Chapter 2

2.1

Consider the geometry shown below: $r_2 = r_1$ ∴ A = 1 and B = 0 since the planes 1 and 2 are spaced an infinitesimal distance apart. Now use Snell's Law:

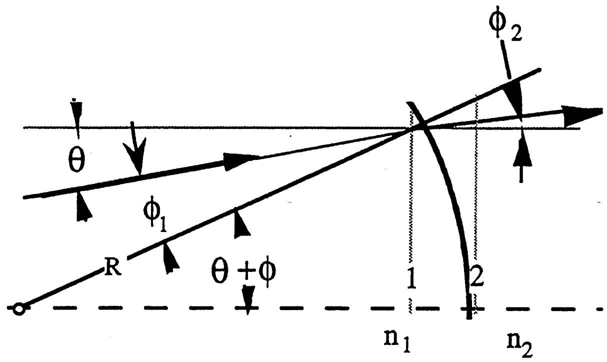
$$\frac{\omega}{c} n_1 \sin\phi_1 = \frac{\omega}{c} n_2 \sin\phi_2; \text{ For small angles: } n_1\phi_1 = n_2\phi_2$$

and $r_2' = (\theta + \phi_1) - \phi_2 = \theta + \phi_1 [1 - (n_1/n_2)]$;

Since $r_1/R = \sin(\theta + \phi_1) \approx \theta + \phi_1$

$\therefore \phi_1 = r_1/R - r_1'$; Hence, $r_2' = r_1' + [1 - \frac{n_1}{n_2}][r_1/R - r_1'] = \left\{\left[1 - \frac{n_1}{n_2}\right]\frac{1}{R}\right\} r_1 + \frac{n_1}{n_2} r_1'$;

Thus, the ray matrix is as shown on the side of the diagram.



$$\begin{bmatrix} 1 & 0 \\ [1 - n_1/n_2]\frac{1}{R} & \frac{n_2}{n_1} \end{bmatrix}$$

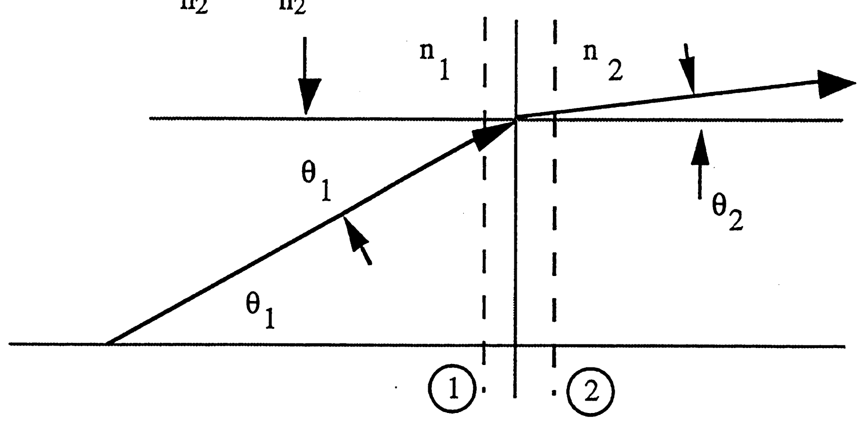
2.2

Consider the geometry shown below:

Since the distance between the two planes → 0, then $r_2 = r_1$ and A = 1,B = 0

Use Snell's law for the interface: $\frac{\omega}{c} n_1 \sin\theta_1 = \frac{\omega}{c} \sin\theta_2$ and now use the small angle approximation:

$\therefore \theta_2 = r_2' = \frac{n_1}{n_2}\theta_1 = \frac{n_1}{n_2} r_1'$ or:



$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$$

Notice that AD–BC ≠1 because of the different indices of refraction.

2.3

Consider the diagram shown below and apply the results of problem 2.2 to reduce the number of matrices to be derived. The matrix for the last interface comes first with the interchange of n_1 and n_2, followed by the matrix for the length d, and finally by the matrix for the first interface.

$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & \frac{n_1}{n_2} d \\ 0 & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{n_1}{n_2} d \\ 0 & 1 \end{bmatrix}$$

Notice that the determinant, AD –BC=1 even though the optical path does include a different index.

