# Solutions to Chapter 2 Problems

*Introduction to Communication Systems,* by Upamanyu Madhow

**Problem 2.1** (a) We do this in two ways. The first is to directly show that $y = x * h$ for some $h$, which shows that the system is LTI with impulse response $h$. We can rewrite

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-u)} I_{\{u \le t\}} x(u) du = \int_{-\infty}^{\infty} e^{-(t-u)} I_{\{t-u \ge 0\}} x(u) du$$

We see that this is in the convolution form

$$y(t) = \int_{-\infty}^{\infty} h(t-u) x(u) du$$

where $h(t) = e^{-t} I_{t \ge 0}$.

The second approach is to show that the system is LTI and then feed in an impulse to find the impulse response. Linearity of $y$ in $x$ is clear, hence let us check for time invariance. Let $x_1(t) = x(t - t_0)$ be a delayed version of the input. The corresponding output is given by

$$y_1(t) = \int_{-\infty}^{t} e^{u-t} x_1(u) du = \int_{-\infty}^{t} e^{u-t} x(u - t_0) du$$

Making the change of variables $v = u - t_0$, we obtain that

$$y_1(t) = \int_{-\infty}^{t-t_0} e^{v+t_0-t} x(v) dv = \int_{-\infty}^{t-t_0} e^{v-(t-t_0)} x(v) dv$$

Comparing with the original expression for $y(t)$, it is clear that we have simply replaced $t$ by $t - t_0$. That is, $y_1(t) = y(t - t_0)$ and the system is time invariant. We now find the impulse response by setting the input to an impulse:

$$h(t) = \int_{-\infty}^{t} e^{u-t} \delta(u) du$$

The impulse at time zero falls into the integration interval only if $t \ge 0$, in which case we select the value of the integrant at $u = 0$. We therefore obtain $h(t) = e^{-t} I_{\{t \ge 0\}}$ as before.

(b) It is easy to see that the Fourier transform of the impulse response is $H(f) = \frac{1}{j2\pi f + 1}$, with magnitude $|H(f)| = \frac{1}{\sqrt{4\pi^2 f^2 + 1}}$. The plot is omitted, but it is clear that this is a sloppy lowpass filter.

(c) We compute the energy in the frequency domain using Parseval's identity. The input $x(t) = 2\text{sinc}(2t) \leftrightarrow X(f) = I_{[-1,1]}(f)$, and the output $Y(f) = H(f)X(f) = \frac{1}{j2\pi f + 1} I_{[-1,1]}(f)$. The energy is given by

$$E_y = \int_{-\infty}^{\infty} |Y(f)|^2 df = \int_{-1}^{1} \frac{1}{4\pi^2 f^2 + 1} df$$

Making the standard substitution $2\pi f = \tan\theta$, so that $2\pi df = \sec^2\theta \; d\theta$ and $4\pi^2 f^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$, we obtain (the limits of the transformed integral are $\pm \tan^{-1}(2\pi) = \pm 1.413$)

$$E_y = \int_{-1.413}^{1.413} \frac{1}{2\pi} d\theta = 0.4498$$

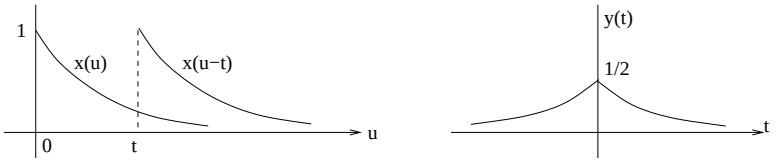



Figure 1: Convolution of a signal and its matched filter in 2.2(a).

**Problem 2.2** (a) The signal $x_2(t) = x_1(-t)$ is the matched filter for $x_1$, and the convolution output at time $t$ is simply the correlation of $x_1$ with itself, delayed by time $t$:

$$y(t) = (x_1 * x_2)(t) = \int x_1(u)x_2(t-u)du = \int x_1(u)x_1(u-t)du$$

It is easy to check that $y(t) = y(-t)$, so we only need to evaluate the convolution for $t \geq 0$, as follows (see Figure 1):

$$y(t) = \int_t^\infty e^{-u}e^{-(u-t)}du = e^t \int_t^\infty e^{-2u}du = e^t \frac{-e^{-2u}}{2}|_t^\infty = \frac{1}{2}e^{-t}$$

Since $y(t)$ is symmetric, we can replace $t$ by $|t$ to obtain

$$y(t) = \frac{1}{2}e^{-|t|}$$

sketched in Figure 1.
(b) A graphical solution is provided in Figure 2.

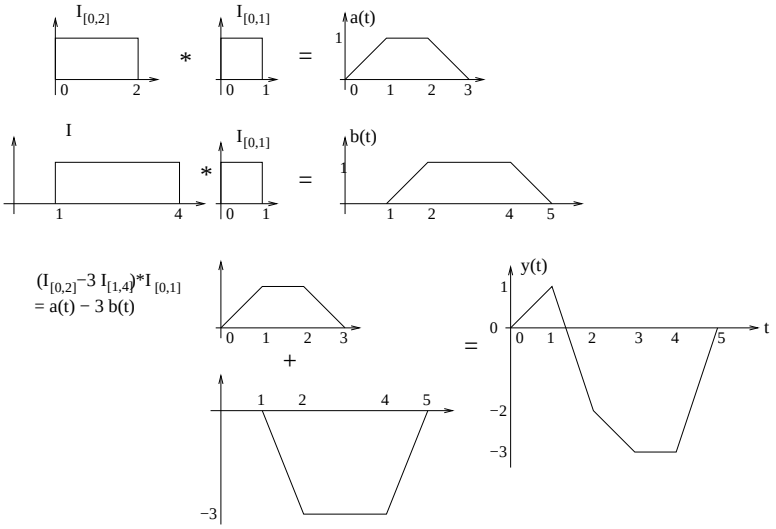



Figure 2: Graphical solution of convolution in Problem 2.2(b).

**Problem 2.3** (a) The signal $u(t)$ and its derivative $v(t) = du/dt$ are sketched in Figure 3. We see that $v(t)$ is the sum of two periodic impulse trains, each of period $T = 0.5$ microseconds, with fundamental frequency $f_0 = \frac{1}{T} = 2$ MHz. Its Fourier series is given by

$$v(t) = \sum_k v_k e^{j2\pi k f_0 t} = \sum_k v_k e^{j4\pi kt}$$

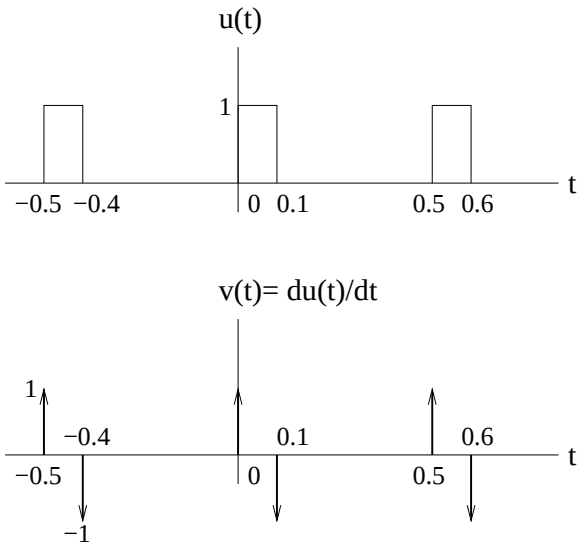



Figure 3: The periodic signal $u(t)$ in Problem 2.3, and its derivative $v(t)$.

