Contents

Preface

These are my own solutions to the problems in *Introduction to Quantum Mechanics, 2nd ed.* I have made every effort to insure that they are clear and correct, but errors are bound to occur, and for this I apologize in advance. I would like to thank the many people who pointed out mistakes in the solution manual for the first edition, and encourage anyone who finds defects in this one to alert me (griffith@reed.edu). I'll maintain a list of errata on my web page (http://academic.reed.edu/physics/faculty/griffiths.html), and incorporate corrections in the manual itself from time to time. I also thank my students at Reed and at Smith for many useful suggestions, and above all Neelaksh Sadhoo, who did most of the typesetting.

At the end of the manual there is a grid that correlates the problem numbers in the second edition with those in the first edition.

David Griffiths

Chapter 1

The Wave Function

Problem 1.1

(a)

$$\langle j\rangle^2 = 21^2 = \boxed{441.}$$

$$\begin{aligned}\langle j^2\rangle &= \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[(14^2)+(15^2)+3(16^2)+2(22^2)+2(24^2)+5(25^2)\right]\\ &= \frac{1}{14}(196+225+768+968+1152+3125) = \frac{6434}{14} = \boxed{459.571.}\end{aligned}$$

(b)

j	$\Delta j = j - \langle j\rangle$
14	$14-21=-7$
15	$15-21=-6$
16	$16-21=-5$
22	$22-21=1$
24	$24-21=3$
25	$25-21=4$

$$\begin{aligned}\sigma^2 &= \frac{1}{N}\sum(\Delta j)^2 N(j) = \frac{1}{14}\left[(-7)^2+(-6)^2+(-5)^2\cdot 3+(1)^2\cdot 2+(3)^2\cdot 2+(4)^2\cdot 5\right]\\ &= \frac{1}{14}(49+36+75+2+18+80) = \frac{260}{14} = \boxed{18.571.}\end{aligned}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571. \qquad \text{[Agrees with (b).]}$$

Problem 1.2

(a)

$$\langle x^2 \rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}} dx = \frac{1}{2\sqrt{h}} \left(\frac{2}{5} x^{5/2} \right) \Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3} \right)^2 = \frac{4}{45} h^2 \ \Rightarrow \ \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h.}$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}} dx = 1 - \frac{1}{2\sqrt{h}} (2\sqrt{x}) \Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}} \left(\sqrt{x_+} - \sqrt{x_-} \right).$$

$$x_+ \equiv \langle x \rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x \rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393.}$$

Problem 1.3

(a)

$$1 = \int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2} dx. \qquad \text{Let } u \equiv x - a,\ du = dx,\ u : -\infty \to \infty.$$

$$1 = A \int_{-\infty}^{\infty} e^{-\lambda u^2} du = A \sqrt{\frac{\pi}{\lambda}} \quad \Rightarrow \quad \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$

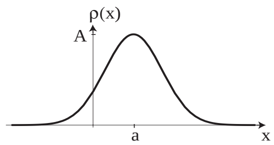
(b)

$$\begin{aligned}
\langle x \rangle &= A \int_{-\infty}^{\infty} x e^{-\lambda(x-a)^2} dx = A \int_{-\infty}^{\infty} (u+a) e^{-\lambda u^2} du \\
&= A \left[\int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right] = A \left(0 + a \sqrt{\frac{\pi}{\lambda}} \right) = \boxed{a.}
\end{aligned}$$

$$\begin{aligned}
\langle x^2 \rangle &= A \int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx \\
&= A \left\{ \int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du + 2a \int_{-\infty}^{\infty} u e^{-\lambda u^2} du + a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du \right\} \\
&= A \left[\frac{1}{2\lambda} \sqrt{\frac{\pi}{\lambda}} + 0 + a^2 \sqrt{\frac{\pi}{\lambda}} \right] = \boxed{a^2 + \frac{1}{2\lambda}.}
\end{aligned}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$

(c)



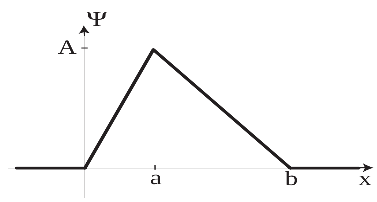
Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2 \left\{ \frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b \right\}$$

$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3} \Rightarrow \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.}\begin{cases} P = 1 & \text{if } \; b = a, \checkmark \\ P = 1/2 & \text{if } \; b = 2a. \checkmark \end{cases}$$

(e)

$$\langle x\rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Big|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

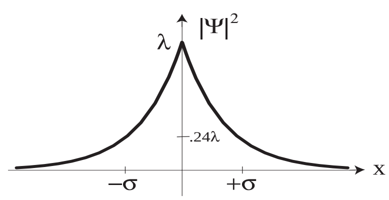
(b)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Big|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.6

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2) dx \neq (x|\Psi|^2)\Big|_a^b.$$

Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar\int \frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$. But, noting that $\frac{\partial^2\Psi}{\partial x\partial t} = \frac{\partial^2\Psi}{\partial t\partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$
$$= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*V\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial V}{\partial x}\Psi = -|\Psi|^2\frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2\frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0t/\hbar}$.

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}$

$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0.$ QED

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \quad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$
$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2x^2.}$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2 \int_0^{\infty} x^2 e^{-2amx^2/\hbar} dx = 2|A|^2 \frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

$$\begin{aligned}
\langle p^2\rangle &= \int \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \Psi dx = -\hbar^2 \int \Psi^* \frac{\partial^2\Psi}{\partial x^2} dx \\
&= -\hbar^2 \int \Psi^* \left[-\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi\right] dx = 2am\hbar \left\{\int |\Psi|^2 dx - \frac{2am}{\hbar}\int x^2|\Psi|^2 dx\right\} \\
&= 2am\hbar\left(1 - \frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1 - \frac{2am}{\hbar}\frac{\hbar}{4am}\right) = 2am\hbar\left(\frac{1}{2}\right) = \boxed{am\hbar.}
\end{aligned}$$

(d)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Longrightarrow \boxed{\sigma_x = \sqrt{\frac{\hbar}{4am}}}; \quad \sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Longrightarrow \boxed{\sigma_p = \sqrt{am\hbar}.}$$

$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar} = \frac{\hbar}{2}$. This *is* (just barely) consistent with the uncertainty principle.

Problem 1.10

From Math Tables: $\pi = 3.141592653589793238462643\cdots$

(a)
$$\boxed{\begin{array}{lllll} P(0)=0 & P(1)=2/25 & P(2)=3/25 & P(3)=5/25 & P(4)=3/25 \\ P(5)=3/25 & P(6)=3/25 & P(7)=1/25 & P(8)=2/25 & P(9)=3/25 \end{array}}$$

In general, $P(j) = \frac{N(j)}{N}$.

(b) *Most probable*: $\boxed{3.}$ *Median*: 13 are ≤ 4, 12 are ≥ 5, so median is $\boxed{4.}$

Average: $\langle j\rangle = \frac{1}{25}[0\cdot 0 + 1\cdot 2 + 2\cdot 3 + 3\cdot 5 + 4\cdot 3 + 5\cdot 3 + 6\cdot 3 + 7\cdot 1 + 8\cdot 2 + 9\cdot 3]$

$= \frac{1}{25}[0 + 2 + 6 + 15 + 12 + 15 + 18 + 7 + 16 + 27] = \frac{118}{25} = \boxed{4.72.}$

(c) $\langle j^2\rangle = \frac{1}{25}[0 + 1^2\cdot 2 + 2^2\cdot 3 + 3^2\cdot 5 + 4^2\cdot 3 + 5^2\cdot 3 + 6^2\cdot 3 + 7^2\cdot 1 + 8^2\cdot 2 + 9^2\cdot 3]$

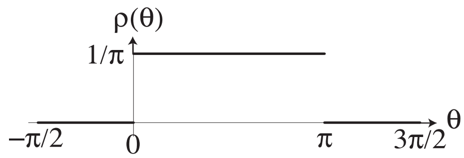
$= \frac{1}{25}[0 + 2 + 12 + 45 + 48 + 75 + 108 + 49 + 128 + 243] = \frac{710}{25} = \boxed{28.4.}$

$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 = 28.4 - 4.72^2 = 28.4 - 22.2784 = 6.1216$; $\quad \sigma = \sqrt{6.1216} = \boxed{2.474.}$

Problem 1.11

(a) Constant for $0 \le \theta \le \pi$, otherwise zero. In view of Eq. 1.16, the constant is $1/\pi$.

$$\boxed{\rho(\theta) = \begin{cases} 1/\pi, & \text{if } 0 \le \theta \le \pi, \\ 0, & \text{otherwise.} \end{cases}}$$



(b)

$$\langle\theta\rangle = \int \theta\rho(\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \theta d\theta = \frac{1}{\pi}\left(\frac{\theta^2}{2}\right)\Big|_0^\pi = \boxed{\frac{\pi}{2}} \quad \text{[of course]}.$$

$$\langle\theta^2\rangle = \frac{1}{\pi}\int_0^\pi \theta^2\,d\theta = \frac{1}{\pi}\left(\frac{\theta^3}{3}\right)\Big|_0^\pi = \boxed{\frac{\pi^2}{3}.}$$

$$\sigma^2 = \langle\theta^2\rangle - \langle\theta\rangle^2 = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{12}; \quad \boxed{\sigma = \frac{\pi}{2\sqrt{3}}.}$$

(c)

$$\langle\sin\theta\rangle = \frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{1}{\pi}\,(-\cos\theta)|_0^\pi = \frac{1}{\pi}(1-(-1)) = \boxed{\frac{2}{\pi}.}$$

$$\langle\cos\theta\rangle = \frac{1}{\pi}\int_0^\pi \cos\theta\,d\theta = \frac{1}{\pi}\,(\sin\theta)|_0^\pi = \boxed{0.}$$

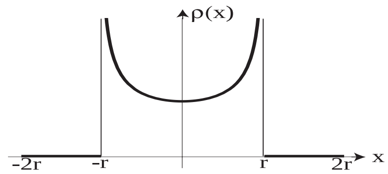
$$\langle\cos^2\theta\rangle = \frac{1}{\pi}\int_0^\pi \cos^2\theta\,d\theta = \frac{1}{\pi}\int_0^\pi (1/2)d\theta = \boxed{\frac{1}{2}.}$$

[Because $\sin^2\theta + \cos^2\theta = 1$, and the integrals of $\sin^2$ and $\cos^2$ are equal (over suitable intervals), one can replace them by 1/2 in such cases.]

