CHAPTER 2
MATRIX ALGEBRA AND GAUSSIAN ELIMINATION

2.1 $\mathbf{A} = \begin{bmatrix} 8 & -2 & 0 \\ -2 & 4 & -3 \\ 0 & -3 & 3 \end{bmatrix}$, $\mathbf{d} = \begin{Bmatrix} 2 \\ -1 \\ 3 \end{Bmatrix}$

(a) $\mathbf{I} - \mathbf{d}\,\mathbf{d}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & -6 \\ 2 & 0 & 3 \\ -6 & 3 & -8 \end{bmatrix}$

(b) det $\mathbf{A}$ = 8 [(4)(3) – (–3) (–3)] – (–2) [(–2)(3) – (0)(–3)] = 12

(c) The characteristic equation is det ($\mathbf{A} - \lambda\, \mathbf{I}$) = 0, or

$$\det \begin{bmatrix} 8-\lambda & -2 & 0 \\ -2 & 4-\lambda & -3 \\ 0 & -3 & 3-\lambda \end{bmatrix} = 0$$

which yields

$$\lambda^3 - 15\lambda^2 + 55\lambda - 12 = 0$$

Handbooks (e.g., CRC Mathematical Handbook) give explicit solutions to cubic equations. Here equations given in Chapter 9 are used, which give formulas for finding the eigenvalues of the (3x3) symmetric stress tensor. Referring to Section 9.3 in the text, we have

$$I_1 = A_{11} + A_{22} + A_{33} = 15, \quad I_2 = 55, \; I_3 = 12$$

Thus, $a = 20$, $b = 13$, $c = 5.164$, $\theta = 37.4\,°$

whence $\lambda_1 = 0.2325$, $\lambda_2 = 5.665$, $\lambda_3 = 9.103$

Note: Since all $\lambda_i > 0$, $\mathbf{A}$ is positive definite.

Now, eigenvector $\mathbf{y}^i$ corresponding to eigenvalue λ_i is obtained from

$$(\mathbf{a} - \lambda_i\, \mathbf{I}\,)\, \mathbf{y}^i = 0, \; i = 1,2,3$$

Thus, $\mathbf{y}^1$ is obtained as

$$\begin{bmatrix} 7.7675 & -2 & 0 \\ -2 & 3.7675 & -3 \\ 0 & -3 & 2.7675 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus,

$7.7675\, y_1 - 2\, y_2 = 0$
$-2\, y_1 + 3.7675\, y_2 - 3\, y_3 = 0$
$-3\, y_2 + 2.7675\, y_3 = 0$

Only two of the above three equations are independent. We have

$y_1 = 0.2575\, y_2$
$y_2 = 0.922\ \ y_3$

Letting $y_3 = 1$, we get $\mathbf{y}^1 = [0.237, 0.922, 1]^T$

The length of the vector is $\|y^1\| = \sqrt{\mathbf{y}^T\mathbf{y}} = 1.381$. Normalizing $\mathbf{y}^1$ to be a unit vector yields

$\mathbf{y}^1 = [0.172, 0.668, 0.724]^T$.

Similarly,

$\mathbf{y}^2 = [0.495, 0.577, -0.650]^T$, $\mathbf{y}^3 = [0.850, -0.470, 0.232]^T$.

(d) **Solution to A x = b using Algorithm 1 for general matrix**:

$n = 3$

First Step ($k = 1$)

***i* = 2** (2nd row)

$$\mathbf{A} = \begin{bmatrix} 8 & -2 & 0 \\ -2 & \mathbf{4} & -\mathbf{3} \\ 0 & -\mathbf{3} & \mathbf{3} \end{bmatrix}$$

$c = a_{21} / a_{11} = -2/8 = -1/4$

$a_{22}^{(1)} = 4 - (-1/4)(-2) = 7/2$, $a_{23}^{(1)} = -3$, $d_2^{(1)} = -1-(-1/4)(2) = -1/2$

$i = 3$ (3rd row)

c = 0

$a_{32}^{(1)} = -3$, $a_{33}^{(1)} = 3$, $d_3^{(1)} = 3$

Thus $\mathbf{A}^{(1)} = \begin{bmatrix} 8 & -2 & 0 \\ 0 & 7/2 & -3 \\ 0 & -3 & 3 \end{bmatrix}$, $\mathbf{d}^{(1)} = \begin{bmatrix} 2 \\ -1/2 \\ 3 \end{bmatrix}$

