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Vector Spaces

1.1 INTRODUCTION

2. **(b)** $x = (2, 4, 0) + t(-5, -10, 0)$ **(d)** $x = (-2, -1, 5) + t(5, 10, 2)$

3. **(b)** $x = (3, -6, 7) + s(-5, 6, -11) + t(2, -3, -9)$
(d) $x = (1, 1, 1) + s(4, 4, 4) + t(-7, 3, 1)$

4. $(0, 0)$

1.2 VECTOR SPACES

2. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

4. **(b)** $\begin{pmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{pmatrix}$ **(d)** $\begin{pmatrix} 30 & -20 \\ -15 & 10 \\ -5 & -40 \end{pmatrix}$

(f) $-x^3 + 7x^2 + 4$ **(h)** $3x^5 - 6x^3 + 12x + 6$

5. $\begin{pmatrix} 8 & 3 & 1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 9 & 1 & 4 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 17 & 4 & 5 \\ 6 & 0 & 0 \\ 4 & 1 & 0 \end{pmatrix}$

16. Yes **18.** No, (VS 1) fails. **19.** No, (VS 8) fails.

1.3 SUBSPACES

2. **(b)** $\begin{pmatrix} 0 & 3 \\ 8 & 4 \\ -6 & 7 \end{pmatrix}$ **(d)** $\begin{pmatrix} 10 & 2 & -5 \\ 0 & -4 & 7 \\ -8 & 3 & 6 \end{pmatrix}$ **(f)** $\begin{pmatrix} -2 & 7 \\ 5 & 0 \\ 1 & 1 \\ 4 & -6 \end{pmatrix}$

The trace is 12.

(h) $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

The trace is 2.

8. **(b)** No **(d)** Yes **(f)** No

9. $\mathsf{W}_1 \cap \mathsf{W}_3 = \{0\}$, $\mathsf{W}_1 \cap \mathsf{W}_4 = \mathsf{W}_1$,
$\mathsf{W}_3 \cap \mathsf{W}_4 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 \colon a_1 = -11a_3 \text{ and } a_2 = -3a_3\}$

1.4 LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

2. **(b)** $(-2, -4, -3)$
(d) $\{x_3(-8, 3, 1, 0) + (-16, 9, 0, 2)\colon\ x_3 \in R\}$
(f) $(3, 4, -2)$

3. **(a)** $(-2, 0, 3) = 4(1, 3, 0) - 3(2, 4, -1)$
(b) $(1, 2, -3) = 5(-3, 2, 1) + 8(2, -1, -1)$
(d) No
(f) $(-2, 2, 2) = 4(1, 2, -1) + 2(-3, -3, 3)$

4. **(a)** $x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1)$
(b) No
(c) $-2x^3 - 11x^2 + 3x + 2 = 4(x^3 - 2x^2 + 3x - 1) - 3(2x^3 + x^2 + 3x - 2)$
(d) $x^3 + x^2 + 2x + 13 = -2(2x^3 - 3x^2 + 4x + 1) + 5(x^3 - x^2 + 2x + 3)$
(f) No

5. **(b)** No **(d)** Yes **(f)** No **(h)** No

11. The span of $\{x\}$ is $\{\mathit{0}\}$ if $x = \mathit{0}$ and is the line through the origin of R^3 in the direction of x if $x \neq \mathit{0}$.

17. if W is finite

1.5 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

2. **(b)** Linearly independent **(d)** Linearly dependent
(f) Linearly independent **(h)** Linearly independent
(j) Linearly dependent

10. $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$

1.6 BASES AND DIMENSION

2. **(b)** Not a basis **(d)** Basis

3. **(b)** Basis **(d)** Basis

4. No, $\dim(\mathsf{P}_3(R)) = 4$. **5.** No, $\dim(\mathsf{R}^3) = 3$.

8. $\{u_1, u_3, u_5, u_7\}$

10. **(b)** $12 - 3x$ **(d)** $-x^3 + 2x^2 + 4x - 5$

14. $\{(0, 1, 0, 0, 0), (0, 0, 0, 0, 1), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0)\}$ and $\{(-1, 0, 0, 0, 1), (0, 1, 1, 1, 0)\}$; $\dim(\mathsf{W}_1) = 4$ and $\dim(\mathsf{W}_2) = 2$.

16. $\dim(\mathsf{W}) = \frac{1}{2}n(n+1)$

18. Let σ_j be the sequence such that

$$\sigma_j(i) = \begin{cases} 0 & i = j \\ 1 & i \neq j. \end{cases}$$

Then $\{\sigma_j\colon\ j = 1, 2, \ldots\}$ is a basis for the vector space in Example 5 of Section 1.2.

22. $\mathsf{W}_1 \subseteq \mathsf{W}_2$

23. **(a)** $v \in \mathsf{W}_1$ **(b)** $\dim(\mathsf{W}_2) = \dim(\mathsf{W}_1) + 1$

25. mn

27. If n is even, then $\dim(\mathsf{W}_1) = \dim(\mathsf{W}_2) = \frac{n}{2}$; and if n is odd, then $\dim(\mathsf{W}_1) = \frac{n+1}{2}$ and $\dim(\mathsf{W}_2) = \frac{n-1}{2}$.

32. **(a)** Take $\mathsf{W}_1 = \mathsf{R}^3$ and $\mathsf{W}_2 = \text{span}(\{e_1\})$.
(b) Take $\mathsf{W}_1 = \text{span}(\{e_1, e_2\})$ and $\mathsf{W}_2 = \text{span}(\{e_3\})$.
(c) Take $\mathsf{W}_1 = \text{span}(\{e_1, e_2\})$ and $\mathsf{W}_2 = \text{span}(\{e_2, e_3\})$.