Problem 1.12

(a) $x = r\cos\theta \Rightarrow dx = -r\sin\theta\,d\theta$. The probability that the needle lies in range $d\theta$ is $\rho(\theta)d\theta = \frac{1}{\pi}d\theta$, so the probability that it's in the range dx is

$$\rho(x)dx = \frac{1}{\pi}\frac{dx}{r\sin\theta} = \frac{1}{\pi}\frac{dx}{r\sqrt{1-(x/r)^2}} = \frac{dx}{\pi\sqrt{r^2-x^2}}.$$



$$\therefore \boxed{\rho(x) = \begin{cases} \frac{1}{\pi\sqrt{r^2-x^2}}, & \text{if } -r < x < r, \\ 0, & \text{otherwise.} \end{cases}}$$ [*Note:* We want the *magnitude* of dx here.]

Total: $\int_{-r}^{r} \frac{1}{\pi\sqrt{r^2-x^2}}dx = \frac{2}{\pi}\int_0^r \frac{1}{\sqrt{r^2-x^2}}dx = \frac{2}{\pi}\sin^{-1}\left.\frac{x}{r}\right|_0^r = \frac{2}{\pi}\sin^{-1}(1) = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.\checkmark$

(b)

$$\langle x\rangle = \frac{1}{\pi}\int_{-r}^{r} x\frac{1}{\sqrt{r^2-x^2}}dx = \boxed{0} \quad \text{[odd integrand, even interval].}$$

$$\langle x^2\rangle = \frac{2}{\pi}\int_0^r \frac{x^2}{\sqrt{r^2-x^2}}dx = \frac{2}{\pi}\left[-\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right]\Bigg|_0^r = \frac{2}{\pi}\frac{r^2}{2}\sin^{-1}(1) = \boxed{\frac{r^2}{2}.}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = r^2/2 \Longrightarrow \boxed{\sigma = r/\sqrt{2}.}$$

To get $\langle x\rangle$ and $\langle x^2\rangle$ from Problem 1.11(c), use $x = r\cos\theta$, so $\langle x\rangle = r\langle\cos\theta\rangle = 0$, $\langle x^2\rangle = r^2\langle\cos^2\theta\rangle = r^2/2$.

Problem 1.13

Suppose the eye end lands a distance y up from a line ($0 \le y < l$), and let x be the projection along that same direction ($-l \le x < l$). The needle crosses the line above if $y + x \ge l$ (i.e. $x \ge l - y$), and it crosses the line below if $y + x < 0$ (i.e. $x < -y$). So for a given value of y, the probability of crossing (using Problem 1.12) is

$$P(y) = \int_{-l}^{-y}\rho(x)dx + \int_{l-y}^{l}\rho(x)dx = \frac{1}{\pi}\left\{\int_{-l}^{-y}\frac{1}{\sqrt{l^2-x^2}}dx + \int_{l-y}^{l}\frac{1}{\sqrt{l^2-x^2}}dx\right\}$$

$$= \frac{1}{\pi}\left\{\sin^{-1}\left(\frac{x}{l}\right)\Big|_{-l}^{-y} + \sin^{-1}\left(\frac{x}{l}\right)\Big|_{l-y}^{l}\right\} = \frac{1}{\pi}\left[-\sin^{-1}(y/l) + 2\sin^{-1}(1) - \sin^{-1}(1-y/l)\right]$$

$$= 1 - \frac{\sin^{-1}(y/l)}{\pi} - \frac{\sin^{-1}(1-y/l)}{\pi}.$$

Now, all values of y are equally likely, so $\rho(y) = 1/l$, and hence the probability of crossing is

$$P = \frac{1}{\pi l}\int_0^l\left[\pi - \sin^{-1}\left(\frac{y}{l}\right) - \sin^{-1}\left(\frac{l-y}{l}\right)\right]dy = \frac{1}{\pi l}\int_0^l\left[\pi - 2\sin^{-1}(y/l)\right]dy$$

$$= \frac{1}{\pi l}\left[\pi l - 2\left(y\sin^{-1}(y/l) + l\sqrt{1-(y/l)^2}\right)\Big|_0^l\right] = 1 - \frac{2}{\pi l}[l\sin^{-1}(1) - l] = 1 - 1 + \frac{2}{\pi} = \boxed{\frac{2}{\pi}.}$$

Problem 1.14

(a) $P_{ab}(t) = \int_a^b |\Psi(x,t)^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (Eq. 1.25):

$$\frac{\partial |\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial t}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)dx = -\left[J(x,t)\right]\Big|_a^b = J(a,t) - J(b,t). \qquad \text{QED}$$

Probability is dimensionless, so J has the dimensions 1/time, and units $\boxed{\text{seconds}^{-1}.}$

(b) Here $\Psi(x,t) = f(x)e^{-iat}$, where $f(x) \equiv Ae^{-amx^2/\hbar}$, so $\Psi\frac{\partial \Psi^*}{\partial x} = fe^{-iat}\frac{df}{dx}e^{iat} = f\frac{df}{dx}$, and $\Psi^*\frac{\partial \Psi}{\partial x} = f\frac{df}{dx}$ too, so $\boxed{J(x,t) = 0.}$

Problem 1.15

(a) Eq. 1.24 now reads $\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V^*\Psi^*$, and Eq. 1.25 picks up an extra term:

$$\frac{\partial}{\partial t}|\Psi|^2 = \cdots + \frac{i}{\hbar}|\Psi|^2(V^* - V) = \cdots + \frac{i}{\hbar}|\Psi|^2(V_0 + i\Gamma - V_0 + i\Gamma) = \cdots - \frac{2\Gamma}{\hbar}|\Psi|^2,$$

and Eq. 1.27 becomes $\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2 dx = -\frac{2\Gamma}{\hbar}P.$ QED

(b)

$$\frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \Longrightarrow \ln P = -\frac{2\Gamma}{\hbar}t + \text{constant} \Longrightarrow \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar},} \text{ so } \boxed{\tau = \frac{\hbar}{2\Gamma}.}$$

Problem 1.16

Use Eqs. [1.23] and [1.24], and integration by parts:

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\Psi_1^*\Psi_2\right)\,dx = \int_{-\infty}^{\infty}\left(\frac{\partial \Psi_1^*}{\partial t}\Psi_2 + \Psi_1^*\frac{\partial \Psi_2}{\partial t}\right)\,dx \\
&= \int_{-\infty}^{\infty}\left[\left(\frac{-i\hbar}{2m}\frac{\partial^2 \Psi_1^*}{\partial x^2} + \frac{i}{\hbar}V\Psi_1^*\right)\Psi_2 + \Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2 \Psi_2}{\partial x^2} - \frac{i}{\hbar}V\Psi_2\right)\right]\,dx \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\frac{\partial^2 \Psi_1^*}{\partial x^2}\Psi_2 - \Psi_1^*\frac{\partial^2 \Psi_2}{\partial x^2}\right)\,dx \\
&= -\frac{i\hbar}{2m}\left[\frac{\partial \Psi_1^*}{\partial x}\Psi_2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial \Psi_1^*}{\partial x}\frac{\partial \Psi_2}{\partial x}\,dx - \Psi_1^*\frac{\partial \Psi_2}{\partial x}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial \Psi_1^*}{\partial x}\frac{\partial \Psi_2}{\partial x}\,dx\right] = 0. \text{ QED}
\end{aligned}$$

Problem 1.17

(a)

$$1 = |A|^2 \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx = 2|A|^2 \int_0^a \left(a^4 - 2a^2x^2 + x^4\right) dx = 2|A|^2 \left[a^4 x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\right]\Bigg|_0^a$$

$$= 2|A|^2 a^5 \left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}a^5|A|^2, \text{ so } \boxed{A = \sqrt{\frac{15}{16a^5}}.}$$

(b)

$$\langle x\rangle = \int_{-a}^{a} x|\Psi|^2\, dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

(c)

$$\langle p\rangle = \frac{\hbar}{i}A^2 \int_{-a}^{a} \left(a^2 - x^2\right) \underbrace{\frac{d}{dx}\left(a^2 - x^2\right)}_{-2x} dx = \boxed{0.} \quad \text{(Odd integrand.)}$$

Since we only know $\langle x\rangle$ at $t = 0$ we cannot calculate $d\langle x\rangle/dt$ directly.