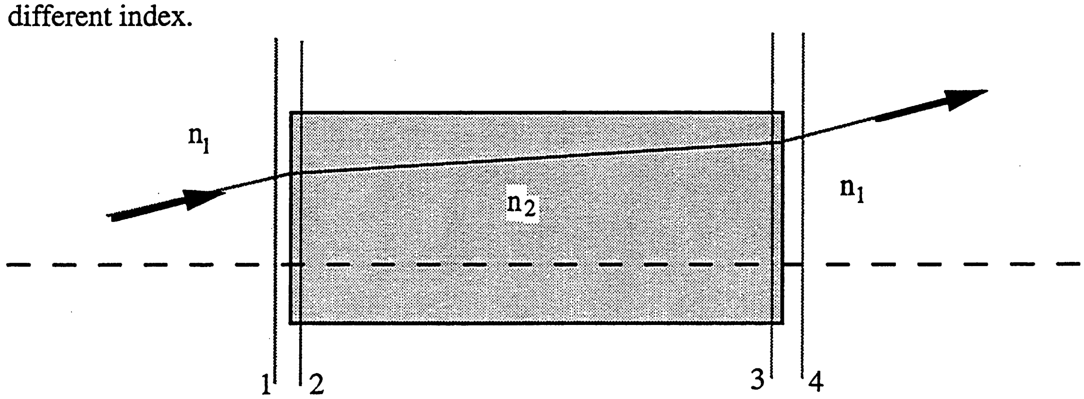


Figure for problem 2.3

2.4

Combine problems 2.1 and 2.3

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \left(1 - \frac{n_1}{n_2}\right)\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & \frac{n_2}{n_1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \left(1 - \frac{n_1}{n_2}\right)\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 + \left(1 - \frac{n_1}{n_2}\right)\frac{d}{R} & \frac{n_1}{n_2} d \\ \left(\frac{n_2}{n_1} - 1\right)\frac{1}{R} & 1 \end{bmatrix} \quad \text{Note: } AD - BC = 1$$

2.5

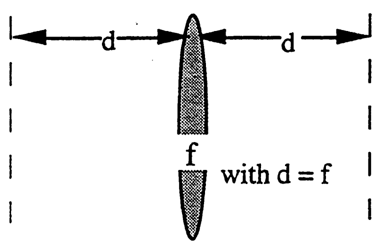
For the purpose of this solution, we will use "t" rather than l ("el") to avoid confusing it with one (1). We include the GRIN-to-air exit interface as the first matrix, then

Eq.2.12.11 for the GRIN lens, and the last matrix represent the air-to-entrance interface where the results of Prob. 2.2 has been used.

$$T = \begin{bmatrix} 1 & 0 \\ 0 & n_0 \end{bmatrix} \cdot \begin{bmatrix} \cos(d/t) & t\sin(d/t) \\ -\frac{1}{t}\sin(d/t) & \cos(d/t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n_0} \end{bmatrix}$$

$$T = \begin{bmatrix} \cos(d/t) & \frac{t}{n_0}\sin(d/t) \\ -\frac{n_0}{t}\sin(d/t) & \cos(d/t) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & \frac{t}{n_0} \\ -\frac{n_0}{t} & 0 \end{bmatrix} \text{for } d = \pi t/2$$

Now consider the following simple lens centered between the input and output planes with d=f and use Eq. 2.3.2 to represent the two components so as to minimize the chore of matrix multiplication.



$$T = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d \\ -\frac{1}{f} & \left(1 - \frac{d}{f}\right) \end{bmatrix}_{d=f} = \begin{bmatrix} 0 & f \\ -\frac{1}{f} & 0 \end{bmatrix}$$

Thus the focal length is $f = t/n_0$

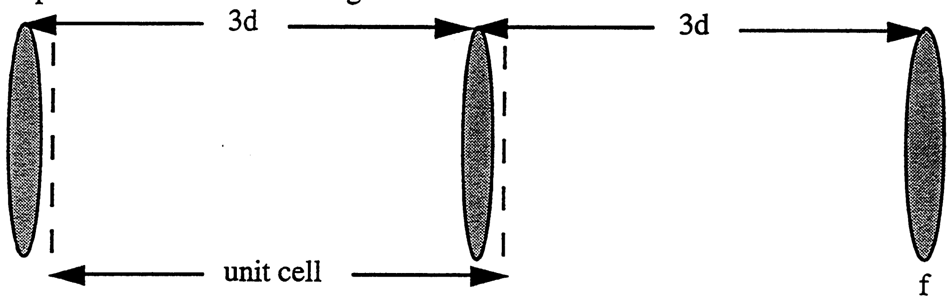
2.6

The only way to have difficulties with this problem is to arrange the matrices in wrong order. Let's evaluate Eq. 2.3.2 for the negative lens + distance d combination (i.e. change the sign on f), multiply by the matrix for the postive lens, and evaluate for d=f.

