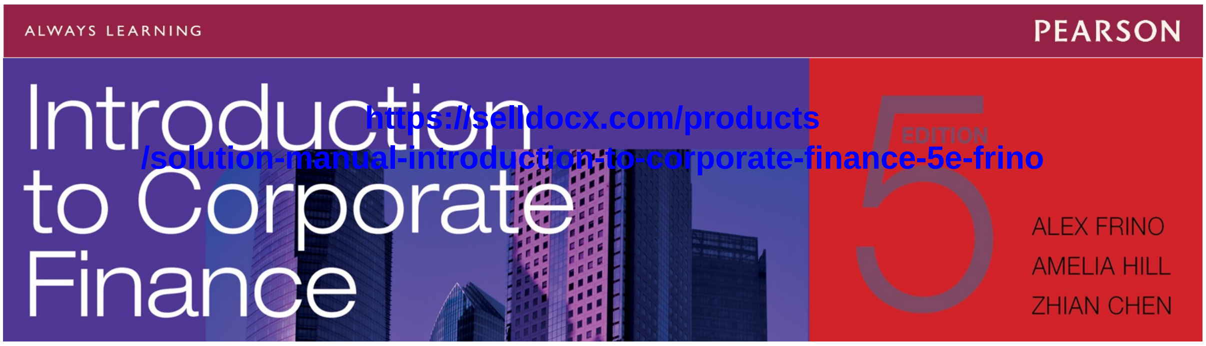


Chapter 2

A review of financial mathematics

Concepts overview

The **important concepts** discussed in Chapter 2 are:

2.1 and 2.2

- The simple, compound and continuously compounded interest rates. These forms of interest calculation vary depending on how the interest is calculated and added to the principal. With simple interest, the interest is calculated based on the principal for one year. The total interest is the year's interest times the number of years. With simple interest, the amount of interest per annum is constant regardless of payments made. Compound interest adds the interest each compounding period. Subsequent interest calculations calculate interest on the principal outstanding and the interest paid or received in prior periods. Thus, the interest on interest compounds over the period of the loan/investment. Continuous compounding is the theoretical case where interest is calculated at every single point in time and added to the balance. Another way of thinking about this is that the compounding period is infinitely small.

2.3

- Future values and present values of single amounts under each interest rate arrangement. The future value is the sum of all individual cash flows calculated forward to a terminal date. In the case of simple interest, this will be the principal at the beginning plus the annual interest times the number of years. With compound interest the future value is the principal plus interest compounded over the period of the loan. The present value is the corollary of the future value. It is the amount of money that needs to be invested at the beginning of the period to be able to have a given cash flow, or set of cash flows, in the future. Both the present and future values embody the concept of the time value of money.

2.4

- Future values and present values of annuities. An annuity is a stream of cash flows where the amounts received are equal for a defined period. The future value of an annuity is the stream of cash flows compounded forward to the end of the defined period. The present value is the amount required now at a given interest rate to be able to replicate that stream of cash flows.

2.5

- Annuities and perpetuities: An ordinary annuity is an annuity where payments occur at the end of each period of the annuity, and hence the first cash flow is at the end of the first period, or one period after the beginning of the annuity period; e.g. interest paid or charged by banks. An annuity due is a special annuity where cash flows occur at the beginning of each period, and hence the first cash flow occurs now rather than in one period's time; e.g. rental and lease agreements are usually annuities due. A deferred annuity is an ordinary annuity except that the first payment is deferred for one or more periods, and hence the first payment is more than one period in the future. A perpetuity is an annuity that continues indefinitely; i.e. a *perpetual annuity.*

- Equivalent annuities: An equivalent annuity requires the present value of a series of cash flows to be converted to an annuity over the same time period. This is an annuity that has the same present value as the original series of cash flows; i.e. the present value of an equivalent annuity is equal to the present value of another series of cash flows.

Suggested answers to concept questions

(p.28)

1. *What are the aims of financial mathematics?*

ANSWER

The broad aim of financial mathematics is to convert single or multiple cash flows, that will be received at different points in time, to one number. This number is the value of all of an asset's cash flows, at a given point in time. Its second aim is to provide the basis for a financially rational choice between different assets. This is because the cash flows of all assets under consideration are stated in 'like' terms (i.e. at the same point in time). Third, this number determines the maximum price that an investor should be willing to pay for an asset. That is, it represents the 'intrinsic value' of an asset.

2. *Would a rational person prefer to receive $100 in one year's time or $100 in five years' time?*

ANSWER

A rational person would prefer $100 in one year's time to $100 in five years' time. This is because money has a time value. That is, the $100 received in one year could be invested at a positive interest rate and thus return an amount greater than $100 four years later. This is the case in developed countries and represents the compensation that is paid for an individual to defer consumption now to a later period. Even in the absence of inflation, there is normally a positive interest rate that compensates for deferring consumption.

(p.37)

3. *Name the three ways of quoting an interest rate.*

ANSWER

1. Simple interest – where interest is earned or paid on the basis of an initial amount invested or borrowed, called the principal. This results in the dollar amount of interest earned or paid being the same each period.

2. Nominal compounding interest – where at the end of each compounding period, the amount of interest earned or accrued is calculated and added to the balance of the principal. When nominal interest rates are quoted, it is normal to include the number of compounding periods per annum; for example 12% p.a. compounding quarterly.

3. Effective interest (per annum) – this is the actual rate of interest paid by the borrower or earned by the lender. It is the interest rate which compounded annually is equivalent to a given nominal compounding interest rate.

4. *What is the difference between a compounding period interest rate and a nominal interest rate?*

ANSWER

The nominal interest rate is the contracted or stated interest rate, which is quoted on a per annum basis and ignores the effect of compounding. For example, a nominal rate of 10% p.a. that is compounding semi-annually and a nominal rate of 10% p.a. compounding weekly have the same nominal rate (10%), but the interest that is effectively paid or received will be different.

The compounding period interest rate is the interest rate applied to a compounding period that is less than one year and requires the nominal quoted rate to be divided by the number of compounding periods within one year. For example the nominal quoted rate of 12% p.a. applied on a quarterly basis (compounding quarterly) is equal to 12%/4 or 3% per quarter.

5. *What is an effective interest rate?*

ANSWER

The effective interest rate is generally expressed as an annual rate. It is the effective rate of interest that is being paid or received, once the effect of compounding is taken into account. For example, a nominal rate of 10% p.a. compounding weekly represents a higher effective rate than 10% p.a. compounding semi-annually.

6. *When compounding occurs more than once per year, will the annual effective interest rate be higher or lower than the nominal rate?*

ANSWER

The effective interest rate is always higher than the quoted nominal rate when compounding occurs more frequently than once a year.

