parameter J_o.

Solution 1.25. a) The parameters are: D, t, Δp, ρ_o, ρ_∞, μ_∞, and d_o. First, create the parameter matrix:

	D	t	Δp	ρ_o	ρ_∞	μ_∞	d_o
Mass:	0	0	1	1	1	1	0
Length:	1	0	-1	-3	-3	-1	1
Time:	0	1	-2	0	0	-1	0

Next, determine the number of dimensionless groups. This rank of this matrix is three so 7 parameters - 3 dimensions = 4 groups, and construct the groups: $\Pi_1 = D/d_o$, $\Pi_2 = \rho_o/\rho_\infty$, $\Pi_3 = \Delta p t^2/\rho_\infty d_o^2$, and $\Pi_4 = \rho_\infty \Delta p d_o^2/\mu_\infty^2$. Now write a dimensionless law:

$$\frac{D}{d_o} = f\left(\frac{\rho_o}{\rho_\infty}, \frac{\Delta p t^2}{\rho_\infty d_o^2}, \frac{\rho_\infty \Delta p d_o^2}{\mu_\infty^2}\right)$$ where f is an unknown function.

b) For high Reynolds number turbulent flow when the reservoir viscosity is no longer a parameter, the above result becomes:

$$\frac{D}{d_o} = g\left(\frac{\rho_o}{\rho_\infty}, \frac{\Delta p t^2}{\rho_\infty d_o^2}\right),$$

where g is an unknown function.

c) When d_o and ρ_∞ are not parameters, there is only one dimensionless group: $\Delta p t^2/\rho_\infty D^2$, so the dimensionless law becomes: $D = const \cdot t\sqrt{\Delta p/\rho_o}$.

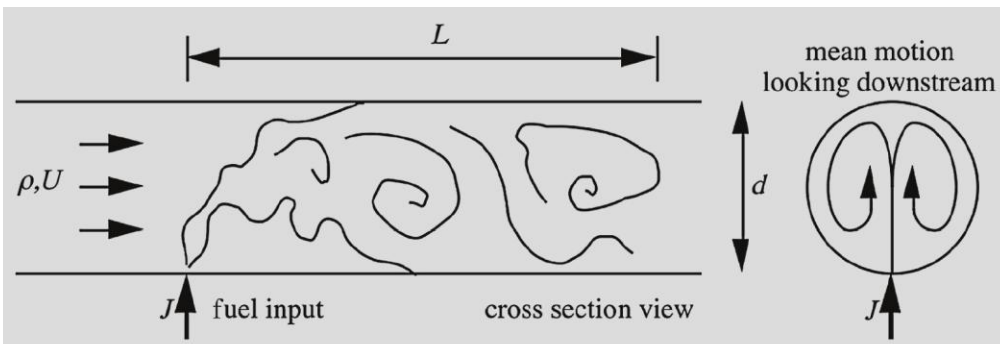
d) When Δp and d_o are replaced by the single parameter J_o = jet momentum flux, there are two dimensionless parameters: $J_o t^2/\rho_\infty D^4$, and ρ_o/ρ_∞, so the dimensionless law becomes:

$$D = \left(J_o t^2/\rho_\infty\right)^{1/4} F(\rho_o/\rho_\infty),$$

where F is an unknown function.

[The results presented here are the fuel-plume penetration scaling laws for fuel injection in Diesel engines where more than half of the world's petroleum ends up being burned.]

Exercise 1.26. One of the simplest types of gasoline carburetors is a tube with small port for transverse injection of fuel. It is desirable to have the fuel uniformly mixed in the passing air stream as quickly as possible. A prediction of the mixing length L is sought. The parameters of this problem are: ρ = density of the flowing air, d = diameter of the tube, μ = viscosity of the flowing air, U = mean axial velocity of the flowing air, and J = momentum flux of the fuel stream.
a) Write a dimensionless law for L.
b) Simplify your result from part a) for turbulent flow where μ must drop out of your dimensional analysis.
c) When this flow is turbulent, it is observed that mixing is essentially complete after one rotation of the counter rotating vortices driven by the injected-fuel momentum (see downstream-view of the drawing for this problem), and that the vortex rotation rate is directly proportional to J. Based on this information, assume that $L \propto$ (rotation time)(U) to eliminate the arbitrary function in the result of part b). The final formula for L should contain an undetermined dimensionless constant.



Solution 1.26. a) The parameters are: L, J, d, μ, ρ, and U. Use these to create the parameter matrix:

	L	J	d	μ	ρ	U
Mass:	0	1	0	1	1	0
Length:	1	1	1	-1	-3	1
Time:	0	-2	0	-1	0	-1

Next, determine the number of dimensionless groups. This rank of this matrix is three so 6 parameters - 3 dimensions = 3 groups, and construct them: $\prod_1 = L/d$, $\prod_2 = \rho Ud/\mu$, $\prod_3 = \rho U^2 d^2/J$. And, finally write a dimensionless law: $L = d\,\Phi(\rho Ud/\mu, \rho U^2 d^2/J)$, where Φ is an unknown function.
b) At high Reynolds numbers, μ must not be a parameter. Therefore: $L = d\Psi(\rho U^2 d^2/J)$ where Ψ is an unknown function.
c) Let Ω = vortex rotation rate. The units of Ω are 1/time and Ω must be proportional to J. Putting this statement in dimensionless terms based on the boundary condition and material parameters of this problem means: $\Omega = \text{const}\dfrac{J}{\rho U d^3} = \text{(rotation time)}^{-1}$

Therefore: $L = \text{const}\,(\Omega^{-1})U = \text{const}\,\dfrac{\rho U^2 d^3}{J}$, or $\dfrac{L}{d} = \text{const}\,\dfrac{\rho U^2 d^2}{J}$. Thus, for transverse injection, more rapid mixing occurs (L decreases) when the injection momentum increases.

