Exercise 2.1. For three spatial dimensions, rewrite the following expressions in index notation and evaluate or simplify them using the values or parameters given, and the definitions of δ_{ij} and ε_{ijk} wherever possible. In b) through e), $\mathbf{x}$ is the position vector, with components x_i.

a) $\mathbf{b}\cdot\mathbf{c}$ where $\mathbf{b} = (1, 4, 17)$ and $\mathbf{c} = (-4, -3, 1)$

b) $(\mathbf{u}\cdot\nabla)\mathbf{x}$ where $\mathbf{u}$ a vector with components u_i.

c) $\nabla\phi$, where $\phi = \mathbf{h}\cdot\mathbf{x}$ and $\mathbf{h}$ is a constant vector with components h_i.

d) $\nabla\times\mathbf{u}$, where $\mathbf{u} = \boldsymbol{\Omega}\times\mathbf{x}$ and $\boldsymbol{\Omega}$ is a constant vector with components Ω_i.

e) $\mathbf{C}\cdot\mathbf{x}$, where $\mathbf{C} = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}$

Solution 2.1. a) $\mathbf{b}\cdot\mathbf{c} = b_i c_i = 1(-4) + 4(-3) + 17(1) = -4 - 12 + 17 = +1$

b) $$(\mathbf{u}\cdot\nabla)\mathbf{x} = u_j \frac{\partial}{\partial x_j} x_i = \left[u_1\left(\frac{\partial}{\partial x_1}\right) + u_2\left(\frac{\partial}{\partial x_2}\right) + u_3\left(\frac{\partial}{\partial x_3}\right)\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1\left(\frac{\partial x_1}{\partial x_1}\right) + u_2\left(\frac{\partial x_1}{\partial x_2}\right) + u_3\left(\frac{\partial x_1}{\partial x_3}\right) \\ u_1\left(\frac{\partial x_2}{\partial x_1}\right) + u_2\left(\frac{\partial x_2}{\partial x_2}\right) + u_3\left(\frac{\partial x_2}{\partial x_3}\right) \\ u_1\left(\frac{\partial x_3}{\partial x_1}\right) + u_2\left(\frac{\partial x_3}{\partial x_2}\right) + u_3\left(\frac{\partial x_3}{\partial x_3}\right) \end{bmatrix} = \begin{bmatrix} u_1\cdot 1 + u_2\cdot 0 + u_3\cdot 0 \\ u_1\cdot 0 + u_2\cdot 1 + u_3\cdot 0 \\ u_1\cdot 0 + u_2\cdot 0 + u_3\cdot 1 \end{bmatrix} = u_j\delta_{ij} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_i$$

c) $$\nabla\phi = \frac{\partial\phi}{\partial x_j} = \frac{\partial}{\partial x_j}(h_i x_i) = h_i\frac{\partial x_i}{\partial x_j} = h_i\delta_{ij} = h_j = \mathbf{h}$$

d) $$\nabla\times\mathbf{u} = \nabla\times(\boldsymbol{\Omega}\times\mathbf{x}) = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}\Omega_l x_m) = \varepsilon_{ijk}\varepsilon_{klm}\Omega_l\delta_{jm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_l\delta_{jm} = (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl})\Omega_l$$

$$= (3\delta_{il} - \delta_{il})\Omega_l = 2\delta_{il}\Omega_l = 2\Omega_l = 2\boldsymbol{\Omega}$$

Here, the following identities have been used: $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, $\delta_{ij}\delta_{jk} = \delta_{ik}$, $\delta_{jj} = 3$, and $\delta_{ij}\Omega_j = \Omega_i$

e) $$\mathbf{C}\cdot\mathbf{x} = C_{ij}x_j = \begin{Bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x_1 + 2x_2 + 3x_3 \\ x_2 + 2x_3 \\ x_3 \end{Bmatrix}$$

Exercise 2.2. Starting from (2.1) and (2.3), prove (2.7).

Solution 2.2. The two representations for the position vector are:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \text{ or } \mathbf{x} = x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'.$$

Develop the dot product of $\mathbf{x}$ with $\mathbf{e}_1$ from each representation,

$$\mathbf{e}_1 \cdot \mathbf{x} = \mathbf{e}_1 \cdot \left(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\right) = x_1\mathbf{e}_1 \cdot \mathbf{e}_1 + x_2\mathbf{e}_1 \cdot \mathbf{e}_2 + x_3\mathbf{e}_1 \cdot \mathbf{e}_3 = x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 = x_1 ,$$

$$\text{and } \mathbf{e}_1 \cdot \mathbf{x} = \mathbf{e}_1 \cdot \left(x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'\right) = x_1'\mathbf{e}_1 \cdot \mathbf{e}_1' + x_2'\mathbf{e}_1 \cdot \mathbf{e}_2' + x_3'\mathbf{e}_1 \cdot \mathbf{e}_3' = x_i'C_{1i},$$

set these equal to find:

$$x_1 = x_i'C_{1i},$$

where $C_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j'$ is a 3 × 3 matrix of direction cosines. In an entirely parallel fashion, forming the dot product of $\mathbf{x}$ with $\mathbf{e}_2$, and $\mathbf{x}$ with $\mathbf{e}_2$ produces:

$$x_2 = x_i'C_{2i} \text{ and } x_3 = x_i'C_{3i}.$$

Thus, for any component x_j, where j = 1, 2, or 3, we have:

$$x_j = x_i'C_{ji},$$

which is (2.7).

Exercise 2.3. For two three-dimensional vectors with Cartesian components a_i and b_i, prove the Cauchy-Schwartz inequality: $(a_ib_i)^2 \leq (a_i)^2(b_i)^2$.

Solution 2.3. Expand the left side term,

$$(a_ib_i)^2 = (a_1b_1 + a_2b_2 + a_3b_3)^2 = a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3,$$

then expand the right side term,

$$\begin{aligned}(a_i)^2(b_i)^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)\\ &= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + (a_1^2b_2^2 + a_2^2b_1^2) + (a_1^2b_3^2 + a_3^2b_1^2) + (a_3^2b_2^2 + a_2^2b_3^2).\end{aligned}$$

Subtract the left side term from the right side term to find:

$$\begin{aligned}&(a_i)^2(b_i)^2 - (a_ib_i)^2\\ &\quad = (a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2) + (a_1^2b_3^2 - 2a_1b_1a_3b_3 + a_3^2b_1^2) + (a_3^2b_2^2 - 2a_2b_2a_3b_3 + a_2^2b_3^2)\\ &\quad = (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_3b_2 - a_2b_3)^2 = |\mathbf{a}\times\mathbf{b}|^2.\end{aligned}$$

Thus, the difference $(a_i)^2(b_i)^2 - (a_ib_i)^2$ is greater than zero unless **a** = (const.)**b** then the difference is zero.

Exercise 2.4. For two three-dimensional vectors with Cartesian components a_i and b_i, prove the triangle inequality: $|\mathbf{a}|+|\mathbf{b}| \geq |\mathbf{a}+\mathbf{b}|$.

Solution 2.4. To avoid square roots, square both side of the equation; this operation does not change the equation's meaning. The left side becomes:

$$\left(|\mathbf{a}|+|\mathbf{b}|\right)^2 = |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 ,$$

and the right side becomes:

$$|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 .$$

So,

$$\left(|\mathbf{a}|+|\mathbf{b}|\right)^2 - |\mathbf{a}+\mathbf{b}|^2 = 2|\mathbf{a}||\mathbf{b}| - 2\mathbf{a}\cdot\mathbf{b} .$$

Thus, to prove the triangle equality, the right side of this last equation must be greater than or equal to zero. This requires:

$$|\mathbf{a}||\mathbf{b}| \geq \mathbf{a}\cdot\mathbf{b} \quad \text{or using index notation:} \quad \sqrt{a_i^2 b_i^2} \geq a_i b_i ,$$

which can be squared to find:

$$a_i^2 b_i^2 \geq (a_i b_i)^2 ,$$

and this is the Cauchy-Schwartz inequality proved in Exercise 2.3. Thus, the triangle equality is proved.

Exercise 2.5. Using Cartesian coordinates where the position vector is $\mathbf{x} = (x_1, x_2, x_3)$ and the fluid velocity is $\mathbf{u} = (u_1, u_2, u_3)$, write out the three components of the vector: $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$.

