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Introduction to Analysis: Solutions Manual

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Preface

This solutions manual is meant for the instructor of a course using the main book as its text. The solutions found here are not the sort of solutions I would expect from a student in my course, rather, these are terse but complete solutions to help you steer the student in the right direction, should they have any difficulty with any of these problems.

All errors in this solutions manual are my own. If you notice any errors or typos, please feel free to contact me and I will be happy to fix them.

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Sets, Functions, and Proofs

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Chapter 1

Section 1.1: Logic, and an Introduction to Proof.

1.1.1. Suppose a is even. Then a^2 is the product of a with a, the product of an even number with another even number. Therefore, a^2 is even.

1.1.2. We prove the contrapositive: Suppose a is odd. Then a^2 is the product of a with a, the product of an odd number with another odd number. Therefore a^2 is odd.

1.1.3. If a is odd, then a^2 as the product of an odd number with another odd number is also odd. The converse is proven in Exercise 2.

1.1.4. The truth table appears below:

P	Q	$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

1.1.5. $\neg(P \Rightarrow Q)$ is true precisely when $P \Rightarrow Q$ is false. This occurs only when P is true and Q is false, so $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge (\neg Q)$:

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$P \wedge (\neg Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

1.1.6. Suppose n is odd. Then $3n$ is also odd because it is the product of two odd numbers. Then $3n + 7$ is also odd because it is the sum of an odd number and another odd number.

1.1.7. We prove both by contradiction. Suppose $\frac{a}{b} + \alpha = \frac{p}{q}$, then $\alpha = \frac{p}{q} - \frac{a}{b} = \frac{pb-qa}{qb}$ is expressible as the quotient of whole numbers, a contradiction. If $\frac{a}{b}\alpha = \frac{p}{q}$, then $\alpha = \frac{pb}{qa}$, the latter is well-defined since $q \neq 0$, and because $x \neq 0$ implies $a \neq 0$. So, we again reach a contradiction.

1.1.8. (a) We prove first that if a^2 is a multiple of 3, then a is a multiple of 3, and we do so by proving the contrapositive. Suppose a is not a multiple of 3. Then a can be expressed as a multiple of 3, with a remainder of either 1 or 2. So suppose that $a = 3k + r$, where k is a whole number, and $r = 1$ or 2. Then $a^2 = (3k + r)(3k + r) = 9k^2 + 6kr + r^2 = 3(3k^2 + 2kr) + r^2$. Thus a^2 is a multiple of 3 plus some remainder r^2: if this remainder not divisible by 3 then a^2 is not divisible by 3, which would complete the proof. If $r = 1$ then r^2 is also 1, and if $r = 2$ then $r^2 = 4$, and neither of these cases therefore result in a^2 being divisible by 3.

Now suppose that $\sqrt{3} = \frac{a}{b}$ for whole numbers a, b, where a and b have no common factors: this can be done since if such whole numbers existed, we could find possibly new whole numbers where the above fraction has been reduced into lowest terms. Then $3b^2 = a^2$, so a^2 is divisible by 3, and so by our previous argument, a is divisible by 3. So there exists a whole number k with $a = 3k$. So now $3b^2 = a^2 = (3k)^2 = 9k^2$, so $b^2 = 3k^2$. But now b^2 is a multiple of 3, so it must be the case that b is a multiple of 3. But now both a and b are multiples of 3, a contradiction to them not having any factors in common.

(b) It is not the case that if a^2 is a multiple of 4, then a is a multiple of 4: consider $a = 6$ and $a^2 = 36$. $a^2 = 36$ is a multiple of 4, but $a = 6$ is not.

Section 1.2: Sets and their Operations.

1.2.1. $\emptyset$ is a set that contains no elements, and $\{\emptyset\}$ is a set that contains one element (that element happens to be the empty set).

1.2.2. Let $s \in S$. Then $s \in \{s\}$, and so $s \in \cup_{x\in S}\{x\}$. Now let $s \in \cup_{x\in S}\{x\}$. Then $s \in \{x\}$ for some $x \in S$, but this set contains only one element, x. So $x = s$, and so since $x \in S$ it follows that $s \in S$. We have proven these sets are equal by double-inclusion.