with $v_0 = 0$ and

$$v_k = \frac{1}{T} - \frac{1}{T}e^{-j2\pi k f_0 \times 0.1} = 2\left(1 - e^{-j0.4\pi k}\right) = \ , \quad k \neq 0$$

(using the Fourier series for the impulse train.)
(b) We have

$$u_k = \frac{v_k}{j2\pi f_0 k} = \frac{1 - e^{-j0.4\pi k}}{j2\pi k} \ , \quad k \neq 0$$

This can be rewritten as a sinc function. Noting that

$$1 - e^{-j0.4\pi k} = e^{-j0.2\pi k}\left(e^{j0.2\pi k} - e^{-j0.2\pi k}\right) = e^{-j0.2\pi k}\ 2j \sin 0.2\pi k$$

we obtain that

$$u_k = \frac{e^{-j0.2\pi k}\ 2j \sin 0.2\pi k}{j2\pi k} = 0.2e^{-j0.2\pi k}\text{sinc}0.2\pi k \ , \quad k \neq 0 \tag{1}$$

The DC coefficient is given by

$$u_0 = \frac{1}{T}\int_0^T u(t)dt = 0.1/0.5 = 0.2$$

Note that this is consistent with what we obtain when we set $k = 0$ in the expression (1) derived for $k \neq 0$, since $\text{sinc}0 = 1$.
(c) Since the fundamental is at 2 MHz, an LPF of bandwidth 100 KHz passes only the DC component, so the output is $u_0 = 0.2$.
(d) Same answer as (c).

**Problem 2.4** (a) The tent signal

$$u(t) = I_{[-0.5,0.5]}(t) * I_{[-0.5,0.5]}(t) \leftrightarrow U(f) = \text{sinc}^2 f$$

(sketch omitted).
(b) We have $\text{sinc}2t \leftrightarrow \frac{1}{2}I_{[-1,1]}(f)$ and $\text{sinc}4t \leftrightarrow \frac{1}{4}I_{[-2,2]}(f)$, so that

$$v(t) = \text{sinc}2t\text{sinc}4t \leftrightarrow V(f) = \frac{1}{2}I_{[-1,1]}(f) * \frac{1}{4}I_{[-2,2]}(f)$$

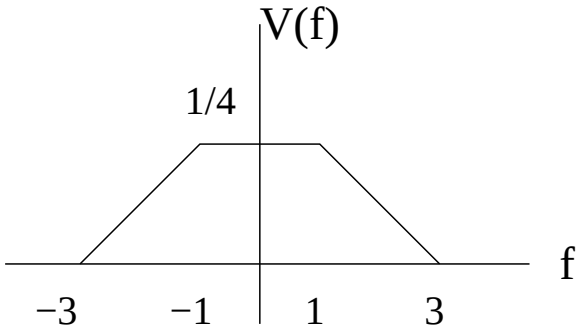



Figure 4: The Fourier transfer $V(f)$ in Problem 2.4(b).

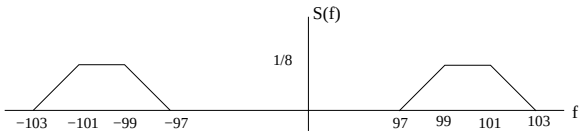



Figure 5: The Fourier transfer $S(f)$ in Problem 2.4(c). Frequency axis not to scale.

sketched in Figure 4.
(c) The spectrum is computed as

$$s(t) = v(t)\cos 200\pi f \leftrightarrow S(f) = \frac{1}{2}V(f-100) + \frac{1}{2}V(f+100)$$

and is sketched in Figure 5, using the results of (b).
(d) While $U(f)$ in (a) is not strictly bandlimited, its energy is concentrated around DC. The signal $V(f)$ in (b) is strictly bandlimited around DC. The signal $S(f)$ in (c) is bandlimited, and has energy concentrated away from DC. Thus, the signals in (a) and (b) are baseband, and the signal in (c) is passband.

**Problem 2.5** (a) Noting that $s(t) = \text{sinc}2t \leftrightarrow S(f) = \frac{1}{2}I_{[-1,1]}(f)$, we have

$$\int_{-\infty}^{\infty} \text{sinc}^2 2t \; dt = \int_{-\infty}^{\infty} |s(t)|^2 \; dt = \int_{-\infty}^{\infty} |S(f)|^2 \; df = \int_{-1}^{1} \frac{1}{4} \; df = \frac{1}{2}$$

(b) Note that $s_1(t) = \text{sinc}t \leftrightarrow S_1(f) = I_{[-0.5,0.5]}(f)$ and $s_2(t) = \text{sinc}2t \leftrightarrow S_2(f) = \frac{1}{2}I_{[-1,1]}(f)$, and that the product of these two signals is even. Thus,

$$\int_{0}^{\infty} \text{sinc}t \; \text{sinc}2t \; dt = \frac{1}{2}\int_{-\infty}^{\infty} s_1(t)s_2^*(t) \; dt = \frac{1}{2}\int_{-\infty}^{\infty} S_1(f) \; S_2^*(f) \; df = \frac{1}{2}\int_{-0.5}^{0.5} \frac{1}{2} \; df = \frac{1}{4}$$

While the waveforms are real-valued in both time and frequency domains, we do put in conjugates to highlight that we are applying Parseval's identity in its correct form.

**Problem 2.6** (a) We have $u(t) = u_1(t)u_2(t)$, where $u_1(t) = \text{sinc}(t) \leftrightarrow U_1(f) = I_{[-\frac{1}{2},\frac{1}{2}]}(f)$ and $u_2(t) = \text{sinc}(2t) \leftrightarrow U_2(f) = \frac{1}{2}I_{[-1,1]}(f)$, so that $U(f) = (U_1 * U_2)(f)$ is a trapezoidal pulse shown in Figure 6. Since the unit of time is microseconds, the unit of frequency is MHz.
(b) The signal $s(t) = u(t)\cos 200\pi t \leftrightarrow (U(f-100) + U(f+100))/2$ is sketched in Figure 6.

**Problem 2.7** The signal $s(t) = \text{sinc}4t \leftrightarrow S(f) = \frac{1}{4}I_{[-2,2]}(f)$. We have $p(t) = \text{sinc}^2 t \leftrightarrow P(f) = (1-|f|)I_{[-1,1]}(f)$ (tent in frequency domain), so that the filter impulse response and transfer

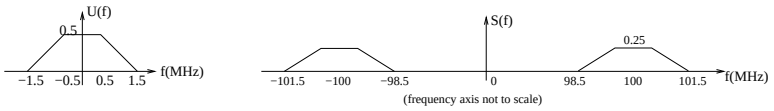



Figure 6: Frequency domain plots for Problem 2.6.

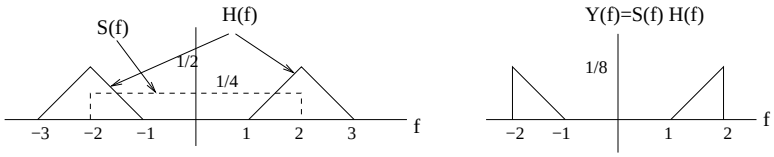



Figure 7: Frequency domain plots for Problem 2.7: filter input $S(f)$ and transfer function $H(f)$ on one plot, and output $Y(f)$ in the other. The amplitude scaling in the two plots is not the same.