(d)

$$\langle x^2\rangle = A^2 \int_{-a}^{a} x^2\left(a^2 - x^2\right)^2 dx = 2A^2 \int_0^a \left(a^4x^2 - 2a^2x^4 + x^6\right) dx$$

$$= 2\frac{15}{16a^5}\left[a^4\frac{x^3}{3} - 2a^2\frac{x^5}{5} + \frac{x^7}{7}\right]\Bigg|_0^a = \frac{15}{8a^5}\left(a^7\right)\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right)$$

$$= \frac{\cancel{15}a^2}{8}\left(\frac{35 - 42 + 15}{\cancel{3}\cdot\cancel{5}\cdot 7}\right) = \frac{a^2}{8}\cdot\frac{8}{7} = \boxed{\frac{a^2}{7}.}$$

(e)

$$\langle p^2\rangle = -A^2\hbar^2 \int_{-a}^{a} \left(a^2 - x^2\right) \underbrace{\frac{d^2}{dx^2}\left(a^2 - x^2\right)}_{-2} dx = 2A^2\hbar^2 2\int_0^a \left(a^2 - x^2\right) dx$$

$$= 4\cdot\frac{15}{16a^5}\hbar^2\left(a^2x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{15\hbar^2}{4a^5}\left(a^3 - \frac{a^3}{3}\right) = \frac{15\hbar^2}{4a^2}\cdot\frac{2}{3} = \boxed{\frac{5}{2}\frac{\hbar^2}{a^2}.}$$

(f)

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{7}a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g)

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5}{2}\frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}}\frac{\hbar}{a}.}$$

(h)

$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}} \cdot \sqrt{\frac{5}{2}}\frac{\hbar}{a} = \sqrt{\frac{5}{14}}\hbar = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$

Problem 1.18

$$\frac{h}{\sqrt{3mk_BT}} > d \;\Rightarrow\; T < \frac{h^2}{3mk_Bd^2}.$$

(a) Electrons ($m = 9.1 \times 10^{-31}$ kg):

$$T < \frac{(6.6\times 10^{-34})^2}{3(9.1\times 10^{-31})(1.4\times 10^{-23})(3\times 10^{-10})^2} = \boxed{1.3\times 10^5 \text{ K.}}$$

Sodium nuclei ($m = 23m_p = 23(1.7\times 10^{-27}) = 3.9\times 10^{-26}$ kg):

$$T < \frac{(6.6\times 10^{-34})^2}{3(3.9\times 10^{-26})(1.4\times 10^{-23})(3\times 10^{-10})^2} = \boxed{3.0 \text{ K.}}$$

(b) $PV = Nk_BT$; volume occupied by one molecule ($N = 1,\ V = d^3$) $\Rightarrow d = (k_BT/P)^{1/3}$.

$$T < \frac{h^2}{2mk_B}\left(\frac{P}{k_BT}\right)^{2/3} \;\Rightarrow\; T^{5/3} < \frac{h^2}{3m}\frac{P^{2/3}}{k_B^{5/3}} \;\Rightarrow T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}.$$

For helium ($m = 4m_p = 6.8\times 10^{-27}$ kg) at 1 atm $= 1.0\times 10^5$ N/m^2:

$$T < \frac{1}{(1.4\times 10^{-23})}\left(\frac{(6.6\times 10^{-34})^2}{3(6.8\times 10^{-27})}\right)^{3/5}(1.0\times 10^5)^{2/5} = \boxed{2.8 \text{ K.}}$$

For hydrogen ($m = 2m_p = 3.4\times 10^{-27}$ kg) with $d = 0.01$ m:

$$T < \frac{(6.6\times 10^{-34})^2}{3(3.4\times 10^{-27})(1.4\times 10^{-23})(10^{-2})^2} = \boxed{3.1\times 10^{-14} \text{ K.}}$$

At 3 K it is definitely in the classical regime.

Chapter 2

Time-Independent Schrödinger Equation

Problem 2.1

(a)

$$\Psi(x,t) = \psi(x)e^{-i(E_0+i\Gamma)t/\hbar} = \psi(x)e^{\Gamma t/\hbar}e^{-iE_0t/\hbar} \Longrightarrow |\Psi|^2 = |\psi|^2 e^{2\Gamma t/\hbar}.$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = e^{2\Gamma t/\hbar} \int_{-\infty}^{\infty} |\psi|^2 dx.$$

The second term is independent of t, so if the product is to be 1 for all time, the first term ($e^{2\Gamma t/\hbar}$) must also be constant, and hence $\Gamma = 0$. QED

(b) If ψ satisfies Eq. 2.5, $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} + V\psi = E\psi$, then (taking the complex conjugate and noting that V and E are real): $-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{dx^2} + V\psi^* = E\psi^*$, so ψ^* *also* satisfies Eq. 2.5. Now, if ψ_1 and ψ_2 satisfy Eq. 2.5, so too does any linear combination of them ($\psi_3 \equiv c_1\psi_1 + c_2\psi_2$):

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\psi_3}{dx^2} + V\psi_3 &= -\frac{\hbar^2}{2m}\left(c_1\frac{\partial^2\psi_1}{dx^2} + c_2\frac{\partial^2\psi_2}{\partial x^2}\right) + V(c_1\psi_1 + c_2\psi_2) \\
&= c_1\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1\right] + c_2\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2\right] \\
&= c_1(E\psi_1) + c_2(E\psi_2) = E(c_1\psi_1 + c_2\psi_2) = E\psi_3.
\end{aligned}$$

Thus, $(\psi + \psi^*)$ and $i(\psi - \psi^*)$ – both of which are *real* – satisfy Eq. 2.5. *Conclusion:* From any complex solution, we can always construct two *real* solutions (of course, if ψ is already real, the second one will be zero). In particular, since $\psi = \frac{1}{2}[(\psi + \psi^*) - i(i(\psi - \psi^*))]$, ψ can be expressed as a linear combination of two real solutions. QED

(c) If $\psi(x)$ satisfies Eq. 2.5, then, changing variables $x \to -x$ and noting that $\partial^2/\partial(-x)^2 = \partial^2/\partial x^2$,

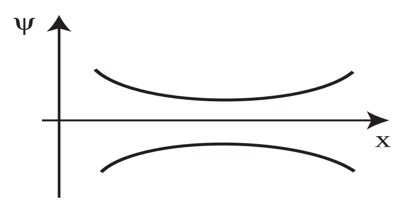
$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x);$$

so if $V(-x) = V(x)$ then $\psi(-x)$ *also* satisfies Eq. 2.5. It follows that $\psi_+(x) \equiv \psi(x) + \psi(-x)$ (which is *even*: $\psi_+(-x) = \psi_+(x)$) and $\psi_-(x) \equiv \psi(x) - \psi(-x)$ (which is *odd*: $\psi_-(-x) = -\psi_-(x)$) both satisfy Eq.

2.5. But $\psi(x) = \frac{1}{2}(\psi_+(x) + \psi_-(x))$, so any solution can be expressed as a linear combination of even and odd solutions. QED

Problem 2.2

Given $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$, if $E < V_{\text{min}}$, then ψ'' and ψ always have the same sign: If ψ is positive(negative), then ψ'' is also positive(negative). This means that ψ always curves away from the axis (see Figure). However, it has got to go to zero as $x \to -\infty$ (else it would not be normalizable). At some point it's got to *depart* from zero (if it *doesn't*, it's going to be identically zero *everywhere*), in (say) the positive direction. At this point its slope is positive, and *increasing*, so ψ gets bigger and bigger as x increases. It can't ever "turn over" and head back toward the axis, because that would requuire a negative second derivative—it always has to bend away from the axis. By the same token, if it starts out heading negative, it just runs more and more negative. In neither case is there any way for it to come back to zero, as it must (at $x \to \infty$) in order to be normalizable. QED



Problem 2.3

Equation 2.20 says $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$; Eq. 2.23 says $\psi(0) = \psi(a) = 0$. If $E = 0$, $d^2\psi/dx^2 = 0$, so $\psi(x) = A + Bx$; $\psi(0) = A = 0 \Rightarrow \psi = Bx$; $\psi(a) = Ba = 0 \Rightarrow B = 0$, so $\psi = 0$. If $E < 0$, $d^2\psi/dx^2 = \kappa^2\psi$, with $\kappa \equiv \sqrt{-2mE}/\hbar$ real, so $\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$. This time $\psi(0) = A + B = 0 \Rightarrow B = -A$, so $\psi = A(e^{\kappa x} - e^{-\kappa x})$, while $\psi(a) = A\left(e^{\kappa a} - e^{i\kappa a}\right) = 0 \Rightarrow$ either $A = 0$, so $\psi = 0$, or else $e^{\kappa a} = e^{-\kappa a}$, so $e^{2\kappa a} = 1$, so $2\kappa a = \ln(1) = 0$, so $\kappa = 0$, and again $\psi = 0$. In all cases, then, the boundary conditions force $\psi = 0$, which is unacceptable (non-normalizable).

