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Vector Spaces

1.1 INTRODUCTION

2. (b) $x = (2, 4, 0) + t(-5, -10, 0)$ (d) $x = (-2, -1, 5) + t(5, 10, 2)$
 3. (b) $x = (3, -6, 7) + s(-5, 6, -11) + t(2, -3, -9)$
 (d) $x = (1, 1, 1) + s(4, 4, 4) + t(-7, 3, 1)$
 4. $(0, 0)$

1.2 VECTOR SPACES

2. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
 4. (b) $\begin{pmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{pmatrix}$ (d) $\begin{pmatrix} 30 & -20 \\ -15 & 10 \\ -5 & -40 \end{pmatrix}$
 (f) $-x^3 + 7x^2 + 4$ (h) $3x^5 - 6x^3 + 12x + 6$
 5. $\begin{pmatrix} 8 & 3 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 9 & 1 & 4 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 17 & 4 & 5 \\ 6 & 0 & 0 \\ 4 & 1 & 0 \end{pmatrix}$
 16. Yes 18. No, (VS 1) fails. 19. No, (VS 8) fails.

1.3 SUBSPACES

2. (b) $\begin{pmatrix} 0 & 3 \\ 8 & 4 \\ -6 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{pmatrix}$ (f) $\begin{pmatrix} -2 & 7 \\ 5 & 0 \\ 1 & 1 \\ 4 & -6 \end{pmatrix}$
 The trace is 12.
 (h) $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$
 The trace is 2.
 8. (b) No (d) Yes (f) No
 9. $W_1 \cap W_3 = \{0\}$, $W_1 \cap W_4 = W_1$,
 $W_3 \cap W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = -11a_3 \text{ and } a_2 = -3a_3\}$

1.4 LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

2. (b) $(-2, -4, -3)$
 (d) $\{x_3(-8, 3, 1, 0) + (-16, 9, 0, 2): x_3 \in R\}$
 (f) $(3, 4, -2)$
3. (a) $(-2, 0, 3) = 4(1, 3, 0) - 3(2, 4, -1)$
 (b) $(1, 2, -3) = 5(-3, 2, 1) + 8(2, -1, -1)$
 (d) No
 (f) $(-2, 2, 2) = 4(1, 2, -1) + 2(-3, -3, 3)$
4. (a) $x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1)$
 (b) No
 (c) $-2x^3 - 11x^2 + 3x + 2 = 4(x^3 - 2x^2 + 3x - 1) - 3(2x^3 + x^2 + 3x - 2)$
 (d) $x^3 + x^2 + 2x + 13 = -2(2x^3 - 3x^2 + 4x + 1) + 5(x^3 - x^2 + 2x + 3)$
 (f) No
5. (b) No (d) Yes (f) No (h) No
11. The span of $\{x\}$ is $\{0\}$ if $x = 0$ and is the line through the origin of R^3 in the direction of x if $x \neq 0$.
17. if W is finite

1.5 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

2. (b) Linearly independent (d) Linearly dependent
 (f) Linearly independent (h) Linearly independent
 (j) Linearly dependent
10. $(1, 0, 0), (0, 1, 0), (1, 1, 0)$

1.6 BASES AND DIMENSION

2. (b) Not a basis (d) Basis
3. (b) Basis (d) Basis
4. No, $\dim(P_3(R)) = 4$. 5. No, $\dim(R^3) = 3$.
8. $\{u_1, u_3, u_5, u_7\}$
10. (b) $12 - 3x$ (d) $-x^3 + 2x^2 + 4x - 5$
14. $\{(0, 1, 0, 0, 0), (0, 0, 0, 0, 1), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0)\}$ and
 $\{(-1, 0, 0, 0, 1), (0, 1, 1, 1, 0)\}$; $\dim(W_1) = 4$ and $\dim(W_2) = 2$.
16. $\dim(W) = \frac{1}{2}n(n+1)$

1.6 Bases and Dimension

18. Let σ_j be the sequence such that

$$\sigma_j(i) = \begin{cases} 0 & i = j \\ 1 & i \neq j. \end{cases}$$

Then $\{\sigma_j: j = 1, 2, \dots\}$ is a basis for the vector space in Example 5 of Section 1.2.

22. $W_1 \subseteq W_2$

23. (a) $v \in W_1$ (b) $\dim(W_2) = \dim(W_1) + 1$

25. mn

27. If n is even, then $\dim(W_1) = \dim(W_2) = \frac{n}{2}$; and if n is odd,
then $\dim(W_1) = \frac{n+1}{2}$ and $\dim(W_2) = \frac{n-1}{2}$.

