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Chapter 2

$$1. P\{X = 0\} = \frac{{}_7H_2}{{}_1O_2} = \frac{14}{30}$$

2. $-n, -n + 2, -n + 4, \dots, n - 2, n$

3. $P\{X = -2\} = \frac{1}{4} = P\{X = 2\}$

$$P\{X = 0\} = \frac{1}{2}$$

4. (a) 1, 2, 3, 4, 5, 6

(b) 1, 2, 3, 4, 5, 6

(c) 2, 3, ..., 11, 12

(d) -5, -4, ..., 4, 5

$$11$$

5. $P\{\max = 6\} = \frac{1}{36} = P\{\min = 1\}$

$$P\{\max = 5\} = \frac{1}{4} = P\{\min = 2\}$$

$$P\{\max = 4\} = \frac{7}{36} = P\{\min = 3\}$$

$$P\{\max = 3\} = \frac{5}{36} = P\{\min = 4\}$$

$$P\{\max = 2\} = \frac{1}{12} = P\{\min = 5\}$$

$$P\{\max = 1\} = \frac{1}{36} = P\{\min = 6\}$$

6. $(H, H, H, H, H), p^5$ if $p = P\{\text{heads}\}$

7. $p(0) = (.3)^3 = .027$

$$p(1) = 3(.3)^2(.7) = .189$$

$$p(2) = 3(.3)(.7)^2 = .441$$

$$p(3) = (.7)^3 = .343$$

8. $p(0) = \frac{1}{2}, p(1) = \frac{1}{2}$

9. $p(0) = \frac{1}{4}, p(1) = \frac{1}{4}, p(2) = \frac{1}{4}, p(3) = \frac{1}{4}$

$$p(3) = \frac{1}{10}, p(3.5) = \frac{1}{10}$$

10. $1 - \frac{[3]!}{2} \frac{[1]!}{6} - \frac{[5]!}{6} \frac{[3]!}{3} \frac{[1]!}{6} = \frac{200}{216}$

11. $\frac{3}{8}$

12. $\frac{[5]!}{4} \frac{[1]!}{3} \frac{[2]!}{3} + \frac{[5]!}{5} \frac{[1]!}{3} = \frac{10+1}{243} = \frac{11}{243}$

$$\begin{aligned}
 13. & \sum_{i=0}^6 (10)_i [1]_{10} \\
 14. & P\{X = 0\} = P\{X = 6\} = \frac{[1]_6}{2^6} = \frac{1}{64} \\
 & P\{X = 1\} = P\{X = 5\} = \frac{6 [1]_6}{2^6} = \frac{6}{64} \\
 & P\{X = 2\} = P\{X = 4\} = \frac{6^2 [1]_6}{2^6} = \frac{15}{64} \\
 & P\{X = 3\} = \frac{6^3 [1]_6}{2^6} = \frac{20}{64} \\
 15. & \frac{P\{X = k\}}{P\{X = k - 1\}} = \frac{\frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}}{\frac{n!}{(n-k+1)!(k-1)!} p^{k-1} (1-p)^{n-k+1}} \\
 & = \frac{(n-k+1)!(k-1)! p}{k! 1-p} = \frac{n-k+1}{k} \frac{p}{1-p}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \frac{P\{X = k\}}{P\{X = k - 1\}} & \geq 1 \leftrightarrow (n - k + 1)p > k(1 - p) \\
 & \leftrightarrow (n + 1)p \geq k
 \end{aligned}$$

The result follows.

$$16. 1 - (.95)^{52} - 52(.95)^{51}(.05)$$

$$17. \text{ Follows since there are } \frac{n!}{x_1! \cdots x_r!}$$

permutations of n objects of which x_1 are alike, x_2 are alike, ..., x_r are alike.

$$\begin{aligned}
 18. (a) & P(X_i = x_i, i = 1, \dots, r-1 | X_r = j) \\
 & = \frac{P(X_i = x_i, i = 1, \dots, r-1, X_r = j)}{P(X_r = j)} \\
 & = \frac{n!}{x_1! \cdots x_{r-1}! j!} p_1^{x_1} \cdots p_{r-1}^{x_{r-1}} p_r^j (1-p_r)^{n-j} \\
 & = \frac{(n-j)!}{x_1! \cdots x_{r-1}!} \prod_{i=1}^{r-1} \frac{p_i}{1-p_r}^{x_i}
 \end{aligned}$$

(b) The conditional distribution of $X_1, \dots, X_{r-1}$ given that $X_r = j$ is multinomial with parameters $n - j, p_i, i = 1, \dots, r - 1$.

