# An Introduction to Numerical Analysis

## Solutions to Exercises

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## Solution to Exercise 1.1

The fixed points are the solutions of

$$q(x) \equiv x^2 - 2x + c = 0.$$

Evidently $q(0) = c > 0$, $q(1) = c - 1 < 0$, $q(x) \to +\infty$ as $x \to \infty$, showing that $0 < \xi_1 < 1 < \xi_2$.

Now

$$\begin{aligned} x_{n+1} &= \tfrac{1}{2}(x_n^2 + c) \\ \xi_1 &= \tfrac{1}{2}(\xi_1^2 + c), \end{aligned}$$

so by subtraction

$$x_{n+1} - \xi_1 = \tfrac{1}{2}(x_n^2 - \xi_1^2) = \tfrac{1}{2}(x_n + \xi_1)(x_n - \xi_1).$$

It follows that if $|x_n + \xi_1| < 2$, then $|x_{n+1} - \xi_1| < |x_n - \xi_1|$. Now $\xi_1 + \xi_2 = 2$, so if $0 \le x_0 < \xi_2$ then $x_0 + \xi_1 < 2$, and evidently $x_0 + \xi_1 > 0$. Hence $x_1$ is closer to $\xi_1$ then was $x_0$, so also $0 \le x_1 < \xi_2$. An induction argument then shows that each $x_n$ satisfies $0 \le x_n < \xi_2$, and

$$|x_n - \xi_1| < \left(\frac{x_0 + \xi_1}{2}\right)^n |x_0 - \xi_1|,$$

and $x_n \to \xi_1$.

Now $x_{n+1}$ is independent of the sign of $x_n$, and is therefore also independent of the sign of $x_0$, and it follows that $x_n \to \xi_1$ for all $x_0$ such that $-\xi_2 < x_0 < \xi_2$.

The same argument shows that if $x_0 > \xi_2$ then $x_1 > x_0 > \xi_2$, and so $x_n \to \infty$. As before this means also that $x_n \to \infty$ if $x_0 < -\xi_2$.

If $x_0 = \xi_2$ then of course $x_n = \xi_2$ for all $n > 0$. If $x_0 = -\xi_2$, then $x_1 = \xi_2$, and again $x_n = \xi_2$ for all $n > 0$.

## Solution to Exercise 1.2

Since $f'(x) = \mathrm{e}^x - 1$ and $f''(x) = \mathrm{e}^x$, $f'(x) > 0$ and $f''(x) > 0$ for all $x > 0$. It therefore follows from Theorem 1.9 that if $x_0 > 0$ then Newton's method converges to the positive root.

Similarly $f'(x) < 0$ and $f''(x) > 0$ in $(-\infty, 0)$ and the same argument shows that the method converges to the negative root if $x_0 < 0$.

If $x_0 = 0$ the method fails, as $f'(0) = 0$, and $x_1$ does not exist.

For this function $f$, Newton's method gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{\exp(x_n) - x_n - 2}{\exp(x_n) - 1} \\ &= x_n - \frac{1 - (x_n + 2)\exp(-x_n)}{1 - \exp(x_n)} \\ &\approx x_n - 1 \quad n > 1. \end{aligned}$$

In fact, $\mathrm{e}^{-100}$ is very small indeed.

In the same way, when $x_0$ is large and negative, say $x_0 = -100$,

$$x_{n+1} \approx x_n - \frac{-x_n - 2}{-1} = -2.$$

Hence when $x_0 = 100$, the first few members of the sequence are $100, 99, 98, \ldots$; after 98 iterations $x_n$ will get close to the positive root, and convergence becomes quadratic and rapid. About 100 iterations are required to give an accurate value for the root.

However, when $x_0 = -100$, $x_1$ is very close to $-2$, and is therefore very close to the negative root. Three, or possibly four, iterations should give the value of the root to six decimal places.