(p.41)

7. How would you define present value?

ANSWER

Present value is an amount applicable to today that is equivalent to a single cash flow or a series of cash flows to be paid or received in the future. For example, it is the amount of money invested now at a given interest rate that would enable you to withdraw funds of the same value and timing as the future set of cash flows.

8. How are present and future values dependent on interest rates?

ANSWER

Interest rates quantify the relationship between present and future value. They specify the rate at which present values grow to become future values, or the rate at which future values must be discounted to find present values. The higher the interest rate, the greater the difference between a given present and future value.

9. What are the two ways of estimating the present value of a cash flow stream that contains multiple cash flows of unequal value?

ANSWER

The first method is to apply the general present value formula to each individual cash flow (Equation 2.8), and then summing the present values. Alternatively, the future value of each of the cash flows can be calculated (Equation 2.9) and summed, and then this single value can be discounted back to the present (Equation 2.7).

10. How would you estimate the future value of a cash flow stream?

ANSWER

The future value is the sum of the individual cash flows compounded forward to a common point in time. The general formula for the future value of a multiple flow cash flow steam is:

$$FV = \sum_{t=1}^{n} X_t (1+r)^t \qquad \text{(Equation 2.9)}$$

where:
FV = the future value of a multiple stream of cash flows
X_t = cash flow received in period t.
r = the compound interest rate on an alternative comparable investment

t = the number of periods before X_t is received

(p.53)

11. What is an annuity?

ANSWER

An annuity is a series of cash flows of equal size that occur at regular time intervals extending into the future.

12. Describe the three basic types of annuity.

ANSWER

1. An ordinary annuity is one where the cash flows are of equal amounts for a defined period with the first cash flow occurring at the end of each period – i.e. the first cash flow is one period in the future.

2. An annuity due is similar to an ordinary annuity except that cash flows occur at the beginning of each period – i.e. the first cash flow occurs immediately.

3. A deferred annuity is an ordinary annuity where the first cash flow has been deferred into the future – i.e. **more than** one period from now.

13. Explain the two ways used to calculate the present value of an ordinary annuity.

ANSWER

The first method is to apply the annuity formula (Equation 2.11). Alternatively, a table that describes the present value of $1 per year for a variety of different years and interest rates, such as that contained in Appendix 2.6, may be used.

14. What is an equivalent annuity?

ANSWER

The formula for the present value of an annuity can be used to convert a cash flow or a series of cash flows into an equivalent annuity. Firstly, the present value of an original series of cash flows is calculated. These are usually of unequal amounts, in which case the present value of each individual cash flow is calculated, and then these present values are summed. This present value is then converted into a series of cash flows of equal amounts – i.e. an equivalent annuity that has the same present value as the original series of cash flows.

15. *What is the value of a perpetuity?*

ANSWER

A perpetuity is a special and fairly common type of annuity. It is a perpetual annuity; i.e., an annuity that continues indefinitely. An example of a perpetuity is a share paying a constant dividend. The formula for a perpetuity is given by Equation 2.14:

$$PV = \frac{A}{r}$$

where:
A = the constant cash flow
r = the interest rate

Appendix 2.4 derives the expression for the present value of a perpetuity.

Discussion questions

1. *What is meant by the 'intrinsic value' of an asset? What action would you take if you observed an asset that was selling at less than its intrinsic value?*

ANSWER

Intrinsic value is the true value of an asset based on the cash flows that will accrue to the holder of the asset. The intrinsic value may also be regarded as the amount someone would pay for the asset now based on the expected income stream in the future. It is generally the present value of future cash flows. Another way of looking at it is that it is the amount you would need to invest now to receive the same income stream at a give interest rate.

2. *Can an individual be indifferent to receiving a dollar now or a dollar one year from now? Explain.*

ANSWER

This is only possible if there is a zero interest rate in the market, which is extremely rare. For all positive interest rates, an individual could invest the $1.00 and receive more than a dollar in a year's time. For example, if the existing interest rate were 10%, a dollar invested now would be worth $1.10 in one year's time. An individual, preferring more to less, would prefer the dollar now so that they could have $1.10 in one year's time. If the interest rate were zero percent, one would in theory be indifferent to having the dollar now, as opposed to a dollar in a year's time. While we do not in general expect interest rates of zero percent to prevail, the inter-bank lending rate in Japan was at zero percent for much of 2000 to 2003. This in part explains the Japanese banks' lack of willingness to deal with non-performing loans. They felt that at zero percent interest they were willing to wait several years for the loans to become performing loans.

3. *Consider Figure 2.1 in the chapter. If you did not wish to spend the extra \$100 until year 5, does this mean that the asset paying \$100 in year 5 is preferred to the asset that pays \$100 in year 3? Why or why not?*

ANSWER

The answer is no. Once again the time value of money is at the heart of our rational choice. The \$100 received in year three can be invested at the current rate of interest to return an amount greater than \$100 in year five. Specifically, if we assume an interest rate of 10%, you could reinvest the \$100 received at the end of year 3 and accumulate an amount of \$121 by year 5 (\$100 invested for one year would yield \$100 plus \$10 interest and this \$110 invested a further year would yield \$110 plus interest of \$11 giving an accumulated amount of \$121).

4. *If you were given the choice of borrowing at an interest rate of 10% p.a. simple interest, or 9.2% p.a. compounded monthly, which should you choose? Why?*

ANSWER

If the nominal rate is held constant, a borrower would prefer less compounding periods per annum to more because we are paying interest on interest as a result of compounding. However, in this case the nominal rates are different and therefore it is necessary to calculate the effective rate of interest in order to make a comparison. We apply Equation 2.3 to determine the effective annual rate of interest for a loan charging 9.2% p.a. interest compounded quarterly.

$$\begin{aligned}\text{Effective Rate} &= \left(1+\frac{r_{nom}}{m}\right)^m - 1\\ &= \left(1+\frac{0.092}{12}\right)^{12} - 1\\ &= 9.6\%\end{aligned}$$

The effective rate of 9.6% is the rate of interest, expressed on an annual basis, when applied to the amount borrowed that gives the same interest as the 9.2% applied to the balance owing on a monthly basis. As the annual effective rate of 9.6% is less than the 10% simple interest option, a rational borrower would prefer the 9.2% rate compounded monthly.

5. *Government pensions in Australia are typically paid fortnightly to those who are of a retirement age (i.e. aged 65 or over). While you can value these pension payments by treating them as annuities, what assumptions are you implicitly making? What difficulties are you expected to encounter when you attempt to calculate a present value?*

ANSWER

One of the most important inputs into the valuation of an annuity is the length of the annuity – i.e. how long the recipient of the pension will live. Clearly this will vary greatly from recipient to recipient. One approach could be to estimate the average life span of recipients, based on actuarial data, and use this to determine the length of the annuity. This could be used to estimate the ***expected*** present value of the annuity, but applying this valuation to any particular pension is problematic.