Solution 2.5.

a) $$(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i\left(\frac{\partial u_j}{\partial x_i}\right) = u_1\left(\frac{\partial u_j}{\partial x_1}\right) + u_2\left(\frac{\partial u_j}{\partial x_2}\right) + u_3\left(\frac{\partial u_j}{\partial x_3}\right) = \begin{Bmatrix} u_1\left(\frac{\partial u_1}{\partial x_1}\right) + u_2\left(\frac{\partial u_1}{\partial x_2}\right) + u_3\left(\frac{\partial u_1}{\partial x_3}\right) \\ u_1\left(\frac{\partial u_2}{\partial x_1}\right) + u_2\left(\frac{\partial u_2}{\partial x_2}\right) + u_3\left(\frac{\partial u_2}{\partial x_3}\right) \\ u_1\left(\frac{\partial u_3}{\partial x_1}\right) + u_2\left(\frac{\partial u_3}{\partial x_2}\right) + u_3\left(\frac{\partial u_3}{\partial x_3}\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y}\right) + w\left(\frac{\partial u}{\partial z}\right) \\ u\left(\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial v}{\partial y}\right) + w\left(\frac{\partial v}{\partial z}\right) \\ u\left(\frac{\partial w}{\partial x}\right) + v\left(\frac{\partial w}{\partial y}\right) + w\left(\frac{\partial w}{\partial z}\right) \end{Bmatrix}$$

The vector in this exercise, $(\mathbf{u}\cdot\nabla)\mathbf{u} = u_i(\partial u_j/\partial x_i)$, is an important one in fluid mechanics. As described in Ch. 3, it is the nonlinear advective acceleration.

Exercise 2.6. Convert $\nabla \times \nabla \rho$ to indicial notation and show that it is zero in Cartesian coordinates for any twice-differentiable scalar function ρ.

Solution 2.6. Start with the definitions of the cross product and the gradient.

$$\nabla \times (\nabla \rho) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \rho)_k = \varepsilon_{ijk} \frac{\partial^2 \rho}{\partial x_j \partial x_k}$$

Write out the vector component by component recalling that $\varepsilon_{ijk} = 0$ if any two indices are equal. Here the "i" index is the free index.

$$\varepsilon_{ijk} \frac{\partial^2 \rho}{\partial x_j \partial x_k} = \left\{ \begin{matrix} \varepsilon_{123} \dfrac{\partial^2 \rho}{\partial x_2 \partial x_3} + \varepsilon_{132} \dfrac{\partial^2 \rho}{\partial x_3 \partial x_2} \\ \varepsilon_{213} \dfrac{\partial^2 \rho}{\partial x_1 \partial x_3} + \varepsilon_{231} \dfrac{\partial^2 \rho}{\partial x_3 \partial x_1} \\ \varepsilon_{312} \dfrac{\partial^2 \rho}{\partial x_1 \partial x_2} + \varepsilon_{321} \dfrac{\partial^2 \rho}{\partial x_2 \partial x_1} \end{matrix} \right\} = \left\{ \begin{matrix} \dfrac{\partial^2 \rho}{\partial x_2 \partial x_3} - \dfrac{\partial^2 \rho}{\partial x_3 \partial x_2} \\ -\dfrac{\partial^2 \rho}{\partial x_1 \partial x_3} + \dfrac{\partial^2 \rho}{\partial x_3 \partial x_1} \\ \dfrac{\partial^2 \rho}{\partial x_1 \partial x_2} - \dfrac{\partial^2 \rho}{\partial x_2 \partial x_1} \end{matrix} \right\} = 0,$$

where the middle equality follows from the definition of ε_{ijk} (2.18), and the final equality follows when ρ is twice differentiable so that $\dfrac{\partial^2 \rho}{\partial x_j \partial x_k} = \dfrac{\partial^2 \rho}{\partial x_k \partial x_j}$.

Exercise 2.7. Using indicial notation, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. [*Hint*: Call $\mathbf{d} \equiv \mathbf{b} \times \mathbf{c}$. Then $(\mathbf{a} \times \mathbf{d})_m = \varepsilon_{pqm} a_p d_q = \varepsilon_{pqm} a_p \varepsilon_{ijq} b_i c_j$. Using (2.19), show that $(\mathbf{a} \times \mathbf{d})_m = (\mathbf{a} \cdot \mathbf{c})\boldsymbol{b_m} - (\mathbf{a} \cdot \mathbf{b})\boldsymbol{c_m}$.]

Solution 2.7. Using the hint and the definition of ε_{ijk} produces:

$$(\mathbf{a} \times \mathbf{d})_m = \varepsilon_{pqm} a_p d_q = \varepsilon_{pqm} a_p \varepsilon_{ijq} b_i c_j = \varepsilon_{pqm} \varepsilon_{ijq} b_i c_j a_p = -\varepsilon_{ijq} \varepsilon_{qpm} b_i c_j a_p.$$

Now use the identity (2.19) for the product of epsilons:

$$(\mathbf{a} \times \mathbf{d})_m = -(\delta_{ip}\delta_{jm} - \delta_{im}\delta_{pj})\, b_i c_j a_p = -b_p c_m a_p + b_m c_p a_p.$$

Each term in the final expression involves a sum over "p", and this is a dot product; therefore

$$(\mathbf{a} \times \mathbf{d})_m = -(\mathbf{a} \cdot \mathbf{b}) c_m + b_m (\mathbf{a} \cdot \mathbf{c}).$$

Thus, for any component m = 1, 2, or 3,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Exercise 2.8. Show that the condition for the vectors **a**, **b**, and **c** to be coplanar is $\varepsilon_{ijk}a_ib_jc_k = 0$.

Solution 2.8. The vector **b** × **c** is perpendicular to **b** and **c**. Thus, **a** will be coplanar with **b** and **c** if it too is perpendicular to **b** × **c**. The condition for a to be perpendicular with **b** × **c** is:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0.$$

In index notation, this is $a_i\varepsilon_{ijk}b_jc_k = 0 = \varepsilon_{ijk}a_ib_jc_k$.

Exercise 2.9. Prove the following relationships: $\delta_{ij}\delta_{ij} = 3$, $\varepsilon_{pqr}\varepsilon_{pqr} = 6$, and $\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$.

Solution 2.9. (*i*) $\delta_{ij}\delta_{ij} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.
For the second two, the identity (2.19) is useful.
(*ii*) $\varepsilon_{pqr}\varepsilon_{pqr} = \varepsilon_{pqr}\varepsilon_{rpq} = \delta_{pp}\delta_{qq} - \delta_{pq}\delta_{pq} = 3(3) - \delta_{pp} = 9 - 3 = 6$.
(*iii*) $\varepsilon_{pqi}\varepsilon_{pqj} = \varepsilon_{ipq}\varepsilon_{pqj} = -\varepsilon_{ipq}\varepsilon_{qpj} = -(\delta_{ip}\delta_{pj} - \delta_{ij}\delta_{pp}) = -\delta_{ij} + 3\delta_{ij} = 2\delta_{ij}$.

Exercise 2.10. Show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where $\mathbf{C}$ is the direction cosine matrix and $\boldsymbol{\delta}$ is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an *orthogonal matrix* because it represents transformation of one set of orthogonal axes into another.

Solution 2.10. To show that $\mathbf{C}\cdot\mathbf{C}^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C} = \boldsymbol{\delta}$, where $\mathbf{C}$ is the direction cosine matrix and $\boldsymbol{\delta}$ is the matrix of the Kronecker delta. Start from (2.5) and (2.7), which are

$$x'_j = x_i C_{ij} \quad \text{and} \quad x_j = x'_i C_{ji},$$

respectively, and change the index "i" into "m" on (2.5): $x'_j = x_m C_{mj}$. Substitute this into (2.7) to find:

$$x_j = x'_i C_{ji} = \left(x_m C_{mi}\right) C_{ji} = C_{mi} C_{ji} x_m.$$

However, we also have $x_j = \delta_{jm} x_m$, so

$$\delta_{jm} x_m = C_{mi} C_{ji} x_m \quad \rightarrow \quad \delta_{jm} = C_{mi} C_{ji},$$

which can be written:

$$\delta_{jm} = C_{mi} C_{ij}^{\mathrm{T}} = \mathbf{C}\cdot\mathbf{C}^{\mathrm{T}},$$

and taking the transpose of the this produces:

$$\left(\delta_{jm}\right)^{\mathrm{T}} = \delta_{mj} = \left(C_{mi} C_{ij}^{\mathrm{T}}\right)^{\mathrm{T}} = C_{mi}^{\mathrm{T}} C_{ij} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{C}.$$

Exercise 2.11. Show that for a second-order tensor **A**, the following quantities are invariant under the rotation of axes:

$$I_1 = A_{ii}\ ,\quad I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}\ , \text{ and } I_3 = \det(A_{ij}).$$

[*Hint*: Use the result of Exercise 2.8 and the transformation rule (2.12) to show that $I'_1 = A'_{ii} = A_{ii} = I_1$. Then show that $A_{ij}A_{ji}$ and $A_{ij}A_{jk}A_{ki}$ are also invariants. In fact, *all* contracted scalars of the form $A_{ij}A_{jk}\ \cdots\ A_{mi}$ are invariants. Finally, verify that $I_2 = \frac{1}{2}\left[I_1^2 - A_{ij}A_{ji}\right]$, $I_3 = \frac{1}{3}\left[A_{ij}A_{jk}A_{ki} - I_1 A_{ij}A_{ji} + I_2 A_{ii}\right]$. Because the right-hand sides are invariant, so are I_2 and I_3.]