Problem 2.4

$$\langle x \rangle = \int x|\psi|^2 dx = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx. \qquad \text{Let } y \equiv \frac{n\pi}{a}x, \text{ so } dx = \frac{a}{n\pi}dy; \quad y: 0 \to n\pi.$$
$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} y\sin^2 y\,dy = \frac{2a}{n^2\pi^2}\left[\frac{y^2}{4} - \frac{y\sin 2y}{4} - \frac{\cos 2y}{8}\right]\Bigg|_0^{n\pi}$$
$$= \frac{2a}{n^2\pi^2}\left[\frac{n^2\pi^2}{4} - \frac{\cos 2n\pi}{8} + \frac{1}{8}\right] = \boxed{\frac{a}{2}.} \quad \text{(Independent of } n.)$$

$$\begin{aligned}\langle x^2\rangle &= \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2}{a}\left(\frac{a}{n\pi}\right)^3\int_0^{n\pi} y^2\sin^2 y\,dy\\ &= \frac{2a^2}{(n\pi)^3}\left[\frac{y^3}{6}-\left(\frac{y^3}{4}-\frac{1}{8}\right)\sin 2y - \frac{y\cos 2y}{4}\right]_0^{n\pi}\\ &= \frac{2a^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6}-\frac{n\pi\cos(2n\pi)}{4}\right] = \boxed{a^2\left[\frac{1}{3}-\frac{1}{2(n\pi)^2}\right]}.\end{aligned}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad (\textit{Note}: \text{Eq. 1.33 is much faster than Eq. 1.35.})$$

$$\begin{aligned}\langle p^2\rangle &= \int \psi_n^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi_n\,dx = -\hbar^2\int\psi_n^*\left(\frac{d^2\psi_n}{dx^2}\right)dx\\ &= (-\hbar^2)\left(-\frac{2mE_n}{\hbar^2}\right)\int\psi_n^*\psi_n\,dx = 2mE_n = \boxed{\left(\frac{n\pi\hbar}{a}\right)^2.}\end{aligned}$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3}-\frac{1}{2(n\pi)^2}-\frac{1}{4}\right) = \frac{a^2}{4}\left(\frac{1}{3}-\frac{2}{(n\pi)^2}\right); \quad \boxed{\sigma_x = \frac{a}{2}\sqrt{\frac{1}{3}-\frac{2}{(n\pi)^2}}.}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \left(\frac{n\pi\hbar}{a}\right)^2; \quad \boxed{\sigma_p = \frac{n\pi\hbar}{a}.} \quad \therefore \sigma_x\sigma_p = \boxed{\frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3}-2}.}$$

The product $\sigma_x\sigma_p$ is $\boxed{\text{smallest for } n=1;}$ in that case, $\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3}-2} = (1.136)\hbar/2 > \hbar/2.$ ✓

Problem 2.5

(a)

$$|\Psi|^2 = \Psi^2\Psi = |A|^2(\psi_1^*+\psi_2^*)(\psi_1+\psi_2) = |A|^2[\psi_1^*\psi_1+\psi_1^*\psi_2+\psi_2^*\psi_1+\psi_2^*\psi_2].$$

$$1 = \int|\Psi|^2dx = |A|^2\int[|\psi_1|^2+\psi_1^*\psi_2+\psi_2^*\psi_1+|\psi_2|^2]dx = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}.}$$

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1t/\hbar}+\psi_2 e^{-iE_2t/\hbar}\right] \quad \left(\text{but } \frac{E_n}{\hbar} = n^2\omega\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t}+\sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right)+\sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right]}.$$

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{-3i\omega t}+e^{3i\omega t}\right)+\sin^2\left(\frac{2\pi}{a}x\right)\right]\\ &= \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin^2\left(\frac{2\pi}{a}x\right)+2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]}.\end{aligned}$$

(c)

$$\langle x\rangle = \int x|\Psi(x,t)|^2 dx$$
$$= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin^2\left(\frac{2\pi}{a}x\right)+2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]dx$$

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right)dx = \left[\frac{x^2}{4}-\frac{x\sin\left(\frac{2\pi}{a}x\right)}{4\pi/a}-\frac{\cos\left(\frac{2\pi}{a}x\right)}{8(\pi/a)^2}\right]\Bigg|_0^a = \frac{a^2}{4} = \int_0^a x\sin^2\left(\frac{2\pi}{a}x\right)dx.$$

$$\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)dx = \frac{1}{2}\int_0^a x\left[\cos\left(\frac{\pi}{a}x\right)-\cos\left(\frac{3\pi}{a}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\cos\left(\frac{\pi}{a}x\right)+\frac{ax}{\pi}\sin\left(\frac{\pi}{a}x\right)-\frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right)-\frac{ax}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\big(\cos(\pi)-\cos(0)\big)-\frac{a^2}{9\pi^2}\big(\cos(3\pi)-\cos(0)\big)\right] = -\frac{a^2}{\pi^2}\left(1-\frac{1}{9}\right) = -\frac{8a^2}{9\pi^2}.$$

$$\therefore \langle x\rangle = \frac{1}{a}\left[\frac{a^2}{4}+\frac{a^2}{4}-\frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \boxed{\frac{a}{2}\left[1-\frac{32}{9\pi^2}\cos(3\omega t)\right].}$$

Amplitude: $\boxed{\frac{32}{9\pi^2}\left(\frac{a}{2}\right) = 0.3603(a/2);}$ *angular frequency:* $\boxed{3\omega = \frac{3\pi^2\hbar}{2ma^2}.}$

(d)

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\left(\frac{a}{2}\right)\left(-\frac{32}{9\pi^2}\right)(-3\omega)\sin(3\omega t) = \boxed{\frac{8\hbar}{3a}\sin(3\omega t).}$$

(e) You could get either $\boxed{E_1 = \pi^2\hbar^2/2ma^2}$ or $\boxed{E_2 = 2\pi^2\hbar^2/ma^2,}$ with equal probability $\boxed{P_1 = P_2 = 1/2.}$

So $\langle H\rangle = \boxed{\frac{1}{2}(E_1+E_2) = \frac{5\pi^2\hbar^2}{4ma^2};}$ it's the *average* of E_1 and E_2.

Problem 2.6

From Problem 2.5, we see that

$$\Psi(x,t) = \boxed{\tfrac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\tfrac{\pi}{a}x\right)+\sin\left(\tfrac{2\pi}{a}x\right)e^{-3i\omega t}e^{i\phi}\right];}$$

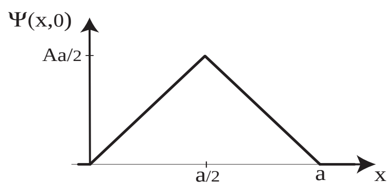
$$|\Psi(x,t)|^2 = \boxed{\tfrac{1}{a}\left[\sin^2\left(\tfrac{\pi}{a}x\right)+\sin^2\left(\tfrac{2\pi}{a}x\right)+2\sin\left(\tfrac{\pi}{a}x\right)\sin\left(\tfrac{2\pi}{a}x\right)\cos(3\omega t-\phi)\right];}$$

and hence $\boxed{\langle x\rangle = \frac{a}{2}\left[1-\frac{32}{9\pi^2}\cos(3\omega t-\phi)\right].}$ This amounts physically to starting the clock at a different time (i.e., shifting the $t=0$ point).

If $\phi = \dfrac{\pi}{2}$, so $\Psi(x,0) = A[\psi_1(x) + i\psi_2(x)]$, then $\cos(3\omega t - \phi) = \sin(3\omega t)$; $\langle x\rangle$ starts at $\dfrac{a}{2}$.

If $\phi = \pi$, so $\Psi(x,0) = A[\psi_1(x) - \psi_2(x)]$, then $\cos(3\omega t - \phi) = -\cos(3\omega t)$; $\langle x\rangle$ starts at $\dfrac{a}{2}\left(1+\dfrac{32}{9\pi^2}\right)$.

Problem 2.7



(a)

$$1 = A^2\int_0^{a/2} x^2dx + A^2\int_{a/2}^a (a-x)^2dx = A^2\left[\left.\frac{x^3}{3}\right|_0^{a/2} - \left.\frac{(a-x)^3}{3}\right|_{a/2}^a\right]$$

$$= \frac{A^2}{3}\left(\frac{a^3}{8}+\frac{a^3}{8}\right) = \frac{A^2a^3}{12} \Rightarrow \boxed{A = \frac{2\sqrt{3}}{\sqrt{a^3}}.}$$

(b)

$$c_n = \sqrt{\frac{2}{a}}\frac{2\sqrt{3}}{a\sqrt{a}}\left[\int_0^{a/2} x\sin\left(\frac{n\pi}{a}x\right)dx + \int_{a/2}^a (a-x)\sin\left(\frac{n\pi}{a}x\right)dx\right]$$

$$= \frac{2\sqrt{6}}{a^2}\left\{\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{a}x\right) - \frac{xa}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Bigg|_0^{a/2}\right.$$

$$\left. + a\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Bigg|_{a/2}^a - \left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{a}x\right) - \left(\frac{ax}{n\pi}\right)\cos\left(\frac{n\pi}{a}x\right)\right]\Bigg|_{a/2}^a\right\}$$

$$= \frac{2\sqrt{6}}{a^2}\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{a^2}{n\pi}\cos n\pi} + \cancel{\frac{a^2}{n\pi}\cos\left(\frac{n\pi}{2}\right)}\right.$$

$$\left. + \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{a^2}{n\pi}\cos n\pi} - \cancel{\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)}\right]$$

$$= \frac{2\sqrt{6}}{\cancel{a^2}}2\frac{\cancel{a^2}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) = \frac{4\sqrt{6}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even}, \\ (-1)^{(n-1)/2}\frac{4\sqrt{6}}{(n\pi)^2}, & n \text{ odd}. \end{cases}$$

So $\boxed{\Psi(x,t) = \frac{4\sqrt{6}}{\pi^2}\sqrt{\frac{2}{a}}\sum_{n=1,3,5,\ldots}(-1)^{(n-1)/2}\frac{1}{n^2}\sin\left(\frac{n\pi}{a}x\right)e^{-E_nt/\hbar},}$ where $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$.