Exercise 1.27. Consider dune formation in large horizontal desert of deep sand.
a) Develop a scaling relationship that describes how the height h of the dunes depends on the average wind speed U, the length of time the wind has been blowing Δt, the average weight and diameter of a sand grain w and d, and the air's density ρ and kinematic viscosity ν.
b) Simplify the result of part a) when the sand-air interface is fully rough and ν is no longer a parameter.
c) If the sand dune height is determined to be proportional to the density of the air, how do you expect it to depend on the weight of a sand grain?

Solution 1.27. a) The boundary condition and material parameters are: U, Δt, w, d, ρ, and ν. The solution parameter is h. First create the parameter matrix:

	h	U	Δt	w	d	ρ	ν
Mass:	0	0	0	1	0	1	0
Length:	1	1	0	1	1	-3	2
Time:	0	-1	1	-2	0	0	-1

Next determine the number of dimensionless groups. This rank of this matrix is three so 7 parameters - 3 dimensions = 4 groups. Construct the dimensionless groups: $\prod_1 = h/d$, $\prod_2 = Ud/\nu$, $\prod_3 = w/\rho U^2 d^2$, and $\prod_4 = U\Delta t/d$. Thus, the dimensionless law is

$$\frac{h}{d} = \Phi\left(\frac{Ud}{\nu}, \frac{w}{\rho U^2 d^2}, \frac{U\Delta t}{d}\right),$$

where Φ is an unknown function.
b) When ν is no longer a parameter, $\prod_2$ drops out:

$$\frac{h}{d} = \Psi\left(\frac{w}{\rho U^2 d^2}, \frac{U\Delta t}{d}\right),$$

where Ψ is another unknown function.
c) When h is proportional to ρ, then

$$\frac{h}{d} = \frac{\rho U^2 d^2}{w}\Theta\left(\frac{U\Delta t}{d}\right),$$

where Θ is another unknown function. Under this condition, dune height will be inversely proportional to w the sand grain weight.

Exercise 1.28. An isolated nominally-spherical bubble with radius R undergoes shape oscillations at frequency f. It is filled with air having density ρ_a and resides in water with density ρ_w and surface tension σ. What frequency ratio should be expected between two isolated bubbles with 2 *cm* and 4 *cm* diameters undergoing geometrically similar shape oscillations? If a soluble surfactant is added to the water that lowers σ by a factor of two, by what factor should air bubble oscillation frequencies increase or decrease?

Solution 1.28. The boundary condition and material parameters are: R, ρ_a, ρ_w, and σ. The solution parameter is f. First create the parameter matrix:

	f	R	ρ_a	ρ_w	σ
Mass:	0	0	1	1	1
Length:	0	1	-3	-3	0
Time:	-1	0	0	0	-2

Next determine the number of dimensionless groups. This rank of this matrix is three, so 5 parameters - 3 dimensions = 2 groups. Construct the dimensionless groups: $\prod_1 = f\sqrt{\rho_w R^3/\sigma}$, and $\prod_2 = \rho_w/\rho_a$. Thus, the dimensionless law is

$$f = \sqrt{\frac{\sigma}{\rho_w R^3}}\Phi\left(\frac{\rho_w}{\rho_a}\right),$$

where Φ is an unknown function. For a fixed density ratio, $\Phi(\rho_w/\rho_a)$ will be constant so f is proportional to $R^{-3/2}$ and to $\sigma^{1/2}$. Thus, the required frequency ratio between different sizes bubbles is:

$$\frac{(f)_{2cm}}{(f)_{4cm}} = \left(\frac{2cm}{4cm}\right)^{-3/2} = 2\sqrt{2} \cong 2.83.$$

Similarly, if the surface tension is decreased by a factor of two, then

$$\frac{(f)_{\sigma/2}}{(f)_{\sigma}} = \left(\frac{1/2}{1}\right)^{-1/2} = \frac{1}{\sqrt{2}} \cong 0.707.$$

Exercise 1.29. In general, boundary layer skin friction, τ_w, depends on the fluid velocity U above the boundary layer, the fluid density ρ, the fluid viscosity μ, the nominal boundary layer thickness δ, and the surface roughness length scale ε.
a) Generate a dimensionless scaling law for boundary layer skin friction.
b) For laminar boundary layers, the skin friction is proportional to μ. When this is true, how must τ_w depend on U and ρ?
c) For turbulent boundary layers, the dominant mechanisms for momentum exchange within the flow do not directly involve the viscosity μ. Reformulate your dimensional analysis without it. How must τ_w depend on U and ρ in when μ is not a parameter?
d) For turbulent boundary layers on smooth surfaces, the skin friction on a solid wall occurs in a viscous sub-layer that is very thin compared to δ. In fact, because the boundary layer provides a buffer between the outer flow and this viscous sub-layer, the viscous sub-layer thickness l_v does not depend directly on U or δ. Determine how l_v depends on the remaining parameters.
e) Now consider non-trivial roughness. When ε is larger than l_v a surface can no longer be considered fluid-dynamically smooth. Thus, based on the results from parts a) through d) and anything you may know about the relative friction levels in laminar and turbulent boundary layer, are high or low speed boundary layer flows more likely to be influenced by surface roughness?

Solution 1.29. a) Construct the parameter & units matrix and recognizing that τ_w is a stress and has units of pressure.