(c) The preceding is true because given that $X_r = j$, each of the $n - j$ trials that did not result in outcome r resulted in outcome i with probability $\frac{p_i}{1-p_r}$, $i = 1, \dots, r - 1$.

$$\begin{aligned}
 19. & P\{X_1 + \cdots + X_k = m\} \\
 & = \binom{n}{m} p_1^{x_1} \cdots p_k^{x_k} (p_{k+1} + \cdots + p_r)^{n-m}
 \end{aligned}$$

$$20. \frac{5!}{2!1!2!} \left[\frac{1}{5} \right]^2 \left[\frac{3}{10} \right]^2 \left[\frac{1}{2} \right]^1 = .054$$

$$21. 1 - \frac{3}{10} - 5 \left[\frac{3}{10} \right]^4 \left[\frac{7}{10} \right] - \frac{1}{2} \left[\frac{3}{10} \right]^3 \left[\frac{7}{10} \right]^2$$

$$22. \frac{1}{32}$$

23. In order for X to equal n , the first $n - 1$ flips must have $r - 1$ heads, and then the n th flip must land heads. By independence the desired probability is thus

$$\binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} p$$

24. It is the number of tails before heads appears for the r th time.

25. A total of 7 games will be played if the first 6 result in 3 wins and 3 losses. Thus,

$$P\{7 \text{ games}\} = \binom{6}{3} p^3 (1-p)^3$$

Differentiation yields

$$\begin{aligned} \frac{d}{dp} P\{7\} &= 20 [3p^2(1-p)^3 - p^3 3(1-p)^2] \\ &= 60 p^2(1-p)^2 [1-2p] \end{aligned}$$

Thus, the derivative is zero when $p = 1/2$. Taking the second derivative shows that the maximum is attained at this value.

26. Let X denote the number of games played.

$$(a) P\{X = 2\} = p^2 + (1-p)^2$$

$$P\{X = 3\} = 2p(1-p)$$

$$\begin{aligned} E[X] &= 2 \{ p^2 + (1-p)^2 \} + 6p(1-p) \\ &= 2 + 2p(1-p) \end{aligned}$$

Since $p(1-p)$ is maximized when $p = 1/2$, we see that $E[X]$ is maximized at that value of p .

$$(b) P\{X = 3\} = p^3 + (1-p)^3$$

$$P\{X = 4\}$$

$$= P\{X = 4, \text{I has 2 wins in first 3 games}\}$$

$$+ P\{X = 4, \text{II has 2 wins in first 3 games}\}$$

$$= 3p^2(1-p)p + 3p(1-p)^2(1-p)$$

$$P\{X = 5\}$$

$$= P\{\text{each player has 2 wins in the first 4 games}\}$$

$$= 6p^2(1-p)^2$$

$$E[X] = 3 \frac{[p^3 + (1-p)^3]}{[p^2 + (1-p)^2]} + 12 p(1-p) + 30 p^2(1-p)^2$$

Differentiating and setting equal to 0 shows that the maximum is attained when $p = 1/2$.

$$\begin{aligned} 27. P\{\text{same number of heads}\} &= \sum_i P\{A=i, B=i\} \\ &= \sum_i \binom{k}{i} (1/2)^k \binom{n-k}{i} (1/2)^{n-k} \end{aligned}$$

$$\begin{aligned}
 &= \sum_i \binom{n-k}{i} (1/2)^n \\
 &= \sum_{k-i}^i \binom{n-k}{i} (1/2)^n \\
 &= \binom{n}{k} (1/2)^n
 \end{aligned}$$

Another argument is as follows:

$$\begin{aligned}
 &P\{\# \text{ heads of } A = \# \text{ heads of } B\} \\
 &= P\{\# \text{ tails of } A = \# \text{ heads of } B\}
 \end{aligned}$$

since coin is fair

$$\begin{aligned}
 &= P\{k - \# \text{ heads of } A = \# \text{ heads of } B\} = \\
 &P\{k = \text{total } \# \text{ heads}\}
 \end{aligned}$$