1.2.3. Let $x \in S_1 \cap S_2)^c$. Then $x \notin S_1 \cap S_2$. Therefore x is either not in S_1 or x is not in S_2. So, $x \in S_1^c$ or $x \in S_2^c$, so that $x \in S_1^c \cup S_2^c$.

To prove the other inclusion, we let $x \in S_1^c \cup S_2^c$. Then $x \in S_1^c$ or $x \in S_2^c$, and so x is either not in S_1 or not in S_2. So, $x \notin S_1 \cap S_2$, and so $x \in (S_1 \cap S_2)^c$. We have proven these sets are equal by double-inclusion.

1.2.4. (a) Let $x \in (\cup_{\alpha \in A} S_\alpha)^c$. Then $x \notin \cup_{\alpha \in A} S_\alpha$. So $x \notin S_\alpha$ for every $\alpha \in A$, and so $x \in \cap_{\alpha \in A} S_\alpha^c$.

To prove the other inclusion, we let $x \in \cap_{\alpha \in A} S_\alpha^c$ Then $x \in S_\alpha^c$ for every α, and so $x \notin S_\alpha$ for every α. Therefore $x \notin \cup_{\alpha \in A} S_\alpha$, and so $x \in (\cup_{\alpha \in A} S_\alpha)^c$.

(b) Let $x \in (\cap_{\alpha \in A} S_\alpha)^c$. Then $x \notin \cap_{\alpha \in A} S_\alpha$. Then $x \notin S_\alpha$ for some $\alpha \in A$, so $x \in \cup_{\alpha \in A} S_\alpha^c$.

To prove the other inclusion, let $x \in \cup_{\alpha \in A} S_\alpha^c$. Then $x \in S_\alpha^c$ for some $\alpha \in A$. Therefore $x \notin S_\alpha$ for some $\alpha \in A$, and so $x \notin \cap_{\alpha \in A} S_\alpha$, so $x \in (\cap_{\alpha \in A} S_\alpha)^c$.

1.2.5. We prove this by double-inclusion. Let $x \in S \times T$. Then there exists $s \in S$ and $t \in T$ with $x = (s, t)$. Then $x \in T_s$, and so $x \in \cup_{x \in S} T_x$. Now let $x \in \cup_{x \in S} T_x$. Then for some $s \in S$, $x \in T_s$, and so there is some t for which $x = (s, t)$. Therefore $x \in S \times T$.

1.2.6. There are 8 elements in $\mathcal{P}(S)$:

$$\begin{array}{cccc} \emptyset, & \{1\}, & \{2\}, & \{3\}, \\ \{1,2\}, & \{1,3\}, & \{2,3\}, & \{1,2,3\}. \end{array}$$

1.2.7. Suppose $S \in S$. Then S is a member of itself, and S is the set of all sets which are not members of themselves. So $S \notin S$. Now suppose $S \notin S$. S is therefore not a member of itself, and so $S \in S$.

1.2.8. This is a paradox: no one on the island may shave the barber. If the barber shaves the barber, then it is the barber who shaves himself, and the barber only shaves those who no *not* shave themselves. If someone else on the island shaves the barber, then this is impossible as well, since only the barber shaves those who do not shave themselves, and this would be a situation where someone else on the island shaves someone who is not shaving themself (even if the person being shaved is the barber).

Section 1.3: Mathematical Induction.

1.3.1 The base case clearly holds: $2 = 2$. If $2 + 4 + \cdots + 2n = n(n+1)$ for some n, then

$$2 + 4 + \cdots + 2n + 2(n+1) = n(n+1) + 2(n+1) = (n+1)(n+2).$$

1.3.2 Since $1^2 = 1$, the base case clearly holds. If the statement is true for some n, then

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}.$$

1.3.3 If S has only one element, then the subsets of S are S itself and the empty set. So there are 2^1 subsets in this case. If any set having n elements has 2^n subsets for some n, then suppose S has $n+1$ elements. Let $x \in S$. Then the set $S - \{x\}$ has n elements, and has 2^n subsets by our induction hypothesis. Place x in each of the subsets of $S - \{x\}$: there are 2^n of these as well. We have now counted every subset of S: if x is not in such a subset, then we counted these first. If x is in a subset, we counted these next. So there are $2 \cdot 2^n = 2^{n+1}$ subsets of S.

