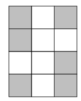
Chapter 0 Problems and Solutions

1. Consider the position:

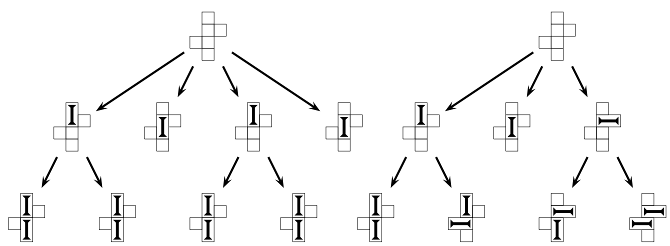


   (a) Draw the complete game trees for both CRAM and DOMINEERING. The leaves (bottoms) of the tree should all be positions in which neither player can move. If two left (or right) options are symmetrically identical, you may omit one.

   (b) In the position above, who wins at DOMINEERING if Vertical plays first? Who wins if Horizontal plays first? Who wins at CRAM?

Solution

   (a) On this particular CRAM board, each horizontal move leaves basically the same position as a vertical move, so we included vertical moves only.



   (b) In DOMINEERING, Vertical wins immediately by playing dead center, while Horizontal has no winning first move. In CRAM, the first player wins by playing vertically in the center.

2. Suppose that you play DOMINEERING (or CRAM) on *two* 8×8 chessboards. At your turn you can move on either chessboard (but not both!). Show that the second player can win.

Solution

The second player can win by using the *Tweedledum-Tweedledee strategy.* He always makes a move that ensures the position on the second board is a 90-degree rotational image of that on the first board. Specifically, whatever move the first player makes (at any turn) he responds on the opposite board with the corresponding move, in order to reestablish the rotational relationship between the two boards.

3. Take the ace through five of a suit from a deck of cards and place them face up on the table. Play a game with these as follows. Players alternately pick a card and add it to the righthand end of a row. If the row ever contains a sequence of three cards in increasing order of rank (ace is low), or in decreasing order of rank, then the game ends and the player who formed that sequence is the winner. Note that the sequence need not be consecutive either in position or value, so for instance, if the play goes $4, 5, 2, 1$ then the $4, 2, 1$ is a decreasing sequence.

   (a) Show that this is a proper combinatorial game (the main issue is to show that draws are impossible).

   (b) Show that the first player can always win.

Solution

(a) Clearly, the game is finite, has two players, the players have perfect information, and there is no chance involved. The game could only be drawn if there were a position with all five numbers placed containing no increasing or decreasing sequence of length three. However, by considering the location of the number 1 (and 5) in such a position, we can show that this is impossible. Suppose that 1 is in the first or second place ($1abcd$ or $d1abc$). Unless abc is a decreasing sequence, the position contains an increasing sequence. Similarly, if 1 were in the fourth or fifth place ($abc1d$ or $abcd1$), we would get a decreasing sequence unless abc is increasing. So, to avoid an increasing or decreasing sequence, 1 would have to be in the middle. Essentially the same argument also shows that 5 would have to be in the middle, so this is impossible. A more general result of this type (any sequence of $nk + 1$ distinct values contains either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $k + 1$) is called the Erdős-Szekeres Theorem.

(b) The first player can win by placing 3 in the first position as her first move. The second player must next play 1 or 5 (or else the first player can win on her next turn). The first player responds to leave either 315 or 351. The second player cannot win on the fourth move so the first player must win when the last number is played.

4. Start with a heap of counters. As a move from a heap of n counters, you may either:

   - assuming n is not a power of 2, remove the largest power of 2 less than n; or
   - assuming n is even, remove half the counters.

   Under normal play, who wins? How about misère play?

Solution

Write the number of counters n in binary form. A move either removes the leading 1, unless n is a power of 2 in which case there are no such moves left, or removes a trailing 0. The number of moves in the game is exactly 1 fewer than the number of 1s plus the number of trailing 0s in the binary expansion of n. Therefore, if this number of moves is odd, the first player wins in normal play and loses in misère play. If it is even, the first player loses in normal play and wins in misère play.