	τ_w	U	ρ	μ	δ	ε
M	1	0	1	1	0	0
L	-1	1	-3	-1	1	1
T	-2	-1	0	1	0	0

This rank of this matrix is three. There are 6 parameters and 3 independent units, thus there will be 6 – 3 = 3 dimensionless groups. By inspection these groups are: a skin-friction coefficient = $\Pi_1 = \dfrac{\tau_w}{\rho U^2}$, a Reynolds number = $\Pi_2 = \dfrac{\rho U\delta}{\mu}$, and the relative roughness = $\Pi_3 = \dfrac{\varepsilon}{\delta}$. Thus the dimensionless law is: $\dfrac{\tau_w}{\rho U^2} = f\left(\dfrac{\rho U\delta}{\mu}, \dfrac{\varepsilon}{\delta}\right)$ where f is an undetermined function.

b) Use the result of part a) and set $\tau_w \propto \mu$. This involves requiring Π_1 to be proportional to $1/\Pi_2$ so the revised form of the dimensionless law in part a) is: $\dfrac{\tau_w}{\rho U^2} = \dfrac{\mu}{\rho U\delta} g\left(\dfrac{\varepsilon}{\delta}\right)$, where g is an undetermined function. Simplify this relationship to find: $\tau_w = \dfrac{\mu U}{\delta} g\left(\dfrac{\varepsilon}{\delta}\right)$. Thus, in laminar boundary layers, τ_w is proportional to U and independent of ρ.

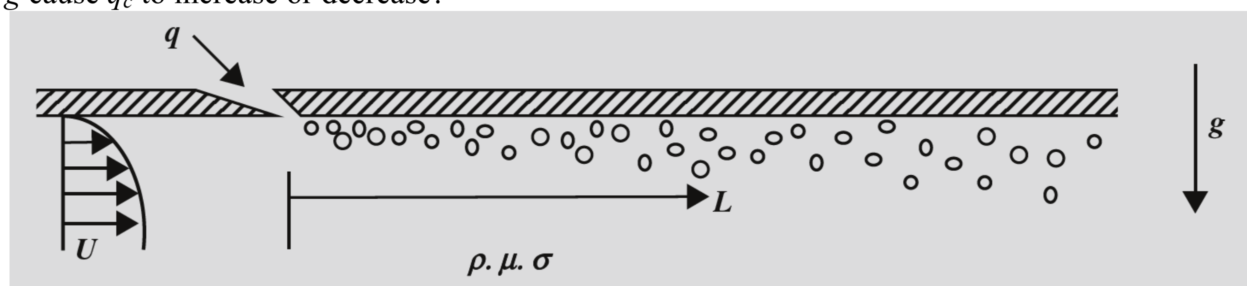
c) When μ is not a parameter the second dimensionless group from part a) must be dropped. Thus, the dimensionless law becomes: $\dfrac{\tau_w}{\rho U^2} = h\left(\dfrac{\varepsilon}{\delta}\right)$ where h is an undetermined function. Here we see that $\tau_w \propto \rho U^2$. Thus, in turbulent boundary layers, τ_w is linearly proportional to ρ and quadratically proportional to U. In reality, completely dropping μ from the dimensional analysis

is not quite right, and the skin-friction coefficient (Π_1 in the this problem) maintains a weak dependence on the Reynolds number when $\varepsilon/\delta << 1$.
d) For this part of this problem, it is necessary to redo the dimensional analysis with the new length scale l_v and the three remaining parameters: τ_w, ρ, and μ. Here there are four parameters and three units, so there is only one dimensionless group: $\Pi = \dfrac{l_v\sqrt{\rho\tau_w}}{\mu}$. This means that:

$$l_v \propto \mu/\sqrt{\rho\tau_w} = \nu/\sqrt{\tau_w/\rho} = \nu/u_* .$$

In the study of wall bounded turbulent flows, the length scale l_v is commonly known as the viscous wall unit and u_* is known as the friction or shear velocity.
e) The results of part b) and part c) both suggest that τ_w will be larger at high flow speeds than at lower flow speeds. This means that l_v will be smaller at high flow speeds for both laminar and turbulent boundary layers. Thus, boundary layers in high-speed flows are more likely to be influenced by constant-size surface roughness.

Exercise 1.30. Turbulent boundary layer skin friction is one of the fluid phenomena that limit the travel speed of aircraft and ships. One means for reducing the skin friction of liquid boundary layers is to inject a gas (typically air) from the surface on which the boundary layer forms. The shear stress, τ_w, that is felt a distance L downstream of such an air injector depends on: the volumetric gas flux per unit span q (in m^2/s), the free stream flow speed U, the liquid density ρ, the liquid viscosity μ, the surface tension σ, and gravitational acceleration g.
a) Formulate a dimensionless law for τ_w in terms of the other parameters.
b) Experimental studies of air injection into liquid turbulent boundary layers on flat plates has found that the bubbles may coalesce to form an air film that provides near perfect lubrication, $\tau_w \to 0$ for $L > 0$, when q is high enough and gravity tends to push the injected gas toward the plate surface. Reformulate your answer to part a) by dropping τ_w and L to determine a dimensionless law for the minimum air injection rate, q_c, necessary to form an air layer.
c) Simplify the result of part c) when surface tension can be neglected.
d) Experimental studies (Elbing et al. 2008) find that q_c is proportional to U^2. Using this information, determine a scaling law for q_c involving the other parameters. Would an increase in g cause q_c to increase or decrease?



Solution 1.30. a) Construct the parameter & units matrix and recognizing that τ_w is a stress and has units of pressure.

	τ_w	L	q	U	ρ	μ	σ	g
M	1	0	0	0	1	1	1	0
L	-1	1	2	1	-3	-1	0	1
T	-2	0	-1	-1	0	-1	-2	-2

This rank of this matrix is three. There are 8 parameters and 3 independent units, thus there will be 8 – 3 = 5 dimensionless groups. By inspection these groups are: a skin-friction coefficient = $\Pi_1 = \frac{\tau_w}{\rho U^2}$, a Reynolds number = $\Pi_2 = \frac{\rho UL}{\mu}$, a Froude number = $\Pi_3 = \frac{U}{\sqrt{gL}}$, a capillary number = $\Pi_4 = \frac{\mu U}{\sigma}$, and flux ratio = $\Pi_5 = \frac{\rho q}{\mu}$. Thus the dimensionless law is:

$$\frac{\tau_w}{\rho U^2} = f\left(\frac{\rho UL}{\mu}, \frac{U}{\sqrt{gL}}, \frac{\mu U}{\sigma}, \frac{\rho q}{\mu}\right)$$ where f is an undetermined function.