$$T = \begin{bmatrix} 1 & d \\ +\frac{1}{f} & \left(1 + \frac{d}{f}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}_{d=f} = \begin{bmatrix} 0 & f \\ -\frac{1}{f} & 2 \end{bmatrix}$$

2.7

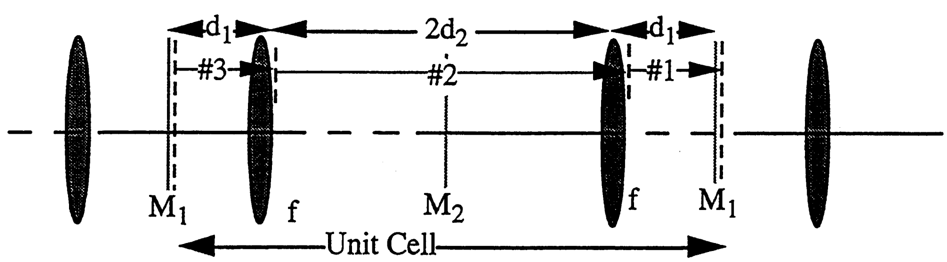
The ray matrix for a flat mirror is A=D=1 and B=C=0 which is the limit for that of a curved mirror with R→∞. Hence one could insert three extra matrices in the unit cell and go through the excercise of matrix multiplication to prove the altermative of ignoring the flat mirrors, as being just re-directors of the optic axis, and measuring distances along that line. This viewpoint leads to the following:



$$T = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3d \\ -\frac{1}{f} & \left(1-\frac{3d}{f}\right) \end{bmatrix};$$

$$S = \frac{A+D+2}{4} = 1 - \frac{3d}{4f} \qquad \therefore \boxed{0 \leq 1 - \frac{3d}{4f} \leq 1}$$

2.8



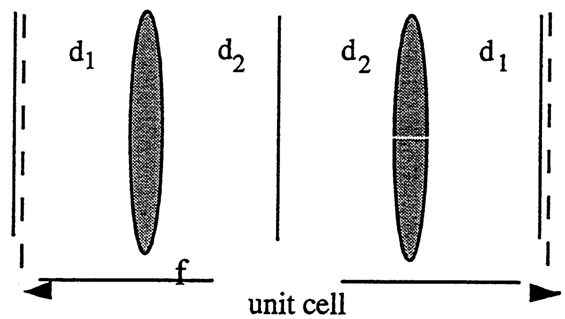
Thus the matrices appear in the order indicated by"#".

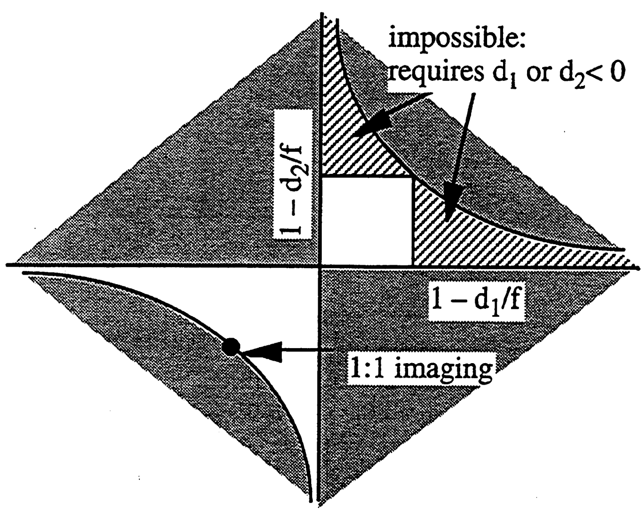
$$T = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2d_2 \\ -\frac{1}{f} & \left(1-\frac{2d_2}{f}\right) \end{bmatrix} \cdot \begin{bmatrix} 1 & d_1 \\ -\frac{1}{f} & \left(1-\frac{d_1}{f}\right) \end{bmatrix}$$