## Solution to Exercise 1.3

Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

To avoid calculating the derivative we might consider approximating the derivative by

$$f'(x_n) \approx \frac{f(x_n+\delta) - f(x_n)}{\delta},$$

where $\delta$ is small. The given iteration uses this approximation, with $\delta = f(x_n)$; if $x_n$ is close to a root then we might expect that $f(x_n)$ is small.

If $x_n - \xi$ is small we can write

$$\begin{aligned} f(x_n) &= f(\xi) + (x_n - \xi)f'(\xi) + \tfrac{1}{2}(x_n-\xi)^2 f''(\xi) + \mathcal{O}(x_n-\xi)^3 \\ &= \eta f' + \tfrac{1}{2}\eta^2 f'' + \mathcal{O}(\eta)^3 \end{aligned}$$

where $\eta = x_n - \xi$, and $f'$ and $f''$ are evaluated at $x = \xi$. Then

$$\begin{aligned} f(x_n + f(x_n)) - f(x_n) &= f(\xi + \eta + \eta f' + \tfrac{1}{2}\eta^2 f'') - f(\xi+\eta) \\ &= \eta(f')^2 + \tfrac{1}{2}\eta^2[3f'f'' + (f')^2 f''] + \mathcal{O}(\eta)^3. \end{aligned}$$

Hence

$$\begin{aligned} x_{n+1} - \xi &= x_n - \xi - \frac{[f(x_n)]^2}{f(x+n+f(x_n)) - f(x_n)} \\ &= \eta - \frac{\eta^2[(f')^2 + \eta f' f'']}{\eta[(f')^2 + \frac{1}{2}\eta f'(3+f')f'']} + \mathcal{O}(\eta)^3 \\ &= \eta^2 \frac{f''(1+f')}{f'} + \mathcal{O}(\eta)^3. \end{aligned}$$

This shows that if $x_0 - \eta$ is sufficiently small the iteration converges quadratically. The analysis here requires that $f'''$ is continuous in a neighbourhood of $\xi$, to justify the terms $\mathcal{O}(\eta^3)$. A more careful analysis might relax this to require only the continuity of $f''$.

The leading term in $x_{n+1} - \xi$ is very similar to that in Newton's method, but with an additional term.

The convergence of this method, starting from a point close to a root, is very similar to Newton's method. But if $x_0$ is some way from the root $f(x_n)$ will not be small, the approximation to the derivative $f'(x_n)$ is very poor, and the behaviour may be very different. For the example

$$f(x) = \mathrm{e}^x - x - 2$$

starting from $x_0 = 1,\ 10$ and $-10$ we find

| | |
|---|---|
| 0 | 1.000000 |
| 1 | 1.205792 |
| 2 | 1.153859 |
| 3 | 1.146328 |
| 4 | 1.146193 |
| 5 | 1.146193 |

| | |
|---|---|
| 0 | 10.000000 |
| 1 | 10.000000 |
| 2 | 10.000000 |
| 3 | 10.000000 |
| 4 | 10.000000 |
| 5 | 10.000000 |

| | |
|---|---|
| 0 | -10.000000 |
| 1 | -1.862331 |
| 2 | -1.841412 |
| 3 | -1.841406 |
| 4 | -1.841406 |

The convergence from $x_0 = 1$ is satisfactory. Starting from $x_0 = -10$ we get similar behaviour to Newton's method, an immediate step to $x_1$ quite close to $-2$, and then rapid convergence to the negative root.

However, starting from $x_0 = 10$ gives a quite different result. This time $f(x_0)$ is roughly 20000 (which is not small), and $f(x_0 + f(x_0))$ is about $10^{9500}$; the difference between $x_0$ and $x_1$ is excessively small. Although the iteration converges, the rate of convergence is so slow that for any practical purpose it is virtually stationary. Even starting from $x_0 = 3$ many thousands of iterations are required for convergence.