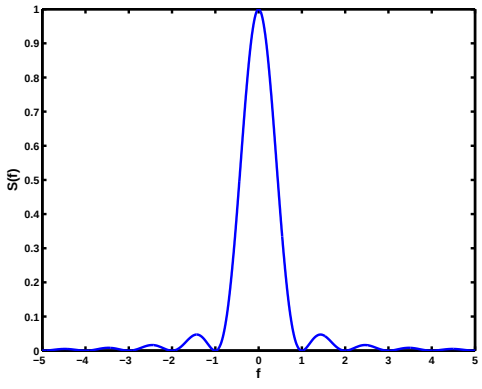



Figure 8: Spectrum of the tent signal in Problem 2.8(a) is $\text{sinc}^2(f)$.

function are given by $h(t) = p(t)\cos 4\pi t \leftrightarrow H(f) = \frac{1}{2}\left(P(f-2) + P(f+2)\right)$. We sketch $S(f)$ and $H(f)$, which then yields $Y(f) = S(f)H(f)$ as shown in Figure 7.

**Problem 2.8** (a) The tent signal can be written as a convolution of two boxes $s(t) = I_{[-1/2,1/2]}(t) * I_{[-1/2,1/2]}(t)$. Since $I_{[-1/2,1/2]}(t) \leftrightarrow \text{sinc}(f)$, we have $S(f) = \text{sinc}^2(f)$, plotted in Figure 8.
(b) The 99% energy containment bandwidth $W$ satisfies

$$\int_{-W}^{W} |S(f)|^2 df = 2\int_{0}^{W} \text{sinc}^4 f df = 0.99E_s$$

where the energy is given by

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_{-\infty}^{\infty} |s(t)|^2 dt = 2\int_{0}^{1} (1-t)^2 dt = 2/3$$

Using the symmetry of $|S(f)|^2$ and plugging in its expression, we see that we need to numerically solve the following equation for $W$:

$$\int_{0}^{W} \text{sinc}^4 f df = 0.33$$

We get $W = 0.58$. Since the unit of time is ms, the unit of frequency is KHz, so the 99% bandwidth is 0.58 KHz.

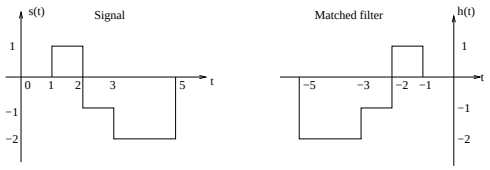



Figure 9: Sketch of signal and its matched filter for Problem 2.12(a).

**Problem 2.9** (a) The cosine pulse

$$p(t) = \cos \pi t \; I_{[-1/2,1/2]}(t) = \frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right) I_{[-1/2,1/2]}(t)$$

We have $I_{[-1/2,1/2]}(t) \leftrightarrow \text{sinc} f$, so that

$$e^{j\pi t} I_{[-1/2,1/2]}(t) \leftrightarrow \text{sinc}(f - \frac{1}{2}) \; , \quad e^{-j\pi t} I_{[-1/2,1/2]}(t) \leftrightarrow \text{sinc}(f + \frac{1}{2})$$

Plugging in, we have

$$P(f) = \frac{1}{2}\left(\text{sinc}(f - \frac{1}{2}) + \text{sinc}(f + \frac{1}{2})\right)$$

We can now simplify further:

$$\text{sinc}(f - \frac{1}{2}) = \frac{\sin\left(\pi(f - \frac{1}{2})\right)}{\pi(f - \frac{1}{2})} = \frac{\sin\left(\pi f - \frac{\pi}{2}\right)}{\pi(f - \frac{1}{2})} = -\frac{\cos \pi f}{\pi(f - \frac{1}{2})}$$

$$\text{sinc}(f + \frac{1}{2}) = \frac{\sin\left(\pi(f + \frac{1}{2})\right)}{\pi(f + \frac{1}{2})} = \frac{\sin\left(\pi f + \frac{\pi}{2}\right)}{\pi(f + \frac{1}{2})} = \frac{\cos \pi f}{\pi(f + \frac{1}{2})}$$

so that

$$P(f) = \frac{1}{2}\left(-\frac{\cos \pi f}{\pi(f-\frac{1}{2})} + \frac{\cos \pi f}{\pi(f+\frac{1}{2})}\right)$$

which simplifies to the desired expression

$$P(f) = \frac{2\cos \pi f}{\pi(1-4f^2)}$$

(b) The sine pulse in Example 2.5.7 is given by $u(t) = \sin \pi t I_{[0,1]}(t) = p(t-\frac{1}{2})$ (i.e., it is a time-shifted version of the cosine pulse in (a)), so that

$$U(f) = P(f)e^{j2\pi(-\frac{1}{2})} = e^{-j\pi f}P(f) = e^{-j\pi f}\frac{2\cos \pi f}{\pi(1-4f^2)}$$

which is the expression in (2.63).

**Problem 2.10** Solutions for the required numerical computations have been skipped.

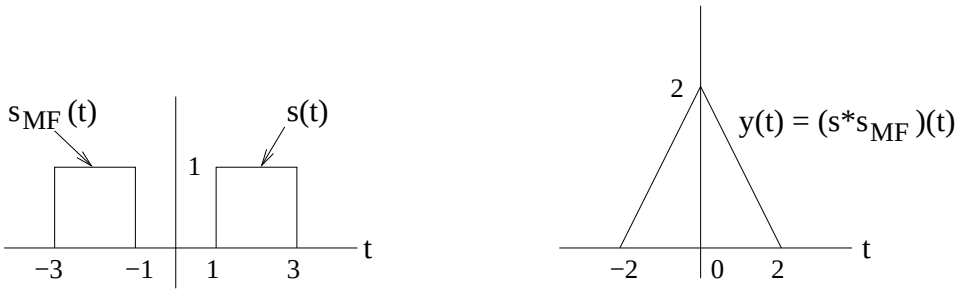



Figure 10: The signal, its matched filter and their convolution in Problem 2.11.

**Problem 2.11** (a) The signal $s(t)$ and its matched filter $h(t) = s_{MF}(t) = s(-t)$ are sketched in the plot on the left in Figure 10.
(b) The convolution of two boxes of equal width gives a triangle, so the only issue is where the triangle is centered. If each box is centered at the origin, then we get a triangle centered at the origin. Now, $s(t) = I_{[1,3]}(t)$ is delayed by 2 relative to a centered box $I_{[-1,1]}(t)$, whereas $s_{MF}(t) = I_{[-3,-1]}(t)$ is delayed by $-2$ (i.e., advanced by 2) relative to a centered box. Thus, when we convolve them, the two delays cancel each other, and we get a tent signal centered at the origin, as shown in the plot on the right in Figure 10. Of course, we would get the same answer by directly convolving the two signals.
(c) Since $s_{MF}(t) = s^*(-t)$, we have $S_{MF}(f) = S^*(f)$. Thus,

$$y(t) = (s * s_{MF})(t) \leftrightarrow Y(f) = S(f)S_{MF}(f) = S(f)S^*(f) = |S(f)|^2 \geq 0 \text{ for all } f$$

**Problem 2.12** (a) The signal $s(t)$ and its matched filter $h(t) = s_{mf}(t) = s(-t)$ are sketched in Figure 11.
(b) Let us break down the convolution into simpler parts: $s(t) = a(t) + b(t)$, where $a(t) = I_{[1,3]}(t)$ and $b(t) = -2I_{[2,5]}(t)$, so that $h(t) = a(-t) + b(-t)$ and

$$y(t) = (s * h)(t) = a(t) * a(-t) + b(t) * b(-t) + a(t) * b(-t) + a(-t) * b(t)$$

All of the terms above are convolutions between boxes. The first two terms give triangles centered at the origin, while the third and fourth terms are trapezoids which are reflections of each other. Figure 12 shows the 4 components of $y(t)$ and then adds them up to obtain the final signal.