b) Dropping τ_w means dropping Π_1. Dropping L means combining Π_2 and Π_3 to form a new dimensionless group: $\Pi_2\Pi_3^2 = \frac{\rho UL}{\mu}\frac{U^3}{gL} = \frac{\rho U^3}{\mu g}$. Thus, with Π_5 as the solution parameter, the scaling law for the minimum air injection rate, q_c, necessary to form an air layer is: $\rho q_c/\mu = \phi\left(\rho U^3/\mu g, \mu U/\sigma\right)$ where ϕ is an undetermined function.

c) When σ is not a parameter, Π_4 can be dropped leaving: $\rho q_c/\mu = \varphi\left(\rho U^3/\mu g\right)$ where φ is an undetermined function.

d) When q_c is proportional to U^2, then dimensional analysis requires:

$$q_c = (\mu/\rho) const.\left(\rho U^3/\mu g\right)^{2/3} = const.U^2\left(\mu/\rho g^2\right)^{1/3}.$$

So, an increase in g would cause q_c to decrease.

Exercise 2.1. For three spatial dimensions, rewrite the following expressions in index notation and evaluate or simplify them using the values or parameters given, and the definitions of δ_{ij} and ε_{ijk} wherever possible. In b) through e), $\mathbf{x}$ is the position vector, with components x_i.

a) $\mathbf{b}\cdot\mathbf{c}$ where $\mathbf{b} = (1, 4, 17)$ and $\mathbf{c} = (-4, -3, 1)$

b) $(\mathbf{u}\cdot\nabla)\mathbf{x}$ where $\mathbf{u}$ a vector with components u_i.

c) $\nabla\phi$, where $\phi = \mathbf{h}\cdot\mathbf{x}$ and $\mathbf{h}$ is a constant vector with components h_i.

d) $\nabla\times\mathbf{u}$, where $\mathbf{u} = \mathbf{\Omega}\times\mathbf{x}$ and $\mathbf{\Omega}$ is a constant vector with components Ω_i.

e) $\mathbf{C}\cdot\mathbf{x}$, where $\mathbf{C} = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}$

Solution 2.1. a) $\mathbf{b}\cdot\mathbf{c} = b_i c_i = 1(-4) + 4(-3) + 17(1) = -4 - 12 + 17 = +1$

b) $$(\mathbf{u}\cdot\nabla)\mathbf{x} = u_j \frac{\partial}{\partial x_j} x_i = \left[u_1\left(\frac{\partial}{\partial x_1}\right) + u_2\left(\frac{\partial}{\partial x_2}\right) + u_3\left(\frac{\partial}{\partial x_3}\right) \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1\left(\frac{\partial x_1}{\partial x_1}\right) + u_2\left(\frac{\partial x_1}{\partial x_2}\right) + u_3\left(\frac{\partial x_1}{\partial x_3}\right) \\ u_1\left(\frac{\partial x_2}{\partial x_1}\right) + u_2\left(\frac{\partial x_2}{\partial x_2}\right) + u_3\left(\frac{\partial x_2}{\partial x_3}\right) \\ u_1\left(\frac{\partial x_3}{\partial x_1}\right) + u_2\left(\frac{\partial x_3}{\partial x_2}\right) + u_3\left(\frac{\partial x_3}{\partial x_3}\right) \end{bmatrix} = \begin{bmatrix} u_1\cdot 1 + u_2\cdot 0 + u_3\cdot 0 \\ u_1\cdot 0 + u_2\cdot 1 + u_3\cdot 0 \\ u_1\cdot 0 + u_2\cdot 0 + u_3\cdot 1 \end{bmatrix} = u_j\delta_{ij} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_i$$

c) $$\nabla\phi = \frac{\partial\phi}{\partial x_j} = \frac{\partial}{\partial x_j}(h_i x_i) = h_i \frac{\partial x_i}{\partial x_j} = h_i\delta_{ij} = h_j = \mathbf{h}$$

d) $$\nabla\times\mathbf{u} = \nabla\times(\mathbf{\Omega}\times\mathbf{x}) = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}\Omega_l x_m) = \varepsilon_{ijk}\varepsilon_{klm}\Omega_l\delta_{jm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_l\delta_{jm} = (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl})\Omega_l$$

$$= (3\delta_{il} - \delta_{il})\Omega_l = 2\delta_{il}\Omega_l = 2\Omega_l = 2\mathbf{\Omega}$$

Here, the following identities have been used: $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, $\delta_{ij}\delta_{jk} = \delta_{ik}$, $\delta_{jj} = 3$, and $\delta_{ij}\Omega_j = \Omega_i$

e) $$\mathbf{C}\cdot\mathbf{x} = C_{ij}x_j = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{Bmatrix}$$

Exercise 2.2. Starting from (2.1) and (2.3), prove (2.7).

Solution 2.2. The two representations for the position vector are:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \text{ or } \mathbf{x} = x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'.$$

Develop the dot product of $\mathbf{x}$ with $\mathbf{e}_1$ from each representation,

$$\mathbf{e}_1 \cdot \mathbf{x} = \mathbf{e}_1 \cdot (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\mathbf{e}_1 \cdot \mathbf{e}_1 + x_2\mathbf{e}_1 \cdot \mathbf{e}_2 + x_3\mathbf{e}_1 \cdot \mathbf{e}_3 = x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 = x_1 ,$$

$$\text{and } \mathbf{e}_1 \cdot \mathbf{x} = \mathbf{e}_1 \cdot (x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3') = x_1'\mathbf{e}_1 \cdot \mathbf{e}_1' + x_2'\mathbf{e}_1 \cdot \mathbf{e}_2' + x_3'\mathbf{e}_1 \cdot \mathbf{e}_3' = x_i'C_{1i},$$

set these equal to find:

$$x_1 = x_i'C_{1i},$$

where $C_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j'$ is a 3 × 3 matrix of direction cosines. In an entirely parallel fashion, forming the dot product of $\mathbf{x}$ with $\mathbf{e}_2$, and $\mathbf{x}$ with $\mathbf{e}_2$ produces:

$$x_2 = x_i'C_{2i} \text{ and } x_3 = x_i'C_{3i}.$$

Thus, for any component x_j, where j = 1, 2, or 3, we have:

$$x_j = x_i'C_{ji},$$

which is (2.7).