(c)

$$P_1 = |c_1|^2 = \frac{16 \cdot 6}{\pi^4} = \boxed{0.9855.}$$

(d)

$$\langle H \rangle = \sum |c_n|^2 E_n = \frac{96}{\pi^4} \frac{\pi^2 \hbar^2}{2ma^2} \Bigg(\underbrace{\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots}_{\pi^2/8} \Bigg) = \frac{48\hbar^2}{\pi^2 m a^2} \frac{\pi^2}{8} = \boxed{\frac{6\hbar^2}{ma^2}.}$$

Problem 2.8

(a)

$$\boxed{\Psi(x,0) = \begin{cases} A, & 0 < x < a/2; \\ 0, & \text{otherwise.} \end{cases}} \quad 1 = A^2 \int_0^{a/2} dx = A^2(a/2) \Rightarrow \boxed{A = \sqrt{\frac{2}{a}}.}$$

(b) From Eq. 2.37,

$$c_1 = A\sqrt{\frac{2}{a}} \int_0^{a/2} \sin\left(\frac{\pi}{a}x\right) dx = \frac{2}{a}\left[-\frac{a}{\pi}\cos\left(\frac{\pi}{a}x\right)\right]\Bigg|_0^{a/2} = -\frac{2}{\pi}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = \frac{2}{\pi}.$$

$$P_1 = |c_1|^2 = \boxed{(2/\pi)^2 = 0.4053.}$$

Problem 2.9

$$\hat{H}\Psi(x,0) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[Ax(a-x)\right] = -A\frac{\hbar^2}{2m}\frac{\partial}{\partial x}(a-2x) = A\frac{\hbar^2}{m}.$$

$$\begin{aligned} \int \Psi(x,0)^* \hat{H}\Psi(x,0)\, dx &= A^2\frac{\hbar^2}{m}\int_0^a x(a-x)\, dx = A^2\frac{\hbar^2}{m}\left(a\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^a \\ &= A^2\frac{\hbar^2}{m}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{30}{a^5}\frac{\hbar^2}{m}\frac{a^3}{6} = \boxed{\frac{5\hbar^2}{ma^2}} \end{aligned}$$

(same as Example 2.3).

Problem 2.10

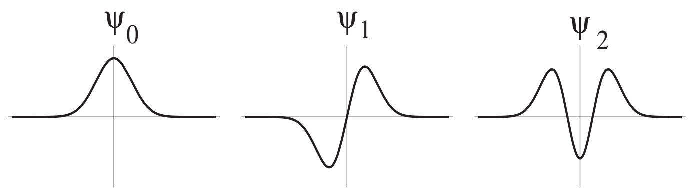
(a) Using Eqs. 2.47 and 2.59,

$$\begin{aligned}
a_+\psi_0 &= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}\\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(-\frac{m\omega}{2\hbar}\right)2x+m\omega x\right]e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}.\\
(a_+)^2\psi_0 &= \frac{1}{2\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}2m\omega\left(-\hbar\frac{d}{dx}+m\omega x\right)xe^{-\frac{m\omega}{2\hbar}x^2}\\
&= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(1-x\frac{m\omega}{2\hbar}2x\right)+m\omega x^2\right]e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega}{2\hbar}x^2}.
\end{aligned}$$

Therefore, from Eq. 2.67,

$$\psi_2 = \frac{1}{\sqrt{2}}(a_+)^2\psi_0 = \boxed{\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega}{2\hbar}x^2}.}$$

(b)



(c) Since ψ_0 and ψ_2 are even, whereas ψ_1 is odd, $\int\psi_0^*\psi_1 dx$ and $\int\psi_2^*\psi_1 dx$ vanish automatically. The only one we need to check is $\int\psi_2^*\psi_0\,dx$:

$$\begin{aligned}
\int\psi_2^*\psi_0\,dx &= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega}{\hbar}x^2}dx\\
&= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\int_{-\infty}^{\infty}e^{-\frac{m\omega}{\hbar}x^2}dx - \frac{2m\omega}{\hbar}\int_{-\infty}^{\infty}x^2e^{-\frac{m\omega}{\hbar}x^2}dx\right)\\
&= -\sqrt{\frac{m\omega}{2\pi\hbar}}\left(\sqrt{\frac{\pi\hbar}{m\omega}}-\frac{2m\omega}{\hbar}\frac{\hbar}{2m\omega}\sqrt{\frac{\pi\hbar}{m\omega}}\right) = 0. \checkmark
\end{aligned}$$

Problem 2.11

(a) Note that ψ_0 is even, and ψ_1 is odd. In either case $|\psi|^2$ is even, so $\langle x\rangle = \int x|\psi|^2dx = \boxed{0.}$ Therefore $\langle p\rangle = md\langle x\rangle/dt = \boxed{0.}$ (These results hold for *any* stationary state of the harmonic oscillator.)

From Eqs. 2.59 and 2.62, $\psi_0 = \alpha e^{-\xi^2/2}$, $\psi_1 = \sqrt{2}\alpha\xi e^{-\xi^2/2}$. So

<u>$n=0$</u>**:**

$$\langle x^2\rangle = \alpha^2\int_{-\infty}^{\infty}x^2e^{-\xi^2/2}dx = \alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^2e^{-\xi^2}d\xi = \frac{1}{\sqrt{\pi}}\left(\frac{\hbar}{m\omega}\right)\frac{\sqrt{\pi}}{2} = \boxed{\frac{\hbar}{2m\omega}.}$$

$$\langle p^2\rangle = \int \psi_0 \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi_0\, dx = -\hbar^2\alpha^2\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty} e^{-\xi^2/2}\left(\frac{d^2}{d\xi^2}e^{-\xi^2/2}\right)d\xi$$
$$= -\frac{m\hbar\omega}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\xi^2-1)e^{-\xi^2/2}d\xi = -\frac{m\hbar\omega}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}-\sqrt{\pi}\right) = \boxed{\frac{m\hbar\omega}{2}.}$$

<u>$n=1$</u>**:**

$$\langle x^2\rangle = 2\alpha^2\int_{-\infty}^{\infty} x^2\xi^2 e^{-\xi^2}dx = 2\alpha^2\left(\frac{\hbar}{m\omega}\right)^{3/2}\int_{-\infty}^{\infty}\xi^4 e^{-\xi^2}d\xi = \frac{2\hbar}{\sqrt{\pi}m\omega}\frac{3\sqrt{\pi}}{4} = \boxed{\frac{3\hbar}{2m\omega}.}$$

$$\langle p^2\rangle = -\hbar^2 2\alpha^2\sqrt{\frac{m\omega}{\hbar}}\int_{-\infty}^{\infty}\xi e^{-\xi^2/2}\left[\frac{d^2}{d\xi^2}\left(\xi e^{-\xi^2/2}\right)\right]d\xi$$
$$= -\frac{2m\omega\hbar}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\xi^4-3\xi^2)e^{-\xi^2}d\xi = -\frac{2m\omega\hbar}{\sqrt{\pi}}\left(\frac{3}{4}\sqrt{\pi}-3\frac{\sqrt{\pi}}{2}\right) = \boxed{\frac{3m\hbar\omega}{2}.}$$

(b) <u>$n=0$</u>**:**

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}};\ \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{m\hbar\omega}{2}};$$

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}. \quad \text{(Right } \textit{at} \text{ the uncertainty limit.)}\checkmark$$

<u>$n=1$</u>**:**

$$\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}};\quad \sigma_p = \sqrt{\frac{3m\hbar\omega}{2}};\quad \sigma_x\sigma_p = 3\frac{\hbar}{2} > \frac{\hbar}{2}.\ \checkmark$$

(c)

$$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \boxed{\left\{\begin{array}{l}\frac{1}{4}\hbar\omega\ (n=0)\\ \\ \frac{3}{4}\hbar\omega\ (n=1)\end{array}\right\}};\quad \langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \boxed{\left\{\begin{array}{l}\frac{1}{4}\hbar\omega\ (n=0)\\ \\ \frac{3}{4}\hbar\omega\ (n=1)\end{array}\right\}}.$$

$$\langle T\rangle + \langle V\rangle = \langle H\rangle = \left\{\begin{array}{l}\frac{1}{2}\hbar\omega\ (n=0) = E_0\\ \\ \frac{3}{2}\hbar\omega\ (n=1) = E_1\end{array}\right\},\ \text{as expected.}$$