## Solution to Exercise 1.4

The number of correct figures in the approximation $x_n$ to the root $\xi$ is

$$D_n = \text{integer part of } \{-\log_{10}|\xi - x_n|\}.$$

From (1.24) for Newton's method we have

$$\frac{|\xi - x_{n+1}|}{|\xi - x_n|^2} \to \frac{|f''(\xi)|}{2|f'(\xi)|}.$$

Hence

$$D_{n+1} \approx 2\,D_n - B,$$

where

$$B = \log_{10} \frac{|f''(\xi)|}{2|f'(\xi)|}.$$

If $B$ is small then $D_{n+1}$ is close to $2\,D_n$, but if $B$ is significantly larger than 1 then $D_{n+1}$ may be smaller than this.

In the example,

$$\begin{aligned} f(x) &= \mathrm{e}^x - x - 1.0000000005 \\ f'(x) &= \mathrm{e}^x - 1 \\ f''(x) &= \mathrm{e}^x, \end{aligned}$$

and $\xi = 0.0001$. Hence

$$B = \log_{10} \frac{\mathrm{e}^{0.0001}}{2(\mathrm{e}^{0.0001} - 1)} = 3.7$$

and the number of significant figures in the next iteration is about $2k-4$, not $2k$.

Starting from $x_0 = 0.0005$ the results of Newton's method are

| | | |
|---|---|---|
| 0 | 0.000500000000000 | 3 |
| 1 | 0.000260018333542 | 3 |
| 2 | 0.000149241714302 | 4 |
| 3 | 0.000108122910746 | 5 |
| 4 | 0.000100303597745 | 6 |
| 5 | 0.000099998797906 | 9 |
| 6 | 0.000099998333362 | 14 |

where the last column shows the number of correct decimal places.

The root is $\xi = 0.000099998333361$ to 15 decimal places.

The number of correct figures increases by a factor quite close to 4.

## Solution to Exercise 1.5

From (1.23) we have

$$\xi - x_{n+1} = -\frac{(\xi - x_n)^2 f''(\eta_n)}{2f'(x_n)}.$$

Now $f'(\xi) = 0$, so by the Mean Value Theorem

$$f'(x_n) - f'(\xi) = (x_n - \xi) f''(\chi_n)$$

for some value of $\chi_n$ between $\xi$ and $x_n$. Hence

$$\xi - x_{n+1} = \frac{(\xi - x_n) f''(\eta_n)}{2f''(\chi_n)}.$$

Now $|f''(\eta_n)| < M$ and $|f''(\chi_n)| > m$, and so

$$|\xi - x_{n+1}| < K|\xi - x_n|,$$

where

$$K = \frac{M}{2m} < 1.$$

Hence if $x_0$ lies in the given interval, all $x_n$ lie in the interval, and $x_n \to \xi$.

Then $\eta_n \to \xi$, $f''(\eta_n) \to f''(\xi)$ and $f''(\chi_n) \to f''(\xi)$. This shows that

$$\frac{\xi - x_{n+1}}{\xi - x_n} \to \tfrac{1}{2}$$

and convergence is linear, with asymptotic rate of convergence $\ln 2$.

For the example $f(x) = \mathrm{e}^x - 1 - x$, $f(0) = 0$, $f'(0) = 0$. Starting from $x_0 = 1$, Newton's method gives

| | |
|---|---|
| 0 | 1.000 |
| 1 | 0.582 |
| 2 | 0.319 |
| 3 | 0.168 |
| 4 | 0.086 |
| 5 | 0.044 |
| 6 | 0.022 |
| 7 | 0.011 |
| 8 | 0.006 |
| 9 | 0.003 |
| 10 | 0.001 |

showing $\xi - x_0$ reducing by a factor close to $\frac{1}{2}$ at each step.