Exercise 2.3. Using Cartesian coordinates where the position vector is $\mathbf{x} = (x_1, x_2, x_3)$ and the fluid velocity is $\mathbf{u} = (u_1, u_2, u_3)$, write out the three components of the vector: $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$.

Solution 2.3.

a) $$(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i\left(\frac{\partial u_j}{\partial x_i}\right) = u_1\left(\frac{\partial u_j}{\partial x_1}\right) + u_2\left(\frac{\partial u_j}{\partial x_2}\right) + u_3\left(\frac{\partial u_j}{\partial x_3}\right) = \begin{Bmatrix} u_1\left(\frac{\partial u_1}{\partial x_1}\right) + u_2\left(\frac{\partial u_1}{\partial x_2}\right) + u_3\left(\frac{\partial u_1}{\partial x_3}\right) \\ u_1\left(\frac{\partial u_2}{\partial x_1}\right) + u_2\left(\frac{\partial u_2}{\partial x_2}\right) + u_3\left(\frac{\partial u_2}{\partial x_3}\right) \\ u_1\left(\frac{\partial u_3}{\partial x_1}\right) + u_2\left(\frac{\partial u_3}{\partial x_2}\right) + u_3\left(\frac{\partial u_3}{\partial x_3}\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z}\right) \\ u\left(\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial v}{\partial y}\right) + w\left(\frac{\partial v}{\partial z}\right) \\ u\left(\frac{\partial w}{\partial x}\right) + v\left(\frac{\partial w}{\partial y}\right) + w\left(\frac{\partial w}{\partial z}\right) \end{Bmatrix}$$

The vector in this exercise, $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$, is an important one in fluid mechanics. As described in Ch. 3, it is the nonlinear advective acceleration.

Exercise 2.4. Convert $\nabla \times \nabla \rho$ to indicial notation and show that it is zero in Cartesian coordinates for any twice-differentiable scalar function ρ.

Solution 2.4. Start with the definitions of the cross product and the gradient.

$$\nabla \times (\nabla \rho) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \rho)_k = \varepsilon_{ijk} \frac{\partial^2 \rho}{\partial x_j \partial x_k}$$

Write out the vector component by component recalling that $\varepsilon_{ijk} = 0$ if any two indices are equal. Here the "i" index is the free index.

$$\varepsilon_{ijk} \frac{\partial^2 \rho}{\partial x_j \partial x_k} = \begin{Bmatrix} \varepsilon_{123} \frac{\partial^2 \rho}{\partial x_2 \partial x_3} + \varepsilon_{132} \frac{\partial^2 \rho}{\partial x_3 \partial x_2} \\ \varepsilon_{213} \frac{\partial^2 \rho}{\partial x_1 \partial x_3} + \varepsilon_{231} \frac{\partial^2 \rho}{\partial x_3 \partial x_1} \\ \varepsilon_{312} \frac{\partial^2 \rho}{\partial x_1 \partial x_2} + \varepsilon_{321} \frac{\partial^2 \rho}{\partial x_2 \partial x_1} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 \rho}{\partial x_2 \partial x_3} - \frac{\partial^2 \rho}{\partial x_3 \partial x_2} \\ -\frac{\partial^2 \rho}{\partial x_1 \partial x_3} + \frac{\partial^2 \rho}{\partial x_3 \partial x_1} \\ \frac{\partial^2 \rho}{\partial x_1 \partial x_2} - \frac{\partial^2 \rho}{\partial x_2 \partial x_1} \end{Bmatrix} = 0,$$

where the middle equality follows from the definition of ε_{ijk} (2.18), and the final equality follows when ρ is twice differentiable so that $\frac{\partial^2 \rho}{\partial x_j \partial x_k} = \frac{\partial^2 \rho}{\partial x_k \partial x_j}$.

Exercise 2.5. Using indicial notation, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. [*Hint*: Call $\mathbf{d} \equiv \mathbf{b} \times \mathbf{c}$. Then $(\mathbf{a} \times \mathbf{d})_m = \varepsilon_{pqm} a_p d_q = \varepsilon_{pqm} a_p \varepsilon_{ijq} b_i c_j$. Using (2.19), show that $(\mathbf{a} \times \mathbf{d})_m = (\mathbf{a} \cdot \mathbf{c})\boldsymbol{b_m} - (\mathbf{a} \cdot \mathbf{b})\boldsymbol{c_m}$.]

Solution 2.5. Using the hint and the definition of ε_{ijk} produces:

$$(\mathbf{a} \times \mathbf{d})_m = \varepsilon_{pqm} a_p d_q = \varepsilon_{pqm} a_p \varepsilon_{ijq} b_i c_j = \varepsilon_{pqm} \varepsilon_{ijq}\, b_i c_j a_p = -\varepsilon_{ijq} \varepsilon_{qpm}\, b_i c_j a_p.$$

Now use the identity (2.19) for the product of epsilons:

$$(\mathbf{a} \times \mathbf{d})_m = -\,(\delta_{ip}\delta_{jm} - \delta_{im}\delta_{pj})\, b_i c_j a_p = -\, b_p c_m a_p + b_m c_p a_p.$$

Each term in the final expression involves a sum over "p", and this is a dot product; therefore

$$(\mathbf{a} \times \mathbf{d})_m = -\,(\mathbf{a} \cdot \mathbf{b}) c_m + b_m (\mathbf{a} \cdot \mathbf{c}).$$

Thus, for any component $m = 1, 2,$ or 3,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\,(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Exercise 2.6. Show that the condition for the vectors **a**, **b**, and **c** to be coplanar is $\varepsilon_{ijk}a_ib_jc_k = 0$.