5. The goal of this problem is to give the reader a taste of what is *not* covered in this book. Two players play a 2×2 *zero-sum matrix game.* (Zero sum means that whatever one person loses, the other gains.) The players are shown a 2×2 matrix of positive numbers. Player A chooses a row of the matrix, and player B simultaneously chooses a column. Their choice determines one matrix entry, that being the number of dollars B must pay A. For example, suppose the matrix is

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

If player A chooses the first row with probability $1/4$, then no matter what player B's strategy is, player A is *guaranteed* to get an average of \$2.50. If, on the other hand, player B chooses the columns with 50-50 odds, then no matter what player A does, player B is *guaranteed* to have to pay an average of \$2.50. Further, neither player can guarantee a better outcome, and so B should pay player A the fair price of \$2.50 to play this game.

In general, if the entries of the matrix game are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

as a function of a, b, c, and d, what is the fair price which B should pay A to play? (Your answer will have several cases.)

Solution

A should choose from the rows so that B cannot gain an advantage from choosing one column over the other. If the probability that A chooses the first row is p and the probability that B chooses the first column is q, A will get

$$q[pa + (1-p)c] + (1-q)[pb + (1-p)d].$$

If A wants a guaranteed average payout, she should assume a worst-case choice of q, which is either 0 or 1 since the payout is a linear function of q. In particular, A is guaranteed an average of

$$\min\{pa + (1-p)c, pb + (1-p)d\}.$$

This function achieves its maximum at one of the two endpoints, when $p = 0$ or $p = 1$, or when

$$pa + (1-p)c = pb + (1-p)d,$$

i.e., when

$$p = \frac{d-c}{a-b-c+d}\,.$$

Plugging this value for p into the previous equality yields $\frac{ad-bc}{a-b-c+d}$ on both sides. Hence, A can guarantee herself an average of

$$\max\begin{cases} \min(a,b) & \text{when } p \text{ is chosen to be } 1,\\ \min(c,d) & \text{when } p \text{ is chosen to be } 0,\\ \frac{ad-bc}{a-b-c+d} & \text{when } p \text{ is chosen to be } \frac{d-c}{a-b-c+d}. \end{cases}$$

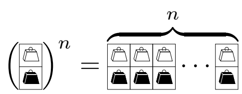
In particular, the fair price is

$$\max\begin{cases} \min(a,b) & \text{when } a \ge c \text{ and } b \ge d,\\ \min(c,d) & \text{when } c \ge a \text{ and } d \ge b,\\ \max(a,c) & \text{when } a \le b \text{ and } c \le d,\\ \max(b,d) & \text{when } b \le a \text{ and } d \le c,\\ \frac{ad-bc}{a-b-c+d} & \text{otherwise.} \end{cases}$$

Chapter 1 Problems and Solutions

Note that a few problems require some familiarity with graph theory. In particular, Euler's Formula, Theorem A.7 on page 265, will come in handy.

1. Consider the $2 \times n$ CLOBBER position



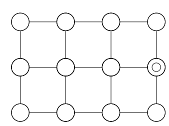
Show that if n is even then

is a second-player win. (By the way, the first player wins when $n \leq 13$ is odd and, we conjecture, for all n odd.)

Solution

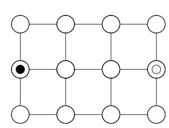
The initial board possesses a symmetry under rotation by 180°, with change of color. The second player can always maintain this symmetry by playing the image of the move made by the first player under this map.

2. Prove that Left to move can win in the COL position



Solution

Left can first move to



and continue similarly thereafter, responding to a Right move with the move obtained by rotating the board through 180°.

3. Suppose two players play STRINGS & COINS with the additional rule that a player, on her turn, can spend a coin to end her turn. The last player to play wins. (Spending a coin means discarding a coin that she has won earlier in the game.)

   (a) Prove that the first player to take any coin wins.

   (b) Suppose the players play on an m-coin by n-coin board with the usual starting position. Prove that if $m + n$ is even, the second player can guarantee a win.

(c) Prove that if $m + n$ is odd, the first player can guarantee a win.

Solution

(a) Suppose Louise takes the first coin. When this happens, she still has to complete her turn. Ignoring the coin for a moment, if, from this position, the first player can win or tie then she completes her turn without spending the coin. If, on the other hand, the second player wins then she finishes her turn by spending the coin and wins.