## Solution to Exercise 1.6

When $f(\xi) = f'(\xi) = f''(\xi) = 0$ we get from the definition of Newton's method, provided that $f'''$ is continuous in some neighbourhood of $\xi$,

$$\begin{aligned}
\xi - x_{n+1} &= \xi - x_n + \frac{f(x_n)}{f'(x_n)} \\
&= \xi - x_n + \frac{\frac{1}{6}(x_n - \xi)^3 f'''(\eta_n)}{\frac{1}{2}(x_n - \xi)^2 f'''(\chi_n)} \\
&= (\xi - x_n)\left\{1 - \frac{f'''(\eta_n)}{3f'''(\chi_n)}\right\}.
\end{aligned}$$

If we now assume that in the neighbourhood $[\xi - k, \xi + k]$ of the root

$$0 < m < |f'''(x)| < M, \quad \text{where } M < 3m,$$

then

$$|\xi - x_{n+1}| < K|\xi - x_n|,$$

where

$$K = 1 - \frac{M}{3m} < 1.$$

Hence if $x_0$ is in this neighbourhood, all the $x_n$ lie in the neighbourhood, and Newton's method converges to $\xi$. Also,

$$\frac{|\xi - x_{n+1}|}{|\xi - x_n|} \to \frac{2}{3},$$

so that convergence is linear, with asymptotic rate of convergence $\ln(3/2)$.

## Solution to Exercise 1.7

The proof follows closely the proof of Theorem 1.9.

From (1.23) it follows that $x_{n+1} < \xi$, provided that $x_n$ lies in the interval $I = [X, \xi]$. Since $f$ is monotonic increasing and $f(\xi) = 0$, $f(x) < 0$ in $I$. Hence if $x_0 \in I$ the sequence $(x_n)$ lies in $I$, and is monotonic increasing. As it is bounded above by $\xi$, it converges; since $\xi$ is the only root of $f(x) = 0$ in $I$, the sequence converges to $\xi$. Since $f''$ is continuous it follows that

$$\frac{\xi - x_{n+1}}{(\xi - x_n)^2} = -\frac{f''(\eta_n)}{2f'(x_n)} \to -\frac{f''(\xi)}{2f'(\xi)},$$

so that convergence is quadratic.

## Solution to Exercise 1.8

Neglecting terms of second order in $\varepsilon$ we get

$$\begin{aligned} x_0 &= 1+\varepsilon \\ x_1 &= -1+\varepsilon \\ x_2 &= \tfrac{1}{2}\varepsilon \\ x_3 &= -1-\varepsilon \\ x_4 &= -1+\varepsilon \\ x_5 &= -1. \end{aligned}$$

Although this value of $x_5$ is not exact, it is clear that for sufficiently small $\varepsilon$ the sequence converges to $-1$.

With $x_0$ and $x_1$ interchanged, the value of $x_2$ is of course the same, but $x_3$ and subsequent values are different:

$$\begin{aligned} x_0 &= -1+\varepsilon \\ x_1 &= 1+\varepsilon \\ x_2 &= \tfrac{1}{2}\varepsilon \\ x_3 &= 1-\varepsilon \\ x_4 &= 1+\varepsilon \\ x_5 &= 1. \end{aligned}$$

## Solution to Exercise 1.9

The function $\varphi$ has the form

$$\varphi(x_n, x_{n-1}) = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n) - \xi(f(x_{n-1}) - f(x_n))}{(x_n - \xi)(x_{n-1} - \xi)(f(x_{n-1}) - f(x_n))}$$

In the limit as $x_n \to \xi$ both numerator and denominator tend to zero, so we apply l'Hopital's rule to give

$$\begin{aligned}\lim_{x_n \to \xi} \varphi(x_n, x_{n-1}) &= \lim \frac{f(x_{n-1}) - x_{n-1} f'(x_n) + \xi f'(x_n))}{-f'(x_n)(x_n - \xi)(x_{n-1} - \xi) + (f(x_{n-1}) - f(x_n))(x_{n-1} - \xi)} \\ &= \frac{f(x_{n-1}) - x_{n-1} f'(\xi) + \xi}{(f(x_{n-1}) - f(\xi))(x_{n-1} - \xi)}\end{aligned}$$

so that

$$\psi(x_{n-1}) = \frac{f(x_{n-1}) - x_{n-1} f'(\xi) + \xi f'(\xi)}{(f(x_{n-1}) - f(\xi))(x_{n-1} - \xi)}.$$