Solution 2.11. First prove I_1 is invariant by using the second order tensor transformation rule (2.12):

$$A'_{mn} = C_{im}C_{jn}A_{ij}.$$

Replace C_{jn} by C^{T}_{nj} and set $n = m$,

$$A'_{mn} = C_{im}C^{\mathrm{T}}_{nj}A_{ij} \rightarrow A'_{mm} = C_{im}C^{\mathrm{T}}_{mj}A_{ij}.$$

Use the result of Exercise 2.8, $\delta_{ij} = C_{im}C^{\mathrm{T}}_{mj} =$, to find:

$$I_1 = A'_{mm} = \delta_{ij}A_{ij} = A_{ii}.$$

Thus, the first invariant is does not depend on a rotation of the coordinate axes.

Now consider whether or not $A_{mn}A_{nm}$ is invariant under a rotation of the coordinate axes. Start with a double application of (2.12):

$$A'_{mn}A'_{nm} = \left(C_{im}C_{jn}A_{ij}\right)\left(C_{pn}C_{qm}A_{pq}\right) = \left(C_{jn}C^{\mathrm{T}}_{np}\right)\left(C_{im}C^{\mathrm{T}}_{mq}\right)A_{ij}A_{pq}.$$

From the result of Exercise 2.8, the factors in parentheses in the last equality are Kronecker delta functions, so

$$A'_{mn}A'_{nm} = \delta_{jp}\delta_{iq}A_{ij}A_{pq} = A_{ij}A_{ji}.$$

Thus, the matrix contraction $A_{mn}A_{nm}$ does not depend on a rotation of the coordinate axes.

The manipulations for $A_{mn}A_{np}A_{pm}$ are a straightforward extension of the prior efforts for A_{ii} and $A_{ij}A_{ji}$.

$$A'_{mn}A'_{np}A'_{pm} = \left(C_{im}C_{jn}A_{ij}\right)\left(C_{qn}C_{rp}A_{qr}\right)\left(C_{sp}C_{tm}A_{st}\right) = \left(C_{jn}C^{\mathrm{T}}_{nq}\right)\left(C_{rp}C^{\mathrm{T}}_{ps}\right)\left(C_{im}C^{\mathrm{T}}_{mt}\right)A_{ij}A_{qr}A_{st}.$$

Again, the factors in parentheses are Kronecker delta functions, so

$$A'_{mn}A'_{np}A'_{pm} = \delta_{jq}\delta_{rs}\delta_{it}A_{ij}A_{qr}A_{st} = A_{iq}A_{qs}A_{si},$$

which implies that the matrix contraction $A_{ij}A_{jk}A_{ki}$ does not depend on a rotation of the coordinate axes.

Now, for the second invariant, verify the given identity, starting from the given definition for I_2.

$$\begin{aligned}
I_2 &= \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \\
&= A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{33} - A_{23}A_{32} + A_{11}A_{33} - A_{13}A_{31} \\
&= A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - \left(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31}\right) \\
&= \tfrac{1}{2}A_{11}^2 + \tfrac{1}{2}A_{22}^2 + \tfrac{1}{2}A_{33}^2 + A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - \left(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31} + \tfrac{1}{2}A_{11}^2 + \tfrac{1}{2}A_{22}^2 + \tfrac{1}{2}A_{33}^2\right) \\
&= \tfrac{1}{2}\left[A_{11} + A_{22} + A_{33}\right]^2 - \tfrac{1}{2}\left(2A_{12}A_{21} + 2A_{23}A_{32} + 2A_{13}A_{31} + A_{11}^2 + A_{22}^2 + A_{33}^2\right)
\end{aligned}$$

$$= \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\left(A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31} + A_{12}A_{21} + A_{22}A_{22} + A_{23}A_{32} + A_{13}A_{31} + A_{23}A_{32} + A_{33}A_{33}\right)$$
$$= \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\left(A_{ij}A_{ji}\right) = \tfrac{1}{2}\left(I_1^2 - A_{ij}A_{ji}\right)$$

Thus, since I_2 only depends on I_1 and $A_{ij}A_{ji}$, it is invariant under a rotation of the coordinate axes because I_1 and $A_{ij}A_{ji}$ are invariant under a rotation of the coordinate axes.

The manipulations for the third invariant are a tedious but not remarkable. Start from the given definition for I_3, and group like terms.

$$I_3 = \det\left(A_{ij}\right) = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$
$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - \left(A_{11}A_{23}A_{32} + A_{22}A_{13}A_{31} + A_{33}A_{12}A_{21}\right) \quad \text{(a)}$$

Now work from the given identity. The triple matrix product $A_{ij}A_{jk}A_{ki}$ has twenty-seven terms:

$$A_{11}^3 + A_{11}A_{12}A_{21} + A_{11}A_{13}A_{31} + A_{12}A_{21}A_{11} + A_{12}A_{22}A_{21} + A_{12}A_{23}A_{31} + A_{13}A_{31}A_{11} + A_{13}A_{32}A_{21} + A_{13}A_{33}A_{31} +$$
$$A_{21}A_{11}A_{12} + A_{21}A_{12}A_{22} + A_{21}A_{13}A_{32} + A_{22}A_{21}A_{12} + A_{22}^3 + A_{22}A_{23}A_{32} + A_{23}A_{31}A_{12} + A_{23}A_{32}A_{22} + A_{23}A_{33}A_{32} +$$
$$A_{31}A_{11}A_{13} + A_{31}A_{12}A_{23} + A_{31}A_{13}A_{33} + A_{32}A_{21}A_{13} + A_{32}A_{22}A_{23} + A_{32}A_{23}A_{33} + A_{33}A_{31}A_{13} + A_{33}A_{32}A_{23} + A_{33}^3$$

These can be grouped as follows:

$$A_{ij}A_{jk}A_{ki} = 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21}) + A_{11}(A_{11}^2 + 3A_{12}A_{21} + 3A_{13}A_{31}) +$$
$$A_{22}(3A_{21}A_{12} + A_{22}^2 + 3A_{23}A_{32}) + A_{33}(3A_{31}A_{13} + 3A_{32}A_{23} + A_{33}^2) \quad \text{(b)}$$

The remaining terms of the given identity are:

$$-I_1A_{ij}A_{ji} + I_2A_{ii} = I_1(I_2 - A_{ij}A_{ji}) = I_1(I_2 + 2I_2 - I_1^2) = 3I_1I_2 - I_1^3,$$

where the result for I_2 has been used. Evaluating the first of these two terms leads to:

$$3I_1I_2 = 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{33} - A_{23}A_{32} + A_{11}A_{33} - A_{13}A_{31})$$
$$= 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33}) - 3(A_{11} + A_{22} + A_{33})(A_{12}A_{21} + A_{23}A_{32} + A_{13}A_{31}).$$

Adding this to (b) produces:

$$A_{ij}A_{jk}A_{ki} + 3I_1I_2 = 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21}) + 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33}) +$$
$$A_{11}(A_{11}^2 - 3A_{23}A_{32}) + A_{22}(A_{22}^2 - 3A_{13}A_{31}) + A_{33}(A_{33}^2 - 3A_{12}A_{21})$$
$$= 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - A_{11}A_{23}A_{32} - A_{22}A_{13}A_{31} - A_{33}A_{12}A_{21}) +$$
$$3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33}) + A_{11}^3 + A_{22}^3 + A_{33}^3 \quad \text{(c)}$$