28. (a) Consider the first time that the two coins give different results. Then

$$\begin{aligned}
 P\{X = 0\} &= P\{(t, h) | (t, h) \text{ or } (h, t)\} \\
 &= \frac{2(1-p)}{2p(1-p)} = \frac{1}{2}
 \end{aligned}$$

(b) No, with this procedure

$$P\{X = 0\} = P\{\text{first flip is a tail}\} = 1 - p$$

29. Each flip after the first will, independently, result in a changeover with probability 1/2. Therefore,

$$P\{k \text{ changeovers}\} = \binom{n-1}{k} (1/2)^{n-1}$$

30.
$$P\{X = i\} = \frac{e^{-\lambda} \lambda^i}{i!} = \lambda^i e^{-\lambda} / i!$$

Hence, $P\{X = i\}$ is increasing for $\lambda \geq i$ and decreasing for $\lambda < i$.

32. (a).394 (b) .303 (c) .091

33. $c \int_1^1 (1 - x^2) dx = 1$

$$c \left[x - \frac{x^3}{3} \right]_{-1}^1 = 1$$

$$c = \frac{3}{4}$$

$$F(y) = \frac{3}{4} \int_{-1}^1 (1 - x^2) dx$$

$$= \frac{3}{4} \left[y - \frac{y^3}{3} \right]_{-1}^1, \quad -1 < y < 1$$

34. $c \int_0^2 (4x - 2x^2) dx = 1$

$$c(2x^2 - 2x^3/3) = 1$$

$$8c/3 = 1$$

$$c = \frac{3}{8}$$

$$P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{3}{8} \int_{1/2}^{3/2} (4x - 2x^2) dx$$

$$= \frac{11}{16}$$

35. $P\{X > 20\} = \int_{20}^{\infty} \frac{10}{x^2} dx = 2^{-20}$

36. $P\{D \leq x\} = \frac{\text{area of disk of radius } x}{\text{area of disk of radius } 1}$

$$= \frac{\pi x^2}{\pi} = x^2$$

37. $P\{M \leq x\} = P\{\max(X_1, \dots, X_n) \leq x\}$

$$= P\{X_1 \leq x, \dots, X_n \leq x\}$$

$$= \prod_{i=1}^n P\{X_i \leq x\}$$

$$= x^n$$

$$f_M(x) = \frac{d}{dx} P\{M \leq x\} = nx^{n-1}$$

38. $c = 2$

39. $E[X] = \frac{31}{6}$

40. Let X denote the number of games played.

$$\begin{aligned}
 P\{X = 4\} &= p^4 + (1 - p)^4 \\
 P\{X = 5\} &= P\{X = 5, \text{ I wins 3 of first 4}\} \\
 &\quad + P\{X = 5, \text{ II wins 3 of first 4}\} \\
 &= 4p^3(1 - p)p + 4(1 - p)^3p(1 - p) \\
 P\{X = 6\} &= P\{X = 6, \text{ I wins 3 of first 5}\} \\
 &\quad + P\{X = 6, \text{ II wins 3 of first 5}\} \\
 &= 10p^3(1 - p)^2p + 10p^2(1 - p)^3(1 - p) \\
 P\{X = 7\} &= P\{\text{first 6 games are split}\} \\
 &= 20p^3(1 - p)^3 \\
 E[X] &= \sum_{i=4}^{\infty} iP\{X = i\}
 \end{aligned}$$

When $p = 1/2$, $E[X] = 93/16 = 5.8125$

41. Let X_i equal 1 if a changeover results from the i th flip and let it be 0 otherwise. Then

$$\text{number of changeovers} = \sum_{i=2}^{\infty} X_i$$

As,

$$\begin{aligned}
 E[X_i] &= P\{X_i = 1\} = P\{\text{flip } i - 1 \neq \text{flip } i\} \\
 &= 2p(1 - p)
 \end{aligned}$$

we see that

$$\begin{aligned}
 E[\text{number of changeovers}] &= \sum_{i=2}^{\infty} E[X_i] \\
 &= 2(n - 1)p(1 - p)
 \end{aligned}$$