Solution 2.6. The vector **b** × **c** is perpendicular to **b** and **c**. Thus, **a** will be coplanar with **b** and **c** if it too is perpendicular to **b** × **c**. The condition for a to be perpendicular with **b** × **c** is:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0.$$

In index notation, this is $a_i\varepsilon_{ijk}b_jc_k = 0 = \varepsilon_{ijk}a_ib_jc_k$.

Exercise 2.7. Prove the following relationships: $\delta_{ij}\delta_{ij} = 3$, $\varepsilon_{pqr}\varepsilon_{pqr} = 6$, and $\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$.

Solution 2.7. (*i*) $\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.
For the second two, the identity (2.19) is useful.
(*ii*) $\varepsilon_{pqr}\varepsilon_{pqr} = \varepsilon_{pqr}\varepsilon_{rpq} = \delta_{pp}\delta_{qq} - \delta_{pq}\delta_{pq} = 3(3) - \delta_{pp} = 9 - 3 = 6$.
(*iii*) $\varepsilon_{pqi}\varepsilon_{pqj} = \varepsilon_{ipq}\varepsilon_{pqj} = -\varepsilon_{ipq}\varepsilon_{qpj} = -(\delta_{ip}\delta_{pj} - \delta_{ij}\delta_{pp}) = -\delta_{ij} + 3\delta_{ij} = 2\delta_{ij}$.

Exercise 2.8. Show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where **C** is the direction cosine matrix and **δ** is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an *orthogonal matrix* because it represents transformation of one set of orthogonal axes into another.

Solution 2.8. To show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where **C** is the direction cosine matrix and **δ** is the matrix of the Kronecker delta. Start from (2.5) and (2.7), which are

$$x'_j = x_i C_{ij} \quad \text{and} \quad x_j = x'_i C_{ji},$$

respectively, and change the index "i" into "m" on (2.5): $x'_j = x_m C_{mj}$. Substitute this into (2.7) to find:

$$x_j = x'_i C_{ji} = (x_m C_{mi}) C_{ji} = C_{mi} C_{ji} x_m.$$

However, we also have $x_j = \delta_{jm} x_m$, so

$$\delta_{jm} x_m = C_{mi} C_{ji} x_m \quad \rightarrow \quad \delta_{jm} = C_{mi} C_{ji},$$

which can be written:

$$\delta_{jm} = C_{mi} C^{\mathrm{T}}_{ij} = \mathbf{C}\cdot\mathbf{C}^{\mathrm{T}},$$

and taking the transpose of the this produces:

$$(\delta_{jm})^{\mathrm{T}} = \delta_{mj} = \left(C_{mi} C^{\mathrm{T}}_{ij}\right)^{\mathrm{T}} = C^{\mathrm{T}}_{mi} C_{ij} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C}.$$

Exercise 2.9. Show that for a second-order tensor **A**, the following quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}\ ,\quad I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}, \text{ and } I_3 = \det(A_{ij}).$$

[*Hint*: Use the result of Exercise 2.8 and the transformation rule (2.12) to show that $I'_1 = A'_{ii} = A_{ii} = I_1$. Then show that $A_{ij}A_{ji}$ and $A_{ij}A_{jk}A_{ki}$ are also invariants. In fact, *all* contracted scalars of the form $A_{ij}A_{jk}\ \cdots\ A_{mi}$ are invariants. Finally, verify that $I_2 = \frac{1}{2}\left[I_1^2 - A_{ij}A_{ji}\right]$, $I_3 = \frac{1}{3}\left[A_{ij}A_{jk}A_{ki} - I_1 A_{ij}A_{ji} + I_2 A_{ii}\right]_i$. Because the right-hand sides are invariant, so are I_2 and I_3.]

Solution 2.9. First prove I_1 is invariant by using the second order tensor transformation rule (2.12):

$$A'_{mn} = C_{im}C_{jn}A_{ij}.$$

Replace C_{jn} by C^{T}_{nj} and set $n = m$,

$$A'_{mn} = C_{im}C^{\mathrm{T}}_{nj}A_{ij} \rightarrow A'_{mm} = C_{im}C^{\mathrm{T}}_{mj}A_{ij}.$$

Use the result of Exercise 2.8, $\delta_{ij} = C_{im}C^{\mathrm{T}}_{mj} =$, to find:

$$I_1 = A'_{mm} = \delta_{ij}A_{ij} = A_{ii}.$$

Thus, the first invariant is does not depend on a rotation of the coordinate axes.

Now consider whether or not $A_{mn}A_{nm}$ is invariant under a rotation of the coordinate axes. Start with a double application of (2.12):

$$A'_{mn}A'_{nm} = \left(C_{im}C_{jn}A_{ij}\right)\left(C_{pn}C_{qm}A_{pq}\right) = \left(C_{jn}C^{\mathrm{T}}_{np}\right)\left(C_{im}C^{\mathrm{T}}_{mq}\right)A_{ij}A_{pq}.$$

From the result of Exercise 2.8, the factors in parentheses in the last equality are Kronecker delta functions, so

$$A'_{mn}A'_{nm} = \delta_{jp}\delta_{iq}A_{ij}A_{pq} = A_{ij}A_{ji}.$$

Thus, the matrix contraction $A_{mn}A_{nm}$ does not depend on a rotation of the coordinate axes.