The last term of the given identity is:

$$I_1^3 = A_{11}^3 + A_{22}^3 + A_{33}^3 + 3(A_{11}^2A_{22} + A_{11}^2A_{33} + A_{22}^2A_{11} + A_{22}^2A_{33} + A_{33}^2A_{11} + A_{33}^2A_{22}) + 6A_{11}A_{22}A_{33}$$
$$= A_{11}^3 + A_{22}^3 + A_{33}^3 + 3(A_{11} + A_{22} + A_{33})(A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33}) - 3A_{11}A_{22}A_{33}$$

Subtracting this from (c) produces:

$$A_{ij}A_{jk}A_{ki} + 3I_1I_2 - I_1^3 = 3(A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} - A_{11}A_{23}A_{32} - A_{22}A_{13}A_{31} - A_{33}A_{12}A_{21} + A_{11}A_{22}A_{33})$$
$$= 3I_3.$$

This verifies that the given identity for I_3 is correct. Thus, since I_3 only depends on I_1, I_2, and $A_{ij}A_{jk}A_{ki}$, it is invariant under a rotation of the coordinate axes because these quantities are invariant under a rotation of the coordinate axes as shown above.

Exercise 2.12. If **u** and **v** are vectors, show that the products $u_i v_j$ obey the transformation rule (2.12), and therefore represent a second-order tensor.

Solution 2.12. Start by applying the vector transformation rule (2.5 or 2.6) to the components of **u** and **v** separately,

$$u'_m = C_{im} u_i \text{ , and } v'_n = C_{jn} v_j .$$

The product of these two equations produces:

$$u'_m v'_n = C_{im} C_{jn} u_i v_j ,$$

which is the same as (2.12) for second order tensors.

Exercise 2.13. Show that δ_{ij} is an isotropic tensor. That is, show that $\delta'_{ij} = \delta_{ij}$ under rotation of the coordinate system. [*Hint*: Use the transformation rule (2.12) and the results of Exercise 2.10.]

Solution 2.13. Apply (2.12) to δ_{ij},

$$\delta'_{mn} = C_{im}C_{jn}\delta_{ij} = C_{im}C_{in} = C^{\mathrm{T}}_{mi}C_{in} = \delta_{mn}.$$

where the final equality follows from the result of Exercise 2.10. Thus, the Kronecker delta is invariant under coordinate rotations.

Exercise 2.14. If **u** and **v** are arbitrary vectors resolved in three-dimensional Cartesian coordinates, use the definition of vector magnitude, $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$, and the Pythagorean theorem to show that $\mathbf{u}\cdot\mathbf{v} = 0$ when **u** and **v** are perpendicular.

Solution 2.14. Consider the magnitude of the sum **u** + **v**,

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2\\ &= u_1^2+u_2^2+u_3^2+v_1^2+v_2^2+v_3^2+2u_1v_1+2u_2v_2+2u_3v_3\\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\cdot\mathbf{v},\end{aligned}$$

which can be rewritten:

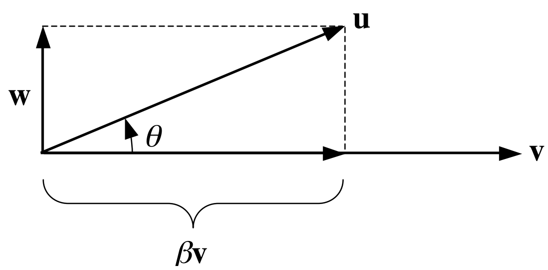
$$\|\mathbf{u}+\mathbf{v}\|^2-\|\mathbf{u}\|^2-\|\mathbf{v}\|^2 = 2\mathbf{u}\cdot\mathbf{v}.$$

When **u** and **v** are perpendicular, the Pythagorean theorem requires the left side to be zero. Thus,

$$\mathbf{u}\cdot\mathbf{v} = 0.$$

Exercise 2.15. If **u** and **v** are vectors with magnitudes u and υ, use the finding of Exercise 2.14 to show that $\mathbf{u}\cdot\mathbf{v} = u\upsilon\cos\theta$ where θ is the angle between **u** and **v**.

Solution 2.15. Start with two arbitrary vectors (**u** and **v**), and view them so that the plane they define is coincident with the page and **v** is horizontal. Consider two additional vectors, $\beta\mathbf{v}$ and **w**, that are perpendicular ($\mathbf{v}\cdot\mathbf{w} = 0$) and can be summed together to produce **u**: $\mathbf{w} + \beta\mathbf{v} = \mathbf{u}$.



Compute the dot-product of **u** and **v**:

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{w} + \beta\mathbf{v})\cdot\mathbf{v} = \mathbf{w}\cdot\mathbf{v} + \beta\mathbf{v}\cdot\mathbf{v} = \beta\upsilon^2.$$

where the final equality holds because $\mathbf{v}\cdot\mathbf{w} = 0$. From the geometry of the figure:

$$\cos\theta \equiv \frac{\|\beta\mathbf{v}\|}{\|\mathbf{u}\|} = \frac{\beta\upsilon}{u}, \text{ or } \beta = \frac{u}{\upsilon}\cos\theta.$$

Insert this into the final equality for $\mathbf{u}\cdot\mathbf{v}$ to find:

$$\mathbf{u}\cdot\mathbf{v} = \left(\frac{u}{\upsilon}\cos\theta\right)\upsilon^2 = u\upsilon\cos\theta.$$

Exercise 2.16. Determine the components of the vector **w** in three-dimensional Cartesian coordinates when **w** is defined by: **u·w** = 0, **v·w** = 0, and **w·w** = $u^2\upsilon^2\sin^2\theta$, where **u** and **v** are known vectors with components u_i and υ_i and magnitudes u and υ, respectively, and θ is the angle between **u** and **v**. Choose the sign(s) of the components of **w** so that **w** = $\mathbf{e}_3$ when **u** = $\mathbf{e}_1$ and **v** = $\mathbf{e}_2$.

Solution 2.16. The effort here is primarily algebraic. Write the three constraints in component form:

$$\mathbf{u}\cdot\mathbf{w} = 0, \text{ or } u_1w_1 + u_2w_2 + u_3w_3 = 0, \tag{1}$$

$$\mathbf{v}\cdot\mathbf{w} = 0, \text{ or } \upsilon_1w_1 + \upsilon_2w_2 + \upsilon_3w_3 = 0, \text{ and} \tag{2}$$

The third one requires a little more effort since the angle needs to be eliminated via a dot product:

$$\mathbf{w}\cdot\mathbf{w} = u^2\upsilon^2\sin^2\theta = u^2\upsilon^2(1-\cos^2\theta) = u^2\upsilon^2 - (\mathbf{u}\cdot\mathbf{w})^2 \text{ or}$$

$$w_1^2 + w_2^2 + w_3^2 = (u_1^2 + u_2^2 + u_3^2)(\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2) - (u_1\upsilon_1 + u_2\upsilon_2 + u_3\upsilon_3)^2, \text{ which leads to}$$

$$w_1^2 + w_2^2 + w_3^2 = (u_1\upsilon_2 - u_2\upsilon_1)^2 + (u_1\upsilon_3 - u_3\upsilon_1)^2 + (u_2\upsilon_3 - u_3\upsilon_2)^2. \tag{3}$$

Equation (1) implies:

$$w_1 = -(w_2u_2 + w_3u_3)/u_1 \tag{4}$$

Combine (2) and (4) to eliminate w_1, and solve the resulting equation for w_2:

$$-\upsilon_1(w_2u_2 + w_3u_3)/u_1 + \upsilon_2w_2 + \upsilon_3w_3 = 0, \text{ or } \left(-\frac{\upsilon_1}{u_1}u_2 + \upsilon_2\right)w_2 + \left(-\frac{\upsilon_1}{u_1}u_3 + \upsilon_3\right)w_3 = 0.$$

Thus:

$$w_2 = +w_3\left(\frac{\upsilon_1}{u_1}u_3 - \upsilon_3\right)\Bigg/\left(-\frac{\upsilon_1}{u_1}u_2 + \upsilon_2\right) = w_3\left(\frac{u_3\upsilon_1 - u_1\upsilon_3}{u_1\upsilon_2 - u_2\upsilon_1}\right). \tag{5}$$