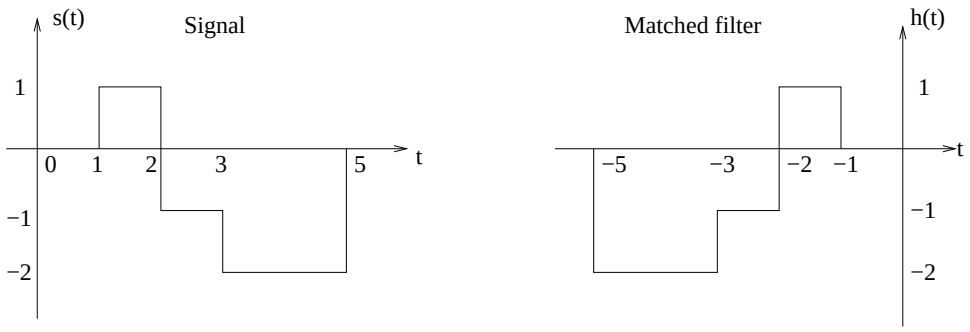



Figure 11: The signal and its matched filter in Problem 2.12.

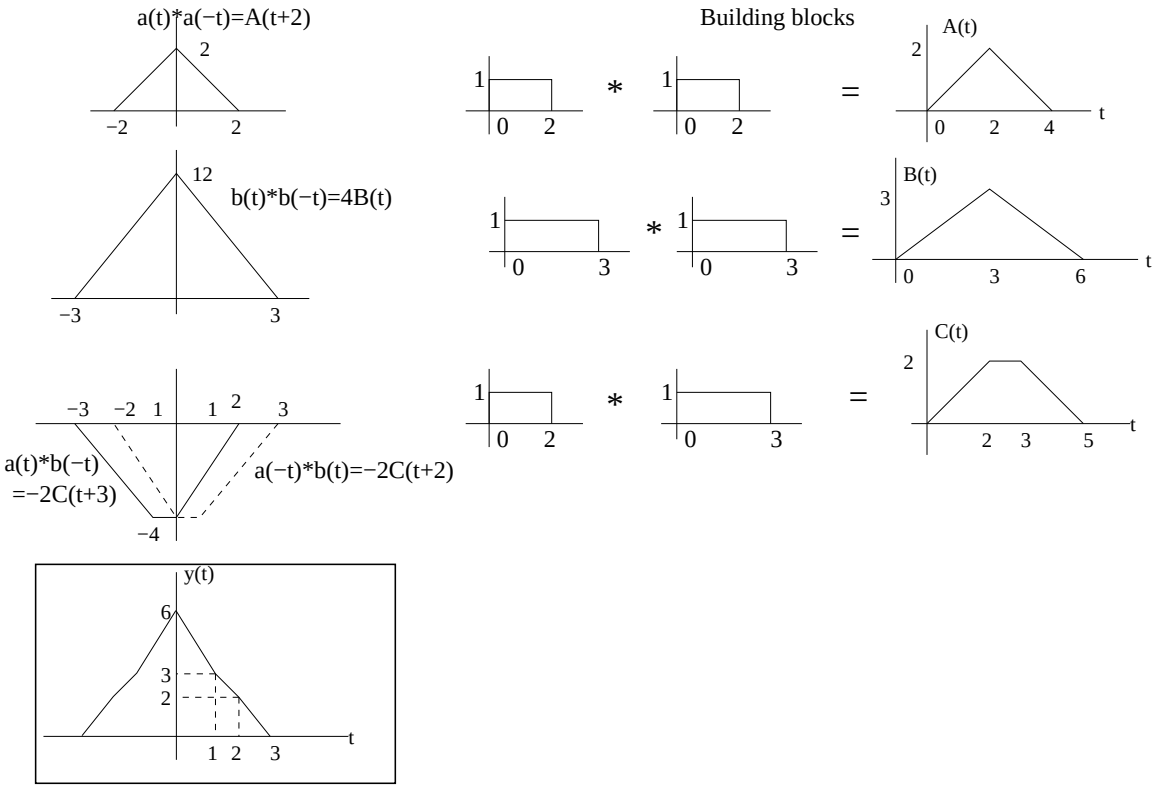
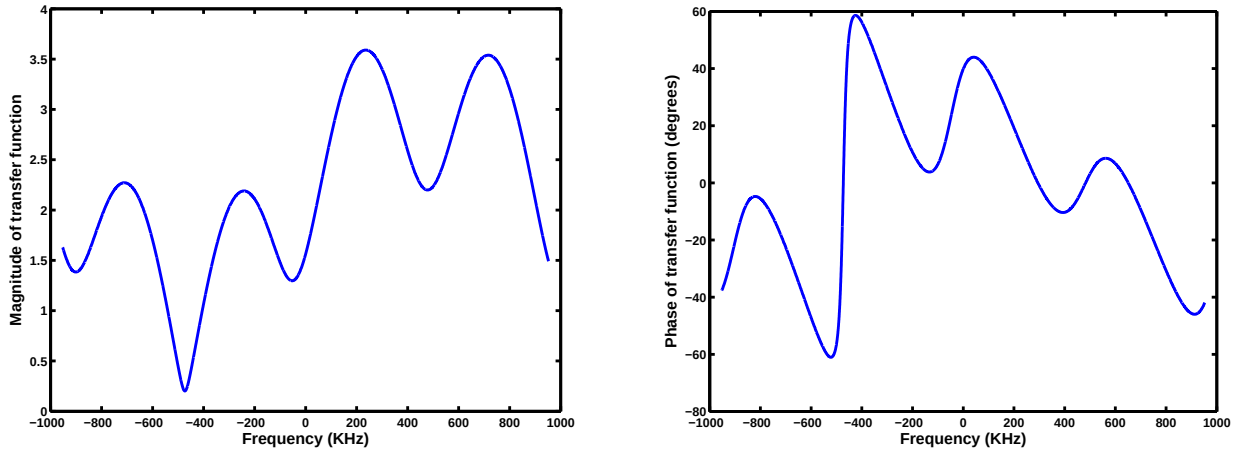



Figure 12: Computing the convolution of the signal with its matched filter in Problem 2.12.

The figure also shows the building blocks (convolution between boxes starting at zero) used to obtain these 4 components (by translation and scaling of these building blocks). In adding up the 4 components of $y(t)$, we can use the simple observation that it is piecewise linear (since its components are piecewise linear), and simply compute $y(t)$ for $t \geq 0$ at the end points of the segments ($t = 0, 1, 2, 3$) and then join them by lines. The signal for $t \leq 0$ can now be obtained by reflection, noting that the convolution of a signal with its matched filter is symmetric (for complex-valued signals, the matched filter is defined as $s_{mf}(t) = s^*(-t)$, and the convolution is conjugate symmetric).
(c) Since $s_{mf}(t) = s^*(-t)$, we have $S_{mf}(f) = S^*(f)$. Thus,

$$y(t) = (s * s_{mf})(t) \leftrightarrow Y(f) = S(f)S_{mf}(f) = S(f)S^*(f) = |S(f)|^2$$

Clearly, $Y(f) \geq 0$ for all $f$.