Problem 2.12

From Eq. 2.69,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-),\quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-),$$

so

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int \psi_n^*(a_+ + a_-)\psi_n\, dx.$$

But (Eq. 2.66)

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\psi_{n-1}.$$

So

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx + \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right] = \boxed{0} \text{ (by orthogonality).}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right).$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int \psi_n^*\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)\psi_n. \quad \text{But}$$

$$\begin{cases} a_+^2\psi_n & = a_+\left(\sqrt{n+1}\psi_{n+1}\right) & = \sqrt{n+1}\sqrt{n+2}\psi_{n+2} & = \sqrt{(n+1)(n+2)}\psi_{n+2}. \\ a_+a_-\psi_n & = a_+\left(\sqrt{n}\psi_{n-1}\right) & = \sqrt{n}\sqrt{n}\psi_n & = n\psi_n. \\ a_-a_+\psi_n & = a_-\left(\sqrt{n+1}\psi_{n+1}\right) & = \sqrt{n+1)}\sqrt{n+1}\psi_n & = (n+1)\psi_n. \\ a_-^2\psi_n & = a_-\left(\sqrt{n}\psi_{n-1}\right) & = \sqrt{n}\sqrt{n-1}\psi_{n-2} & = \sqrt{(n-1)n}\psi_{n-2}. \end{cases}$$

So

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[0 + n\int |\psi_n|^2 dx + (n+1)\int |\psi_n|^2\,dx + 0\right] = \frac{\hbar}{2m\omega}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}.}$$

$$p^2 = -\frac{\hbar m\omega}{2}(a_+ - a_-)^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right) \Rightarrow$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\left[0 - n - (n+1) + 0\right] = \frac{\hbar m\omega}{2}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)m\hbar\omega.}$$

$$\langle T\rangle = \langle p^2/2m\rangle = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega}.$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{\frac{\hbar}{m\omega}}; \quad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{m\hbar\omega}; \quad \sigma_x\sigma_p = \left(n+\frac{1}{2}\right)\hbar \geq \frac{\hbar}{2}. \checkmark$$

Problem 2.13

(a)

$$\begin{aligned} 1 &= \int |\Psi(x,0)|^2 dx = |A|^2\int \left(9|\psi_0|^2 + 12\psi_0^*\psi_1 + 12\psi_1^*\psi_0 + 16|\psi_1|^2\right)dx \\ &= |A|^2(9+0+0+16) = 25|A|^2 \Rightarrow \boxed{A = 1/5.} \end{aligned}$$

(b)

$$\Psi(x,t) = \frac{1}{5}\left[3\psi_0(x)e^{-iE_0t/\hbar} + 4\psi_1(x)e^{-iE_1t/\hbar}\right] = \boxed{\frac{1}{5}\left[3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}\right].}$$

(Here ψ_0 and ψ_1 are given by Eqs. 2.59 and 2.62; E_1 and E_2 by Eq. 2.61.)

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{25}\left[9\psi_0^2 + 12\psi_0\psi_1 e^{i\omega t/2}e^{-3i\omega t/2} + 12\psi_0\psi_1 e^{-i\omega t/2}e^{3i\omega t/2} + 16\psi_1^2\right] \\ &= \boxed{\frac{1}{25}\left[9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1\cos(\omega t)\right].}\end{aligned}$$

(c)

$$\langle x\rangle = \frac{1}{25}\left[9\int x\psi_0^2\,dx + 16\int x\psi_1^2\,dx + 24\cos(\omega t)\int x\psi_0\psi_1\,dx\right].$$

But $\int x\psi_0^2\,dx = \int x\psi_1^2\,dx = 0$ (see Problem 2.11 or 2.12), while

$$\begin{aligned}\int x\psi_0\psi_1\,dx &= \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\int xe^{-\frac{m\omega}{2\hbar}x^2}xe^{-\frac{m\omega}{2\hbar}x^2}\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)\int_{-\infty}^{\infty} x^2e^{-\frac{m\omega}{\hbar}x^2}\,dx \\ &= \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)2\sqrt{\pi}2\left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^3 = \sqrt{\frac{\hbar}{2m\omega}}.\end{aligned}$$

So

$$\langle x\rangle = \boxed{\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t);}\quad \langle p\rangle = m\frac{d}{dt}\langle x\rangle = \boxed{-\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t).}$$

(With ψ_2 in place of ψ_1 the frequency would be $(E_2 - E_0)/\hbar = [(5/2)\hbar\omega - (1/2)\hbar\omega]/\hbar = 2\omega$.)

Ehrenfest's theorem says $d\langle p\rangle/dt = -\langle\partial V/\partial x\rangle$. Here

$$\frac{d\langle p\rangle}{dt} = -\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\omega\cos(\omega t),\quad V = \frac{1}{2}m\omega^2x^2 \Rightarrow \frac{\partial V}{\partial x} = m\omega^2 x,$$

so

$$-\langle\frac{\partial V}{\partial x}\rangle = -m\omega^2\langle x\rangle = -m\omega^2\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t),$$

so Ehrenfest's theorem holds.

(d) You could get $\boxed{E_0 = \frac{1}{2}\hbar\omega,}$ with probability $|c_0|^2 = \boxed{9/25,}$ or $\boxed{E_1 = \frac{3}{2}\hbar\omega,}$ with probability $|c_1|^2 = \boxed{16/25.}$

Problem 2.14

The new allowed energies are $E_n' = (n+\frac{1}{2})\hbar\omega' = 2(n+\frac{1}{2})\hbar\omega = \hbar\omega, 3\hbar\omega, 5\hbar\omega, \dots$. So the probability of getting $\frac{1}{2}\hbar\omega$ is $\boxed{\text{zero.}}$ The probability of getting $\hbar\omega$ (the new ground state energy) is $P_0 = |c_0|^2$, where $c_0 = \int\Psi(x,0)\psi_0'\,dx$, with

$$\Psi(x,0) = \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2},\quad \psi_0(x)' = \left(\frac{m2\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m2\omega}{2\hbar}x^2}.$$

So

$$c_0 = 2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{3m\omega}{2\hbar}x^2}\,dx = 2^{1/4}\sqrt{\frac{m\omega}{\pi\hbar}}2\sqrt{\pi}\left(\frac{1}{2}\sqrt{\frac{2\hbar}{3m\omega}}\right) = 2^{1/4}\sqrt{\frac{2}{3}}.$$

Therefore

$$P_0 = \boxed{\frac{2}{3}\sqrt{2} = 0.9428.}$$

Problem 2.15

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}, \text{ so } P = 2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{x_0}^{\infty} e^{-\xi^2}dx = 2\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\int_{\xi_0}^{\infty} e^{-\xi^2}d\xi.$$

Classically allowed region extends out to: $\frac{1}{2}m\omega^2x_0^2 = E_0 = \frac{1}{2}\hbar\omega$, or $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, so $\xi_0 = 1$.

$$P = \frac{2}{\sqrt{\pi}}\int_1^{\infty} e^{-\xi^2}d\xi = 2(1 - F(\sqrt{2})) \text{ (in notation of CRC Table)} = \boxed{0.157.}$$

Problem 2.16

<u>$n = 5$:</u> $j = 1 \Rightarrow a_3 = \frac{-2(5-1)}{(1+1)(1+2)}a_1 = -\frac{4}{3}a_1; j = 3 \Rightarrow a_5 = \frac{-2(5-3)}{(3+1)(3+2)}a_3 = -\frac{1}{5}a_3 = \frac{4}{15}a_1;\ j = 5 \Rightarrow a_7 = 0$. So $H_5(\xi) = a_1\xi - \frac{4}{3}a_1\xi^3 + \frac{4}{15}a_1\xi^5 = \frac{a_1}{15}(15\xi - 20\xi^3 + 4\xi^5)$. By convention the coefficient of ξ^5 is 2^5, so $a_1 = 15\cdot 8$, and $\boxed{H_5(\xi) = 120\xi - 160\xi^3 + 32\xi^5}$ (which agrees with Table 2.1).