Combine (4) and (5) to find:

$$w_1 = -\frac{w_3}{u_1}\left(\left(\frac{\upsilon_1u_3 - \upsilon_3u_1}{\upsilon_2u_1 - \upsilon_1u_2}\right)u_2 + u_3\right) = -\frac{w_3}{u_1}\left(\frac{\upsilon_1u_3u_2 - \upsilon_3u_1u_2 + \upsilon_2u_1u_3 - \upsilon_1u_2u_3}{\upsilon_2u_1 - \upsilon_1u_2} +\right)$$

$$= -\frac{w_3}{u_1}\left(\frac{-\upsilon_3u_1u_2 + \upsilon_2u_1u_3}{\upsilon_2u_1 - \upsilon_1u_2}\right) = w_3\left(\frac{u_2\upsilon_3 - u_3\upsilon_2}{u_1\upsilon_2 - u_2\upsilon_1}\right). \tag{6}$$

Put (5) and (6) into (3) and factor out w_3 on the left side, then divide out the extensive common factor that (luckily) appears on the right and as the numerator inside the big parentheses.

$$w_3^2\left(\frac{(u_2\upsilon_3 - u_3\upsilon_2)^2 + (u_3\upsilon_1 - u_1\upsilon_3)^2 + (u_1\upsilon_2 - u_2\upsilon_1)^2}{(u_1\upsilon_2 - u_2\upsilon_1)^2}\right) = (u_1\upsilon_2 - u_2\upsilon_1)^2 + (u_1\upsilon_3 - u_3\upsilon_1)^2 + (u_2\upsilon_3 - u_3\upsilon_2)^2$$

$$w_3^2\left(\frac{1}{(u_1\upsilon_2 - u_2\upsilon_1)^2}\right) = 1, \text{ so } w_3 = \pm(u_1\upsilon_2 - u_2\upsilon_1).$$

If **u** = (1,0,0), and **v** = (0,1,0), then using the plus sign produces $w_3 = +1$, so $w_3 = +(u_1\upsilon_2 - u_2\upsilon_1)$.
Cyclic permutation of the indices allows the other components of w to be determined:

$$w_1 = u_2\upsilon_3 - u_3\upsilon_2,$$
$$w_2 = u_3\upsilon_1 - u_1\upsilon_3,$$
$$w_3 = u_1\upsilon_2 - u_2\upsilon_1.$$

Exercise 2.17. If a is a positive constant and **b** is a constant vector, determine the divergence and the curl of $\mathbf{u} = a\mathbf{x}/x^3$ and $\mathbf{u} = \mathbf{b}\times(\mathbf{x}/x^2)$ where $x = \sqrt{x_1^2 + x_2^2 + x_3^2} \equiv \sqrt{x_i x_i}$ is the length of **x**.

Solution 2.17. Start with the divergence calculations, and use $x = \sqrt{x_1^2 + x_2^2 + x_3^2}$ to save writing.

$$\nabla\cdot\left(\frac{a\mathbf{x}}{x^3}\right) = a\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\cdot\left(\frac{x_1,x_2,x_3}{\left[x_1^2+x_2^2+x_3^2\right]^{3/2}}\right) = a\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\cdot\left(\frac{x_1,x_2,x_3}{x^3}\right)$$

$$= a\left(\frac{\partial}{\partial x_1}\left(\frac{x_1}{x^3}\right)+\frac{\partial}{\partial x_2}\left(\frac{x_2}{x^3}\right)+\frac{\partial}{\partial x_3}\left(\frac{x_3}{x^3}\right)\right) = a\left(\frac{1}{x^3}-\frac{3}{2}\frac{x_1}{x^5}(2x_1)+\frac{1}{x^3}-\frac{3}{2}\frac{x_2}{x^5}(2x_2)+\frac{1}{x^3}-\frac{3}{2}\frac{x_3}{x^5}(2x_3)\right)$$

$$= a\left(\frac{3}{x^3}-\frac{3\left(x_1^2+x_2^2+x_3^2\right)}{x^5}\right) = a\left(\frac{3}{x^3}-\frac{3}{x^3}\right) = 0.$$

Thus, the vector field $a\mathbf{x}/x^3$ is divergence free even though it points away from the origin everywhere.

$$\nabla\cdot\left(\frac{\mathbf{b}\times\mathbf{x}}{x^2}\right) = \left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\cdot\left(\frac{b_2x_3-b_3x_2,b_3x_1-b_1x_3,b_1x_3-b_2x_1}{x_1^2+x_2^2+x_3^2}\right)$$

$$= \left(\frac{\partial}{\partial x_1}\left(\frac{b_2x_3-b_3x_2}{x^2}\right)+\frac{\partial}{\partial x_2}\left(\frac{b_3x_1-b_1x_3}{x^2}\right)+\frac{\partial}{\partial x_3}\left(\frac{b_1x_2-b_2x_1}{x^2}\right)\right)$$

$$= (b_2x_3-b_3x_2)\left(-\frac{2}{x^4}(2x_1)\right)+(b_3x_1-b_1x_3)\left(-\frac{2}{x^4}(2x_2)\right)+(b_1x_2-b_2x_1)\left(-\frac{2}{x^4}(2x_3)\right)$$

$$= -\frac{4}{x^4}\left(b_2x_3x_1-b_3x_2x_1+b_3x_1x_2-b_1x_3x_2+b_1x_2x_3-b_2x_1x_3\right) = 0.$$

This field is divergence free, too. The curl calculations produce:

$$\nabla\times\left(\frac{a\mathbf{x}}{x^3}\right) = a\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\times\left(\frac{x_1,x_2,x_3}{x^3}\right) = a\left(x_3\frac{\partial x^{-3}}{\partial x_2}-x_2\frac{\partial x^{-3}}{\partial x_3},x_1\frac{\partial x^{-3}}{\partial x_3}-x_3\frac{\partial x^{-3}}{\partial x_1},x_2\frac{\partial x^{-3}}{\partial x_1}-x_1\frac{\partial x^{-3}}{\partial x_2}\right)$$

$$= a\left(-\frac{3}{2}\frac{x_3}{x^5}(2x_2)+\frac{3}{2}\frac{x_2}{x^5}(2x_3),-\frac{3}{2}\frac{x_1}{x^5}(2x_3)+\frac{3}{2}\frac{x_3}{x^5}(2x_1),-\frac{3}{2}\frac{x_2}{x^5}(2x_1)+\frac{3}{2}\frac{x_1}{x^5}(2x_2)\right) = (0,0,0)$$

Thus, thus the vector field $a\mathbf{x}/x^3$ is also irrotational.

$$\nabla\times\left(\frac{\mathbf{b}\times\mathbf{x}}{x^2}\right) = \left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\times\left(\frac{b_2x_3-b_3x_2,b_3x_1-b_1x_3,b_1x_2-b_2x_1}{x_1^2+x_2^2+x_3^2}\right).$$

There are no obvious simplifications here. Therefore, compute the first component and obtain the others by cyclic permutation of the indices.

$$\nabla\times\left(\frac{\mathbf{b}\times\mathbf{x}}{x^2}\right)_1 = \frac{\partial}{\partial x_2}\left(\frac{b_1x_2-b_2x_1}{x^2}\right)-\frac{\partial}{\partial x_3}\left(\frac{b_3x_1-b_1x_3}{x^2}\right)$$

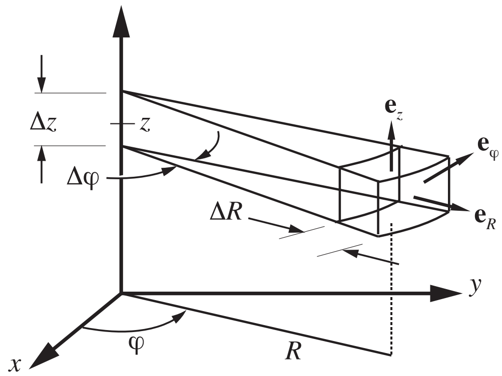
$$= \frac{b_1}{x^2}+(b_1x_2-b_2x_1)\left(\frac{-2}{x^4}\right)2x_2+\frac{b_1}{x^2}-(b_3x_1-b_1x_3)\left(\frac{-2}{x^4}\right)2x_3$$

$$= \frac{2b_1x^2-4b_1x_2^2+4b_2x_1x_2+4b_3x_1x_3-4b_1x_3^2}{x^4} = -\frac{2b_1}{x^2}+\frac{4x_1}{x^4}\left(b_1x_1+b_2x_2+b_3x_3\right)$$

This field is rotational. The other two components of its curl are:

$$\nabla\times\left(\frac{\mathbf{b}\times\mathbf{x}}{x^2}\right)_2=-\frac{2b_2}{x^2}+\frac{4x_2}{x^4}\left(b_1x_1+b_2x_2+b_3x_3\right)\ ,\quad \nabla\times\left(\frac{\mathbf{b}\times\mathbf{x}}{x^2}\right)_3=-\frac{2b_3}{x^2}+\frac{4x_3}{x^4}\left(b_1x_1+b_2x_2+b_3x_3\right).$$

Exercise 2.18. Obtain the recipe for the gradient of a scalar function in cylindrical polar coordinates from the integral definition (2.32).