$$T = \begin{bmatrix} 1 - \frac{2d_1}{f} - \frac{2d_2}{f} + \frac{2d_1d_2}{f^2} & \left(1 - \frac{d_1}{f}\right)\left(2d_1 + 2d_2 - \frac{2d_1d_2}{f}\right) \\ -\frac{2}{f}\left(1 - \frac{d_2}{f}\right) & 1 - \frac{2d_1}{f} - \frac{2d_2}{f} + \frac{2d_1d_2}{f^2} \end{bmatrix}$$

Note: $A = D$ as it should since there is a plane of symmetry (M_2).

Stability: $0 \le 1 - \frac{d_1}{f} - \frac{d_2}{f} + \frac{d_1d_2}{f^2} < 1$ or: $0 \le \left(1 - \frac{d_1}{f}\right)\left(1 - \frac{d_2}{f}\right) \le 1$

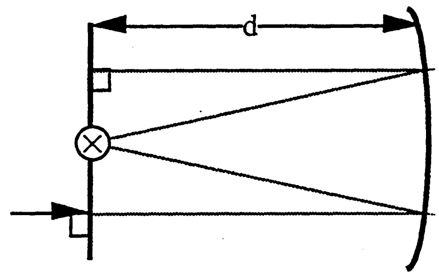


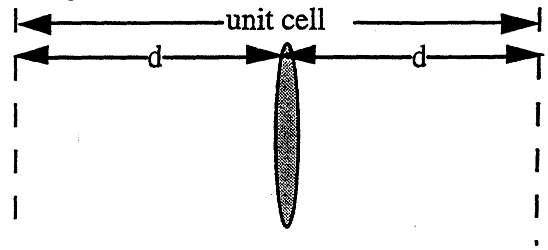


Figures for problem 2.8

2.9

$(d/R) = (1/2)$; $\therefore$ $d = f$; 4–Round trips as the figure below indicates





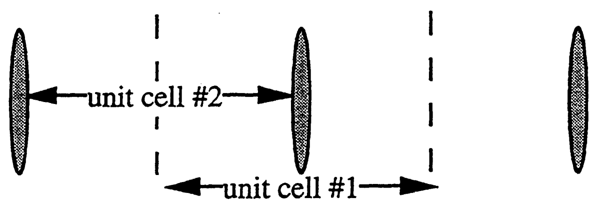
One can do this the hard way using the figure at the right: $T = \begin{bmatrix} 1 & 2d \\ \frac{1}{f} & \left(1 - \frac{2d}{f}\right) \end{bmatrix}$

$$\cos\theta = \frac{A+D}{2} = 0; \ \therefore \theta = \pi/2; \ \alpha = \tan^{-1}\left\{\frac{a\left[1 - \left(\frac{A+D}{2}\right)^2\right]^{1/2}}{a\left(\frac{A-D}{2}\right) + Bm}\right\}$$

For an input slope $m = 0$; $\frac{A+D}{2} = 0$; $\frac{A-D}{2} = \frac{d}{f} = 1$; $\alpha = \tan^{-1}(1) = \pi/4$;

$r_{initial} = r_{max} \sin \alpha = r_{max}/\sqrt{2}$; $r_{max} = -\sqrt{2}\, r_0 \sin\left[s\frac{\pi}{2} + \frac{\pi}{4}\right]$

2.10



$$T_1 = \begin{bmatrix} 1 - \frac{d}{f} & d\left(2 - \frac{d}{f}\right) \\ -\frac{1}{f} & \left(1 - \frac{d}{f}\right) \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 - \frac{2d}{f} & 2d \\ -\frac{1}{f} & 1 \end{bmatrix}$$

$$F_{1,2} = \left[\frac{A + D}{2}\right] \pm \left\{\left[\frac{A + D}{2}\right]^2 - 1\right\}^{1/2};$$

$\left[\frac{A + D}{2}\right] = 1 - \frac{d}{f}$ for both cases;