(b) The second player wins by playing a 180° rotationally symmetric strategy until a coin is offered, and then he wins as above.

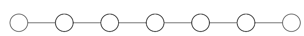
(c) The first player should cut the central string and then play a rotationally symmetric strategy as in (b).

4. Two players play the following game on a round tabletop of radius R. Players take turns placing pennies (of unit radius) on the tabletop, but no penny is allowed to touch another or to project beyond the edge of the table. The first player who cannot legally play loses. Determine who should win as a function of R.[1]

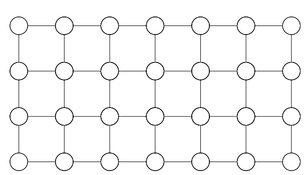
Solution

The first player wins when $R \geq 1$ by placing a penny at the exact center of the board and then continuing with a 180° rotationally symmetric strategy.

5. Who wins SNORT when played on a path of length n?



How about an $m \times n$ grid?



Solution

The first player wins if either m or n is odd, which includes the case of a path, by playing the central vertex (if the number of vertices is odd) or one of the central vertices (if the number of vertices is even) and then a rotationally symmetric strategy. If, however, both m and n are even, the second player can play from the outset in a rotationally symmetric fashion and win.

[1] The players are assumed to have perfect fine motor control!

6. The game of ADD-TO-15 is the same as 3-TO-15 (page 21) except that the first player to get *any* number of cards adding to 15 wins. Under perfect play, is ADD-TO-15 a first-player win, second-player win, or draw?

Solution

The first player, Alice, wins: one strategy is to take 6, Bob must take 9 or lose on the next move; Alice now takes 8 and takes either 1 or 7 on the next move to win.

7. The following vertex deletion game is played on a directed graph. A player's turn consists of removing any single vertex with even indegree (and any edges into or out of that vertex.) Determine the winner if the start position is a directed tree, with all edges pointing toward the root.

Solution

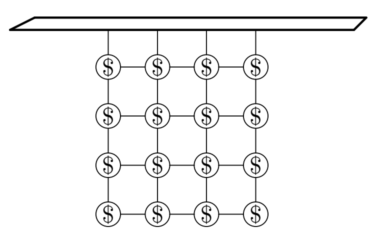
If any vertices remain, then there is always a leaf which has indegree 0, so there is always a move. Thus, the second player wins if and only if the number of vertices is even.

8. Two players play a vertex deletion game on an undirected graph. A turn consists of removing exactly one vertex of even degree (and all edges incident to it.) Determine the winner.

Solution

In any graph, there is an even number of vertices of odd degree. So, if a graph has an odd number of vertices then it has at least one of even degree, and there is a move available. Note that a solitary vertex has degree zero which is even. Therefore, the first player wins if and only if the number of vertices in the graph is odd.

9. A bunch of coins is dangling from the ceiling. The coins are tied to one another and to the ceiling by strings as pictured below. Players alternately cut strings, and a player whose cut causes any coins to drop to the ground loses. If both players play well, who wins?



Solution

The position can be thought of as a graph whose vertices are the coins and whose edges are the strings, with one extra vertex for the ceiling. Just before the last move is played (that is, the move that must drop some of the coins), the position is a spanning tree of the original graph. Since the graph has 17 vertices, the spanning tree will have exactly 16 edges and so there have been $28 - 16 = 12$ moves. Since the first player of the game will make the seventeenth move, the first player loses.

10. The game of SPROUTLETTES is played like SPROUTS, only the maximum allowed degree of a node is only 2. Who wins in SPROUTLETTES?

Solution

Let n be the number of vertices. During a game, every vertex is part of a path or a cycle. Hence, a degree-1 vertex v can never be isolated for there is another degree-1 vertex at the other end of its path. Since each move uses up 2 slots, and doesn't create any, the number of moves is exactly n. Therefore, the game is a variant of SHE LOVES ME SHE LOVES ME NOT, won by the first player if and only if n is odd.

11. Find a winning strategy for BRUSSELS SPROUTS. (*Hint*: Describe how end positions must look and deduce how many moves the game lasts.)