Solution 2.18. Start from the appropriate form of (2.32), $\nabla\Psi = \lim\limits_{V\to 0}\frac{1}{V}\iint\limits_A \Psi\mathbf{n}dA$, where Ψ is a scalar function of position $\mathbf{x}$. Here we choose a nearly rectangular volume $V = (R\Delta\varphi)(\Delta R)(\Delta z)$ centered on the point $\mathbf{x} = (R, \varphi, z)$ with sides aligned perpendicular to the coordinate directions. Here the $\mathbf{e}_\varphi$ unit vector depends on φ so its direction is slightly different at $\varphi \pm \Delta\varphi/2$. For small $\Delta\varphi$, this can be handled by keeping the linear term of a simple Taylor series: $\left[\mathbf{e}_\varphi\right]_{\varphi\pm\Delta\varphi/2} \cong \mathbf{e}_\varphi \pm (\Delta\varphi/2)(\partial\mathbf{e}_\varphi/\partial\varphi) = \mathbf{e}_\varphi \mp (\Delta\varphi/2)\mathbf{e}_R$. Considering the drawing and noting that $\mathbf{n}$ is an outward normal, there are six contributions to $\mathbf{n}dA$:

outside = $\left(R + \frac{\Delta R}{2}\right)\Delta\varphi\Delta z\mathbf{e}_R$, inside = $-\left(R - \frac{\Delta R}{2}\right)\Delta\varphi\Delta z\mathbf{e}_R$,

close vertical side = $\Delta R\Delta z\left(-\mathbf{e}_\varphi - \frac{\Delta\varphi}{2}\mathbf{e}_R\right)$, more distant vertical side = $\Delta R\Delta z\left(\mathbf{e}_\varphi - \frac{\Delta\varphi}{2}\mathbf{e}_R\right)$,

top = $R\Delta\varphi\Delta R\mathbf{e}_z$, and bottom = $-R\Delta\varphi\Delta R\mathbf{e}_z$.

Here all the unit vectors are evaluated at the center of the volume. Using a two term Taylor series approximation for Ψ on each of the six surfaces, and taking the six contributions in the same order, the integral definition becomes a sum of six terms representing $\Psi\mathbf{n}dA$.

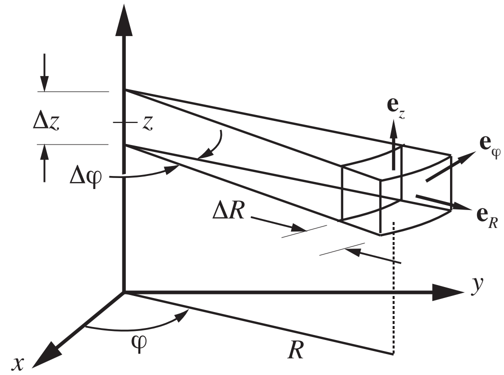
$$\nabla\Psi = \lim_{\substack{\Delta R\to 0\\ \Delta\varphi\to 0\\ \Delta z\to 0}} \frac{1}{R\Delta\varphi\Delta R\Delta z}\left\{\begin{array}{l} \left[\left(\Psi + \frac{\Delta R}{2}\frac{\partial\Psi}{\partial R}\right)\left(R + \frac{\Delta R}{2}\right)\mathbf{e}_R\Delta\varphi\Delta z\right] - \left[\left(\Psi - \frac{\Delta R}{2}\frac{\partial\Psi}{\partial R}\right)\left(R - \frac{\Delta R}{2}\right)\mathbf{e}_R\Delta\varphi\Delta z\right] + \\ \left[\left(\Psi - \frac{\Delta\varphi}{2}\frac{\partial\Psi}{\partial\varphi}\right)\left(-\mathbf{e}_\varphi - \frac{\Delta\varphi}{2}\mathbf{e}_R\right)\Delta R\Delta z\right] + \left[\left(\Psi + \frac{\Delta\varphi}{2}\frac{\partial\Psi}{\partial\varphi}\right)\left(\mathbf{e}_\varphi - \frac{\Delta\varphi}{2}\mathbf{e}_R\right)\Delta R\Delta z\right] + \\ \left[\left(\Psi + \frac{\Delta z}{2}\frac{\partial\Psi}{\partial z}\right)\mathbf{e}_z R\Delta\varphi\Delta R\right] - \left[\left(\Psi - \frac{\Delta z}{2}\frac{\partial\Psi}{\partial z}\right)\mathbf{e}_z R\Delta\varphi\Delta R\right] + \ldots \end{array}\right\}$$

Here the mean value theorem has been used and all listings of Ψ and its derivatives above are evaluated at the center of the volume. The largest terms inside the big {,}-brackets are proportional to $\Delta\varphi\Delta R\Delta z$. The remaining higher order terms vanish when the limit is taken.

$$\nabla\Psi = \lim_{\substack{\Delta R\to 0\\ \Delta\varphi\to 0\\ \Delta z\to 0}} \frac{1}{R\Delta\varphi\Delta R\Delta z}\left\{\begin{array}{l} \left[\frac{\Psi}{2}\mathbf{e}_R + \frac{R}{2}\frac{\partial\Psi}{\partial R}\mathbf{e}_R\right]\Delta\varphi\Delta R\Delta z - \left[-\frac{\Psi}{2}\mathbf{e}_R - \frac{R}{2}\frac{\partial\Psi}{\partial R}\mathbf{e}_R\right]\Delta\varphi\Delta R\Delta z + \\ \left[\frac{\mathbf{e}_\varphi}{2}\frac{\partial\Psi}{\partial\varphi} - \frac{\mathbf{e}_R}{2}\Psi\right]\Delta\varphi\Delta R\Delta z + \left[\frac{\mathbf{e}_\varphi}{2}\frac{\partial\Psi}{\partial\varphi} - \frac{\mathbf{e}_R}{2}\Psi\right]\Delta\varphi\Delta R\Delta z + \\ \left[\frac{R}{2}\frac{\partial\Psi}{\partial z}\mathbf{e}_z\right]\Delta\varphi\Delta R\Delta z - \left[-\frac{R}{2}\frac{\partial\Psi}{\partial z}\mathbf{e}_z\right]\Delta\varphi\Delta R\Delta z + \ldots \end{array}\right\}$$

$$\nabla\Psi = \left\{\frac{\Psi}{R}\mathbf{e}_R + \frac{\partial\Psi}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial\Psi}{\partial\varphi}\mathbf{e}_\varphi - \frac{\Psi}{R}\mathbf{e}_R + \frac{\partial\Psi}{\partial z}\mathbf{e}_z\right\} = \mathbf{e}_R\frac{\partial\Psi}{\partial R} + \mathbf{e}_\varphi\frac{1}{R}\frac{\partial\Psi}{\partial\varphi} + \mathbf{e}_z\frac{\partial\Psi}{\partial z}$$

Exercise 2.19. Obtain the recipe for the divergence of a vector function in cylindrical polar coordinates from the integral definition (2.32).