1.3.4 When $n = 1$, there is nothing to prove, and when $n = 2$ the DeMorgan's laws assert this is the case. Now suppose $n \geq 2$. By grouping the first n terms and using DeMorgan's laws and the induction hypothesis,

$$(\cap_{k=1}^{n+1} S_k)^c = (\cap_{k=1}^{n} S_k \cap S_{n+1})^c = (\cap_{k=1}^{n} S_k)^c \cup S_{n+1}^c = \cup_{k=1}^{n+1} S_k^c.$$

1.3.5 Suppose there exists a nonempty $S \subset \mathbb{N}$ that has no smallest element. We use Mathematical Induction to prove that S must be empty by proving: $k \notin S$ for all $k \leq n$, with $k, n \in \mathbb{N}$. The first statement reads $1 \notin S$. If this were not true and $1 \in S$, then S would have a smallest element. So, $1 \notin S$. Now suppose $1, 2, \ldots, n \notin S$. Could $n+1 \in S$? If so, then $n+1$ would be the smallest element of S, and we assumed that S has no smallest element. So $n+1 \notin S$.

1.3.6 The argument need not be valid when $n = 2$. A black dog and a white dog satisfy the condition that when you remove one the remaining set has the same color.

Section 1.4: Functions.

1.4.1 (a) $f(S) = [1, 16]$. (b) $f^{-1}(T) = [-2, 2]$. (c) $f^{-1}(T) = \emptyset$. (d) Yes.

1.4.2 (a) Let $y \in f(S_1) \cap f(S_2)$. Then $y \in f(S_1)$ and $y \in f(S_2)$. So there exists and $x_i \in S_i$ with $f(x_i) = y$. But f is injective, so $x_1 = x_2$, and this element must be in $S_1 \cap S_2$. So, $y \in f(S_1 \cap S_2)$. (b) Let $f(x) = x^2$, and $S_1 = \{-1\}$ and $S_2 = \{1\}$.

1.4.3 (a) Let $y \in f(X - S)$. So there exists an $x \in X - S$ with $f(x) = y$. So, $y \in f(X)$. If $y \in f(S)$, then there exists an $s \in S$ with $f(s) = y$, and since $f(x) = f(s) = y$ and f is injective, $x = s$. But $x \in X - S$ and $s = x \in S$, a contradiction. So $y \notin f(S)$, and so $y \in f(X) - f(S)$. (b) Let $f(x) = x^2$, $X = \{-1, 1\}$ and $S = \{-1\}$. $f(X - S) = \{1\}$ but $f(X) - f(S) = \{1\} - \{1\} = \emptyset$.

1.4.4 For any $x \in D$, if $f(x) \geq g(x)$, then $\max\{f, g\}(x) = f(x)$, and $\min\{f, g\}(x) = g(x)$. Then $|f(x) - g(x)| = f(x) - g(x)$, and the formulas hold. A similar analysis demonstrates the formulas hold if $f(x) \leq g(x)$.

1.4.5 Let $x \in f^{-1}(T_1 \cap T_2)$. Then $f(x) \in T_1 \cap T_2$. So, $f(x)$ is in both of these sets, and as a result $x \in f^{-1}(T_1)$ and $x \in f^{-1}(T_2)$.

1.4.6 If $x \in S$, then $f(x) \in f(S)$, and x is an element that maps to $f(x)$, so $x \in f^{-1}(f(S))$. If f is injective, then x is the only element that does this, so if $x \in f^{-1}(f(S))$, $f(x) \in f(S)$, and so $x \in S$.

1.4.7 Let $y \in f(f^{-1}(T))$. There exists an $x \in f^{-1}(T)$ with $f(x) = y$. Since $x \in f^{-1}(T)$, $f(x) = y \in T$. If f is surjective, and $y \in T$, then there exists an $x \in f^{-1}(T)$ with $f(x) = y$. So, $f(x) = y \in f(f^{-1}(T))$.

1.4.8 Let $z \in Z$. Since g is surjective, there exists a $y \in Y$ with $g(y) = z$. Since f is surjective, there exists an $x \in X$ with $f(x) = y$, so $(g \circ f)(x) = z$.

1.4.9 If $g \circ f$ is injective but f were not injective, there would exist nonequal $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Then $(g \circ f)(x_1) = (g \circ f)(x_2)$, which is a contradiction.