<u>$n = 6$:</u> $j = 0 \Rightarrow a_2 = \frac{-2(6-0)}{(0+1)(0+2)}a_0 = -6a_0;\ j = 2 \Rightarrow a_4 = \frac{-2(6-2)}{(2+1)(2+2)}a_2 = -\frac{2}{3}a_2 = 4a_0; j = 4 \Rightarrow a_6 = \frac{-2(6-4)}{(4+1)(4+2)}a_4 = -\frac{2}{15}a_4 = -\frac{8}{15}a_0;\ j = 6 \Rightarrow a_8 = 0$. So $H_6(\xi) = a_0 - 6a_0\xi^2 + 4a_0\xi^4 - \frac{8}{15}\xi^6 a_0$. The coefficient of ξ^6 is 2^6, so $2^6 = -\frac{8}{15}a_0 \Rightarrow a_0 = -15\cdot 8 = -120$. $\boxed{H_6(\xi) = -120 + 720\xi^2 - 480\xi^4 + 64\xi^6.}$

Problem 2.17

(a)

$$\frac{d}{d\xi}(e^{-\xi^2}) = -2\xi e^{-\xi^2};\ \left(\frac{d}{d\xi}\right)^2 e^{-\xi^2} = \frac{d}{d\xi}(-2\xi e^{-\xi^2}) = (-2 + 4\xi^2)e^{-\xi^2};$$

$$\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \frac{d}{d\xi}\left[(-2 + 4\xi^2)e^{-\xi^2}\right] = \left[8\xi + (-2 + 4\xi^2)(-2\xi)\right]e^{-\xi^2} = (12\xi - 8\xi^3)e^{-\xi^2};$$

$$\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \frac{d}{d\xi}\left[(12\xi - 8\xi^3)e^{-\xi^2}\right] = \left[12 - 24\xi^2 + (12\xi - 8\xi^3)(-2\xi)\right]e^{-\xi^2} = (12 - 48\xi^2 + 16\xi^4)e^{-\xi^2}.$$

$$H_3(\xi) = -e^{\xi^2}\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \boxed{-12\xi + 8\xi^3;}\ H_4(\xi) = e^{\xi^2}\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \boxed{12 - 48\xi^2 + 16\xi^4.}$$

(b)

$$H_5 = 2\xi H_4 - 8H_3 = 2\xi(12 - 48\xi^2 + 16\xi^4) - 8(-12\xi + 8\xi^3) = \boxed{120\xi - 160\xi^3 + 32\xi^5.}$$

$$H_6 = 2\xi H_5 - 10H_4 = 2\xi(120\xi - 160\xi^3 + 32\xi^5) - 10(12 - 48\xi^2 + 16\xi^4) = \boxed{-120 + 720\xi^2 - 480\xi^4 + 64\xi^6.}$$

(c)

$$\frac{dH_5}{d\xi} = 120 - 480\xi^2 + 160\xi^4 = 10(12 - 48\xi^2 + 16\xi^4) = (2)(5)H_4. \checkmark$$

$$\frac{dH_6}{d\xi} = 1440\xi - 1920\xi^3 + 384\xi^5 = 12(120\xi - 160\xi^3 + 32\xi^5) = (2)(6)H_5. \checkmark$$

(d)

$$\frac{d}{dz}(e^{-z^2+2z\xi}) = (-2z + \xi)e^{-z^2+2z\xi}; \text{ setting } z = 0, \ \boxed{H_0(\xi) = 2\xi.}$$

$$\left(\frac{d}{dz}\right)^2 (e^{-z^2+2z\xi}) = \frac{d}{dz}\left[(-2z + 2\xi)e^{-z^2+2z\xi}\right]$$
$$= \left[-2 + (-2z + 2\xi)^2\right]e^{-z^2+2z\xi}; \text{ setting } z = 0, \ \boxed{H_1(\xi) = -2 + 4\xi^2.}$$

$$\left(\frac{d}{dz}\right)^3 (e^{-z^2+2z\xi}) = \frac{d}{dz}\left\{\left[-2 + (-2z + 2\xi)^2\right]e^{-z^2+2z\xi}\right\}$$
$$= \left\{2(-2z + 2\xi)(-2) + \left[-2 + (-2z + 2\xi)^2\right](-2z + 2\xi)\right\}e^{-z^2+2z\xi};$$

setting $z = 0$, $H_2(\xi) = -8\xi + (-2 + 4\xi^2)(2\xi) = \boxed{-12\xi + 8\xi^3.}$

Problem 2.18

$$Ae^{ikx} + Be^{-ikx} = A(\cos kx + i\sin kx) + B(\cos kx - i\sin kx) = (A + B)\cos kx + i(A - B)\sin kx$$
$$= C\cos kx + D\sin kx, \text{ with } \boxed{C = A + B; \ D = i(A - B).}$$

$$C\cos kx + D\sin kx = C\left(\frac{e^{ikx} + e^{-ikx}}{2}\right) + D\left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) = \frac{1}{2}(C - iD)e^{ikx} + \frac{1}{2}(C + iD)e^{-ikx}$$
$$= Ae^{ikx} + Be^{-ikx}, \text{ with } \boxed{A = \frac{1}{2}(C - iD); \ B = \frac{1}{2}(C + iD).}$$

Problem 2.19

Equation 2.94 says $\Psi = Ae^{i(kx-\frac{\hbar k^2}{2m}t)}$, so

$$J = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i\hbar}{2m}|A|^2\left[e^{i(kx-\frac{\hbar k^2}{2m}t)}(-ik)e^{-i(kx-\frac{\hbar k^2}{2m}t)} - e^{-i(kx-\frac{\hbar k^2}{2m}t)}(ik)e^{i(kx-\frac{\hbar k^2}{2m}t)}\right]$$
$$= \frac{i\hbar}{2m}|A|^2(-2ik) = \boxed{\frac{\hbar k}{m}|A|^2.}$$

It flows in the positive (x) direction (as you would expect).

Problem 2.20

(a)

$$f(x) = b_0 + \sum_{n=1}^{\infty}\frac{a_n}{2i}\left(e^{in\pi x/a} - e^{-in\pi x/a}\right) + \sum_{n=1}^{\infty}\frac{b_n}{2}\left(e^{in\pi x/a} + e^{-in\pi x/a}\right)$$
$$= b_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{in\pi x/a} + \sum_{n=1}^{\infty}\left(-\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{-in\pi x/a}.$$

Let

$$\boxed{c_0 \equiv b_0;\ c_n = \tfrac{1}{2}\left(-ia_n + b_n\right),\ \text{for } n = 1, 2, 3, \ldots;\ c_n \equiv \tfrac{1}{2}\left(ia_{-n} + b_{-n}\right),\ \text{for } n = -1, -2, -3, \ldots.}$$

Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$. QED

(b)

$$\int_{-a}^{a} f(x)e^{-im\pi x/a}dx = \sum_{n=-\infty}^{\infty} c_n\int_{-a}^{a} e^{i(n-m)\pi x/a}dx. \quad \text{But for } n \neq m,$$

$$\int_{-a}^{a} e^{i(n-m)\pi x/a}dx = \left.\frac{e^{i(n-m)\pi x/a}}{i(n-m)\pi/a}\right|_{-a}^{a} = \frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{i(n-m)\pi/a} = \frac{(-1)^{n-m} - (-1)^{n-m}}{i(n-m)\pi/a} = 0,$$

whereas for $n = m$,

$$\int_{-a}^{a} e^{i(n-m)\pi x/a}dx = \int_{-a}^{a} dx = 2a.$$

So all terms except $n = m$ are zero, and

$$\int_{-a}^{a} f(x)e^{-im\pi x/a} = 2ac_m,\ \text{so } c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx. \quad \text{QED}$$

(c)

$$f(x) = \sum_{n=-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{1}{a}F(k)e^{ikx} = \frac{1}{\sqrt{2\pi}}\sum F(k)e^{ikx}\Delta k,$$

where $\boxed{\Delta k \equiv \frac{\pi}{a}}$ is the increment in k from n to $(n+1)$.

$$F(k) = \sqrt{\frac{2}{\pi}}a\frac{1}{2a}\int_{-a}^{a} f(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-ikx}dx.$$

(d) As $a \to \infty$, k becomes a continuous variable,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk; \;\; F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}dx.$$

Problem 2.21

(a)

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 2|A|^2\int_0^{\infty} e^{-2ax}dx = 2|A|^2 \left.\frac{e^{-2ax}}{-2a}\right|_0^{\infty} = \frac{|A|^2}{a} \Rightarrow A = \boxed{\sqrt{a}.}$$

(b)

$$\phi(k) = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{-ikx}\,dx = \frac{A}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}(\cos kx - i\sin kx)dx.$$

The cosine integrand is even, and the sine is odd, so the latter vanishes and

$$\begin{aligned}
\phi(k) &= 2\frac{A}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}\cos kx\,dx = \frac{A}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}\left(e^{ikx}+e^{-ikx}\right)\,dx \\
&= \frac{A}{\sqrt{2\pi}}\int_0^{\infty}\left(e^{(ik-a)x}+e^{-(ik+a)x}\right)dx = \frac{A}{\sqrt{2\pi}}\left.\left[\frac{e^{(ik-a)x}}{ik-a}+\frac{e^{-(ik+a)x}}{-(ik+a)}\right]\right|_0^{\infty} \\
&= \frac{A}{\sqrt{2\pi}}\left(\frac{-1}{ik-a}+\frac{1}{ik+a}\right) = \frac{A}{\sqrt{2\pi}}\frac{-ik-a+ik-a}{-k^2-a^2} = \boxed{\sqrt{\frac{a}{2\pi}}\frac{2a}{k^2+a^2}.}
\end{aligned}$$

(c)

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}2\sqrt{\frac{a^3}{2\pi}}\int_{-\infty}^{\infty}\frac{1}{k^2+a^2}e^{i(kx-\frac{\hbar k^2}{2m}t)}dk = \boxed{\frac{a^{3/2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{k^2+a^2}e^{i(kx-\frac{\hbar k^2}{2m}t)}dk.}$$

(d) For *large* a, $\Psi(x,0)$ is a sharp narrow spike whereas $\phi(k) \cong \sqrt{2/\pi a}$ is broad and flat; position is well-defined but momentum is ill-defined. For *small* a, $\Psi(x,0)$ is a broad and flat whereas $\phi(k) \cong (\sqrt{2a^3/\pi})/k^2$ is a sharp narrow spike; position is ill-defined but momentum is well-defined.