A second input is the discount rate. This could be based on current interest rates, but these are likely to change over the life of the recipient. One would need to estimate the expected interest rates over the period of the annuity, but interest rate forecasts more than a few years in the future are very difficult to estimate with accuracy.

Thirdly, the size of the payment is likely to vary over time, as governments vary pension payments as a result of inflation, budgetary constraints and political considerations. The size of the payments over the life of the recipient is difficult to predict.

Practical questions

1. *Find the simple interest earned on $900 at 4.5% from April 1 to May 16 (both inclusive).*

ANSWER

The interest rate of 4.5% is an annual rate. Before we can find the formula, we need to determine the interest rate applicable to this period of time (46 days). The period of time is 46/365 of one year.

$$
\begin{aligned}
FV &= PV \times r \times dtm/365 \\
&= 900 \times 0.045 \times 46/365 \\
&= \$5.10
\end{aligned}
$$

2. *Would you rather have a savings account the pays 6% interest compounded semi-annually or one that pays 6% compounded monthly? Why?*

ANSWER

All other things being equal, an investor would prefer more compounding period per annum to less, because interest is being paid on previously earned interest more frequently and the effective interest rate will be greater. The savings account paying 6% interest compounded monthly would generate a greater return.

3. *You're a financial advisor working for the Golden Eggs Financial Advisory Limited. You come across a particular security that pays $10,000 in four years' time. Currently, an investment opportunity of similar risk is available to you, paying an interest rate of 6% p.a. compounded annually. What is the fair value (present value) of this security?*

ANSWER

$$PV = \frac{FV}{(1+r)^n} = \frac{10,000}{1.06^4} = \$7,920.94$$

4. *What is the present value of $121 received in two years if the nominal interest rates were:*

(a) 6% with annual compounding?
(b) 6% with semi-annual compounding?
(c) 6% with monthly compounding?

ANSWER

(a) The length of each compounding period is one year, the interest rate per period is 6% and the number of periods is 2.

$$PV = \frac{FV}{(1+r)^n} = \frac{121}{1.06^n} = \$107.69$$

(b) The length of each compounding period is six months, the interest rate per period is 3% and the number of periods in 2 years is 4.

$$PV = \frac{FV}{(1+r)^n} = \frac{121}{1.03^4} = \$107.51$$

(c) The length of each compounding period is one month, the interest rate per period is 0.5% and the number of compounding periods in 2 years is 24.

$$PV = \frac{FV}{(1+r)^n} = \frac{121}{1.005^{24}} = \$107.35$$

5. *How long will it take for $100 to accumulate to $200 at 8% p.a. compound?*

ANSWER

We use the formula that relates PV to FV, and solve for *n*.

$$FV = PV(1+r)^n$$
$$\therefore (1+r)^n = \frac{FV}{PV}$$
$$\therefore \ln(1+r)^n = \ln\frac{FV}{PV}$$
$$\therefore n\ln(1+r) = \ln\frac{FV}{PV}$$
$$\therefore n = \frac{\ln(FV/PV)}{\ln(1+r)} = \frac{\ln(200/100)}{\ln(1.08)} = 9 \text{ years}$$

6. *An investment costs $150 and pays $200 in two years. What is the effective annual compound interest rate? What is the nominal rate if the investment is compounded monthly?*

ANSWER

We use the formula that relates PV to FV, and solve for *r*.

$$FV = PV(1+r)^n$$
$$\therefore (1+r)^n = \frac{FV}{PV}$$
$$\therefore (1+r) = \sqrt[n]{\frac{FV}{PV}}$$
$$\therefore r = \sqrt[n]{\frac{FV}{PV}} - 1 = \sqrt{\frac{200}{150} - 1} = 15.47\%$$

This gives us the effective annual interest rate. To find the nominal rate, if this represents monthly compounding, we use the formula for effective interest rate and solve for r_{nom}.

$$\text{Effective Rate} = \left(1 + \frac{r_{nom}}{m}\right)^m - 1$$
$$\therefore 0.1547 = \left(1 + \frac{r_{nom}}{12}\right)^{12} - 1$$
$$\therefore 1.1547 = \left(1 + \frac{r_{nom}}{12}\right)^{12}$$
$$\therefore \left(1 + \frac{r_{nom}}{12}\right) = \sqrt[12]{1.1547} = 1.0121$$
$$\therefore \frac{r_{nom}}{12} = 1.012 - 1 = 0.0121$$
$$\therefore r_{nom} = 0.0121 \times 12 = 14.52\%$$

7. *'Mad Dog' McNamara wishes to accumulate $9,500 at the end of three years. How much does he need to deposit now if the interest rate is*

(a) 10% per annum compounded annually?
(b) 8% per annum compounded annually?
(c) 8% per annum compounded monthly?

Can you explain why the amounts differ in each case?

ANSWER

We need to find the present value of the future amount of $9,500.

(a) $$PV = \frac{FV}{(1+r)^n} = \frac{9500}{1.10^3} = \$7,137.49$$

(b) $$PV = \frac{FV}{(1+r)^n} = \frac{9500}{1.08^3} = \$7,541.41$$

(c) With monthly compounding, the interest rate must be divided by 12 to find the interest rate per compounding period (*r*), and the number of years must be multiplied by 12 to find the total number of periods (*n*).

$$PV = \frac{FV}{(1+r)^n} = \frac{9500}{1.0067^{36}} = \$7,470.01$$

The answer to (b) is greater than the answer to (a) because the interest rate is lower. The deposit will not earn as much interest under the scenario in part (b), so more needs to be deposited in order to grow to the desired amount.

The answer to (c) is less than the answer to (b) because interest is compounded more frequently. This means that the deposit earns 'interest on interest' more often. More frequently throughout the year, interest is calculated and added to the deposit, which means that next time interest is calculated it is calculated on a larger amount. As a result, less needs to be deposited in order to grow to the desired amount.

8. *What is the present value of a cash flow of $10,000 to be received in five years given an interest rate of 6% p.a. over five years (a) compounded annually, (b) compounded monthly, and (c) continuously compounded?*

ANSWER

(a) $$PV = \frac{FV}{(1+r)^n} = \frac{10000}{1.06^5} = \$7,472.58$$

For part (b) we have to divide the nominal annual interest rate by *m*, number of compounding periods in a year, to find the interest rate per compounding period (*r*), and we have to multiply the number of years by *m* to find the total number of periods.

(b) $$PV = \frac{FV}{(1+r)^n} = \frac{10{,}000}{1.005^{60}} = \$7{,}413.72$$

The present value is smaller with monthly compounding. This is because the initial amount would grow more quickly over the period of an investment because interest on interest is calculated and paid more frequently.