Now $d/R = 1.01$; $d/f = 2.02$ $\therefore$ $\left[\frac{A + D}{2}\right] = -1.02$;

$F_1 = -0.8190;\ F_2 = -1.2210;$

Let $r = r_a(F_1)^s + r_b(F_2)^s;$

Unit Cell #1: $r_b = \frac{1}{F_1 - F_2}\{a(F_1 - A)\} = 0.5\times10^{-2};$

$r_a = \frac{1}{F_1 - F_2}\{a(F_2 - A)\} = 0.5\times10^{-2};$

Thus the position of the ray after s round-trips is:

$r_s = 0.5\times10^{-2}\{(0.819)^s + (-1.221)^s\};\ r_s > 1$ cm after s = 15.

Unit Cell #2: $r_b = 0.5525;\ r_a = -0.4525;$

$r_s = -0.4525\,(0.819)^s + 0.5525(-1.221)^s$

at s = 6; r = 1.694 cm ← misses spherical mirror after 12 round-trips plus 1 more pass to the spherical mirror; ∴ $P_{out} = 1\ \mu W \times G^{13} = 1220$ watts!

2.11

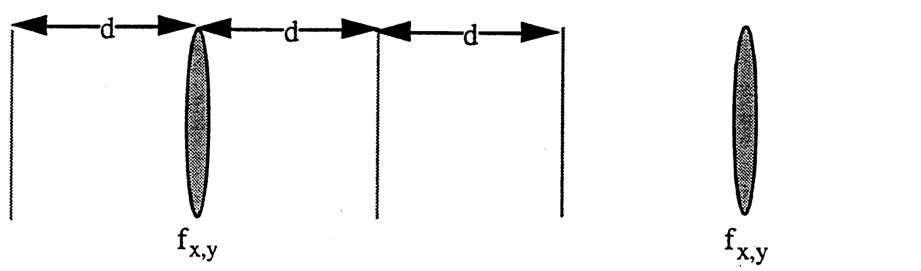
The effective focal lengths are:

$f_x = \frac{R}{2}\cos\theta = \frac{\sqrt{3}}{4}R\ (\theta = 30°);$ $f_y = \frac{R}{2\cos\theta} = \frac{R}{\sqrt{3}};$

Stability: $0 < 1 - 4d/3f_{x,y} < 1;$

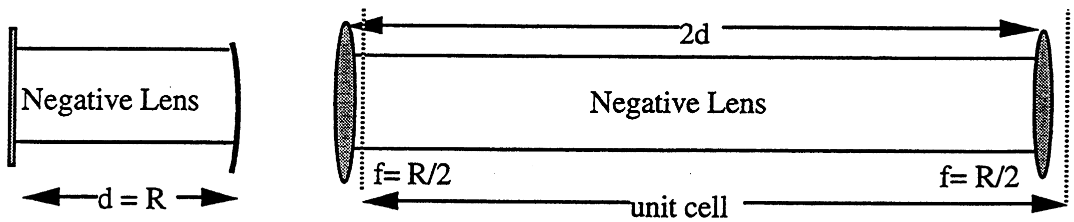
$d < \frac{R}{\sqrt{3}}$ or $\frac{4}{3}\frac{R}{\sqrt{3}} = 0.577\,R$ or $0.7698\,R;$

$\therefore\ d < 0.577\,R$



2.12

This is a situation that can and has happened in a gas discharge excited laser with the current heating the gas on the axis to a higher temperature than at the walls which are cooled by convection or by an intentional water jacket. In order for the pressure to be a constant across the radius, there is a greater density of atoms near the wall than on the axis and hence one has a negative gas lens. At first glance, one would guess that this would push the system towards instability away from the borderline situation as specified. Not so!



$$T = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} \cosh\frac{2d}{L} & L\sinh\frac{2d}{L} \\ \frac{1}{L}\sinh\frac{2d}{L} & \cosh\frac{2d}{L} \end{bmatrix} = \begin{bmatrix} \cosh\frac{2d}{L} & \text{(not needed)} \\ \text{(not needed)} & \cosh\frac{2d}{L} - \frac{L}{f}\sinh\frac{2d}{L} \end{bmatrix}$$

$$\frac{A+D+2}{4} = S = \frac{1}{2}\left\{1 + \cosh\frac{2d}{L} - 2\left(\frac{L}{2d}\right)\sinh\frac{2d}{L}\right\}$$

Let $\frac{2d}{L} = \theta$ and recognize that the lens is a "small" effect implying that L is large and that θ is small. For small θ:

$$\cosh\theta = 1 + \frac{\theta^2}{2} + \frac{\theta^4}{4!} \text{ and } \sinh\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$$

$$S = \frac{1}{2}\left\{1 + 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{2}{\theta}\left[\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5}\right]\right\} = \frac{1}{2}\left[\frac{\theta^2}{2} - \frac{\theta^2}{3}\right] = \frac{\theta^2}{12} = \frac{1}{3}\left(\frac{d}{L}\right)^2$$

Stability becomes positive, i.e. the cavity is **more stable** by virtue of the de-focusing element.

2.13

Temperature distribution: $T(r) = T_w + (T_c - T_w)\left[1 - (r/a)^2\right]$;

Density of [He] at 1 Torr,23°C $= \dfrac{2.69\times10^{+19}}{760} \cdot \dfrac{273}{296} = 3.26\times10^{+16}\ \text{cm}^{-3}$

Specific refractivity of a Helium atom $= \dfrac{n-1}{2.69^{+19}\text{cm}^{-3}} = 1.338\times10^{-24}\ \text{cm}^3$

Atoms must be conserved:

$$\therefore \quad N(r) = \frac{N_1}{1 - \left[\frac{T_c - T_w}{T_c}\right]\left(\frac{r}{a}\right)^2}; \text{ and } 2\pi\int N(r)rdr = \pi a^2 N_0 \text{ in all cases}$$

$$\therefore\ N_1 = \frac{T_c - T_w}{T_c}\ \frac{1}{\ln (T_c/T_w)}\, N_0 = 0.834\ N_0 = 0.834 \text{ x } 3.26\times10^{+16} = 2.7\times10^{+16}\ \text{cm}^{-3}$$

For helium: $n(r) - 1 = N_1\left\{1 + \frac{(T_c - T_w)}{T_c}\left(\frac{r}{a}\right)^2\right\}\times1.338\times10^{-24}$

$$n(r) = 1+1.58\times10^{-8}\left(\frac{r}{a}\right)^2 \stackrel{\Delta}{=} 1+\frac{r^2}{2L^2};\ \therefore L^2 = \frac{a^2}{2}\ \frac{1}{1.58\times10^{-8}} = 7.88\times10^{+6};\ L = 2.81\times10^{+3}\ \text{cm}$$

For CO_2, the specific refractivity $= 1.67\times10^{-23}\text{cm}^3$; $N_1 = 2.7\times10^{+18}\ \text{cm}^{-3}$ (100 Torr)

$n(r) = 1 + 1.96\times10^{-5}\,(r/a)^2 \stackrel{\Delta}{=} 1 + \frac{r^2}{2L^2}$; $L = 79.7$ cm; gas heating is more pronounced.

2.14

$$n(r) = n_0 - \Delta n\left(\frac{r}{a}\right)^2 \stackrel{\Delta}{=} n_0\left[1 - \frac{r^2}{2l_f^2}\right];\ \frac{n_0}{2l_f^2}\, r^2 = \Delta n\, \frac{r^2}{a^2};$$

$l_f^2 = \frac{n_0}{2\Delta n}\, a^2$; $l_f = 9.68a = 1.94\times10^{-2}$ cm

$r(z) = r_1 \cos\frac{z}{l_f} + r' l_f \sin\frac{z}{l_f}$; Thus, when $z \sim \pi\, l_f$, the ray crosses axis or $z \sim 6.08\times10^{-2}$ cm.

Thus, in 1 km = 10^5 cm, the ray would cross ~ $1.64\times10^{+6}$ times (more–or–less).

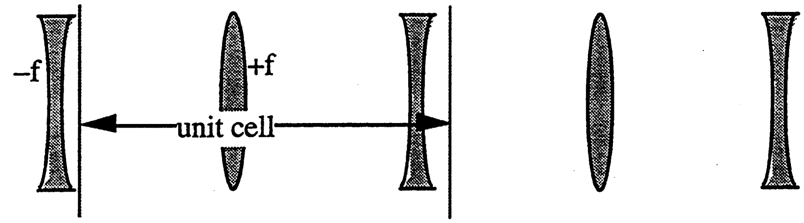
2.15

One always starts and stops at the same point. Hence, $AD - BC \equiv 1$

2.16

Excellent paper and is highly recommended in order to introduce the students to the literature.

2.17



One can use the arithmetic in Sec. 2.4 by substituting $f_1 = -f$, and $f_2 = +f$.

Eq. 2.5.3 becomes: $0 < S = 1 - \left[\frac{d}{|R|}\right]^2 < 1$

2.18

A simple sketch of a general optical system points out that $r'_a(b \to a) = -\, r'_a(a \to b)$ and likewise for $r'_b(b \to a) = -\, r'_b(a \to b)$. Use that fact and solve for the matrix for the reverse direction to obtain the desired result.

2.19

For a system with a plane of symmetry, one can use the result of problem 2.18.

$$T = T_{12}\,T_{21} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \cdot \begin{bmatrix} D_1 & B_1 \\ C_1 & A_1 \end{bmatrix} = \begin{bmatrix} A_1D_1 + B_1C_1 & 2A_1B_1 \\ 2C_1D_1 & A_1D_1 + B_1C_1 \end{bmatrix}$$

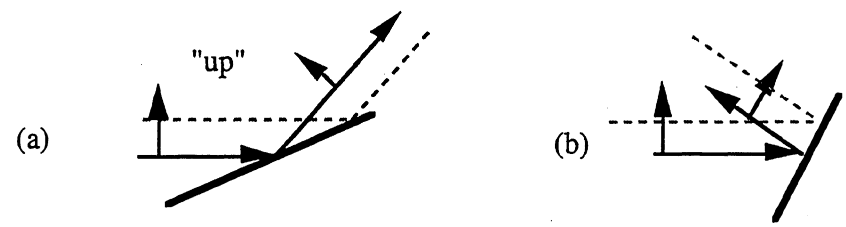
2.20

$$T = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \delta d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} 1-\delta d/f & \delta d \\ (-1/f)(1-\delta d/f) - 1/f & 1-\delta d/f \end{bmatrix}$$

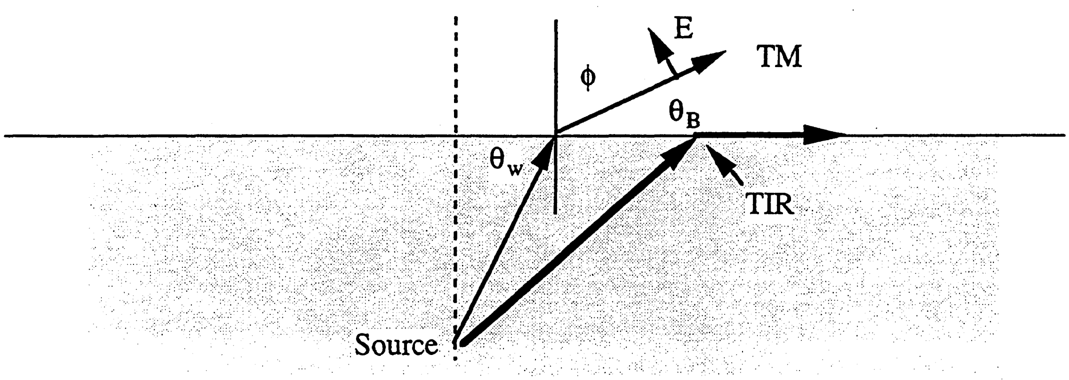
Take the limit as $\delta d/f \to 0$ yields: $T = \begin{bmatrix} 1 & 0 \\ -2/f & 1 \end{bmatrix}$

2.21

Consider the diagram shown below. If we consider a positive r_2 as being measured counter-clockwise with respect to the chief ray as in (a), then A = D = –1 is the choice. If we consider a positive r_2 to be above the incident chief ray as in (b), then A = D = 1 is the choice. Both have selling points. For instance, using the A = D = –1 is most logical for grazing incidence on a mirror because it indicates an output position on the "other side" of the chief ray. Choice (b) is more intuitive for mirrors excited at near normal incidence since the ray stays on the same side of the axis. In any case, stability always involves the product of AD and is not affect by which option is chosen, except to require that the choice be maintained throughout.



2.22



At TIR: $\frac{\omega}{c} n_w \cos\theta = \frac{\omega}{c} (n_a{=}1) \cos(\theta{=}0)$ ∴ $\cos\theta = 1/n$. $\theta = 41.2°$.

(a) (10/a) = tan θ ; a = (10/tan θ) = a = 11.4 cm. (b) The Fresnel reflectivity depends on polarization but at the Brewster's angle, there is no reflection. Thus the transmission for TM waves is a maximum. (c) tanθ = n for Brewster's angle. $\phi = 53.1°$, $\theta_a = 36.9°$.

2.23

$$T_{\text{unit cell}} = \begin{bmatrix} D_a & B_a \\ C_a & A_a \end{bmatrix} \cdot \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} = \begin{bmatrix} A_aD_a+B_aC_a & 2B_aD_a \\ 2A_aC_a & A_aD_a+B_aC_a \end{bmatrix}$$

$$0 < \frac{A_T+D_T+2}{4} < 1 \quad \Rightarrow \quad 0 < \frac{A_aD_a+B_aC_a+1}{2} < 1$$

$A_aD_a - B_aC_a = 1 \quad \therefore B_aC_a = A_aD_a - 1;$ $\boxed{\text{Stability} \rightarrow 0< A_aD_a<1}$

2.24

We use the results of the last problem and abbreviate:

$$T = \left\{ \begin{bmatrix} A_1D_1+B_1C_1 & 2B_1D_1 \\ 2A_1C_1 & A_1D_1+B_1C_1 \end{bmatrix} \triangleq \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \right\}^2 = \begin{bmatrix} A_a^2+B_aC_a & A_aB_a+B_aD_a \\ A_aC_a+D_aC_a & D_a^2+B_aC_a \end{bmatrix}$$

where $A_a = D_a$; Now stability require that: $0 \le \frac{A_T+D_T+2}{4} < 1;\ 0 \le \frac{A_a^2+D_a^2+2B_aC_a+2}{4} < 1;$

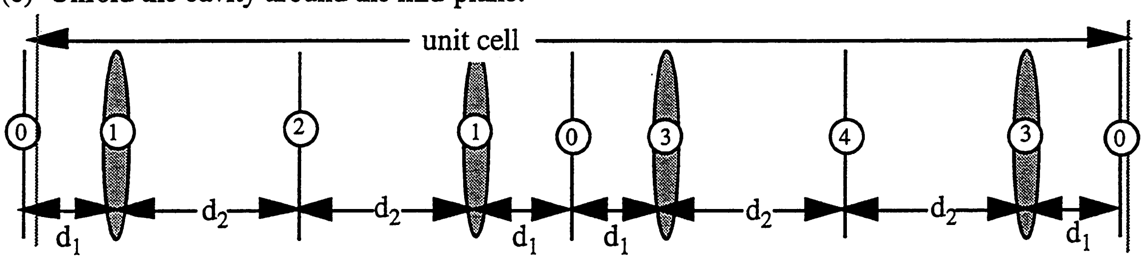
Substitute $BC = AD - 1$; $A_a^2 + D_a^2 + 2(A_aD_a-1) + 2 = (A_a + D_a)^2$

(b) $0 \le \left[\frac{A_a + D_a}{2}\right]^2 < 1 \Rightarrow 0 < A_a^2 < 1$ or $-1 < A_a < 1$; Since $A_a = D_a$

Now $A_a = A_1D_1+B_1C_1 = 2A_1D_1-1$; $\therefore -1 < 2A_1D_1-1 < 1$; or $0 < 2A_1D_1-1 < 2$;

$$\boxed{0 < A_1D_1 < 1}$$

(c) Unfold the cavity around the mid-plane:



(d) Evaluate the transmission matrix between the mid-plane 0 and 2 and apply the above results:

$$T_1 = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ -\frac{1}{f} & \left(1 - \frac{d_1}{f}\right) \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1-d_2/f & d_1+d_2(1-d_1/f) \\ -\frac{1}{f} & (1-d_1,f) \end{bmatrix} \quad \text{or} \quad \boxed{0 < \left(1 - \frac{d_1}{f}\right)\left(1 - \frac{d_2}{f}\right) < 1}$$