The manipulations for $A_{mn}A_{np}A_{pm}$ are a straightforward extension of the prior efforts for A_{ii} and $A_{ij}A_{ji}$.

$$A'_{mn}A'_{np}A'_{pm} = \left(C_{im}C_{jn}A_{ij}\right)\left(C_{qn}C_{rp}A_{qr}\right)\left(C_{sp}C_{tm}A_{st}\right) = \left(C_{jn}C^{\mathrm{T}}_{nq}\right)\left(C_{rp}C^{\mathrm{T}}_{ps}\right)\left(C_{im}C^{\mathrm{T}}_{mt}\right)A_{ij}A_{qr}A_{st}.$$

Again, the factors in parentheses are Kronecker delta functions, so

$$A'_{mn}A'_{np}A'_{pm} = \delta_{jq}\delta_{rs}\delta_{it}A_{ij}A_{qr}A_{st} = A_{iq}A_{qs}A_{si},$$

which implies that the matrix contraction $A_{ij}A_{jk}A_{ki}$ does not depend on a rotation of the coordinate axes.

Now, for the second invariant, verify the given identity, starting from the given definition for I_2.

$$\begin{aligned}
I_2 &= \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \\
&= A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{33} - A_{23}A_{32} + A_{11}A_{33} - A_{13}A_{31} \\
&= A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - \left(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31}\right) \\
&= \tfrac{1}{2}A_{11}^2 + \tfrac{1}{2}A_{22}^2 + \tfrac{1}{2}A_{33}^2 + A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - \left(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31} + \tfrac{1}{2}A_{11}^2 + \tfrac{1}{2}A_{22}^2 + \tfrac{1}{2}A_{33}^2\right) \\
&= \tfrac{1}{2}\left[A_{11} + A_{22} + A_{33}\right]^2 - \tfrac{1}{2}\left(2A_{12}A_{21} + 2A_{23}A_{32} + 2A_{13}A_{31} + A_{11}^2 + A_{22}^2 + A_{33}^2\right) \\
&= \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\left(A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31} + A_{12}A_{21} + A_{22}A_{22} + A_{23}A_{32} + A_{13}A_{31} + A_{23}A_{32} + A_{33}A_{33}\right)
\end{aligned}$$

$$=\tfrac{1}{2}I_1^2-\tfrac{1}{2}\left(A_{ij}A_{ji}\right)=\tfrac{1}{2}\left(I_1^2-A_{ij}A_{ji}\right)$$

Thus, since I_2 only depends on I_1 and $A_{ij}A_{ji}$, it is invariant under a rotation of the coordinate axes because I_1 and $A_{ij}A_{ji}$ are invariant under a rotation of the coordinate axes.

The manipulations for the third invariant are a tedious but not remarkable. Start from the given definition for I_3, and group like terms.

$$I_3=\det\left(A_{ij}\right)=A_{11}(A_{22}A_{33}-A_{23}A_{32})-A_{12}(A_{21}A_{33}-A_{23}A_{31})+A_{13}(A_{21}A_{32}-A_{22}A_{31})$$
$$=A_{11}A_{22}A_{33}+A_{12}A_{23}A_{31}+A_{13}A_{32}A_{21}-\left(A_{11}A_{23}A_{32}+A_{22}A_{13}A_{31}+A_{33}A_{12}A_{21}\right)\qquad\text{(a)}$$

Now work from the given identity. The triple matrix product $A_{ij}A_{jk}A_{ki}$ has twenty-seven terms:

$$A_{11}^3+A_{11}A_{12}A_{21}+A_{11}A_{13}A_{31}+A_{12}A_{21}A_{11}+A_{12}A_{22}A_{21}+A_{12}A_{23}A_{31}+A_{13}A_{31}A_{11}+A_{13}A_{32}A_{21}+A_{13}A_{33}A_{31}+$$
$$A_{21}A_{11}A_{12}+A_{21}A_{12}A_{22}+A_{21}A_{13}A_{32}+A_{22}A_{21}A_{12}+A_{22}^3+A_{22}A_{23}A_{32}+A_{23}A_{31}A_{12}+A_{23}A_{32}A_{22}+A_{23}A_{33}A_{32}+$$
$$A_{31}A_{11}A_{13}+A_{31}A_{12}A_{23}+A_{31}A_{13}A_{33}+A_{32}A_{21}A_{13}+A_{32}A_{22}A_{23}+A_{32}A_{23}A_{33}+A_{33}A_{31}A_{13}+A_{33}A_{32}A_{23}+A_{33}^3$$

These can be grouped as follows:

$$A_{ij}A_{jk}A_{ki}=3(A_{12}A_{23}A_{31}+A_{13}A_{32}A_{21})+A_{11}(A_{11}^2+3A_{12}A_{21}+3A_{13}A_{31})+$$
$$A_{22}(3A_{21}A_{12}+A_{22}^2+3A_{23}A_{32})+A_{33}(3A_{31}A_{13}+3A_{32}A_{23}+A_{33}^2)\qquad\text{(b)}$$

The remaining terms of the given identity are:

$$-I_1A_{ij}A_{ji}+I_2A_{ii}=I_1(I_2-A_{ij}A_{ji})=I_1(I_2+2I_2-I_1^2)=3I_1I_2-I_1^3,$$

where the result for I_2 has been used. Evaluating the first of these two terms leads to:

$$3I_1I_2=3(A_{11}+A_{22}+A_{33})(A_{11}A_{22}-A_{12}A_{21}+A_{22}A_{33}-A_{23}A_{32}+A_{11}A_{33}-A_{13}A_{31})$$
$$=3(A_{11}+A_{22}+A_{33})(A_{11}A_{22}+A_{22}A_{33}+A_{11}A_{33})-3(A_{11}+A_{22}+A_{33})(A_{12}A_{21}+A_{23}A_{32}+A_{13}A_{31}).$$