(a) Transfer Function Magnitude (b) Transfer Function Phase

Figure 13: Both magnitude and phase vary across frequency, with small magnitudes corresponding to large phase variations.

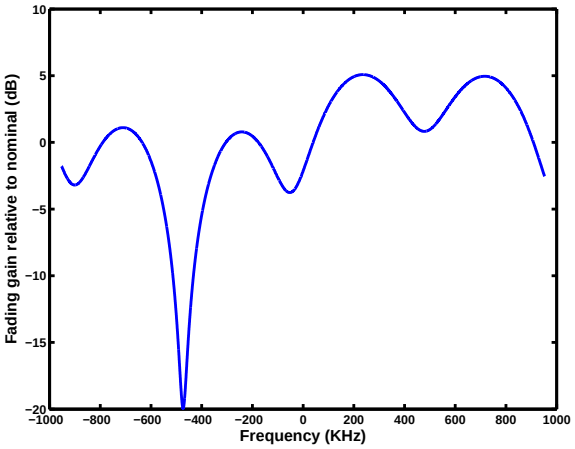



Figure 14: Fading with respect to nominal single path channel.

**Problem 2.13**(a) The delay spread is $2.2 - 0.1 = 2.1$ microseconds. The coherence bandwidth is $1/2.1 = 0.476$ MHz or 476 KHz.

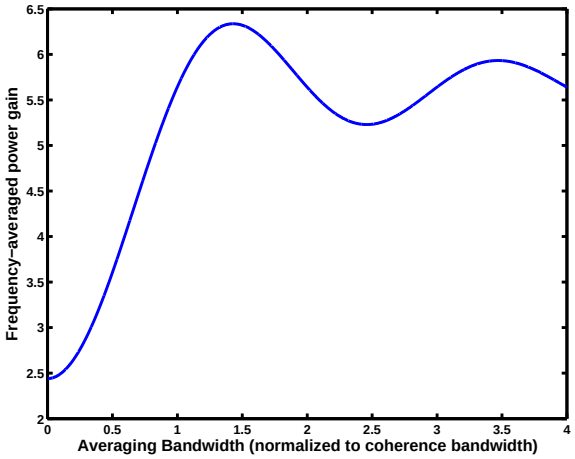



Figure 15: Fading with respect to nominal single path channel.

(b) Dropping the delay corresponding to the first path as in the text, the channel transfer function is given by

$$H(f) = 2 + je^{-j2\pi f\Delta_1} - 0.8e^{-j2\pi f\Delta_2}$$

where $\Delta_1 = 0.54$ microseconds and $\Delta_2 = 2.1$ microseconds are the delay differences for the second and third paths with respect to the first. The magnitude and phase of the transfer function are plotted in Figure 13. When the magnitude is small, small changes in the real and imaginary parts can cause large changes in the angle. Thus, there are rapid phase changes around deep fades.
(c) The fading with respect to the nominal single path channel is given by $20\log_{10}\frac{|H(f)|}{2}$ dB, and is plotted in Figure 14. We see fading depths in excess of 20 dB.
(d) The frequency-averaged power gain is plotted as a function of normalized bandwidth (normalized by coherence bandwidth) in Figure 15. We see that the gain quickly exceeds the nominal single path gain of 4 as the averaging bandwidth approaches the coherence bandwidth. It is left as an exercise to argue that the asymptotic value for large bandwidths is 5.64.
*Hint:* The transfer function $H(f)$ is comprised of a constant plus two frequency domain sinusoids. What is the average power of $|H(f)|^2$?

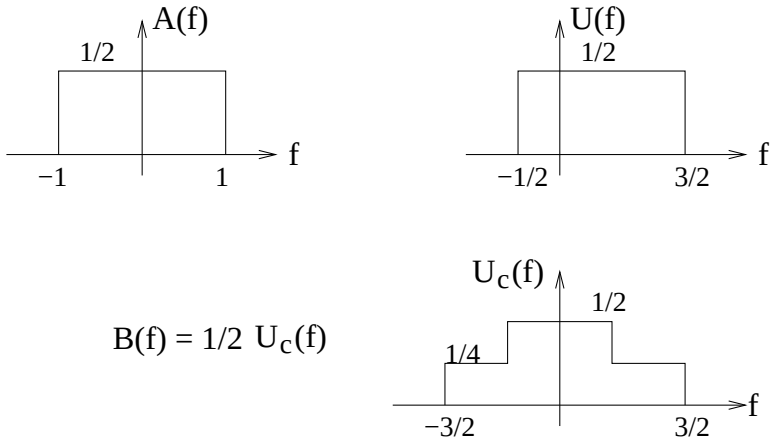



Figure 16: Relevant spectra for Problem 2.14(a)-(c).

**Problem 2.14** (a) We have $a(t) = \text{sinc}(2t) \leftrightarrow A(f) = \frac{1}{2}I_{[-1,1]}(f)$. We can see that $u_p(t) =$

$a(t)\cos 200\pi t \leftrightarrow U_p(f) = \frac{1}{2}\left(A(f-100)+A(f+100)\right)$. This occupies the band 99-101 MHz (since the unit of time is microseconds, the unit of frequency is MHz).
Remark: At the risk of belaboring the obvious, recall that for physical (real-valued) signals, we consider only positive frequencies in defining the band being occupied. Negative frequencies carry no additional information, since the spectrum is constrained to be conjugate symmetric.
(b) The output $b(t) = \frac{1}{2}u_c(t)$, where $u_c$ is the I component of $u_p$ with respect to the reference $199\pi t$. We know that the complex envelope of $u_p$ for reference $200\pi t$ is $a(t)$. In order to retard the reference phase by $\pi t$, we must advance the complex envelope's phase by $\pi t$. Thus, $u(t) = a(t)e^{j\pi t}$. For completeness, we also provide the algebra behind this intuitive statement (it is recommended that you do this until you are very familiar with complex baseband):

$$u_p(t) = \mathrm{Re}\left(a(t)e^{j200\pi t}\right) = \mathrm{Re}\left(a(t)e^{j\pi t}e^{j199\pi t}\right)$$

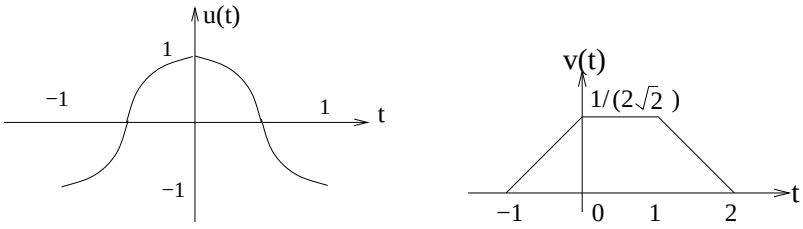
from which we can read off that the complex envelope with respect to reference $199\pi t$ is $a(t)e^{j\pi t}$. We can now compute

$$b(t) = \frac{1}{2}u_c(t) = \frac{1}{2}\mathrm{Re}(u(t)) = \frac{1}{2}a(t)\cos\pi t = \frac{1}{2}\mathrm{sinc}(2t)\cos\pi t$$

Thus, $B(f) = \frac{1}{4}\left(A(f-\frac{1}{2})+A(f+\frac{1}{2})\right)$, with $A(f) = \frac{1}{2}I_{[-1,1]}(f)$. The spectrum $B(f)$ and the other relevant spectra for this problem are shown in Figure 16. Another way in which we could have found $B(f)$ would have been to note that $U(f) = A(f-\frac{1}{2})$ and that the frequency domain expression for the I component is given by $U_c(f) = \left(U(f)+U^*(-f)\right)/2$. This gives the same result as before for $B(f) = \frac{1}{2}U_c(f)$.
(c) This downconversion gives

$$c(t) = -\frac{1}{2}u_s(t) = -\frac{1}{2}\mathrm{Im}(u(t)) = -\frac{1}{2}a(t)\sin\pi t = -\frac{1}{2}\mathrm{sinc}(2t)\sin\pi t$$

We can now compute $C(f) = -\frac{1}{4j}\left(A(f-\frac{1}{2})-A(f+\frac{1}{2})\right)$. Since $A(f)$ is real-valued, we have $\mathrm{Re}\left(C(f)\right) = 0$ and $\mathrm{Im}\left(C(f)\right) = \frac{1}{4j}\left(A(f-\frac{1}{2})-A(f+\frac{1}{2})\right)$, which is also sketched in Figure 16.
(d) The complex envelope $u(t) = u_c(t) + ju_s(t) = 2\left(b(t)-jc(t)\right)$. Since $u(t) = a(t)e^{j\pi t}$, we have $a(t) = u(t)e^{-j\pi t} = 2\left(b(t)-jc(t)\right)(\cos\pi t - j\sin\pi t)$. Since $a(t)$ is real-valued, the imaginary part has to drop out (check!), and real part gives $a(t) = 2\left(b(t)\cos\pi t - c(t)\sin\pi t\right)$. The block diagram is straightforward to draw, so we skip it.




(a) Downconverter output in 2.15(a)

(b) Downconverter output in 2.15(b)

Figure 17: Sketches for Problem 2.15.

**Problem 2.15** (a) This problem does not have subscripts denoting passband signals, but clearly, $s(t)$ is passband, and $u(t)$ is the I component with respect to the $401\pi t$ reference. The complex envelope with respect to the $400\pi t$ carrier reference is clearly $\tilde{s}_1(t) = I_{[-1,1]}(t)$. When we advance the carrier reference by $\pi t$ to $401\pi t$, we must correspondingly retard the complex envelope by $\pi t$ to get $\tilde{s}_2(t) = I_{[-1,1]}(t)e^{-j\pi t}$. We now have

$$u(t) = \tilde{s}_{2c}(t) = \mathrm{Re}\left(I_{[-1,1]}(t)e^{-j\pi t}\right) = I_{[-1,1]}(t)\cos\pi t$$

which is sketched in Figure (a).
(b) When the passband signal $s(t)$ is passed through the passband filter $h(t)$, we get a passband signal $y(t)$. The output of the downconverter is $v(t) = -y_s(t)$, where $y_s$ is the Q component of the filter output $y$, with respect to the carrier reference $400\pi t + \frac{\pi}{4}$. Let us therefore find the complex envelope of $y$ with respect to this reference, by finding the complex envelopes of $s$ and $h$ with respect to the reference and then convolving them. The complex envelope of $s(t)$ with respect to the $400\pi t$ carrier reference is $\tilde{s}_1(t) = I_{[-1,1]}(t)$, so the complex envelope with respect to the reference $400\pi t + \frac{\pi}{4}$ is $\tilde{s}_3(t) = I_{[-1,1]}(t)e^{-j\frac{\pi}{4}}$. The complex envelope of $h(t)$ with respect to the $400\pi t + \frac{\pi}{4}$ reference is $\tilde{h}(t) = -jI_{[0,1]}(t)$ (we can read off from the expression for $h$ that $h_c = 0$ and $h_s = -I_{[0,1]}(t)$). The complex envelope of the filter output is therefore given by

$$\tilde{y} = \frac{1}{2}\tilde{s}_3 * \tilde{h} = \frac{1}{2}\left(I_{[-1,1]}(t)e^{-j\frac{\pi}{4}}\right) * \left(-jI_{[0,1]}(t)\right) = \frac{1}{2}p(t)e^{-j3\pi/4}$$

where we have noted that $-j = e^{-j\pi/2}$, and where $p(t) = I_{[-1,1]} * I_{[0,1]}$ is a trapezoidal pulse. We therefore obtain

$$v(t) = -y_s(t) = -\mathrm{Im}\left(\tilde{y}(t)\right) = \frac{1}{2}p(t)\sin 3\pi/4 = \frac{1}{2\sqrt{2}}p(t)$$

which is sketched in Figure (b).

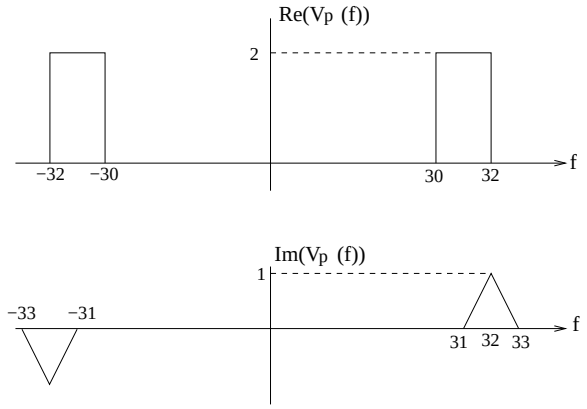



Figure 18: Spectrum of passband waveform in Problem 2.17 (frequency axis not to scale).

**Problem 2.16** (a) The inner product is zero by I-Q orthogonality. Alternatively, let $\tilde{u}_i$ denote the complex envelope for the passband signals $u_i$, $i = 1, 2$ with respect to the $100\pi t$ reference. Then $\tilde{u}_1(t) = I_{[0,1]}(t)$ and $\tilde{u}_2(t) = -jI_{[0,1]}(t)$. The passband inner product is related to the complex baseband inner product as follows: $\langle u_1, u_2\rangle = \frac{1}{2}\mathrm{Re}\left(\langle\tilde{u}_1, \tilde{u}_2\rangle\right) = 0$, since

$$\langle\tilde{u}_1, \tilde{u}_2\rangle = \int_{-\infty}^{\infty}\tilde{u}_1(t)\tilde{u}_2^*(t)dt = j\int_0^1 dt = j$$

(b) The complex envelope $\tilde{y} = \frac{1}{2}\tilde{u}_1 * \tilde{u}_2 = -j\frac{1}{2}s(t) = y_c + jy_s$, where $s(t) = I_{[0,1]} * I_{[0,1]}$ is a triangular pulse over $[0, 2]$. Thus, the passband signal $y(t) = \frac{1}{2}s(t)\sin 100\pi t$ ($y_c = 0$, $y_s = -\frac{1}{2}s(t)$.
(c) Since $Y(f) = \frac{1}{2}\tilde{Y}(f - f_c) + \frac{1}{2}\tilde{Y}^*(-f - f_c)$, the magnitude spectrum is given by

$$|Y(f)| = \frac{1}{2}|\tilde{Y}(f - f_c)| + \frac{1}{2}|\tilde{Y}(-f - f_c)|$$

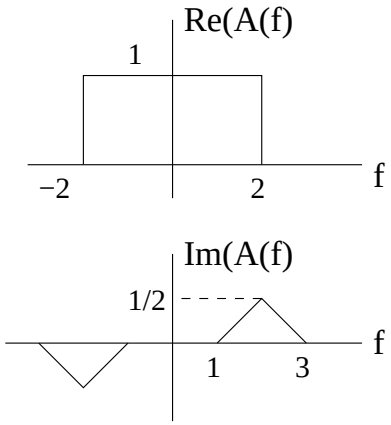



Figure 19: Spectrum of downconverter output in Problem 2.17(b).