Solution 2.19. Start from the appropriate form of (2.32), $\nabla\cdot\mathbf{Q} = \lim_{V\to 0}\frac{1}{V}\iint_A \mathbf{n}\cdot\mathbf{Q}dA$, where $\mathbf{Q} = (Q_R, Q_\varphi, Q_z)$ is a vector function of position $\mathbf{x}$. Here we choose a nearly rectangular volume $V = (R\Delta\varphi)(\Delta R)(\Delta z)$ centered on the point $\mathbf{x} = (R, \varphi, z)$ with sides aligned perpendicular to the coordinate directions. Here the $\mathbf{e}_\varphi$ unit vector depends on φ so its direction is slightly different at $\varphi \pm \Delta\varphi/2$. Considering the drawing and noting that $\mathbf{n}$ is an outward normal, there are six contributions to $\mathbf{n}dA$:

outside = $\left(R + \frac{\Delta R}{2}\right)\Delta\varphi\Delta z\mathbf{e}_R$, inside = $-\left(R - \frac{\Delta R}{2}\right)\Delta\varphi\Delta z\mathbf{e}_R$, close vertical side $= -\Delta R\Delta z\left[\mathbf{e}_\varphi\right]_{\varphi-\Delta\varphi/2}$,

more distant vertical side = $\Delta R\Delta z\left[\mathbf{e}_\varphi\right]_{\varphi+\Delta\varphi/2}$, top = $R\Delta\varphi\Delta R\mathbf{e}_z$, and bottom = $-R\Delta\varphi\Delta R\mathbf{e}_z$.

Here the unit vectors are evaluated at the center of the volume unless otherwise specified. Using a two-term Taylor series approximation for the components of $\mathbf{Q}$ on each of the six surfaces, and taking the six contributions to $\mathbf{n}\cdot\mathbf{Q}dA$ in the same order, the integral definition becomes:

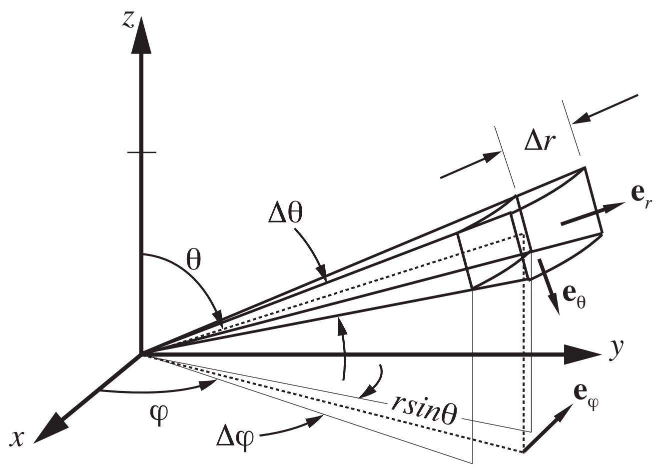
$$\nabla\cdot\mathbf{Q} = \lim_{\substack{\Delta R\to 0\\ \Delta\varphi\to 0\\ \Delta z\to 0}}\frac{1}{R\Delta\varphi\Delta R\Delta z}\left\{\begin{array}{l}\left[\left(Q_R + \frac{\Delta R}{2}\frac{\partial Q_R}{\partial R}\right)\left(R + \frac{\Delta R}{2}\right)\Delta\varphi\Delta z\right] - \left[\left(Q_R - \frac{\Delta R}{2}\frac{\partial Q_R}{\partial R}\right)\left(R - \frac{\Delta R}{2}\right)\Delta\varphi\Delta z\right] + \\ \left[\left(Q_\varphi - \frac{\Delta\varphi}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right)(-\Delta R\Delta z)\right] + \left[\left(Q_\varphi + \frac{\Delta\varphi}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right)\Delta R\Delta z\right] + \\ \left[\left(Q_z + \frac{\Delta z}{2}\frac{\partial Q_z}{\partial z}\right)R\Delta\varphi\Delta R\right] - \left[\left(Q_z - \frac{\Delta z}{2}\frac{\partial Q_z}{\partial z}\right)R\Delta\varphi\Delta R\right] + \ldots\end{array}\right\}$$

Here the mean value theorem has been used and all listings of the components of $\mathbf{Q}$ and their derivatives are evaluated at the center of the volume. The largest terms inside the big {,}-brackets are proportional to $\Delta\varphi\Delta R\Delta z$. The remaining higher order terms vanish when the limit is taken.

$$\nabla\cdot\mathbf{Q} = \lim_{\substack{\Delta R\to 0\\ \Delta\varphi\to 0\\ \Delta z\to 0}}\frac{1}{R\Delta\varphi\Delta R\Delta z}\left\{\begin{array}{l}\left[\frac{Q_R}{2} + \frac{R}{2}\frac{\partial Q_R}{\partial R}\right]\Delta\varphi\Delta R\Delta z - \left[-\frac{Q_R}{2} - \frac{R}{2}\frac{\partial Q_R}{\partial R}\right]\Delta\varphi\Delta R\Delta z + \\ -\left[-\frac{1}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right]\Delta\varphi\Delta R\Delta z + \left[\frac{1}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right]\Delta\varphi\Delta R\Delta z + \\ \left[\frac{R}{2}\frac{\partial\Psi}{\partial z}\right]\Delta\varphi\Delta R\Delta z - \left[-\frac{R}{2}\frac{\partial\Psi}{\partial z}\right]\Delta\varphi\Delta R\Delta z + \ldots\end{array}\right\}$$

$$\nabla\cdot\mathbf{Q} = \left\{\frac{Q_R}{R} + \frac{\partial Q_R}{\partial R} + \frac{1}{R}\frac{\partial Q_\varphi}{\partial\varphi} + \frac{\partial Q_z}{\partial z}\right\} = \frac{1}{R}\frac{\partial}{\partial R}\left(RQ_R\right) + \frac{1}{R}\frac{\partial Q_\varphi}{\partial\varphi} + \frac{\partial Q_z}{\partial z}$$

Exercise 2.20. Obtain the recipe for the divergence of a vector function in spherical polar coordinates from the integral definition (2.32).



Solution 2.20. Start from the appropriate form of (2.32), $\nabla\cdot\mathbf{Q} = \lim_{V\to 0}\frac{1}{V}\iint_A \mathbf{n}\cdot\mathbf{Q}dA$, where $\mathbf{Q}$ = (Q_r, Q_θ, Q_φ) is a vector function of position $\mathbf{x}$. Here we choose a nearly rectangular volume $V = (r\Delta\theta)(r\sin\theta\Delta\varphi)(\Delta r)$ centered on the point $\mathbf{x} = (r, \theta, \varphi)$ with sides aligned perpendicular to the coordinate directions. Here the unit vectors depend on θ and φ so directions are slightly different at $\theta \pm \Delta\theta/2$, and $\varphi \pm \Delta\varphi/2$. Considering the drawing and noting that $\mathbf{n}$ is an outward normal, there are six contributions to $\mathbf{n}dA$:

outside $=\left(r+\frac{\Delta r}{2}\right)\Delta\theta\left(r+\frac{\Delta r}{2}\right)\sin\theta\Delta\varphi(\mathbf{e}_r)$, inside $=\left(r-\frac{\Delta r}{2}\right)\Delta\theta\left(r-\frac{\Delta r}{2}\right)\sin\theta\Delta\varphi(-\mathbf{e}_r)$,

bottom $=\left[r\sin(\theta+\Delta\theta/2)\Delta\varphi\Delta r\right](\mathbf{e}_\theta)_{\theta+\Delta\theta/2}$, top $=\left[r\sin(\theta-\Delta\theta/2)\Delta\varphi\Delta r\right](-\mathbf{e}_\theta)_{\theta-\Delta\theta/2}$,

close vertical side $= r\Delta\theta\Delta r\left(-\mathbf{e}_\varphi\right)_{\varphi-\Delta\varphi/2}$, and more distant vertical side $= r\Delta\theta\Delta r\left(+\mathbf{e}_\varphi\right)_{\varphi+\Delta\varphi/2}$.

Here the unit vectors are evaluated at the center of the volume unless otherwise specified. Using a two-term Taylor series approximation for the corresponding components of $\mathbf{Q}$ on each of the six surfaces produces:

outside: $\left(Q_r+\frac{\Delta r}{2}\frac{\partial Q_r}{\partial r}\right)\mathbf{e}_r$, inside: $\left(Q_r-\frac{\Delta r}{2}\frac{\partial Q_r}{\partial r}\right)\mathbf{e}_r$, bottom: $\left(Q_\theta+\frac{\Delta\theta}{2}\frac{\partial Q_\theta}{\partial\theta}\right)(\mathbf{e}_\theta)_{\theta+\Delta\theta/2}$,

top: $\left(Q_\theta-\frac{\Delta\theta}{2}\frac{\partial Q_\theta}{\partial\theta}\right)(\mathbf{e}_\theta)_{\theta-\Delta\theta/2}$, close vertical side : $\left(Q_\varphi-\frac{\Delta\varphi}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right)\left(-\mathbf{e}_\varphi\right)_{\varphi-\Delta\varphi/2}$, and

more distant vertical side : $\left(Q_\varphi+\frac{\Delta\varphi}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right)\left(\mathbf{e}_\varphi\right)_{\varphi+\Delta\varphi/2}$.

Collecting and summing the six contributions to $\mathbf{n}\cdot\mathbf{Q}dA$, the integral definition becomes:

$$\nabla\cdot\mathbf{Q} = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0\\ \Delta\varphi\to 0}}\frac{1}{(r\Delta\theta)(r\sin\theta\Delta\varphi)\Delta r}\times$$

$$\left\{\begin{array}{l}\left[\left(Q_r+\frac{\Delta r}{2}\frac{\partial Q_r}{\partial r}\right)\Delta\theta\left(r+\frac{\Delta r}{2}\right)^2\sin\theta\Delta\varphi\right]-\left[\left(Q_r-\frac{\Delta r}{2}\frac{\partial Q_r}{\partial r}\right)\Delta\theta\left(r-\frac{\Delta r}{2}\right)^2\sin\theta\Delta\varphi\right]\\ +\left[\left(Q_\theta+\frac{\Delta\theta}{2}\frac{\partial Q_\theta}{\partial\theta}\right)r\sin\left(\theta+\frac{\Delta\theta}{2}\right)\Delta\varphi\Delta r\right]-\left[\left(Q_\theta-\frac{\Delta\theta}{2}\frac{\partial Q_\theta}{\partial\theta}\right)r\sin\left(\theta-\frac{\Delta\theta}{2}\right)\Delta\varphi\Delta r\right]\\ -\left[\left(Q_\varphi-\frac{\Delta\varphi}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right)r\Delta\theta\Delta r\right]+\left[\left(Q_\varphi+\frac{\Delta\varphi}{2}\frac{\partial Q_\varphi}{\partial\varphi}\right)r\Delta\theta\Delta r\right]+\ldots\end{array}\right\}$$

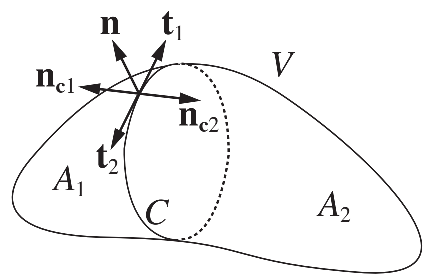
The largest terms inside the big {,}-brackets are proportional to $\Delta\theta\Delta\varphi\Delta r$. The remaining higher order terms vanish when the limit is taken.

$$\nabla\cdot\mathbf{Q} = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0\\ \Delta\varphi\to 0}} \frac{1}{(r\Delta\theta)(r\sin\theta\Delta\varphi)\Delta r}\times\left\{\begin{array}{l}\left[r^2\frac{\partial Q_r}{\partial r}+2rQ_r\right]\Delta\theta\sin\theta\Delta\varphi\Delta r\\ +\left[\sin\theta\frac{\partial Q_\theta}{\partial\theta}+\cos\theta Q_\theta\right]r\Delta\theta\Delta\varphi\Delta r\\ +\left[\frac{\partial Q_\varphi}{\partial\varphi}\right]r\Delta\theta\Delta\varphi\Delta r+\ldots\end{array}\right\}$$

Cancel the common factors and take the limit, to find:

$$\begin{aligned}\nabla\cdot\mathbf{Q} &= \frac{1}{(r)(r\sin\theta)}\times\left\{\left[r^2\frac{\partial Q_r}{\partial r}+2rQ_r\right]\sin\theta+\left[\sin\theta\frac{\partial Q_\theta}{\partial\theta}+\cos\theta Q_\theta\right]r+\left[\frac{\partial Q_\varphi}{\partial\varphi}\right]r\right\}\\ &= \frac{1}{r^2\sin\theta}\times\left\{\frac{\partial}{\partial r}\left(r^2Q_r\right)\sin\theta+r\frac{\partial}{\partial\theta}\left(\sin\theta Q_\theta\right)+r\frac{\partial Q_\varphi}{\partial\varphi}\right\}\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2Q_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta Q_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial Q_\varphi}{\partial\varphi}\end{aligned}$$

Exercise 2.21. Use the vector integral theorems to prove that $\nabla\cdot(\nabla\times\mathbf{u})=0$ for any twice-differentiable vector function $\mathbf{u}$ regardless of the coordinate system.



Solution 2.21. Start with the divergence theorem for a vector function $\mathbf{Q}$ that depends on the spatial coordinates,

$$\iiint_V \nabla\cdot\mathbf{Q}dV = \iint_A \mathbf{n}\cdot\mathbf{Q}dA$$

where the arbitrary closed volume V has surface A, and the outward normal is $\mathbf{n}$. For this exercise, let $\mathbf{Q}=\nabla\times\mathbf{u}$ so that

$$\iiint_V \nabla\cdot(\nabla\times\mathbf{u})dV = \iint_A \mathbf{n}\cdot(\nabla\times\mathbf{u})dA\,.$$

Now split V into two sub-volumes V_1 and V_2, where the surface of V_1 is A_1 and the surface of V_2 is A_2. Here A_1 and A_2 are not closed surfaces, but $A_1 + A_2 = A$ so:

$$\iiint_V \nabla\cdot(\nabla\times\mathbf{u})dV = \iint_{A_1} \mathbf{n}\cdot(\nabla\times\mathbf{u})dA + \iint_{A_2} \mathbf{n}\cdot(\nabla\times\mathbf{u})dA\,.$$

where $\mathbf{n}$ is the same as when the surfaces were joined. However, the bounding curve C for A_1 and A_2 is the same, so Stokes theorem produces:

$$\iiint_V \nabla\cdot(\nabla\times\mathbf{u})dV = \int_C \mathbf{u}\cdot\mathbf{t}_1\,ds + \int_C \mathbf{u}\cdot\mathbf{t}_2 ds\,.$$

Here the tangent vectors $\mathbf{t}_1=\mathbf{n}_{c1}\times\mathbf{n}$ and $\mathbf{t}_2=\mathbf{n}_{c2}\times\mathbf{n}$ have opposite signs because $\mathbf{n}_{c1}$ and $\mathbf{n}_{c2}$, the normals to C that are tangent to surfaces A_1 and A_2, respectively, have opposite sign. Thus, the two terms on the right side of the last equation are equal and opposite, so

$$\iiint_V \nabla\cdot(\nabla\times\mathbf{u})dV = 0\,. \qquad (i)$$

For an arbitrary closed volume of any size, shape, or location, this can only be true if $\nabla\cdot(\nabla\times\mathbf{u})=0$. For example, if $\nabla\cdot(\nabla\times\mathbf{u})$ were nonzero at some location, then integration in small volume centered on this location would not be zero. Such a nonzero integral is not allowed by (i); thus, $\nabla\cdot(\nabla\times\mathbf{u})$ must be zero everywhere because V is arbitrary.

Exercise 2.22. Use Stokes' theorem to prove that $\nabla\times(\nabla\phi)=0$ for any single-valued twice-differentiable scalar ϕ regardless of the coordinate system.

Solution 2.22. From (2.34) Stokes Theorem is:

$$\iint_A (\nabla\times\mathbf{u})\cdot\mathbf{n}dA = \int_C \mathbf{u}\cdot\mathbf{t}ds.$$

Let $\mathbf{u}=\nabla\phi$, and note that $\nabla\phi\cdot\mathbf{t}ds=(\partial\phi/\partial s)ds=d\phi$ because the **t** vector points along the contour C that has path increment ds. Therefore:

$$\iint_A \left(\nabla\times[\nabla\phi]\right)\cdot\mathbf{n}dA = \int_C \nabla\phi\cdot\mathbf{t}ds = \int_C d\phi = 0, \qquad (ii)$$

where the final equality holds for integration on a closed contour of a single-valued function ϕ.

For an arbitrary surface A of any size, shape, orientation, or location, this can only be true if $\nabla\times(\nabla\phi)=0$. For example, if $\nabla\times(\nabla\phi)=0$ were nonzero at some location, then an area integration in a small region centered on this location would not be zero. Such a nonzero integral is not allowed by (*ii*); thus, $\nabla\times(\nabla\phi)=0$ must be zero everywhere because A is arbitrary.