To find the present value using continuous compounding, we use Equation 2.6.

(c) $$FV = PVe^{rt} \therefore PV = \frac{FV}{e^{rt}} = \frac{10{,}000}{e^{(0.06)(5)}} = \$7{,}408.18$$

Clearly compounding occurs even more frequently under continuous compounding, and therefore the present value is even smaller than it was with monthly compounding.

9. *What is the present value of $50,000 due in 15 years if the interest rate is 14% p.a. compounded semi-annually?*

ANSWER

$$PV = \frac{FV}{(1+r)^n} = \frac{50{,}000}{1.07^{30}} = \$6{,}568.36$$

10. *An antique vase your grandfather bought 25 years ago for $5,000 has now appreciated in value to $16,000. Do you consider this to be a large gain? What if you were told that the average interest rate over the 25 years was 5% p.a.?*

ANSWER

In dollar terms, the gain of $11,000 would appear at first glance to be large. However, the future value of an investment of $5,000 at 5% p.a. for 25 years would be:

$$FV = PV(1+r)^n = 5000(1+.05)^{25} = \$16{,}931.77$$

Compared to investing in an interest-earning bank account, the increase in value of the vase is not particularly large.

11. *The price of a Beatles recording during their historic first visit to Australia has tripled in value in the past 40 years. Since you could have earned 3% per annum in a savings account over the same period, was your investment in the recording (a) a terrific success, (b) okay, or (c) a disaster? At what interest rate would you be indifferent to these investments?*

ANSWER

If I paid say $10 for the Beatles record, it would be worth $30 today (having tripled in value). If the best alternative investment was a savings account at 3%, the future value of my investment would be:

$$FV = PV(1+r)^n = 10(1+.03)^{40} = 32.62$$

Buying the record was not a good financial investment.

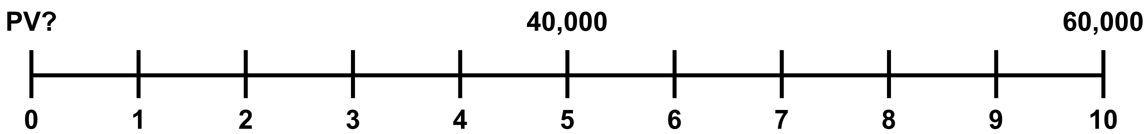
To find the interest rate at which I would be indifferent:

$$FV = PV(1+r)^n \therefore r = \left(\frac{FV}{PV}\right)^{\frac{1}{n}} - 1 = \left(\frac{30}{10}\right)^{\frac{1}{40}} - 1 = 2.78\%$$

12. *An investment repays $40,000 in 5 years and a further $60,000 in 10 years. If the interest rate over the period of the investment is 12% p.a. compounded monthly, what is the investment's present value?*

ANSWER

Whenever you have a problem involving multiple cash flows, it is often advisable to draw a diagram showing all cash flows and identifying the value to be calculated.



There are three steps required to solve this problem. First, you must find the interest rate per compounding period. Second, the cash flows of $40,000 and $60,000 are discounted using the monthly rate of interest over 60 months (5 years × 12 compounding periods per year) and 120 months (10 years × 12 compounding periods per year). In the third step the two discounted values are added together

Monthly discounting rate = $\frac{0.12}{12} = 0.01 \text{ or } 1\% \text{ per month}$

Present value of the cash flows $PV = \frac{40{,}000}{(1+.01)^{60}} + \frac{60{,}000}{(1+.01)^{120}} = \$40{,}197.67$

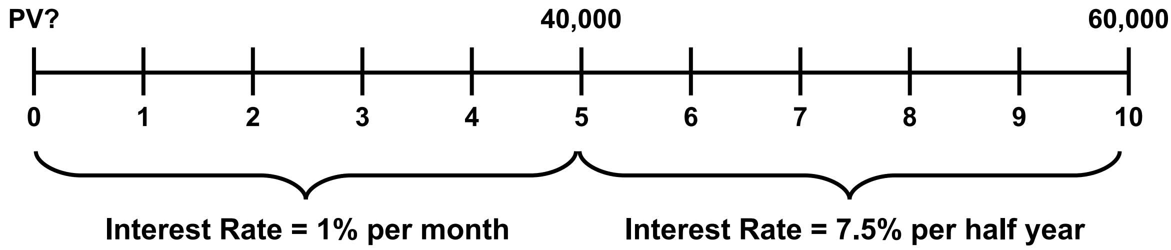
13. *If, in Question 12, the interest rate for the first five years had been 12% p.a. compounded monthly but had increased to 15% p.a. compounding semi-annually, what would the investment's present value be?*

ANSWER

The second cash flow of $60,000 must be discounted to Year 5 using the interest rate of 15% p.a. compounding semi-annually, and then both cash flows must be discounted from Year 5 to Year 0 using the interest rate of 12% p.a. compounding monthly.

Interest rate per compounding period (Years 6 - 10) =

$$\frac{0.15}{2} = 0.075 \;=\; 7.5\% \text{ per half year}$$



The present value of $60,000 discounted from Year 10 to Year 5 is given by:

$$PV_5 = \frac{FV}{(1+r)^n} = \frac{60{,}000}{(1+.075)^{10}} = \$29{,}111.63$$

Since $60,000 in 10 years is equivalent to $29,111.63 in 5 years, the problem becomes the present value of $40,000 + $29,111.63 to be received in 5 years.

$$PV_0 = \frac{FV}{(1+r)^n} = \frac{69{,}111.63}{(1+.01)^{60}} = \$38{,}042.47$$

14. *An investor is earning $15,000 this year but expects to be earning $60,000 next year. What is the maximum amount that the investor can consume today if the interest rate is 10%? If the investor decides to consume zero this year, how much could she consume next year?*

ANSWER

The investor can borrow against the future income of $60,000. At a borrowing rate of 10%, the amount that could be borrowed would be the present value of the future sum of $60,000, discounted at an interest rate of 10%.

$$PV = \frac{FV}{(1+r)^n} = \frac{60{,}000}{(1+.10)^1} = \$54{,}545.45$$

Thus the investor could consume $54,545 now plus the $15,000 income from this year, giving a total consumption of $69,545.

If the investor chose to consume nothing this year, then the future consumption would be the future value of $15,000 at 10% interest plus next year's salary of $60,000. Total consumption in one year's time would be:

$$FV = PV_0(1+r)^n + PV_1 = 15{,}000(1+.10)^1 + 60{,}000 = 16{,}500 + 60{,}000 = \$76{,}500$$

15. *You wish to place $1,000 in a bank account for five years, at an interest rate of 7% p.a. How much would you accumulate after five years assuming (a) simple interest, and (b) compound interest? Explain the difference between (a) and (b).*

ANSWER

(a) $FV = PV(1+r) = 1{,}000 \times (1 + 0.07 \times 5) = \$1{,}350$

(b) $FV = PV(1+r)^n = 1{,}000(1+0.07)^5 = \$1{,}402.55$

The difference between the future value under simple interest ($1,350) and the future value under compound interest of ($1,402.55) is the fact that under compound interest, interest is paid on interest paid in prior periods.