In the limit as $x_{n-1} \to \xi$ the numerator and denominator of $\psi(x_{n-1})$ both tend to zero, so again we use l'Hopital's rule to give

$$\lim_{x_{n-1} \to \xi} \psi(x_{n-1}) = \lim_{x_{n-1} \to \xi} \frac{f'(x_{n-1}) - f'(\xi)}{f'(x_{n-1})(x_{n-1} - \xi) + (f(x_{n-1}) - f(\xi))}.$$

We must now use l'Hopital's rule again, to give finally

$$\begin{aligned}\lim_{x_{n-1} \to \xi} \psi(x_{n-1}) &= \lim \frac{f''(x_{n-1})}{f''(x_{n-1})(x_{n-1} - \xi) + f'(x_{n-1}) + f'(x_{n-1})} \\ &= \frac{f''(\xi)}{2f'(\xi)}.\end{aligned}$$

Now the limit of $\varphi$ does not depend on the way in which $x_n$ and $x_{n-1}$ tend to $\xi$, so finally we have

$$\frac{x_{n+1} - \xi}{(x_n - \xi)(x_{n-1} - \xi)} \to \frac{f''(\xi)}{2f'(\xi)}.$$

Now assume that

$$\frac{x_{n+1} - \xi}{(x_n - \xi)^q} \to A;$$

then

$$\frac{x_n - \xi}{(x_{n-1} - \xi)^q} \to A;$$

or

$$\frac{(x_n - \xi)^{1/q}}{x_{n-1} - \xi} \to A^{1/q},$$

and so

$$\frac{x_{n+1} - \xi}{(x_n - \xi)^{q-1/q}(x_{n-1} - \xi)} \to A^{1+1/q}.$$

Comparing with the previous limit, we require

$$q - 1/q = 1, \quad \text{and} \quad A^{1+1/q} = \frac{f''(\xi)}{2f'(\xi)}.$$

This gives a quadratic equation for $q$, and since we clearly require that $\theta$ is positive we obtain $q = \frac{1}{2}(1 + \sqrt{5})$, giving the required result.

## Solution to Exercise 1.10

Fig. 1.6 shows a typical situation with $f''(x) > 0$, so the graph of $f$ lies below the line $PQ$. Here $P$ and $Q$ are the points corresponding to $u_n$ and $v_n$. Also $R$ is the point corresponding to $\theta$, so that $f(\theta) < 0$. Hence in the next iteration $u_{n+1} = \theta$ and $v_{n+1} = v_n$.

The same picture applies to the next step, and again $v_{n+2} = v_{n+1}$, and so on. Thus if $f'' > 0$ in $[u_N, v_N]$, and $f(u_N) < 0 < f(v_N)$ then $v_n = v_N$ for all $n \geq N$.

If on the other hand $f(u_N) > 0$ and $f(v_N) < 0$ we see in the same way that $u_n = u_N$ for all $n \geq N$.

Similar results are easily deduced if $f'' < 0$ in $[u_N, v_N]$; it is only necessary to replace $f$ by the function $-f$.

Now returning to the situation in Fig. 1.6, the point $v_n$ remains fixed, and the points $u_n$ are monotonically increasing. Hence the sequence $(u_n)$ is monotonically increasing for $n \geq N$, and is bounded above by $v_N$, and is therefore convergent to the unique solution of $f(x) = 0$ in the interval $[u_N, v_N]$. In the general situation, we see that one end of the interval $[u_n, v_n]$ eventually remains fixed, and the other end converges to the root.