Problem 2.22

(a)

$$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = |A|^2 \sqrt{\frac{\pi}{2a}}; \quad \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}.}$$

(b)

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)} \frac{1}{\sqrt{a}}\, dy = \frac{1}{\sqrt{a}} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\frac{\pi}{a}} e^{b^2/4a}.$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} e^{-k^2/4a} = \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} \underbrace{e^{-k^2/4a} e^{i(kx-\hbar k^2 t/2m)}}_{e^{-[(\frac{1}{4a}+i\hbar t/2m)k^2 - ixk]}} dk$$

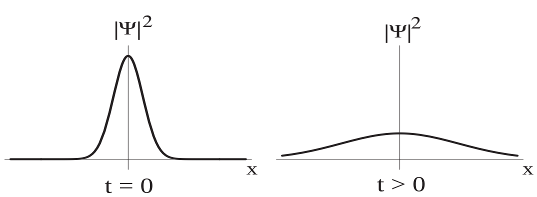
$$= \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a} + i\hbar t/2m}} e^{-x^2/4(\frac{1}{4a}+i\hbar t/2m)} = \boxed{\left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+2i\hbar a t/m)}}{\sqrt{1+2i\hbar a t/m}}.}$$

(c)

Let $\theta \equiv 2\hbar a t/m$. Then $|\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-ax^2/(1+i\theta)} e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}}$. The exponent is

$$-\frac{ax^2}{(1+i\theta)} - \frac{ax^2}{(1-i\theta)} = -ax^2 \frac{(1-i\theta+1+i\theta)}{(1+i\theta)(1-i\theta)} = \frac{-2ax^2}{1+\theta^2}; \; |\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}.$$

Or, with $w \equiv \sqrt{\frac{a}{1+\theta^2}}$, $\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}.}$ As t increases, the graph of $|\Psi|^2$ flattens out and broadens.



(d)

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0} \text{ (odd integrand)}; \; \langle p \rangle = m\frac{d\langle x \rangle}{dt} = \boxed{0.}$$

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx = \sqrt{\frac{2}{\pi}} w \frac{1}{4w^2} \sqrt{\frac{\pi}{2w^2}} = \boxed{\frac{1}{4w^2}.} \; \langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{d^2\Psi}{dx^2} dx.$$

Write $\Psi = Be^{-bx^2}$, where $B \equiv \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+i\theta}}$ and $b \equiv \frac{a}{1+i\theta}$.

$$\frac{d^2\Psi}{dx^2} = B\frac{d}{dx}\left(-2bxe^{-bx^2}\right) = -2bB(1-2bx^2)e^{-bx^2}.$$

$$\Psi^*\frac{d^2\Psi}{dx^2} = -2b|B|^2(1-2bx^2)e^{-(b+b^*)x^2};\ b+b^* = \frac{a}{1+i\theta} + \frac{a}{1-i\theta} = \frac{2a}{1+\theta^2} = 2w^2.$$

$$|B|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\theta^2}} = \sqrt{\frac{2}{\pi}}w. \quad \text{So } \Psi^*\frac{d^2\Psi}{dx^2} = -2b\sqrt{\frac{2}{\pi}}w(1-2bx^2)e^{-2w^2x^2}.$$

$$\begin{aligned}\langle p^2\rangle &= 2b\hbar^2\sqrt{\frac{2}{\pi}}w\int_{-\infty}^{\infty}(1-2bx^2)e^{-2w^2x^2}dx \\ &= 2b\hbar^2\sqrt{\frac{2}{\pi}}w\left(\sqrt{\frac{\pi}{2w^2}} - 2b\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}\right) = 2b\hbar^2\left(1-\frac{b}{2w^2}\right).\end{aligned}$$

But $1 - \frac{b}{2w^2} = 1 - \left(\frac{a}{1+i\theta}\right)\left(\frac{1+\theta^2}{2a}\right) = 1 - \frac{(1-i\theta)}{2} = \frac{1+i\theta}{2} = \frac{a}{2b}$, so

$$\langle p^2\rangle = 2b\hbar^2\frac{a}{2b} = \boxed{\hbar^2 a.} \qquad \boxed{\sigma_x = \frac{1}{2w};} \qquad \boxed{\sigma_p = \hbar\sqrt{a}.}$$

(e)

$$\sigma_x\sigma_p = \frac{1}{2w}\hbar\sqrt{a} = \frac{\hbar}{2}\sqrt{1+\theta^2} = \frac{\hbar}{2}\sqrt{1+(2\hbar at/m)^2} \geq \frac{\hbar}{2}. \checkmark$$

Closest at $\boxed{t = 0,}$ at which time it is right *at* the uncertainty limit.

Problem 2.23

(a)

$$(-2)^3 - 3(-2)^2 + 2(-2) - 1 = -8 - 12 - 4 - 1 = \boxed{-25.}$$

(b)

$$\cos(3\pi) + 2 = -1 + 2 = \boxed{1.}$$

(c)

$\boxed{0}$ ($x = 2$ is outside the domain of integration).

Problem 2.24

(a) Let $y \equiv cx$, so $dx = \frac{1}{c}dy$. $\left\{ \begin{array}{l} \text{If } c > 0,\ y : -\infty \to \infty. \\ \text{If } c < 0,\ y : \infty \to -\infty. \end{array} \right\}$

$$\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \begin{cases} \frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = \frac{1}{c}f(0) \quad (c>0);\ \text{or} \\ \frac{1}{c}\int_{\infty}^{-\infty} f(y/c)\delta(y)dy = -\frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = -\frac{1}{c}f(0)\ (c<0). \end{cases}$$

In either case, $\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \frac{1}{|c|}f(0) = \int_{-\infty}^{\infty} f(x)\frac{1}{|c|}\delta(x)dx$. So $\delta(cx) = \frac{1}{|c|}\delta(x)$. ✓

(b)

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f\theta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx}\theta dx \quad \text{(integration by parts)}$$
$$= f(\infty) - \int_{0}^{\infty} \frac{df}{dx}dx = f(\infty) - f(\infty) + f(0) = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx.$$

So $d\theta/dx = \delta(x)$. ✓ [Makes sense: The θ function is constant (so derivative is zero) except at $x = 0$, where the derivative is infinite.]

Problem 2.25

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2} = \frac{\sqrt{m\alpha}}{\hbar}\begin{cases} e^{-m\alpha x/\hbar^2}, & (x \geq 0), \\ e^{m\alpha x/\hbar^2}, & (x \leq 0). \end{cases}$$

$$\langle x \rangle = 0 \text{ (odd integrand).}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2|\psi|^2 dx = 2\frac{m\alpha}{\hbar^2}\int_0^{\infty} x^2 e^{-2m\alpha x/\hbar^2}dx = \frac{2m\alpha}{\hbar^2}2\left(\frac{\hbar^2}{2m\alpha}\right)^3 = \frac{\hbar^4}{2m^2\alpha^2}; \quad \sigma_x = \frac{\hbar^2}{\sqrt{2}m\alpha}.$$

$$\frac{d\psi}{dx} = \frac{\sqrt{m\alpha}}{\hbar}\left\{ \begin{array}{ll} -\frac{m\alpha}{\hbar^2}e^{-m\alpha x/\hbar^2}, & (x \geq 0) \\ \frac{m\alpha}{\hbar^2}e^{m\alpha x/\hbar^2}, & (x \leq 0) \end{array} \right\} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[-\theta(x)e^{-m\alpha x/\hbar^2} + \theta(-x)e^{m\alpha x/\hbar^2}\right].$$

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[-\delta(x)e^{-m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(x)e^{-m\alpha x/\hbar^2} - \delta(-x)e^{m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(-x)e^{m\alpha x/\hbar^2}\right] \\ &= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[-2\delta(x) + \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\right]. \end{aligned}$$

In the last step I used the fact that $\delta(-x) = \delta(x)$ (Eq. 2.142), $f(x)\delta(x) = f(0)\delta(x)$ (Eq. 2.112), and $\theta(-x) + \theta(x) = 1$ (Eq. 2.143). Since $d\psi/dx$ is an odd function, $\langle p \rangle = 0$.

$$\begin{aligned} \langle p^2 \rangle &= -\hbar^2 \int_{-\infty}^{\infty} \psi \frac{d^2\psi}{dx^2}dx = -\hbar^2 \frac{\sqrt{m\alpha}}{\hbar}\left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2}\left[-2\delta(x) + \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\right]dx \\ &= \left(\frac{m\alpha}{\hbar}\right)^2 \left[2 - 2\frac{m\alpha}{\hbar^2}\int_0^{\infty} e^{-2m\alpha x/\hbar^2}dx\right] = 2\left(\frac{m\alpha}{\hbar}\right)^2\left[1 - \frac{m\alpha}{\hbar^2}\frac{\hbar^2}{2m\alpha}\right] = \left(\frac{m\alpha}{\hbar}\right)^2. \end{aligned}$$

Evidently

$$\sigma_p = \frac{m\alpha}{\hbar}, \quad \text{so} \quad \sigma_x\sigma_p = \frac{\hbar^2}{\sqrt{2}m\alpha}\frac{m\alpha}{\hbar} = \sqrt{2}\frac{\hbar}{2} > \frac{\hbar}{2}. \ ✓$$