16. *You have $10,000 to invest for one year and the following choices are offered by the banks in your area:*

(a) 6% p.a. compounded annually,
(b) 5.8% p.a. compounded quarterly, or
(c) 5.65% p.a. continuously compounded.

Which of the alternatives would you choose? Why?

ANSWER

We need to convert these rates to effective annual interest rates so that we have a common basis for comparison.

(a) The effective annual rate equals the nominal rate for annual compounding: 6%.

(b) Effective Rate $= \left(1 + \frac{r_{nom}}{m}\right)^m - 1 = \left(1 + \frac{0.058}{4}\right)^4 - 1 = 5.93\%$

(c) The formula for future value under continuous compounding is $FV = PVe^{rt}$. Hence, the future value of $1 under continuous compounding for 1 year is e^{r}. Since the rate of return over an investment period is given by:

Rate of return $= \frac{FV - PV}{PV}$,

the rate of return on $1 invested for 1 year at 5.65% compounded continuously is:

Rate of return $= \frac{FV - PV}{PV} = \frac{e^{r} - 1}{1} = e^{0.0565} - 1 = 5.81\%$

The account paying 6% p.a. compounded annually offers the highest effective interest rate.

17. *You have fallen seriously ill, and would not be able to work for the next two years. Fortunately, your insurance company has agreed to pay you $2,000 a month for the next two years, with the first payment in one month's time. If the current interest rate is 6.17% p.a., compounding annually, what is the present value of this stream of cash flows (to the nearest dollar)?*

ANSWER

In order to calculate the present value of this annuity, we need to determine the relevant interest rate per compounding period – in this case, per month (*r*). Given that the effective interest rate is the annually compounding rate given, we can do this by rearranging the formula for the effective annual rate and solving for *r*:

$$\text{Effective rate} = \left(1 + \frac{r_{nom}}{m}\right)^{m} - 1 = \left(1 + r\right)^{m} - 1$$

$$\therefore r = \sqrt[m]{1 + \text{Effective rate}} - 1 = \sqrt[12]{1.0617} - 1 = 0.005 = 0.5\%$$

The present value of the annuity is therefore:

$$PV = A\left[\frac{1 - \left(1 + r\right)^{-n}}{r}\right] = 2000\left[\frac{1 - 1.005^{-24}}{0.005}\right] = \$45{,}126$$

18. *Your grandmother has a debt that she may repay by paying $5,000 now or $10,000 in four years' time. If the interest rate is 14% p.a. compounded monthly, would you advise her to repay the debt now or in four years?*

ANSWER

We need to compare the present value of $10,000 to be paid in 4 years time with $5,000 now. The interest rate per compounding period is 0.14/12 = 0.0117, and the number of compounding periods in four years is 4 $\times$ 12 = 48.

$$PV = \frac{PV}{(1+r)^n} = \frac{10,000}{1.0117^{48}} = \$5,721.58$$

It would be better to pay $5,000 now rather than $10,000 in 4 years, because paying $10,000 in 4 years is equivalent to paying $5,730.64 now.

19. *If you borrowed $10,000 from a bank for five years at an interest rate of 10% p.a. compounded quarterly, how much would you owe the bank when the loan matures?*

ANSWER

The interest rate per compounding period is 0.10/4 = 0.025, and the number of compounding periods in five years is 5 × 4 = 20.

$$FV = PV(1+r)^n = 10{,}000(1.025)^{20} = \$16{,}386.16$$

20. *What is the continuously compounded return equivalent of a return of 10% p.a. compounded annually? Explain the difference between the two.*

ANSWER

The future value of $1 at 10% p.a. compounded annually for one year is given by:

$$FV = PV(1+r)^n = (1.10)^1 = \$1.10$$

The future value of $1 compounded continuously for one year is given by:

$$FV = PVe^{rt} = e^r$$

Hence the continuously compounded return equivalent to 10% p.a. compounded annually is given by:

$$e^r = 1.10 \therefore r = \ln(1.10) = 9.53\%$$

21. *If you purchased NAB shares for $28.95 and later sold them for $29.60, what continuously compounded rate of return have you realised?*

ANSWER

$$FV = PVe^{rt}$$

$$\therefore 29.60 = 28.95e^{r \times 1}$$

$$\therefore e^r = \frac{29.60}{28.95}$$

$$\therefore r = \ln\left(\frac{29.60}{28.95}\right) = 2.22\%$$

22. *An investor can buy a government bond for $45,000 that will pay $50,000 in a year.*

(a) What is the rate of return on this bond?

(b) If the interest rate is 9% per annum compounded annually, what is the present value of the bond?

(c) What is the net present value of the bond?

(d) Should the investor buy the bond?

(e) If the market interest rate rises to 12% p.a., should the investor buy the bond?

ANSWER

(a) Rate of return $= \frac{FV - PV}{PV} = \frac{50{,}000 - 45{,}000}{45{,}000} = 11.1\%$

(b) $PV = \frac{FV}{(1+r)^n} = \frac{50{,}000}{1.09} = \$45{,}871.56$

(c) The net present value is the present value of future cash flows minus the initial investment.

$$NVP = PV - \text{Investment} = 45{,}871.56 - 45{,}000 = \$871.56$$

(d) Yes, at an interest rate of 9% the NPV is positive. The investor should buy the bond.

(e) $NPV = \frac{FV}{(1+r)^n} - \text{Investment} = \frac{50{,}000}{1.12} - 45{,}000 = -\$357.14$

At an interest rate of 12% the NPV is negative. The investor would lose $357 in present value terms by investing the bond, and therefore should not buy the bond.

23. *Consider Figure 2.2 in the chapter. Given the cash flow structure of the asset – that is, $2.3 million per year from years 1 to 6 – answer the following, assuming an interest rate of 10% p.a. compounded annually.*

(a) What is the accumulated (future) value of this asset?

(b) What is the present value of this asset?

Now assume that the interest rate of 10% p.a. is compounded monthly.

(c) What is the accumulated (future) value of this asset?

(d) *What is the present value of this asset?*

ANSWER

(a) $$PV = A\left[\frac{(1+r)^n - 1}{r}\right] = 2,300,000\left[\frac{1.10^6 - 1}{0.10}\right] = \$17,745,903$$

(b) $$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] = 2,300,000\left[\frac{1-1.10^{-6}}{0.10}\right] = \$10,017,100$$

For parts (c) and (d), we need to convert the nominal rate of 10% p.a. compounding monthly to an effective annual interest rate in order to apply it to annual cash flows.