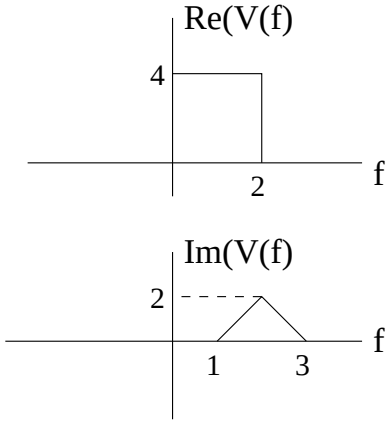



Figure 20: Complex envelope with respect to $f_c = 30$ for Method 2 in Problem 2.17(b).

Now plug in $\tilde{Y}(f) = \frac{1}{2}|S(f)| = \frac{1}{2}\text{sinc}^2(f)$ (sketch omitted).

**Problem 2.17**(a) Since $v_p(t)$ is real-valued, $V_p(f)$ is conjugate symmetric, hence $\text{Re}\,(V_p(f))$ is symmetric and $\text{Im}\,(V_p(f))$ is antisymmetric. We use this, together with the given information, to plot the spectrum in Figure 18.
(b) Denote the output by $a(t) \leftrightarrow A(f)$. Let us do this in two ways.
*Method 1:* The first does not invoke the complex envelope, and uses the result that $v_p \cos(60\pi t) \leftrightarrow \frac{1}{2}\left(V_p(f-30) + V_p(f+30)\right)$, with the LPF rejecting the spectrum component around $2f_c = 60$. The resulting spectrum $A(f)$ is shown in Figure 19. We have $\text{Re}(A(f)) \leftrightarrow 4\text{sinc}(4t)$, and $j\text{Im}(A(f)) \leftrightarrow j\frac{1}{2}\text{sinc}^2 t\left(e^{j4\pi t} - e^{-j4\pi t}\right) = -\text{sinc}^2 t \sin 4\pi t$. Adding up these two contributions, we have $\text{Re}(A(f)) + j\text{Im}(A(f)) \leftrightarrow a(t) = 4\text{sinc}(4t) - \text{sinc}^2 t \sin 4\pi t$.
*Method 2:* We recognize that $a(t) = \frac{1}{2}v_c(t) = \frac{1}{2}\text{Re}(v(t))$ is the output of a downconversion operation, where $v(t)$ is the complex envelope with respect to $f_c = 30$, and $v_c(t)$ the I component. We know that we can get $V(f)$ from $V_p(f)$ by throwing away the spectrum for negative frequencies, and moving the spectrum for positive frequencies to the left by $f_c = 30$ and doubling it. The result is shown in Figure 20. We see that $\text{Re}(V(f)) \leftrightarrow 8\text{sinc}2te^{j2\pi t}$ and $j\text{Im}(V(f)) \leftrightarrow j2\text{sinc}^2 te^{j4\pi t}$ so that $V(f) = \text{Re}(V(f)) + j\text{Im}(V(f)) \leftrightarrow v(t) = 8\text{sinc}2te^{j2\pi t} + j2\text{sinc}^2 te^{j4\pi t}$. We can now take the real part to find the I component as follows:

$$v_c(t) = \text{Re}(v(t)) = 8\text{sinc}2t \cos 2\pi t - 2\text{sinc}^2 t \sin 4\pi t$$

Check that $a(t) = \frac{1}{2}v_c(t)$ matches the answer from Method 1. We can also find the I component in the frequency domain directly from $V(f)$: $V_c(f) = \frac{V(f)+V^*(-f)}{2}$, which means that $\text{Re}(V_c(f)) = \frac{\text{Re}(V(f))+\text{Re}(V(-f))}{2}$ and $\text{Im}(V_c(f)) = \frac{\text{Im}V(f))+\text{Im}(V(-f))}{2}$. Applying this to the spectrum in Figure 20, check that the answer you get for $A(f) = \frac{1}{2}V_c(f)$ matches Figure 19.

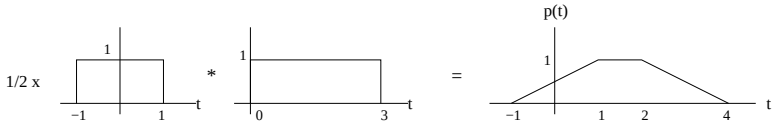



Figure 21: Computing the complex envelope of filter output in Problem 2.18.

**Problem 2.18:** We compute the complex envelope of the output first. With respect to the $e^{j100\pi t}$ reference, the complex envelopes of the signal is $\tilde{u}(t) = I_{[-1,1]}(t)$ (I component is $I_{[-1,1]}(t)$) and the complex envelope of the filter is $\tilde{h}(t) = -jI_{[0,3]}(t)$ (Q component is $-I_{[0,3]}(t)$). The complex envelope of the output is given by

$$\tilde{y} = \frac{1}{2}\tilde{u} * \tilde{h} = -jp(t)$$

where $p(t) = \frac{1}{2}I_{[-1,1]}(t) * I_{[0,3]}(t)$ is a trapezoidal pulse obtained by convolving two boxes as shown in Figure 21. The passband output is therefore given by $y(t) = p(t)\sin 100\pi t$ (Q component is $-p(t)$).

**Problem 2.19:** Matlab problem, solution omitted.

**Problem 2.20** (a) For a frequency reference of $f_c = 50$, or a phase reference of $100\pi t$, the complex envelope of $u_p(t)$ can be read off as $u(t) = \text{sinc}(2t)$. The complex envelope of $v_p(t)$ with respect to $101\pi t + \frac{\pi}{4}$ can be read off as $\tilde{v}(t) = -j\text{sinc}t$. Switching the reference to $100\pi t$ means retarding it by $\pi t + \frac{\pi}{4}$, hence we must advance the complex envelope by that amount to get

$$v(t) = -j\text{sinc}te^{j(\pi t + \frac{\pi}{4})} = \text{sinc}te^{j(\pi t - \frac{\pi}{4})}$$

as the complex envelope with respect to $100\pi t$.
(b) The bandwidth of the passband signal equals the two-sided bandwidth of the complex envelope. We have $U(f) = \frac{1}{2}I_{[-1,1]}(f)$, which has two-sided bandwidth 2, and $V(f) = e^{-j\frac{\pi}{4}}I_{[0,1]}(f)$, which has two-sided bandwidth 1. Thus, $u_p$ has bandwidth 2 and $v_p$ has bandwidth 1.
(c) The passband inner product is related to the complex baseband inner product as follows:

$$\langle u_p, v_p\rangle = \frac{1}{2}\mathrm{Re}\langle u, v\rangle$$

Since the complex envelopes are bandlimited, it is convenient to use Parseval's identity to compute the inner product in the frequency domain:

$$\langle u, v\rangle = \langle U, V\rangle = \int U(f)V^*(f)df = \int \frac{1}{2}I_{[-1,1]}(f)e^{j\frac{\pi}{4}}I_{[0,1]}(f)df = \frac{e^{j\frac{\pi}{4}}}{2}$$