1.4.10 (a) $f(x) = \frac{1}{x}$. (b) $g(x) = x - 1$. (c) As a composition of bijections, use $g \circ f$ from parts (a) and (b).

1.4.11 Suppose an inverse exists. f is surjective since if $y \in Y$, $g(y) \in X$ with $f(g(y)) = y$. f is injective since if we consider $f(x_1) = f(x_2)$, then

$$g(f(x_1)) = x_1 = x_2 = g(f(x_2)).$$

If f is a bijection, for $y \in Y$, define $g(y)$ to be the unique number $x \in X$ for which $f(x) = y$. Then $g(y) = g(f(x)) = x$ by definition. Also, for $y \in Y$, $g(y) = x$, where $f(x) = y$, so that $f(g(y)) = f(x) = y$.

Section 1.5: Cardinality.

1.5.1 This is true by definition if S is countably infinite (g is a bijection). If $S = \{a_1, \ldots, a_n\}$ is finite then define the surjection $g(k) = a_k$ for $k = 1, \ldots, n$, and $g(k) = a_n$ for $k > n$.

1.5.2 We first assume S and T are disjoint. There are bijections $f : \{1, \ldots, N_S\} \to S$ and $g : \{1, \ldots, N_T\}$. Define $h : \{1, \ldots, N_S + N_T\} \to S \cup T$ as $h(x) = f(x)$ for $x = 1, \ldots, N_S$, and $h(x) = g(x - N_S)$ for $x = N_S + 1, \ldots, N_S + N_T$. If S and T are not disjoint, replace every element in $S \cap T$ in T with a duplicate element (that is a different element) and call this new set $\tilde{T}$. Then S and $\tilde{T}$ are disjoint, and both of them still finite, and $S \cup T \subseteq S \cup \tilde{T}$. Now by Proposition 1.44, $S \cup T$ is finite.

1.5.3 Define $f(x) = x$ if x is not of the form $\frac{1}{2^k}$ for $k = 0, 1, \ldots$. Then define $f(\frac{1}{2^k}) = \frac{1}{2^{k+1}}$.

1.5.4 (a) False, $\{1\}$ is not equivalent to $\{1, 2\}$. (b) False, $\mathbb{Q}$ is not equivalent to $\mathbb{R}$. (c) False, choose S to be finite and T to be countably infinite. (d) True, both are equivalent to $\mathbb{N}$, and set equivalence is an equivalence relation. (e) False, $\mathbb{R}$ is not equivalent to $\mathcal{P}(\mathbb{R})$, but both are uncountable since $\mathbb{R} \subseteq \mathcal{P}(\mathbb{R})$.

1.5.5 The bijection $f(x) = (b-a)x + a$ shows $[0,1] \sim [a,b]$, so $[a,b]$ is uncountable. Each of the following sets has a subset of the form $[c,d]$ for $c < d$, so each of these are uncountable by Proposition 1.51.

1.5.6 We can define the injection $g(x) = \{x\}$. Suppose there were a surjection $f : X \to \mathcal{P}(X)$. Define $T = f(y)$ as suggested in the hint. If $y \in T$, then since $f(y) = T$, we contradict the definition of T. If $y \notin T$, then by definition of T, $y \in T$, also a contradiction.

1.5.7 The false statement is that S and T are equivalent to $\mathbb{N}$. It requires some explanation that the cross product of two countable sets is countable.

1.5.8 Since S and T are countable, there exists surjections f, g from $\mathbb{N}$ to S and T, respectively, by Exercise 1. The function $f \times g : \mathbb{N} \times \mathbb{N} \to S \times T$ defined by $(f \times g)(n,m) = (f(n), g(m))$ is a surjection from the countable $N \times N$ onto $S \times T$. $S \times T$ is now countable by Proposition 1.53.

1.5.9 If they were countable, then $\mathbb{R}$ would be countable as the union of the rationals and the irrationals.

1.5.10 It needn't be. If $S_i = \emptyset$, then such a cross product is empty. But if infinitely many of the S_i contain more than one element, the cross product is uncountable. For example, if $S_i = \{0, 1, \ldots, 9\}$, then this countable cross product is in bijection with the decimal expansions in $\mathbb{R}$ occurring after a decimal point, which is $[0,1]$.