Second Step ($k = 2$)

$i = 3$ (3rd row)

c = -6/7, $a_{33}^{(2)} = 3 - (-6/7)(-3) = 3/7$, $d_3^{(2)} = 3 - (-6/7)(-1/2) = 18/7$

Thus $\mathbf{A}^{(2)} = \begin{bmatrix} 8 & -2 & 0 \\ 0 & 7/2 & -3 \\ 0 & 0 & 3/7 \end{bmatrix}$, $\mathbf{d}^{(1)} = \begin{bmatrix} 2 \\ -1/2 \\ 18/7 \end{bmatrix}$

Back-Substitution

$$\begin{bmatrix} 8 & -2 & 0 \\ 0 & -7/2 & -3 \\ 0 & 0 & 3/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/2 \\ 18/7 \end{bmatrix}$$

Third row gives: 3/7 x_3 = 18/7 whence x_3 = 6
2nd row then gives x_2 = 5,
1st row gives x_1 = 1.5
Thus, solution is $\mathbf{x} = [1.5, 5, 6]^T$.

Solution to A x = b using Algorithm 2 for symmetric, banded matrix

A is stored as $\begin{bmatrix} 8 & -2 \\ 4 & -3 \\ 3 & 0 \end{bmatrix}$

$n = 3$, $nbw = 2$

First Step ($k = 1$)
nbk = min (3,2) = 2
2nd row ($i = 2$):
$i1 = 2$ c = a_{12}/a_{11} = -1/4
$j1 = 1$ $j2 = 2$
$a_{21} = 4 - (-1/4)(-2) = 7/2$
Thus $\mathbf{A}^{(1)} = \begin{bmatrix} 8 & -2 \\ 7/2 & -3 \\ 3 & 0 \end{bmatrix}$

Second Step ($k = 2$)
$nbk = 2$
3rd row ($i = 3$):
c = $-6/7$
$j1 = 1$ $j2 = 2$
$a_{31} = 3/7$
Thus $\mathbf{A}^{(2)} = \begin{bmatrix} 8 & -2 \\ 7/2 & -3 \\ 3/7 & 0 \end{bmatrix}$

reduction of right-hand-side vector **d** and back-substitution is same as in Algorithm 1 above, resulting in the the same solution $\mathbf{x} = [1.5, 5, 6]^T$. ■

2.2 $\mathbf{N} = \begin{bmatrix} \xi & 1-\xi^2 \end{bmatrix}$

(a) $\int_{-1}^{1} \mathbf{N}\, d\xi = \begin{bmatrix} 0 & \frac{4}{3} \end{bmatrix}$

(b) $\int_{-1}^{1} \mathbf{N}^{\mathrm{T}}\ \mathbf{N}\ d\xi = \begin{bmatrix} \int \xi^2 & \int \xi(1-\xi^2) \\ \int \xi(1-\xi^2) & \int (1-\xi^2)^2 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 16/15 \end{bmatrix}$ ■

2.3 $q = x_1 - 6\,x_2 + 3\,x_1^2 + 5\,x_1\,x_2$

$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 3 & 2.5 \\ 2.5 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\equiv \quad \mathbf{x}^{\mathrm{T}}\, \mathbf{Q}\, \mathbf{x} + \mathbf{c}^{\mathrm{T}}\, \mathbf{x}$

where

$$\mathbf{Q} = \begin{bmatrix} 3 & 2.5 \\ 2.5 & 0 \end{bmatrix} \mathit{and}\ \mathbf{c} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$ ■

2.4 The detailed algorithm is given in the text. This is an excercise in computer programming. The solutions are (a) (–2.25, –11.5, –10.5) (b) (1.55, 5.1, 6.1) ■

2.5 The minors are

$M_{11} = \det \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = -1$

$M_{12} = \det \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = -5$

$M_{13} = \det \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = -1$

$M_{21} = 1$; $M_{22} = -7$; $M_{23} = -5$

$M_{31} = 3$; $M_{32} = -3$; $M_{33} = -3$

Co factor

$C_{ij} = (-1)^{i+j}\, M_{ij}$

Thus, the co-factor matrix is

$$A_c = \begin{bmatrix} -1 & 5 & -1 \\ -1 & -7 & -5 \\ 3 & 3 & -3 \end{bmatrix}$$

The inverse of matrix A

$$A^{-1} = = \frac{1}{\det A}\; Ac^T \text{ yields}$$

det (A) = (2 – 3) – 2 (4 – 9) + 3 (2 – 3) = 6

$$\therefore\; A^{-1} = \frac{1}{6}\begin{bmatrix} -1 & -1 & 3 \\ 5 & -7 & 3 \\ -1 & 5 & 3 \end{bmatrix}$$

2.6 Area = $\frac{1}{2}$ det $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 7 & 8 \\ 1 & 11 & 12 \end{bmatrix}$

$= [6]$

= 6 square units.

2.7 $A_1 = \frac{1}{2} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1.5 \\ 1 & 2.5 & 5 \end{bmatrix} = 1.625$

$A_2 = \frac{1}{2} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2.5 & 5 \\ 1 & 1 & 1 \end{bmatrix} = 1.25$

$A_3 = \frac{1}{2} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 1.5 \end{bmatrix} = 0.75$

$A = A_1 + A_2 + A_3 = 3.625$

$A_1 / A = 0.448$, $A_2 / A = 0.345$, $A_3 / A = 0.207$ ■

2.8 $A_{i,j} = B_{i,\, j-i+1}$ for $j \geq i$

Thus, $A_{11,14}$ corresponds to $B_{11,4}$

and $B_{6,1}$ ″ ″ $A_{6,6}$ ■

2.9 Full (10x10) matrix

BANDED

$n = 10$, $nbw = 10$

No. of storage locations = (n) (nbw)

= 100

SKYLINE

```
x  x  x  x ...
   x  x  x ...
      x  x ...
         x ...
```

No. of storage locations = no. of column entries

= 1 + 2 + 3 + ... + 10

= (10) (10+1) / 2

= 55 ■

2.10 Based on Eq. 2.2;

$K = 1$ (1st row)

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$K = 2$ (2nd row)

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{2}{2} = 1$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = 3$$

$K = 3$ (3rd row)

$$l_{31} = \frac{a_{31}}{l_{11}} = -3$$

$$l_{32} = \frac{a_{32} - l_{31} l_{21}}{l_{32}} = 4$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = 1$$

Thus

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

2.11 By expanding the matrix equations, we obtain a set of simultaneous equations for the

ℓ and u coefficients, resulting in the solution

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } \boldsymbol{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

2.12 The quadrilateral *ABCD* is divided in to two triangles with corners *ABC* and *ACD*.

Area of triangle *ABC*

$$A_1 = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 2 \\ 1 & 6 & 6 \end{bmatrix} = 12.5$$

Area of triangle *ACD*

$$A_2 = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 6 \\ 1 & 3 & 7 \end{bmatrix} = 10$$

$\therefore$ Area of the quadrilateral *ABCD*

$A = A_1 + A_2$

$A = 22.5$ square unit.

2.13 Setting $\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, it is easy to show that $\mathbf{TT}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. ■

2.14 a) The minors of the given matrix are:

$M_{11} = 7;\ M_{12} = 4;\ M_{13} = -1$

$M_{21} = 11;\ M_{22} = 17;\ M_{23} = 7$

$M_{31} = 3;\ M_{32} = 6;\ M_{33} = 6$

b) The co factors of the given matrix are:

$C_{11} = 7;\ C_{12} = -4;\ C_{13} = -1$

$C_{21} = -11;\ C_{22} = 17;\ C_{33} = -7$

$C_{31} = 3;\ C_{32} = -6;\ C_{33} = 6$

c) The adjoint matrix is

$$\begin{bmatrix} 7 & -11 & 3 \\ -4 & 17 & -6 \\ -1 & -7 & 6 \end{bmatrix}$$

c) The determinant of the matrix is 15

e) The inverse of the given matrix is

$$\frac{1}{15}\begin{bmatrix} 7 & -11 & 3 \\ -4 & 17 & -6 \\ -1 & -7 & 6 \end{bmatrix}$$