We therefore obtain

$$\langle u_p, v_p\rangle = \frac{1}{2}\mathrm{Re}\left(\frac{e^{j\frac{\pi}{4}}}{2}\right) = \frac{1}{4}\cos\frac{\pi}{4} = \frac{1}{4\sqrt{2}}$$

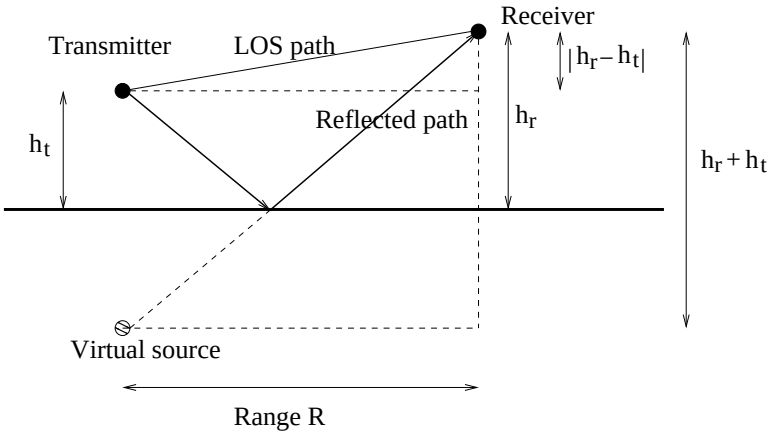



Figure 22: Geometry for computing delay spread in Problem 2.21.

**Problem 2.21** (a) From Figure 22, we see that the length of the LOS path is given by $\sqrt{R^2 + (h_r - h_t)^2}$, and that the length of the reflected path, which equals that of the line joining the virtual source to the receiver, is given by $\sqrt{R^2 + (h_r + h_t)^2}$. The difference in these lengths is given by

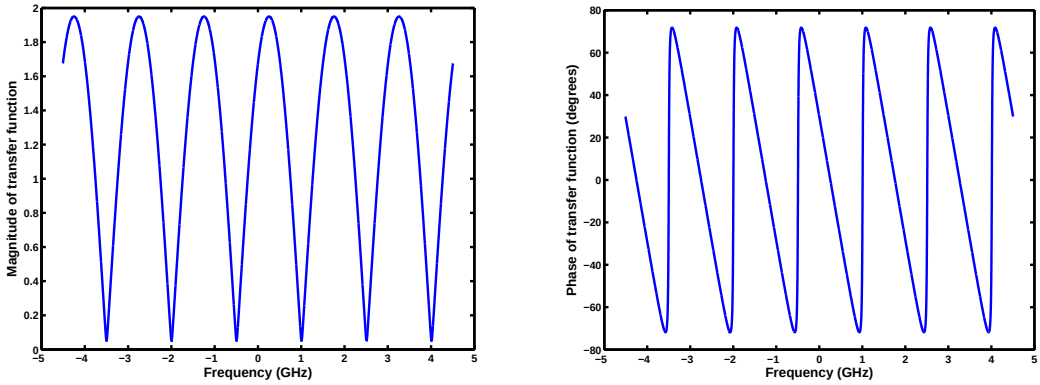
$$\Delta L = \sqrt{R^2 + (h_t + h_r)^2} - \sqrt{R^2 + (h_r - h_t)^2} = R\left(1 + \frac{(h_r + h_t)^2}{R^2}\right)^{\frac{1}{2}} - R\left(1 + \frac{(h_r - h_r)^2}{R^2}\right)^{\frac{1}{2}}$$

Using the approximation $(1 + x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x$, we obtain

$$\Delta L \approx R\left(1 + \frac{(h_r + h_t)^2}{2R^2}\right) - R\left(1 + \frac{(h_r - h_t)^2}{2R^2}\right) = R\left(\frac{(h_r + h_t)^2 - (h_r - h_t)^2}{2R^2}\right) = \frac{2h_t h_r}{R}$$

The corresponding delay spread is given by $\tau_d = \frac{\Delta L}{c} \approx \frac{2h_t h_r}{Rc}$.
(b) For $R = 200$m, $h_t = 2$m, $h_r = 10$m, we obtain a delay spread of 0.666 nanoseconds through

exact computation, and 0.667 nanoseconds through the approximation in (a). Thus, the approximation is very accurate when $R \gg h_t, h_r$.
(c) The coherence bandwidth is $1/\tau_d$, which evaluates to about 1.5 GHz.
(d) For $R = 10$m, $h_t = h_r = 2$m, modeling an indoor link, we obtain a delay spread of 2.57 nanoseconds (corresponding to coherence bandwidth 389 MHz) through exact computation, and 2.67 nanoseconds (corresponding to coherence bandwidth 375 MHz) using the approximation. The approximation is not as good as for the larger range, but is still effective in giving a quick estimate.




(a) Transfer Function Magnitude (b) Transfer Function Phase

Figure 23: Both magnitude and phase vary across frequency, with small magnitudes corresponding to large phase variations.

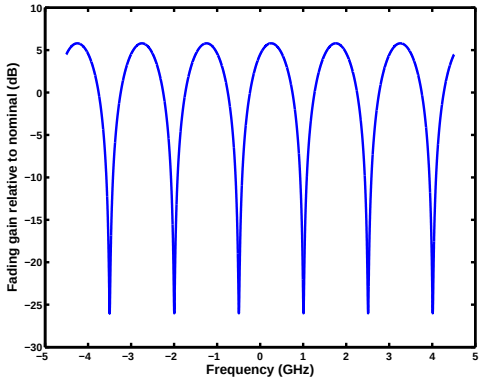



Figure 24: Fading with respect to nominal single path channel.

**Problem 2.22**(a) Normalizing the LOS path to unit magnitude and phase, the channel impulse response is given by

$$h(t) = 1 + A_2 e^{-j(2\pi f_c \tau_d + \phi_2)} \delta(t - \tau_d)$$

We can now plug in $A_2 = 0.95$, $\phi_2 = \pi$, and, $\tau_d \approx 2/3$ nanoseconds and the coherence bandwidth is 1.5 GHz (from Problem 2.21(a)).
(b) The magnitude and phase of the channel transfer function are plotted in Figure 23.
(c) The fading gain in dB relative to a nominal LOS channel is plotted in Figure 24. The fading depth is about 25 dB.
(d) The averaged gain is plotted in Figure 25. We see that it stabilizes at a little less than one coherence bandwidth, which means that about 1-1.5 GHz suffices to average out frequency-selective fading. Note that we can average out fading relatively easily because there are only two

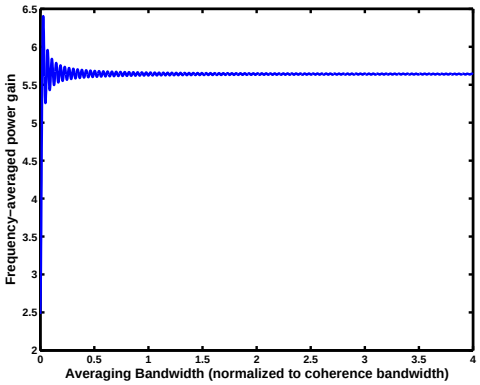



Figure 25: Fading with respect to nominal single path channel.

paths; when there are many paths, it may take a larger multiple of the coherence bandwidth to provide effective averaging.