42. Suppose the coupon collector has i different types. Let X_i denote the number of additional coupons collected until the collector has $i + 1$ types. It is easy to see that the X_i are independent geometric random variables with respective parameters $(n - i)/n$, $i = 0, 1, \dots, n - 1$. Therefore,

$$\begin{aligned}
 \sum_{i=0}^{\infty} [\sum_{n=1}^{\infty} n^{-1}]^i &= \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} n^{-1} \\
 &= \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n - i)} \\
 &= \sum_{j=1}^{\infty} \frac{1}{j}
 \end{aligned}$$

43. (a) $X = \sum_{i=1}^n Xi$

(b) $E[X_i] = P\{X_i = 1\}$
 $= P\{\text{red ball } i \text{ is chosen before all } n \text{ black balls}\}$
 $= 1/(n + 1)$ since each of these $n + 1$ balls is equally likely to be the one chosen earliest

Therefore,

$$E[X] = \sum_{i=1}^n E[X_i] = n/(n + 1)$$

44. (a) Let Y_i equal 1 if red ball i is chosen after the first but before the second black ball, $i = 1, \dots, n$. Then

$$Y = \sum_{i=1}^n Y_i$$

(b) $E[Y_i] = P\{Y_i = 1\}$
 $= P\{\text{red ball } i \text{ is the second chosen from a set of } n + 1 \text{ balls}\}$
 $= 1/(n + 1)$ since each of the $n + 1$ is equally likely to be the second one chosen.

Therefore,

$$E[Y] = n/(n + 1)$$

- (c) Answer is the same as in Problem 41.

- (d) We can let the outcome of this experiment be the vector $(R_1, R_2, \dots, R_n)$ where R_i is the number of red balls chosen after the $(i - 1)$ st but before the i th black ball. Since all orderings of the $n + m$ balls are equally likely it follows that all different orderings of $R_1, \dots, R_n$ will have the same probability distribution. For instance,

$$P\{R_1 = a, R_2 = b\} = P\{R_2 = a, R_1 = b\}$$

From this it follows that all the R_i have the same distribution and thus the same mean.

45. Let N_i denote the number of keys in box i , $i = 1, \dots, k$. Then, with X equal to the number of collisions we have that $X = \sum_{i=1}^k (N_i - 1)_+ = \sum_{i=1}^k (N_i - 1 + I_{\{N_i = 0\}})$ where $I_{\{N_i = 0\}}$ is equal to 1 if $N_i = 0$ and is equal to 0 otherwise.

Hence,

$$E[X] = \sum_{i=1}^k (r p_i - 1 + (1 - p_i)^r) = r - k + \sum_{i=1}^k (1 - p_i)^r$$

Another way to solve this problem is to let Y denote the number of boxes having at least one key, and then use the identity $X = r - Y$, which is true since only the first $\sum_{i=1}^k$ key put in each box does not result in a collision. Writing $Y = \sum_{i=1}^k I\{N_i > 0\}$ and taking expectations yields

$$\begin{aligned} E[X] &= r - E[Y] = r - \sum_{i=1}^k [1 - (1 - p_i)^r] \\ &= r - k + \sum_{i=1}^k (1 - p_i)^r \end{aligned}$$

46. Using that $X = \sum_{n=1}^{\infty} I_n$, we obtain

$$E[X] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P\{X \geq n\}$$

Making the change of variables $m = n - 1$ gives

$$E[X] = \sum_{m=0}^{\infty} P\{X \geq m + 1\} = \sum_{m=0}^{\infty} P\{X > m\}$$

(b) Let

$$I_n = \begin{cases} 1, & \text{if } n \leq X \\ 0, & \text{if } n > X \end{cases}$$

$$J_m = \begin{cases} 1, & \text{if } m \leq Y \\ 0, & \text{if } m > Y \end{cases}$$

Then

$$XY = \sum_{n=1}^{\infty} I_n \sum_{m=1}^{\infty} J_m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} I_n J_m$$

Taking expectations now yields the result

$$\begin{aligned} E[XY] &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E[I_n J_m] \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m) \end{aligned}$$