35. **(b)** $\dim(\mathsf{V}) = \dim(\mathsf{W}) + \dim(\mathsf{V}/\mathsf{W})$

2

Linear Transformations and Matrices

2.1 LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

3. The nullity is 0, and the rank is 2. Thus T is one-to-one, but not onto.

6. The nullity is $n-1$, and the rank is 1. Thus T is not one-to-one unless $n=1$, and T is not onto unless $n=1$.

11. $\mathsf{T}(8,11)=(5,-3,16)$

18. $\mathsf{T}(a,b)=(b,0)$. $\mathsf{N}(\mathsf{T})=\operatorname{span}\{(1,0)\}=\mathsf{R}(\mathsf{T})$.

19. Define $\mathsf{T}=\mathsf{I}$ and $\mathsf{U}=2\mathsf{I}$.

23. All of R^3 or a plane in R^3 through the origin

24. **(a)** $\mathsf{T}(a,b)=(0,b)$ **(b)** $\mathsf{T}(a,b)=(0,b-a)$

25. **(b)** $\mathsf{T}(a,b,c)=(0,0,c)$

26. **(a)** $\mathsf{T}=\mathsf{I}_\mathsf{V}$ **(d)** $\mathsf{T}=\mathsf{T}_0$

27. **(b)** See (a) and (b) of Exercise 22.

31. **(c)** Let $\mathsf{V}=\mathsf{P}(F)$ and $\mathsf{V}=\operatorname{span}(\{1\})$. Define T first on the standard basis of V by $\mathsf{T}(1)=\mathsf{T}(x)=\mathit{0}$, and $\mathsf{T}(x^k)=x^{k-1}$ for $k\geq 2$. Now extend T to a linear transformation from V to V. Then $\mathsf{N}(\mathsf{T})=\operatorname{span}(\{1,x\})$, and $\mathsf{R}(\mathsf{T})=\operatorname{span}(\{x^k\colon k\geq 1\})$. So $\mathsf{V}=\mathsf{R}(\mathsf{T})\oplus\mathsf{W}$, but $\mathsf{W}\neq\mathsf{N}(\mathsf{T})$.

2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

2. **(b)** $\begin{pmatrix}2 & 3 & -1\\ 1 & 0 & 3\end{pmatrix}$ **(e)** $\begin{pmatrix}1 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\\ \vdots & \vdots & & \vdots\\ 1 & 0 & \dots & 0\end{pmatrix}$

4. $\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\end{pmatrix}$

5. **(c)** $(1\ 0\ 0\ 1)$ **(d)** $(1\ 2\ 4)$ **(f)** $\begin{pmatrix}3\\ -6\\ 1\end{pmatrix}$ **(g)** (a)

2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

2. **(a)** $(AB)D = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

(b) $A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

3. **(b)** $[h]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $[\mathsf{U}(h)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

4. **(b)** $\begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$ **(d)** (12)

9. $\mathsf{T}(a_1, a_2) = (0, a_1 + a_2)$, $\mathsf{U}(a_1, a_2) = (0, a_1)$,

$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

20. **(a)** $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix}$ There are no cliques.

(b) $B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{pmatrix}$

Persons 1, 3, and 4 belong to a clique.

23. $\dfrac{n^2 - n}{2}$

2.4 INVERTIBILITY AND ISOMORPHISMS

14. $\mathsf{T}\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$

2.5 THE CHANGE OF COORDINATE MATRIX

2. **(b)** $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

3. **(b)** $\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

6. **(b)** $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $[\mathsf{L}_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

(d) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$, $[\mathsf{L}_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$

7. **(b)** $\mathsf{T}(x, y) = \frac{1}{m^2+1}(x + my, mx + m^2 y)$

2.6 DUAL SPACES

3. **(b)** $f_1(a + bx + cx^2) = a$, $f_2(a + bx + cx^2) = b$, $f_3(a + bx + cx^2) = c$

4. The basis for V is $\{(.4, -.3, -.1), (.6, .3, .1), (.2, .1, -.3)\}$

6. **(a)** $\mathsf{T}^t(f)(x, y) = 7x + 4y$ **(b)** $[\mathsf{T}^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

(c) $[\mathsf{T}]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $([\mathsf{T}]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

2.7 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

3. **(b)** $\{1, e^t\}$ **(d)** $\{e^{-t}, te^{-t}\}$

4. $\{t, te^t, t^2 e^t\}$

16. **(a)** $\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}t\right)$

(b) $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$

17. $y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$

18. **(a)** *Case 1:* $r^2 = 4km$. $y(t) = e^{-(r/2m)t}[c_1 + c_2 t]$

Case 2: $r^2 > 4km$. $y(t) = c_1 e^{at} + c_2 e^{bt}$, where

$$a = \frac{-r}{2m} + \frac{\sqrt{r^2 - 4mk}}{2m}, \quad b = \frac{-r}{2m} - \frac{\sqrt{r^2 - 4mk}}{2m}$$

Case 3: $r^2 < 4km$. $y(t) = e^{at}[c_1 \cos bt + c_2 \sin bt]$, where

$$a = \frac{-r}{2m}, \quad b = \frac{\sqrt{4mk - r^2}}{2m}$$

(b) Referring to the three cases listed in (a):

Case 1: $y(t) = v_0 te^{-(r/2m)t}$

Case 2: $y(t) = \dfrac{v_0 m}{\sqrt{r^2 - 4mk}}[e^{at} - e^{bt}]$

Case 3: $y(t) = \dfrac{v_0}{b} e^{at} \sin bt$