Effective rate $= \left(1+\frac{r_{nom}}{m}\right)^m - 1 = \left(1+\frac{0.10}{12}\right)^{12} - 1 = 10.47\%$

(c) $$PV = A\left[\frac{(1+r)^n - 1}{r}\right] = 2,300,000\left[\frac{1.1047^6 - 1}{0.1047}\right] = \$17,957,690$$

(d) $$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] = 2,300,000\left[\frac{1-1.1047^{-6}}{0.1047}\right] = \$9.880,623$$

24. *A four-year security is selling today for $25,000. The applicable interest rate is 8% p.a. and the security offers an equal annual cash flow at the end of each year. What cash flow would the purchaser of this security receive each year?*

ANSWER

$$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right]$$

$$\therefore A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{25,000}{\left[\frac{1-1.08^{-4}}{0.08}\right]} = \$7,548.02$$

25. *You wish to borrow $20,000 for five years, and your bank will charge interest at 10% p.a. compounded annually.*

(a) *If you were to repay the loan in equal annual instalments, what annual payment would you be making for the next five years?*

(b) *If you were to repay the loan in equal monthly instalments, what monthly payment would you be making for the next five years?*

ANSWER

(a) $$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right]$$

$$\therefore\ A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{20{,}000}{\left[\frac{1-1.10^{-5}}{0.10}\right]} = \$5{,}275.95$$

(b) The interest rate per compounding period is 0.10/12 = 0.0083% and the number of compounding periods in five years is 5 × 12 = 60.

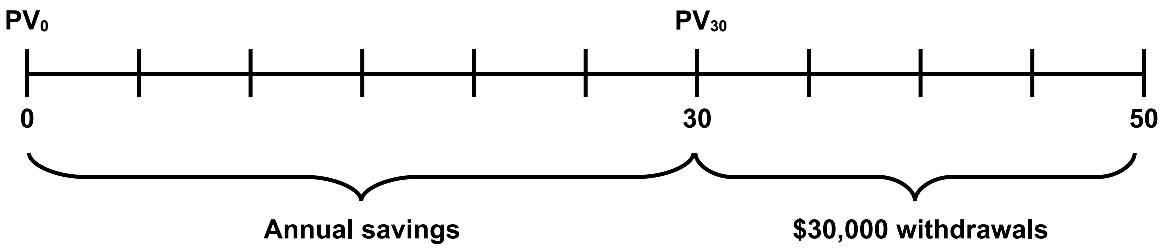
$$A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{20{,}000}{\left[\frac{1-1.0083^{-60}}{0.0083}\right]} = \$424.55$$

26. *Your sister has just graduated from university and has begun employment with an investment bank. She intends to retire in 30 years from now and would like to be able to withdraw $30,000 per year from her savings for a period of 20 years after retirement. She expects to earn 9% annually on her savings.*

Assuming end-of-year cash flows, what equal annual amount must your sister save during her 30 years of employment in order to be able to withdraw the desired annual amount during the 20 years of retirement?

ANSWER

There are 2 steps involved in solving this problem. We need to find the present value (as at year 30) of the 20 years of withdrawals from year 30 to year 50. We then treat this amount as the future value of 30 years of saving to find the amount that must be saved each year to accumulate this amount.



The present value of the 20 withdrawals of $30,000 is given by:

$$PV_{30} = A\left[\frac{1-(1+r)^{-n}}{r}\right] = 30000\left[\frac{1-1.09^{-20}}{0.09}\right] = \$273{,}856.37$$

If we set this amount equal to the future value of 30 years of saving, the annual amount to be saved is given by:

$$FV = A\left[\frac{(1+r)^n - 1}{r}\right] \therefore A = \frac{FV}{\left[\frac{(1+r)^n - 1}{r}\right]} = \frac{273856.37}{\left[\frac{1.09^{30} - 1}{0.09}\right]} = \$2,009.11$$

27. (a) *You have the opportunity to purchase security Y that will pay $2,000 per year forever. At an interest rate of 8% p.a., what is this security worth?*

(a) *An alternative security, Z, will pay $2,000 per year for the next 20 years. Assuming the same interest rate compounded annually, what is this security worth?*

(b) *Explain the difference between the values of securities Y and Z.*

ANSWER

(a) $$PV_Y = \frac{A}{r} = \frac{2,000}{0.08} = \$25,000$$

(b) $$PV_Z = A\left[\frac{1-(1+r)^{-n}}{r}\right] = 2,000\left[\frac{1-1.08^{-20}}{0.08}\right] = \$19,636.29$$

(b) Y is more valuable than Z because it has more cash flows. The cash flows continue forever, rather than just 20 years. However, it may seem surprising that an infinite stream of cash flows beyond 20 years adds very little to the value of the security – just over $5,000. This is because cash flows in the distant future are discounted very heavily, and are worth very little in present value terms.

28. *Your company is deciding between two investment choices: one that pays $100 per year in perpetuity, and another that pays $100 per year for 100 years. The current market interest rate for investments of similar risk is at 10% p.a. What is the present value of these two investments? Why are they so similar?*

ANSWER

PV of the perpetuity: $PV = \frac{A}{r} = \frac{100}{0.1} = \$1,000$

PV of the annuity: $PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] = 100\left[\frac{1-1.10^{-100}}{0.10}\right] = \$999.93$

Given that there are an infinite number of payments for the first investment, its value is not much greater than that of the second investment. This is because cash flows in the far future are discounted very heavily (because of the time value of money) and add very little to the value of the investment.

29. *You work at Rest Assured Insurance Co. and have just received a claim from an individual involved in a work-related accident. The claim has a value of $200,000. It is the company's policy to provide multiple payment options with various maturities. The claimant may choose the cash out the claim now, or cash out with monthly payments over a span of three, five or ten years. If the current market interest rate is 6% p.a. (compounded monthly), what should be the monthly payments in each case?*

ANSWER

$$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] \therefore A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]}$$

Payment over three years: $$A = \frac{200,000}{\left[\frac{1-(1.005)^{-36}}{0.005}\right]} = \$6,084.39$$

Payment over five years: $$A = \frac{200,000}{\left[\frac{1-(1.005)^{-60}}{0.005}\right]} = \$3,866.56$$

Payment over ten years: $$A = \frac{200,000}{\left[\frac{1-(1.005)^{-120}}{0.005}\right]} = \$2,220.41$$

30. *You are a private equity firm that invests in brand-new and high-risk projects. You are considering investing in a new tollway. The tollway will take five years to build and require $1 billion in outlays in each of those five years. The tollway is expected to produce revenue of $1 billion in years 6 and 7, and then $2 billion per year in perpetuity from year 8 onwards. What is the value of the investment if the interest rate is 10% p.a. compounded annually?*

ANSWER

We need to calculate the present value of all of these cash flows, keeping in mind that cash outlays are negative. The first five years of cash flows constitute an annuity.

$$PV_{(1-5)} = A\left[\frac{1-(1+r)^{-n}}{r}\right] = -1\left[\frac{1-1.10^{-5}}{0.10}\right] = -\$3.79 \text{ billion}$$

The cash flows in years 6 and 7 can be discounted separately to find their present value.

$$PV_{(6)} = \frac{FV}{(1+r)^n} = \frac{1}{1.1^6} = \$0.56 \text{ billion}$$ and

$$PV_{(7)} = \frac{FV}{(1+r)^n} = \frac{1}{1.1^7} = \$0.51 \text{ billion}$$

The cash flows from year 8 onward constitute a **deferred perpetuity**. Since the first payment of a perpetuity occurs one period after the beginning of the perpetuity period, if the first cash flow under this perpetuity occurs in year 8 then the beginning of the perpetuity is at year 7. Hence, the present value of the perpetuity must be discounted by 7 years to find the present value today.

As at year 7: $PV_{(8+)} = \frac{A}{r} = \frac{2}{0.10} = \$20 \text{ billion}$

As at year 0: $PV_{(8+)} = \frac{FV}{(1+r)^n} = \frac{20}{1.1^7} = \$10.26 \text{ billion}$

These separate amounts can then be added together:

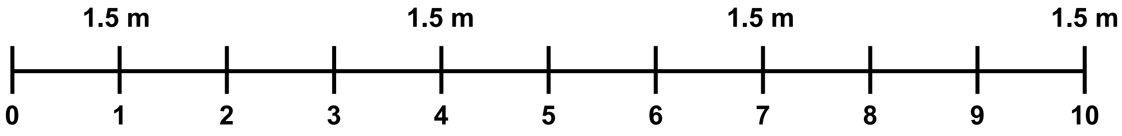
$$PV = -3.79 + 0.56 + 0.51 + 10.26 = \$7.54 \text{ billion}$$

31. *You have just bought a mine with a proven gold reserve. You have two sets of machinery which you can use to extract and process the gold in the mine. The first requires an outlay of $1.5 million every three years, with the first payment to be made one year's time. The second set of machinery requires an outlay of $5 million every seven years, also with the first payment to be made in one year's time. Which set of machinery is cheaper to operate, if the interest rate is 10% p.a. compounded annually?*

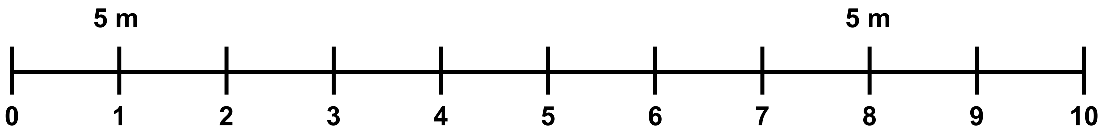
ANSWER

Time lines of cash flows are extremely useful in visualising problems such as these.

Machine 1:



Machine 2:



To solve this problem we need to convert each of these cash flow patterns into an equivalent annuity – an annual payment that has the same present value as the above cash flow. The machinery with the lowest equivalent annuity is the one that should be selected.

Machine 1:

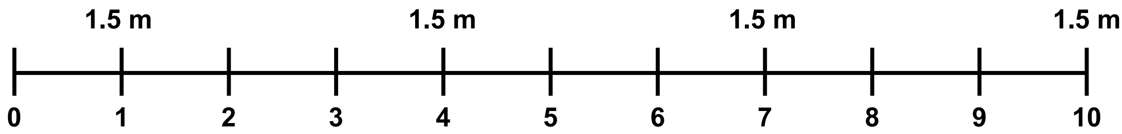
We first find the present value of $1.5m in one year.

$$PV = \frac{FV}{(1+r)^n} = \frac{1.5}{1.10} = \$1.36 \text{ million}$$

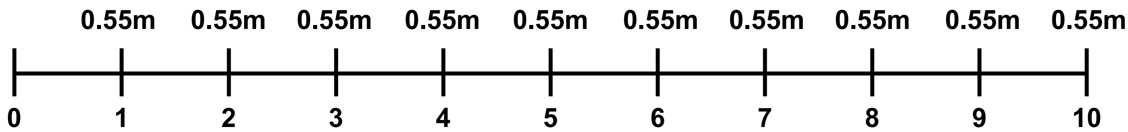
If we set this equal to the present value of a 3-year annuity, we can find the equivalent annual amount over this period of time.

$$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] \therefore A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{1.36}{\left[\frac{1-1.10^{-3}}{0.10}\right]} = \$0.55 \text{ million}$$

Hence,



is equivalent to:



Machine 2:

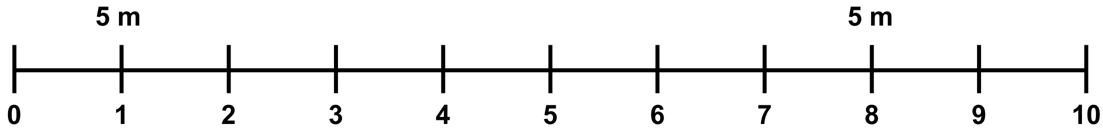
We then find the present value of $5m in one year.

$$PV = \frac{FV}{(1+r)^n} = \frac{5}{1.10} = \$4.55 \text{ million}$$

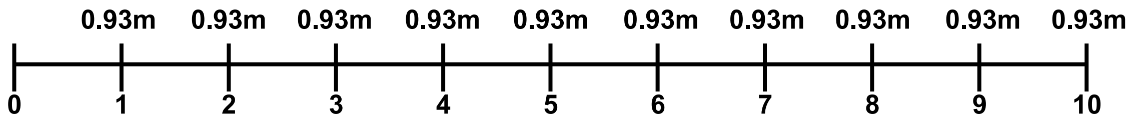
If set this equal to the present value of a 7-year annuity, we can find the equivalent annual amount over this period of time.

$$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] \therefore A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{4.55}{\left[\frac{1-1.10^{-7}}{0.10}\right]} = \$0.93 \text{ million}$$

Hence,



is equivalent to:



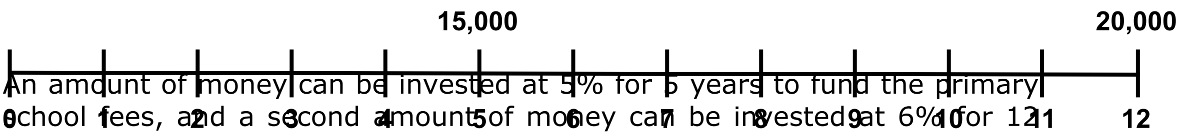
Clearly Machine 1 is cheaper to operate.

32. *Mr Thought has just had a son, and wants to send him to a private school. School fees from kindergarten through to year 6 are expected to total \$15,000, while those from year 7 to year 12 are expected to total \$20,000. Mr Thought's son is due to start school when he is five years old.*

How much money must Mr Thought set aside now if fixed-term deposit rates from five to 10 years are currently 5% p.a. compounded annually, and from 10 to 20 years are currently 6% p.a. compounded annually? There is a penalty of 3% for withdrawal of funds from a fixed-term deposit before maturity.

ANSWER

If there are 7 years of school from kindergarten to year 6 inclusive, and assuming that the fees must be paid at the beginning of both primary and secondary school, the cash flows are as follows:



An amount of money can be invested at 5% for 5 years to fund the primary school fees, and a second amount of money can be invested at 6% for 12 years to fund the secondary school fees. Hence, the total amount that must be set aside is given by:

$$PV = \frac{FV}{(1+r)^n} = \frac{15,000}{1.05^5} + \frac{20,000}{1.06^{12}} = 11,752.89 + 9,939.39 = \$21,692.28$$

33. *Today is your 30th birthday. Given the recent changes in the tax law relating to retirement, the earliest you can retire is your 60th birthday. Your employer contributions to your superannuation are \$2,500 p.a. on each birthday, starting immediately. You estimate that you will need \$60,000 per annum to live on from your 61st birthday to your 90th birthday (inclusive). You estimate that you can earn 12% p.a. between now and your 90th birthday.*

(a) *Can you afford to retire on your 60th birthday? Demonstrate your answer with calculations.*

(b) *Given your savings plan, how much can you expect to receive as income each year when you retire – that is, from your 61st to your 90th birthday?*

ANSWER

The first step is to calculate the amount of money that will have been accumulated by your 60th birthday. Assuming that there is a payment on your 60th birthday, and and also a payment on your 30th birthday, this is a 31-year annuity ordinary annuity.

$$FV = A\left[\frac{(1+r)^n - 1}{r}\right] = 2500\left[\frac{1.12^{31} - 1}{0.12}\right] = \$678.231/52$$

This amount is the present value of the 30-year annuity that comprises the annual withdrawals from your 61st birthday to your 90th birthday.

(a) $$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] = 60,000\left[\frac{1-1.12^{-30}}{0.12}\right] = \$483,311.04$$

You would only require \$483,311 to be accumulated by your 60th birthday to be able to withdraw \$60,000 per year from your 61st birthday to your 90th birthday, so you can afford to retire on your 60th birthday.

(b)

$$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] \therefore A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{678,231.52}{\left[\frac{1-1.12^{-30}}{0.12}\right]} = \$84,198.14$$

This is the amount that can be withdrawn each year from your 61st birthday to your 90th birthday.

34. *A close friend who is financially naive is trying to compare two loan contracts. Your friend wishes to borrow $15,000 over the next 5 years and make monthly payments.*

Loan 1

South West Bank is offering a loan at 10% p.a. (monthly compounding) over five years.

(a) *What are the monthly payments on this loan?*

(b) *What is the effective annual rate?*

Loan 2

Local Credit Company has told your friend that the monthly payments he will make are $311.20. However, Local Credit Company has not revealed the interest rate it used to calculate this payment. Local Credit Company did state that all loans are calculated using monthly compounding.

(a) *What is the annual nominal interest rate on this loan?*

(b) *What is the effective annual rate?*

Which loan should your friend choose, and why?

ANSWER

Loan 1

(a) The interest rate per compounding period is 0.10/12 = 0.00833 and the number of compounding periods in 5 years is 12 × 5 = 60.

$$PV = A\left[\frac{1-(1+r)^{-n}}{r}\right] \therefore A = \frac{PV}{\left[\frac{1-(1+r)^{-n}}{r}\right]} = \frac{15,000}{\left[\frac{1-1.00833^{-60}}{0.00833}\right]} = \$318.68$$

(b) Effective Rate $= \left(1+\frac{r_{nom}}{m}\right)^{m} - 1 = \left(1+\frac{0.10}{12}\right)^{12} - 1 = 10.47\%$

Loan 2

(a) $PV = A\left[\frac{1-(1+r)^{-n}}{r}\right]$

$$\therefore 15000 = 311.20\left[\frac{1-(1+r)^{-60}}{r}\right] \therefore \left[\frac{1-(1+r)^{-60}}{r}\right] = \frac{15,000}{311.20} = 48.2$$

This can be solved using trial and error, using a financial calculator or using present value tables. The answer using either of the first two methods is 0.748% per month (the third method will give an approximate value). Hence, the nominal rate is 0.748 × 12 = 8.98% p.a.

(b) Effective Rate $= \left(1 + \frac{r_{nom}}{m}\right)^m - 1 = \left(1 + \frac{0.0898}{12}\right)^{12} - 1 = 9.35\%$

The second loan has the lowest effective interest and the smallest payments.

35. The current interest rate on two-year Australian treasury bonds is 5.5% nominal with interest paid every six months. Your bank account offers a two-year fixed-term deposit of 5% p.a. compounded six-monthly, and a bank account that pays interest of 3% p.a. compounded six-monthly and which is expected to pay interest at this rate for the foreseeable future. Should you invest your money in treasury bonds or the fixed-term deposit? (Careful).

In order to make a comparison between the treasury bonds and the fixed term deposit, we need to consider all of the returns on the investment, including the return from reinvesting the coupon payments in the bank account.

Let us assume that we buy a bond with a face value of $100, and that the bond is trading at par (i.e. the price is $100).

Coupon payments of $2.75 (5.5% × $100 / 2) will be received in 6, 12, 18 and 24 months, along with the face value of $100 on maturity in 24 months (2 years). If these cash flows are invested in the bank account paying 3% p.a. (or 1.5% per six months), the coupon payment in 6 months will accrue interest for 18 months, the coupon payment in 12 months will accrue interest for 12 months, the coupon payment in 18 months will accrue interest for 6 months, and the final coupon payment and the face value will not have time to accrue any interest.

Since the bank account pays interest semi-annually, the interest rate per compounding period is 3%/2 = 1.5%.

The total proceeds after 2 years will be:

$$FV = 2.75(1.015)^3 + 2.75(1.015)^2 + 2.75(1.015) + 102.75 = \$111.25$$

The total proceeds from investing $100 in the two-year fixed-term deposit at 5% p.a. (or 2.5% per six months) is:

$$FV = PV(1+r)^n = 100(1.025)^4 = \$110.38$$

You should invest your money in treasury bonds.