47. Let X_i be 1 if trial i is a success and 0 otherwise.

- (a) The largest value is .6. If $X_1 = X_2 = X_3$, then $1.8 = E[X] = 3E[X_1] = 3P\{X_1 = 1\}$ and so

$$P\{X = 3\} = P\{X_1 = 1\} = .6$$

That this is the largest value is seen by Markov's inequality, which yields

$$P\{X \geq 3\} \leq E[X]/3 = .6$$

- (b) The smallest value is 0. To construct a probability scenario for which $P\{X = 3\} = 0$ let U be a uniform random variable on $(0, 1)$, and define

$$X_1 = \begin{cases} 1 & \text{if } U \leq .6 \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{if } U \geq .4 \\ 0 & \text{otherwise} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if either } U \leq .3 \text{ or } U \geq .7 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that

$$P\{X_1 = X_2 = X_3 = 1\} = 0$$

49. $E[X^2] - (E[X])^2 = \text{Var}(X) = E(X - E[X])^2 \geq 0$. Equality when $\text{Var}(X) = 0$, that is, when X is constant.

$$\begin{aligned} 50. \quad \text{Var}(cX) &= E[(cX - E[cX])^2] \\ &= E[c^2(X - E(X))^2] \\ &= c^2 \text{Var}(X) \end{aligned}$$

$$\begin{aligned} \text{Var}(c + X) &= E[(c + X - E[c + X])^2] \\ &= E[(X - E[X])^2] \\ &= \text{Var}(X) \end{aligned}$$

51. $N = \sum_{i=1}^r X_i$ where X_i is the number of flips between the $(i-1)$ st and i th head. Hence, X_i is geometric with mean $1/p$. Thus,

$$E[N] = \sum_{i=1}^r E[X_i] = \frac{r}{p}$$

52. (a) $\frac{-n}{n+1}$
 (b) 0
 (c) 1

$$53. \frac{1}{n+1} \cdot \frac{1}{2n+1} - \frac{1}{n+1} \quad \left[\quad \right]_2$$

54. (a) Using the fact that $E[X + Y] = 0$ we see that $0 = 2p(1, 1) - 2p(-1, -1)$, which gives the result.
 (b) This follows since

$$0 = E[X - Y] = 2p(1, -1) - 2p(-1, 1)$$

$$(c) \text{Var}(X) = E[X^2] = 1$$

$$(d) \text{Var}(Y) = E[Y^2] = 1$$

(e) Since

$$\begin{aligned} 1 &= p(1, 1) + p(-1, 1) + p(1, -1) + p(-1, -1) \\ &= 2p(1, 1) + 2p(1, -1) \end{aligned}$$

we see that if $p = 2p(1, 1)$ then $1 - p = 2p(1, -1)$

Now,

$$\begin{aligned} \text{Cov}(X, Y) &= E[XY] \\ &= p(1, 1) + p(-1, -1) \\ &\quad - p(1, -1) - p(-1, 1) \\ &= p - (1 - p) = 2p - 1 \end{aligned}$$

$$\begin{aligned} 55. (a) P(Y = j) &= \sum_{i=0}^j \binom{j}{i} e^{-3\lambda} \lambda^j / j! \\ &= e^{-2\lambda} \lambda^j \sum_{i=0}^j \binom{j}{i} 1^i 1^{j-i} \\ &= e^{-2\lambda} \lambda^j \sum_{i=0}^j \binom{j}{i} \\ &= e^{-2\lambda} \lambda^j \sum_{i=0}^j \binom{j}{i} \\ &= e^{-2\lambda} \lambda^j \sum_{i=0}^j \binom{j}{i} \end{aligned}$$

$$\begin{aligned} (b) P(X = i) &= \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda} \lambda^j / j! \\ &= \frac{1}{i!} \sum_{j=i}^{\infty} \frac{j!}{(j-i)!} e^{-2\lambda} \lambda^j \end{aligned}$$

$$\begin{aligned}
 & \frac{i! e^{-2\lambda} \sum_{j=i} (j-i)! \lambda^j}{i!} \\
 &= \lambda^i e^{-2\lambda} \sum_{k=0} \lambda^k / k! \\
 &= e^{-\lambda} \lambda^i / i!
 \end{aligned}$$

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