Write $u_n = \xi + \delta$, and

$$\frac{u_{n+1} - \xi}{\delta} = \frac{(\xi + \delta) f(v_N) - v_N f(\xi + \delta) - \xi(f(v_N) - f(\xi + \delta))}{\delta(f(v_N) - f(\xi + \delta))}.$$

In the limit as $\delta \to 0$ the numerator and denominator both tend to zero, so we apply l'Hopital's rule to give

$$\begin{aligned}\lim_{\delta \to 0} \frac{u_{n+1} - \xi}{\delta} &= \lim_{\delta \to 0} \frac{f(v_N) - v_N f'(\xi + \delta) + \xi f'(\xi + \delta)}{f(v_n) - f(\xi + \delta) - \delta f'(\xi + \delta)} \\ &= \frac{f(v_N) - v_N f'(\xi) + \xi f'(\xi)}{f(v_N)}\end{aligned}$$

Hence the sequence $(u_n)$ converges linearly to $\xi$, and the asymptotic rate of convergence is

$$-\ln\left\{1 - \frac{(v_N - \xi) f'(\xi)}{f(v_N)}\right\}$$

This may also be written

$$-\ln\left\{1 - \frac{f'(\xi)}{f'(\eta_N)}\right\}$$

for some $\eta_N$ lying between $\xi$ and $v_N$. Since $f(\xi) = 0$, it follows that

$\eta_N > \xi$. Evidently the closer $v_N$ is to the root $\xi$, the closer $f'(\eta_N)$ is to $f'(\xi)$, and the more rapidly the iteration converges.

Asymptotically this method converges more slowly than the standard secant method. Its advantage is that if $f(u_0)$ and $f(v_0)$ have opposite signs the iteration is guaranteed to converge to a root lying in $[u_0, v_0]$; the method is therefore robust. However, it is easy to draw a situation where $v_0$ is far from $\xi$, and where the bisection method is likely to be more efficient.

## Solution to Exercise 1.11

The sequence $(x_n)$ converges to the two-cycle $a, b$ if $x_{2n} \to a$ and $x_{2n+1} \to b$, or equivalently with $a$ and $b$ interchanged. So $a$ and $b$ are fixed points of the composite iteration $x_{n+1} = h(x_n)$, where $h(x) = g(g(x))$, and we define a stable two-cycle to be one which corresponds to a stable fixed point of $h$. Now

$$h'(x) = g'(g(x))\, g'(x);$$

if $h'(a) < 1$ the fixed point $a$ of $h$ is stable; since $g(a) = a$ it follows that if $|g'(a)g'(b)| < 1$ then the two-cycle $a, b$ is stable. In the same way, if $|g'(a)g'(b)| > 1$ then the two-cycle is not stable.

For Newton's method

$$x_{n+1} = x_n - f(x_m)/f'(x_n),$$

and the corresponding function $g$ is defined by

$$g(x) = x - f(x)/f'(x).$$

In this case

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Hence, if

$$\left|\frac{f(a)f''(a)}{[f'(a)]^2}\right| \left|\frac{f(b)f''(b)}{[f'(b)]^2}\right| < 1$$

the two-cycle is stable.

Newton's method for the solution of $x^3 - x = 0$ has the two-cycle $a, -a$ if

$$-a = a - \frac{a^3 - a}{3a^2 - 1}$$
$$a = -a - \frac{-a^3 + a}{3a^2 - 1}.$$

These equations have the solution

$$a = \frac{1}{\sqrt{5}}.$$

Here $f'(a) = 3a^2 - 1 = -2/5$ and $f''(a) = 6a = 6/\sqrt{5}$. So

$$\left|\frac{f(a)f''(a)}{[f'(a)]^2}\right| \left|\frac{f(b)f''(b)}{[f'(b)]^2}\right| = 36,$$

and the two-cycle is not stable.