Adding this to (b) produces:

$$A_{ij}A_{jk}A_{ki}+3I_1I_2=3(A_{12}A_{23}A_{31}+A_{13}A_{32}A_{21})+3(A_{11}+A_{22}+A_{33})(A_{11}A_{22}+A_{22}A_{33}+A_{11}A_{33})+$$
$$A_{11}(A_{11}^2-3A_{23}A_{32})+A_{22}(A_{22}^2-3A_{13}A_{31})+A_{33}(A_{33}^2-3A_{12}A_{21})$$
$$=3(A_{12}A_{23}A_{31}+A_{13}A_{32}A_{21}-A_{11}A_{23}A_{32}-A_{22}A_{13}A_{31}-A_{33}A_{12}A_{21})+$$
$$3(A_{11}+A_{22}+A_{33})(A_{11}A_{22}+A_{22}A_{33}+A_{11}A_{33})+A_{11}^3+A_{22}^3+A_{33}^3\qquad\text{(c)}$$

The last term of the given identity is:

$$I_1^3=A_{11}^3+A_{22}^3+A_{33}^3+3(A_{11}^2A_{22}+A_{11}^2A_{33}+A_{22}^2A_{11}+A_{22}^2A_{33}+A_{33}^2A_{11}+A_{33}^2A_{22})+6A_{11}A_{22}A_{33}$$
$$=A_{11}^3+A_{22}^3+A_{33}^3+3(A_{11}+A_{22}+A_{33})(A_{11}A_{22}+A_{11}A_{33}+A_{22}A_{33})-3A_{11}A_{22}A_{33}$$

Subtracting this from (c) produces:

$$A_{ij}A_{jk}A_{ki}+3I_1I_2-I_1^3=3(A_{12}A_{23}A_{31}+A_{13}A_{32}A_{21}-A_{11}A_{23}A_{32}-A_{22}A_{13}A_{31}-A_{33}A_{12}A_{21}+A_{11}A_{22}A_{33})$$
$$=3I_3.$$

This verifies that the given identity for I_3 is correct. Thus, since I_3 only depends on I_1, I_2, and $A_{ij}A_{jk}A_{ki}$, it is invariant under a rotation of the coordinate axes because these quantities are invariant under a rotation of the coordinate axes as shown above.

Exercise 2.10. If **u** and **v** are vectors, show that the products $u_i \upsilon_j$ obey the transformation rule (2.12), and therefore represent a second-order tensor.

Solution 2.10. Start by applying the vector transformation rule (2.5 or 2.6) to the components of **u** and **v** separately,

$$u'_m = C_{im} u_i \text{ , and } v'_n = C_{jn} v_j .$$

The product of these two equations produces:

$$u'_m v'_n = C_{im} C_{jn} u_i v_j ,$$

which is the same as (2.12) for second order tensors.

Exercise 2.11. Show that δ_{ij} is an isotropic tensor. That is, show that $\delta'_{ij} = \delta_{ij}$ under rotation of the coordinate system. [*Hint*: Use the transformation rule (2.12) and the results of Exercise 2.8.]

Solution 2.11. Apply (2.12) to δ_{ij},

$$\delta'_{mn} = C_{im}C_{jn}\delta_{ij} = C_{im}C_{in} = C^{\mathrm{T}}_{mi}C_{in} = \delta_{mn}\,.$$

where the final equality follows from the result of Exercise 2.8. Thus, the Kronecker delta is invariant under coordinate rotations.

Exercise 2.12. If **u** and **v** are arbitrary vectors resolved in three-dimensional Cartesian coordinates, show that **u**·**v** = 0 when **u** and **v** are perpendicular.

Solution 2.12. Consider the magnitude of the sum **u** + **v**,

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2\\ &= u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2+2u_1v_1+2u_2v_2+2u_3v_3\\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\cdot\mathbf{v},\end{aligned}$$

which can be rewritten:

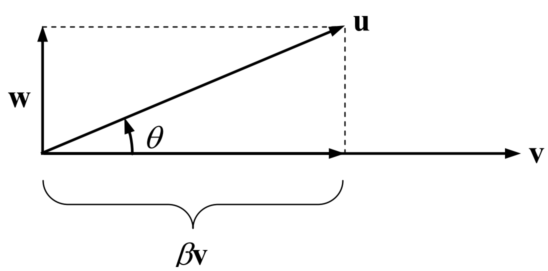
$$\|\mathbf{u}+\mathbf{v}\|^2-\|\mathbf{u}\|^2-\|\mathbf{v}\|^2=2\mathbf{u}\cdot\mathbf{v}.$$

When **u** and **v** are perpendicular, the Pythagorean theorem requires the left side to be zero. Thus,

$$\mathbf{u}\cdot\mathbf{v}=0.$$

Exercise 2.13. If **u** and **v** are vectors with magnitudes u and υ, use the finding of Exercise 2.12 to show that $\mathbf{u}\cdot\mathbf{v} = u\upsilon\cos\theta$ where θ is the angle between **u** and **v**.

Solution 2.13. Start with two arbitrary vectors (**u** and **v**), and view them so that the plane they define is coincident with the page and **v** is horizontal. Consider two additional vectors, $\beta\mathbf{v}$ and **w**, that are perpendicular ($\mathbf{v}\cdot\mathbf{w} = 0$) and can be summed together to produce **u**: $\mathbf{w} + \beta\mathbf{v} = \mathbf{u}$.



Compute the dot-product of **u** and **v**:

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{w} + \beta\mathbf{v})\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{v} + \beta\mathbf{v}\cdot\mathbf{v} = \beta\upsilon^2.$$

where the final equality holds because $\mathbf{v}\cdot\mathbf{w} = 0$. From the geometry of the figure:

$$\cos\theta \equiv \frac{\|\beta\mathbf{v}\|}{\|\mathbf{u}\|} = \frac{\beta\upsilon}{u}, \text{ or } \beta = \frac{u}{\upsilon}\cos\theta.$$

Insert this into the final equality for $\mathbf{u}\cdot\mathbf{v}$ to find:

$$\mathbf{u}\cdot\mathbf{v} = \left(\frac{u}{\upsilon}\cos\theta\right)\upsilon^2 = u\upsilon\cos\theta.$$