32. (a) Take $W_1 = \mathbb{R}^3$ and $W_2 = \text{span}(\{e_1\})$.
(b) Take $W_1 = \text{span}(\{e_1, e_2\})$ and $W_2 = \text{span}(\{e_3\})$.
(c) Take $W_1 = \text{span}(\{e_1, e_2\})$ and $W_2 = \text{span}(\{e_2, e_3\})$.

35. (b) $\dim(V) = \dim(W) + \dim(V/W)$

Linear Transformations and Matrices

2.1 LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

3. The nullity is 0, and the rank is 2. Thus T is one-to-one, but not onto.
6. The nullity is $n - 1$, and the rank is 1. Thus T is not one-to-one unless $n = 1$, and T is not onto unless $n = 1$.
11. $T(8, 11) = (5, -3, 16)$
18. $T(a, b) = (b, 0)$. $N(T) = \text{span}\{(1, 0)\} = R(T)$.
19. Define $T = I$ and $U = 2I$.
23. All of \mathbb{R}^3 or a plane in \mathbb{R}^3 through the origin
24. (a) $T(a, b) = (0, b)$ (b) $T(a, b) = (0, b - a)$
25. (b) $T(a, b, c) = (0, 0, c)$
26. (a) $T = I_V$ (d) $T = T_0$
27. (b) See (a) and (b) of Exercise 22.
31. (c) Let $V = P(F)$ and $V = \text{span}(\{1\})$. Define T first on the standard basis of V by $T(1) = T(x) = 0$, and $T(x^k) = x^{k-1}$ for $k \geq 2$. Now extend T to a linear transformation from V to V . Then $N(T) = \text{span}(\{1, x\})$, and $R(T) = \text{span}(\{x^k : k \geq 1\})$. So $V = R(T) \oplus W$, but $W \neq N(T)$.

2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

2. (b) $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$
4. $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
5. (c) $(1 \ 0 \ 0 \ 1)$ (d) $(1 \ 2 \ 4)$ (f) $\begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$ (g) (a)

2.3 Composition of Linear Transformations and Matrix Multiplication

2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

2. (a) $(AB)D = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

(b) $A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

3. (b) $[h]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $[U(h)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

4. (b) $\begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$ (d) (12)

9. $T(a_1, a_2) = (0, a_1 + a_2)$, $U(a_1, a_2) = (0, a_1)$,

$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

20. (a) $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix}$ There are no cliques.

(b) $B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{pmatrix}$

Persons 1, 3, and 4 belong to a clique.

23. $\frac{n^2 - n}{2}$

2.4 INVERTIBILITY AND ISOMORPHISMS

14. $T \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$

2.5 THE CHANGE OF COORDINATE MATRIX

2. (b) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

3. (b) $\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

6. (b) $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$
- (d) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$, $[L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$
7. (b) $T(x, y) = \frac{1}{m^2+1}(x + my, mx + m^2y)$

2.6 DUAL SPACES

3. (b) $f_1(a + bx + cx^2) = a$, $f_2(a + bx + cx^2) = b$, $f_3(a + bx + cx^2) = c$
4. The basis for V is $\{(.4, -.3, -.1), (.6, .3, .1), (.2, .1, -.3)\}$
6. (a) $T^t(f)(x, y) = 7x + 4y$ (b) $[T^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$
- (c) $[T]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $([T]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

2.7 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

3. (b) $\{1, e^t\}$ (d) $\{e^{-t}, te^{-t}\}$
4. $\{t, te^t, t^2e^t\}$
16. (a) $\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}t\right)$
- (b) $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$
17. $y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$
18. (a) Case 1: $r^2 = 4km$. $y(t) = e^{-(r/2m)t}[c_1 + c_2t]$
Case 2: $r^2 > 4km$. $y(t) = c_1e^{at} + c_2e^{bt}$, where

$$a = \frac{-r}{2m} + \frac{\sqrt{r^2 - 4mk}}{2m}, \quad b = \frac{-r}{2m} - \frac{\sqrt{r^2 - 4mk}}{2m}$$

Case 3: $r^2 < 4km$. $y(t) = e^{at}[c_1 \cos bt + c_2 \sin bt]$, where

$$a = \frac{-r}{2m}, \quad b = \frac{\sqrt{4mk - r^2}}{2m}$$
- (b) Referring to the three cases listed in (a):
Case 1: $y(t) = v_0te^{-(r/2m)t}$
Case 2: $y(t) = \frac{v_0m}{\sqrt{r^2 - 4mk}}[e^{at} - e^{bt}]$
Case 3: $y(t) = \frac{v_0}{b}e^{at} \sin bt$