Solution

At the end of the game, the position is a planar graph. Let m be the number of moves and c the number of initial vertices (= crosses). On each move, one new vertex and two new edges (each connecting the newly created vertex to another one) are created. Thus, at any stage of the game, the number of edges is $E = 2m$, and the number of vertices is $V = c + m$. Note that the number of free arms remains constant, so that at the end of the game each face has exactly one arm of a cross inside it and the number of regions is $R = 4c$. Substituting these numbers into Euler's Formula (Theorem A.7 on page 265), $E - V - R + 2 = 0$, gives $2m - (c + m) - 4c + 2 = 0$, showing that $m = 5c - 2$. Suprisingly, the number of initial crosses determines the exact number of moves! So, the first player wins if and only if the number of initial crosses is odd.

12. How many moves does a game of SPROUTS last as a function of both the number of initial dots and the number of isolated degree two nodes at the end of the game? Give a rule for playing SPROUTS analogous to the number of long chains in DOTS & BOXES.

Solution

Let m be the number of moves, s the number of initial dots, and d be the number of degree-2 dots at the end. At the end of the game, the number of edges is $E = 2m$; every move has created a new vertex so that $V = s+m$. In a graph, the sum of the degrees of the vertices is twice the number of edges, so $4m = 3(s+m) - d$ or $m = 3s - d$. If s is even, then the first player should try to make d odd and the second player should try to make d even. If s is odd, then the desired parities for each player are reversed.

13. Prove that the first player wins at HEX. You are free to find and present a proof you find in the literature, but be sure to cite your source and rephrase the argument in your own words.

Solution

The solution parallels the proof of Theorem 1.12. The key is to argue (as cleanly as you can) that the game cannot end in a draw.

14. SQUEX is a game like HEX but is played on a square board. A player makes a turn by placing a checker of her own color on the board. Squares on the board are *adjacent* if they share a side. Black's goal is to connect the top and bottom edges with a path of black checkers, while White wishes to connect the left and right edges with white checkers.

    (a) Prove that the first player should win or draw an $n \times n$ SQUEX position.

    (b) For what values of n is $n \times n$ SQUEX a win for the first player, and when is it a draw? Prove your answer by giving an explicit strategy for the first player to win or the second player to draw as appropriate.

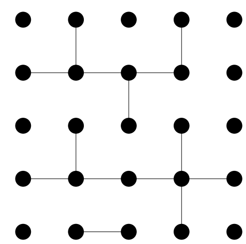
    (c) How about $m \times n$ SQUEX?

Solution

One way to show that the first player wins or draws SQUEX is to use the argument from the alternative proof of Theorem 1.10.

On a 1×1 board the first player wins trivially; it is also easy to see who wins for $1 \times n$ and $m \times 1$. For all other sizes the second player can force a draw. Suppose Black moves first. White selects any two adjacent rows and responds to any Black move on those two rows by the corresponding move in the other row (either above or below). This prevents any path of black stones from the top to the bottom being formed and, thus, guarantees at least a draw for White.

15. Alice is about to make a move in the following DOTS & BOXES position:



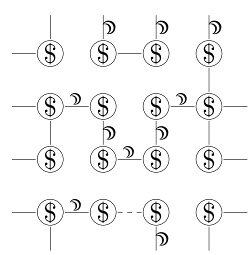
(a) Construct the equivalent STRINGS & COINS position.

(b) Determine if Alice wants an even or odd number of long chains.

(c) Determine all of Alice's winning first moves from this position.

Solution

(a) Here is the STRINGS & COINS position, labeled with loony moves and the winning move (dashed).



(b) Alice wants an odd number of long chains. There are Two ways to see this. First, since the number of moves so far is 14 (which is even), Alice must have started the game. From an empty board, the number of boxes plus moves available is even, so player 1 wants an odd number of long chains. Alternatively, treat the current position as the start position. There are 16 boxes (even) and 26 moves left (also even) so the player to move wants an odd number of chains.

(c) There is no way to break up the middle chain without making a loony move. However, there is one non-loony move that breaks up the lower-left chain, leaving exactly one chain. So Alice should make this move and sacrifice two boxes. Note that if Alice fails to make this move, Bob can take one of the two loose strings hanging off the lower-left coin, making a second long chain.

16. You are about to make a move from the following STRINGS & COINS position: