Instructor's Guide to Accompany

Linear Algebra

A Modern Introduction

Third Edition

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Preface

The purpose of this *Instructor's Guide* is to save you time while helping you to teach an honest, interesting, student-centered course. For each section, there are suggested additions to your lecture that can supplement (but not replace) things like like taking the inverse of $\begin{bmatrix} 2 & 1 & 6 \\ 3 & 1 & -1 \\ 5 & 2 & -1 \end{bmatrix}$. Speaking of problems like that, for each section we have some worked-out routine examples that you can use straight out of our guide without having to do the computations yourself ahead of time.

Lecturing is not your only option, of course. This guide provides group activities, ready for copying, that will allow your students to discover and explore the concepts of linear algebra. You may find that your classes become more "fun", but we assure you that this unfortunate by-product of an engaged student population can't always be avoided.

This guide was designed to be used with *Linear Algebra, Third Edition* as a source of both supplementary and complementary material. Depending on your preferences, you can occasionally glance through the *Guide* for content ideas and alternate approaches, or you can use it as a major component in planning your day-to-day classes. In addition to lecture notes and group activities, each section has technology tips, sample homework assignments, and sample quiz questions.

The unfortunate among us remember two linear algebra courses: the dry, computational one where we do Gaussian elimination by hand on larger and larger systems, and the dry, theoretical one where students do the same induction proof on the same abstract vector spaces repeatedly, occasionally taking a break to find yet another method of putting things in row-echelon form. Poole's book finds a third way, bringing to life the beautiful structures that we linear algebra fans love so much, and adding accessible applications that show how this wonderful subject can be applied. We were proud to write this *Instructor's Guide* in that spirit.

We value reactions from all of our colleagues who are teaching from this guide, both to correct any errors and to suggest additional material for future editions. We are especially interested in which features of the guide are the most and the least useful. Please email any feedback to `linearalgebra@dougshaw.com`.

We would like to thank David Poole, John-Paul Ramin and Bob Pirtle for giving us this opportunity. Stacy Green has been a wonderful editor, and her guidance throughout this project has been greatly appreciated. The "Find the Error" problems came from a raid of Dr. Poole's notes (we added the moon pie taunts). Melissa Potter's help has been invaluable on this and other projects. She always does wonderful work — catching errors, offering suggestions, and reminding us that students are people, and that we should stop using the definite article to describe them. When deadlines slipped, she was always ready to work on short notice. Thanks again, Melissa.

Andy Bulman-Fleming, one of the best typesetters in the business, again did a stellar job, with turnaround times that we had no right to ask for. When we agreed to write this book, our first question was, “Can we have Andy?” because we love the clarity of his graphics and the smoothness of his design. We think it makes a big difference, and we’re sure you will agree.

We would also like to thank the University of Northern Iowa mathematics department, for always encouraging its professors to try new things and to pursue their passions. Our colleagues, department head, and dean have been very supportive.

This *Instructor’s Guide* has meant many late evenings, weekend meetings, sudden phone calls, more late evenings, and bringing laptop computers to the Village Inn during breakfast. Our wives, Laurel and Margaret, have never complained about being Brooks/Cole Widows, and we hope they will be happy to have us back. We are proud to dedicate this book to them.

Michael Prophet
Douglas Shaw

Contents

How to Use the Instructor's Guide

For each section of *Linear Algebra, Third Edition*, this *Instructor's Guide* provides information on the items listed below.

1. **Suggested Time and Emphasis** These suggestions assume that the class is fifty minutes long. They also advise whether or not the material is essential to the rest of the course. If a section is labeled "optional", the time range given is the amount of time for the material in the event that it is covered.
2. **Points to Stress** This is a short summary of the big ideas to be covered.
3. **Sample Questions**
   - **Drill Questions:** Some instructors have reported that they like to open or close class by handing out a single question. These questions are designed to be straightforward "right down the middle" questions for students who have read but not yet mastered the material.
   - **Discussion Questions:** These questions are more open-ended questions designed to provoke a lively conversation among the students, or between the students and the instructor. While most of them have answers, some of them do not, and some of them will be answered later on in the course. The idea here is to get the students talking mathematics, as opposed to talking about mathematics.
   - **Test Questions:** These questions are meant to be interesting ones to add to an exam. We do not reccommend making all of the questions like ours — the questions we provide are meant to add spice to a more routine test.
4. **Lecture Notes** These suggestions are meant to work along with the text to create a classroom atmosphere of experimentation and inquiry.
5. **Lecture Examples** These are routine examples with all the computations worked out, designed to save a bit of time in class preparation.
6. **Tech Tips** Many students have access to symbolic algebra packages like Maple, Mathematica, Matlab, and the TI-89 or TI-92 calculators. These tips are meant to give you ideas on incorporating technology into your course.
7. **Group Work** One of the main difficulties instructors have in presenting group work to their classes is that of choosing an appropriate group task. Suggestions for implementation and answers to the group activities are provided first, followed by photocopy-ready handouts on separate pages. This guide's main philosophy of group work is that there should be a solid introduction to each exercise ("What are we supposed to do?") and good closure before class is dismissed ("Why did we just do that?")

8. **Sample Core Assignment** Every teacher has a different philosophy on assigning homework. These problems have been selected to form the basis for a homework assignment. Many instructors will want to assign a superset of our problems, and some might want to trim them slightly. The problems that require proofs as answers are marked with a superscript P.

1 Vectors

1.1 The Geometry and Algebra of Vectors

Suggested Time and Emphasis

1 class. Essential material.

Points to Stress

1. A vector must have an initial point and a terminal point.
2. Translation invariance of vectors, including the concept of standard position.
3. The geometric interpretation of vector addition.
4. Linear combinations.

Drill Question

Find a vector $\mathbf{v}$ that can be written as a linear combination of $[1, 1, 1]$ and $[2, 0, 3]$.

Answer Answers will vary

Discussion Question

Is there a vector $\mathbf{w}$ in $\mathbb{R}^n$, other than $\mathbf{0}$, that has the property $\mathbf{v} + \mathbf{w} = \mathbf{v}$ for every $\mathbf{v}$ in $\mathbb{R}^n$?

Answer No

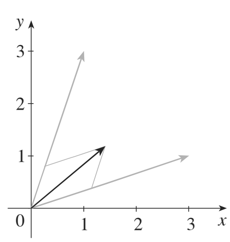
Test Question

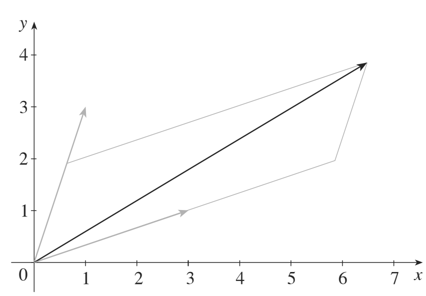
Can every element of $\mathbb{R}^3$ be written as a linear combination of $\mathbf{v}_1 = [2, 1, 0]$, $[0, 1, 0]$ and $[2, 2, 0]$?

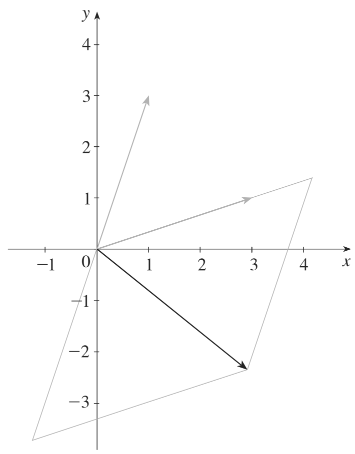
Answer No; any linear combination of these vectors has z-component 0.

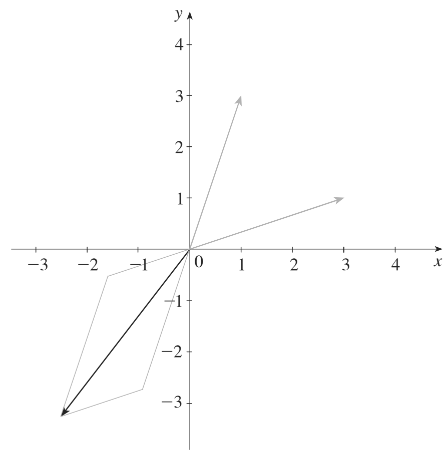
Lecture Notes

- Draw $[3, 1]$ and $[1, 3]$ on the board in standard position. Have the students draw vectors in standard position, and show them how they can always be "swallowed" by linear combinations of these two vectors. Examples are given below.





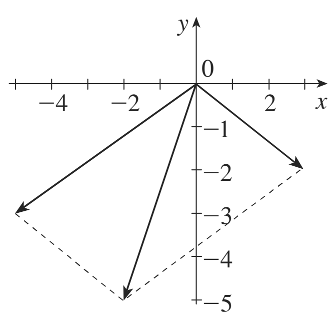




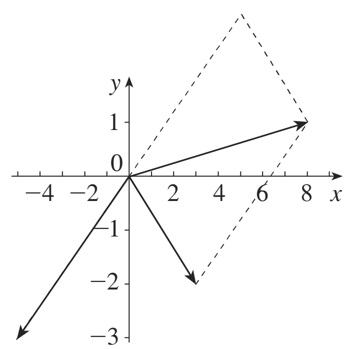
- Assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are in standard position. Demonstrate that the vector $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$ can be viewed as the vector with initial point $\mathbf{v}_2$ and terminal point $\mathbf{v}_1$. Draw this picture in $\mathbb{R}^2$ and (if you can) in $\mathbb{R}^3$.
- Ask the students to describe the set of all possible linear combinations of $[1, 0]$ and $[0.1]$. Extend to all linear combinations of $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. Then ask them to explore the linear combinations of $[1, 0, 0]$, $[0, 0, 1]$, and $[-2, 0, 2]$.
- Let $\mathbf{v}_1 = [1, 2, 3]$ and let $\mathbf{v}_2 = [4, 5, 6]$. Find a third vector $\mathbf{v}_3$, such as $[5, 7, 9]$, so that $\mathbb{R}^3$ is *not* the set of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Lecture Examples

- Geometric and algebraic addition and subtraction:



$[3, -2] + [-5, -3] = [-2, -5]$



$[3, -2] - [-5, -3] = [8, 1]$

- Addition of 4-vectors: $[4, 2, 9, \pi] + [-4, 1, 1, e] = [0, 3, 10, e + \pi]$

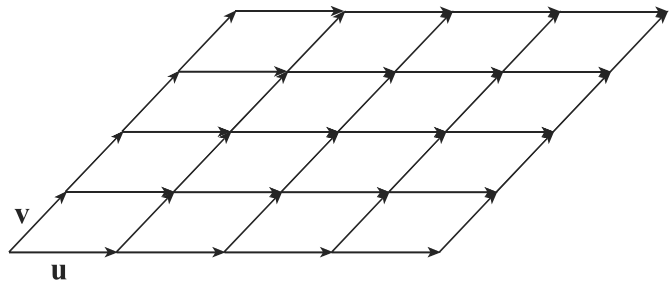
Tech Tip

Show students how to draw vectors in Maple using the `arrow` command. Consider drawing a sequence of vectors of the form $[\cos t, \sin t]$, where t ranges between 0 and 2π.

Group Work 1: The Spanning Set

The purpose of this activity is to give the students a sense of how two non-parallel, two-dimensional vectors span an entire plane.

Start by giving each student or group of students a sheet of regular graph paper and a transparent grid of parallelograms formed by two vectors $\mathbf{u}$ and $\mathbf{v}$.



Next give each student a point (x, y) in the plane. By placing the grid over the graph paper they should estimate values of r and s such that $[x, y] = r\mathbf{u} + s\mathbf{v}$. Repeat for several other points, including some that require negative values of r and/or s, until the students have convinced themselves that every point in the plane can be expressed in this manner.

Now repeat the activity with different vectors $\mathbf{u}$ and $\mathbf{v}$, perhaps using the same points as before.

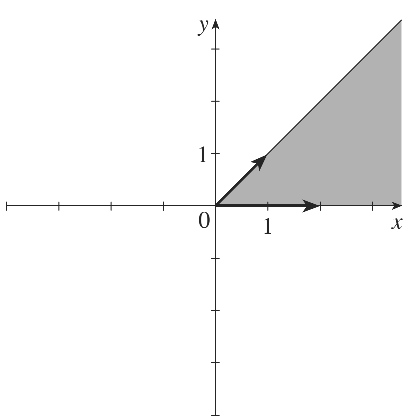
As a wrap-up, give the students specific vectors $\mathbf{u}$ and $\mathbf{v}$, such as $\mathbf{u} = [3, 1]$ and $\mathbf{v} = [-1, -2]$, and have them determine values of r and s for several points. See if they can find general formulas for r and s in terms of the point (x, y). What “goes wrong” algebraically if $\mathbf{u}$ and $\mathbf{v}$ are parallel?

Group Work 2: Cones in General

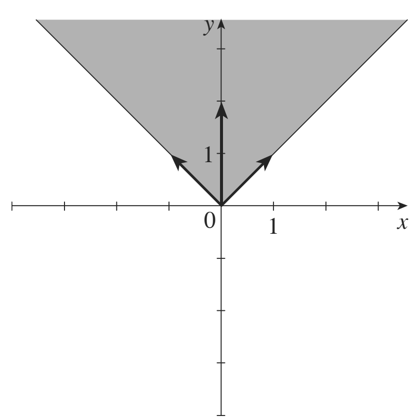
This activity will introduce students to the concept of the cone of a set of vectors. You might want to stop after the first question and make sure they all understand the definition, or give a different example to do all together before setting them loose.

Answers

1.



2.



3. Yes. $[0, 7, 3] = 2\,[1, 1, 1] + 2\,[-1, 2, 1] + (-1)\,[0, -1, 1]$

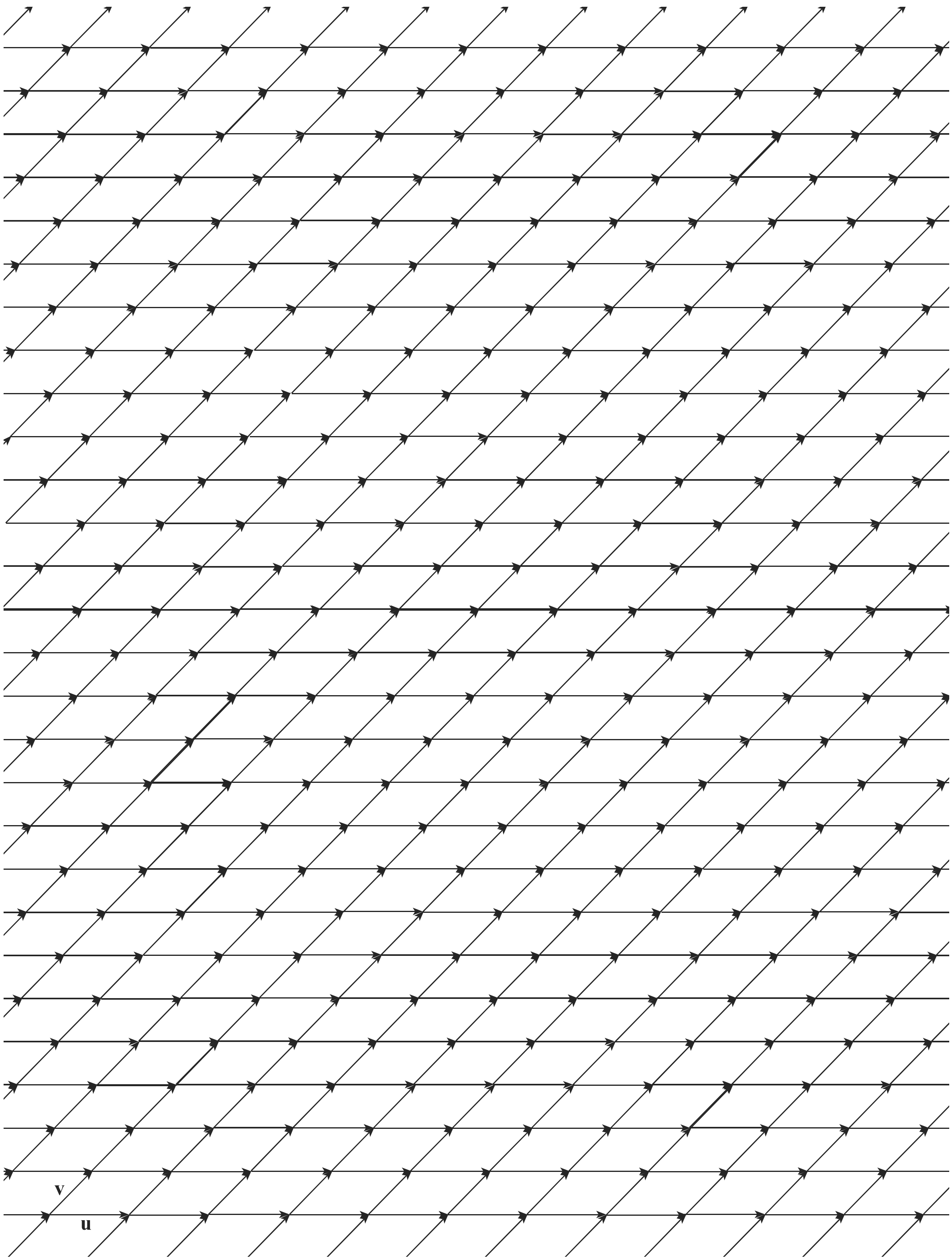
4. Yes. $[0, 7, 3] = 2\,[1, 1, 1] + 2\,[-1, 2, 1] + (-1)\,[0, -1, 1] + 0\,[0, 1, 0]$

Suggested Core Assignment

Note: Exercises requiring proofs are marked with a superscript P.

Exercises 2, 5, 6, 12, 14, 18, 22, 24$^{\mathrm{P}}$

Group Work 1, Section 1.1
The Spanning Set



Group Work 2, Section 1.1
Cones in General

The ***cone*** in $\mathbb{R}^n$ generated by a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$ is the set of all nonnegative linear combinations of these vectors, drawn in standard position. (A nonnegative linear combination is of the form $\sum_{i=1}^{k} c_i \mathbf{v}_i$, where $c_i \geq 0$, i.e. a linear combination where all of the coefficients are nonnegative.)

1. Cones in $\mathbb{R}^2$ are easy to identify. Shade in the cone generated by $\mathbf{v}_1 = [1, 1]$ and $\mathbf{v}_2 = [2, 0]$.

2. Shade in the cone generated by $\mathbf{v}_1 = [-1, 1]$, $\mathbf{v}_2 = [0, 2]$, and $\mathbf{v}_3 = [1, 1]$.

3. Does the vector $[0, 7, 3]$ belong to the cone generated by $[1, 1, 1]$, $[-1, 2, 1]$, and $[0, -1, 1]$?

4. Does the vector $[0, 7, 3]$ belong to the cone generated by $[1, 1, 1]$, $[-1, 2, 1]$, $[0, -1, 1]$, and $[0, 1, 0]$?

1.2 Length and Angle: The Dot Product

Suggested Time and Emphasis

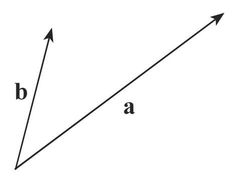
1 class. Essential material.

Points to Stress

1. The dot product and its properties.
2. Definition of orthogonality.
3. Length of a vector.
4. Projections.

Drill Question

Given the vectors $\mathbf{a}$ and $\mathbf{b}$ below, draw $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$ and $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$.

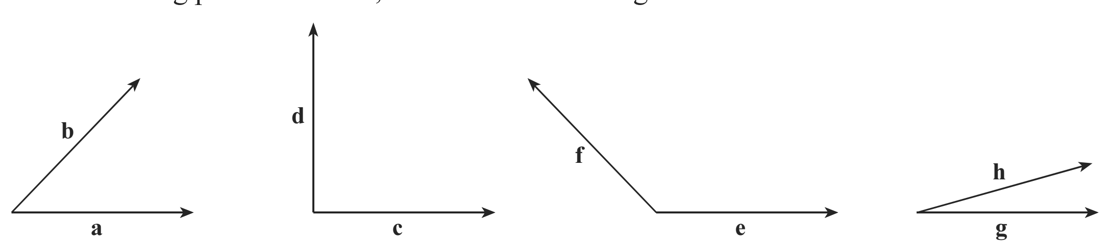


Discussion Question

Investigate the following question: What is the maximum number of mutually orthogonal nonzero vectors in $\mathbb{R}^k$? It's easy to see geometrically that in $\mathbb{R}^2$, there exist two mutually orthogonal vectors and there cannot exist three. In $\mathbb{R}^3$, sketch the lines determined by three mutually orthogonal vectors. Notice that this creates a set of coordinate axes just like the standard ones. Every element of $\mathbb{R}^3$ belongs to one of the "octants" created by your new axes, and thus every element of $\mathbb{R}^3$ is a linear combination of the three mutually orthogonal vectors. Show why this implies that there cannot be a fourth nonzero vector that is orthogonal to the other three.

Test Question

Consider the following pairs of vectors, all of which have length 1:



Put the following quantities in order, from smallest to largest:

$$\mathbf{a} \cdot \mathbf{b} \qquad \mathbf{c} \cdot \mathbf{d} \qquad \mathbf{e} \cdot \mathbf{f} \qquad \mathbf{g} \cdot \mathbf{h}$$

Answer $\mathbf{e} \cdot \mathbf{f}$, $\mathbf{c} \cdot \mathbf{d}$, $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{g} \cdot \mathbf{h}$

Lecture Notes

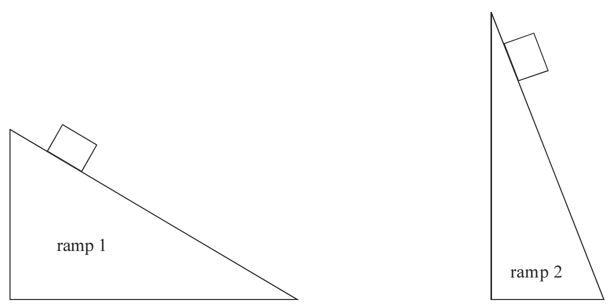
- Assume that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \neq 0$. Pose the question, "Is it necessarily true that $\mathbf{b} = \mathbf{c}$?" When you've convinced them (perhaps by example) that the answer is "no", the next logical question to ask is, "What can we say about $\mathbf{b}$ and $\mathbf{c}$?" It can be shown that $\mathbf{b}$ and $\mathbf{c}$ have the same projection onto $\mathbf{a}$, since $\mathbf{a} \perp (\mathbf{b} - \mathbf{c})$.

- Derive the formula for $\text{proj}_{\mathbf{u}}\, \mathbf{v}$ as in Exercise 57: The vector $\text{proj}_{\mathbf{u}}\, \mathbf{v}$ must be a scalar multiple of $\mathbf{u}$, and thus we know that $\text{proj}_{\mathbf{u}}\, \mathbf{v} = c\mathbf{u}$ for some c. By definition, $\mathbf{v} - c\mathbf{u}$ is orthogonal to $\mathbf{u}$, so we can use this fact to solve for c.

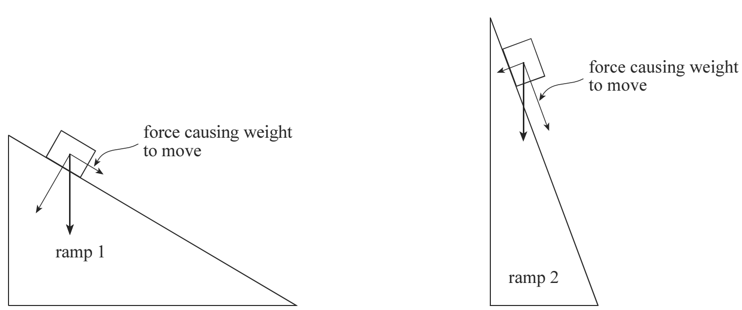
- Demonstrate the distributive property in general. Note that while the dot product is commutative and distributive, the associative property makes no sense, as it is not possible to take the dot product of three vectors.

- Demonstrate the proper formation of statements involving dot products. For example, the statement $c\,(\mathbf{a} \cdot \mathbf{b})$ makes sense, while the statements $\mathbf{d} \cdot (\mathbf{a} \cdot \mathbf{b})$ and $c \cdot \mathbf{a}$ do not.

- This is a nice, direct application of vector projections: It is clear that a weight will slide more quickly down ramp 2 than down ramp 1:



Gravity is the same in both cases, yet there is a definite difference in speed. The reason behind this is interesting. Gravity is doing two things at once: it is letting the weight slide down, and it is also preventing the weight from floating off the ramp and flying into outer space. We can draw a "free body diagram" that shows how the gravity available to let the weight slide down is affected by the angle of the ramp.



A block slides faster on a steeper slope because the projection of the gravitational force in the direction of the slope is larger. There is more force pushing the block down the slope, and less of a force holding it to the surface of the slope.

Lecture Examples

- $[5, 6, 2] \cdot [-3, 1, 0] = -9$
- Two vectors that are orthogonal: $\mathbf{a} = [5, 6, 2]$, $\mathbf{b} = \left[1, -1, \frac{1}{2}\right]$
- Projections: If $\mathbf{a} = [2, 1, -1]$ and $\mathbf{b} = [3, 2, 7]$, then $\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left[\frac{1}{3}, \frac{1}{6}, -\frac{1}{6}\right]$, $\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \left[\frac{3}{62}, \frac{1}{31}, \frac{7}{62}\right]$, and $||\mathbf{a}|| = \sqrt{6}$.

Tech Tips

- Have the students write a routine that takes, as its input, two non-zero vectors in $\mathbb{R}^3$ and computes the angle between them.
- A more advanced challenge would be to have the students devise a routine that finds a vector perpendicular to two given vectors. (In essence, the students are being asked to derive something like the cross product.)

Group Work 1: Orthogonal Projections Rule

This worksheet extends what the students know about orthogonal projections to $\mathbb{R}^3$. If they have a strong conceptual knowledge of projections, this should come easily.

Answers

1. P is a plane through the origin.
2. Answers will vary.
3. $[u_1 - c_1 - c_2, u_2 - c_1, u_3 - c_1]$
4. $-3c_1 - c_2 + u_1 + u_2 + u_3 = 0, u_1 - c_1 - c_2 = 0$
5. $c_1 = \frac{1}{2}(u_1 + u_2)$, $c_2 = \frac{1}{2}(2u_1 - u_2 - u_3)$

Group Work 2: A Glimpse of Things to Come

This activity foreshadows later work, expressing a given vector as a linear combination of two other vectors. Students may find this difficult because the vectors and components are given in general. One of the main hurdles for a linear algebra student is making the transition to thinking abstractly.

Answers

1. We know that $\mathbf{v} \cdot \mathbf{u} = 0$, which implies $v_1 u_1 + v_2 u_2 = 0$.
2. This is clear if one thinks about it geometrically. There is also an algebraic solution:

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} = (0)\,\mathbf{u} = 0$$

3. The easiest way to see this is geometrically. If $\mathbf{v}$ and $\mathbf{u}$ are orthogonal, then $\mathbf{v}$ and $[u_2, -u_1]$ are parallel. The dot product of two nonzero parallel vectors cannot be 0. And $\mathbf{v} \cdot [u_2, -u_1] = v_1 u_2 - v_2 u_1$. There is a more cumbersome algebraic proof:

$$\begin{aligned}
v_1 &= -\frac{v_2 u_2}{u_1} \\
v_1 u_2 &= -\frac{v_2 (u_2)^2}{u_1} \\
v_1 u_2 - v_2 u_1 &= -\frac{v_2 (u_2)^2}{u_1} - \frac{v_2 (u_1)^2}{u_1} \\
&= -\frac{v_2}{u_1}\left(u_2^2 + u_1^2\right)
\end{aligned}$$

The only way that $-\frac{v_2}{u_1}\left(u_2^2+u_1^2\right)$ can be zero is if $v_2=0$, but then orthogonality forces u_1 to be zero as well, causing the expression to be undefined. The case $v_2=u_1=0$ is easy to check directly.

4. Again, this is easy to see geometrically. To obtain an algebraic solution, we choose an arbitrary vector $[x_1, x_2]$ and solve the system

$$\begin{aligned} x_1 &= au_1+bv_1 \\ x_2 &= au_2+bv_2 \end{aligned}$$

This system has a unique solution provided $u_1v_2-u_2v_1$ is not zero, which was shown in the previous question.

Group Work 3: The Right Stuff

Give each group of students a different set of three points, and have them use vectors to determine if they form a right triangle. (It is easiest to write the points directly on the pages before handing them out.) They can do this using dot products, by calculating side lengths and using the Pythagorean Theorem, or by calculating the slopes of lines between the pairs of points. Have the students whose points are in $\mathbb{R}^2$ carefully graph their points to provide a visual check. At the end of the exercise, point out that using the dot product is the easier method.

Sample triples:

$(-2,-1)$, $(-2,8)$, $(8,-1)$	Right	$(3,4)$, $(3,12)$, $(6,5)$	Not right
$(0,0)$, $(10,7)$, $(-14,20)$	Right	$(2,1,2)$, $(3,3,1)$, $(2,2,4)$	Right
$(-1,-2,-3)$, $(0,0,-4)$, $(-1,-1,-1)$	Right	$(2,3,6)$, $(3,4,7)$, $(3,3,6)$	Right

Group Work 4: The Regular Hexagon

If the students have trouble with this one, copy the figure onto the blackboard. Then draw a point at its center, and draw lines from this point to every vertex. This modified figure should make the exercise more straightforward.

Answers

1. 1, 1, 1 **2.** 120° **3.** $\cos 60^\circ = \frac{1}{2}$ **4.** $-\frac{1}{2}$ **5.** $\left[-\frac{1}{2}, 0\right]$, $\left[-\frac{1}{4}, \frac{\sqrt{3}}{4}\right]$ **6.** 1

Group Work 5: Find the Error (Part 1)

Answer

There is no cancellation law for dot products. For example, $\begin{bmatrix}-1\\6\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=11$ and $\begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=11$, but it is not the case that $\begin{bmatrix}-1\\6\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}$. Some students may make the argument that no two vectors exist that meet the requirements of this problem. Examples of two such vectors are $\mathbf{u}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}2\\0\end{bmatrix}$.

Group Work 6: Find the Error (Part 2)

Answer

No two such vectors exist. This can be shown by Cauchy-Schwarz, or by the fact that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$.

Suggested Core Assignment

Exercises 4, 10, 16, 19, 25, 30, 36, 43, 44, 54^{P}, 59^{P}, 63^{P}, 65^{P}

Group Work 1, Section 1.2
Orthogonal Projections Rule

We know how to orthogonally project one vector onto another. Let's try to extend this procedure in the setting of $\mathbb{R}^3$.

1. Let $\mathbf{v}_1 = [1, 1, 1]$ and $\mathbf{v}_2 = [1, 0, 0]$ be vectors (in standard position) in $\mathbb{R}^3$ and let P denote the set of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. Describe what P looks like.

2. With P fixed, we want to orthogonally project a given vector $\mathbf{u}$ onto P. That is, we want the orthogonal component of $\mathbf{u}$ that belongs to P. Sketch P and orthogonal projections for a couple of different $\mathbf{u}$'s. Let's let $\mathrm{proj}_P\,\mathbf{u}$ denote the projection.

3. Now since $\mathrm{proj}_P\,\mathbf{u}$ belongs to P, we can write $\mathrm{proj}_P\,\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ for some constants c_1 and c_2. Our goal is to obtain formulas for c_1 and c_2. Expressing $\mathbf{u}$ using coordinates, we write $\mathbf{u} = [u_1, u_2, u_3]$. Now express, in coordinate form, the vector $\mathbf{u} - \mathrm{proj}_P\,\mathbf{u}$. Make sure to use the equation for $\mathrm{proj}_P\,\mathbf{u}$ and the coordinates of $\mathbf{v}_1$ and $\mathbf{v}_2$ above.

4. From our work in Problem 2, we know that $\mathbf{u} - \mathrm{proj}_P\,\mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Combine this fact with your work from Problem 3. to find 2 equations involving the unknowns c_1 and c_2.

5. Finally, solve the above equations to give formulas for c_1 and c_2.

Group Work 2, Section 1.2
A Glimpse of Things to Come

Let $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$ be *orthogonal* elements of $\mathbb{R}^2$.

1. Why do we know that $v_1 u_1 + v_2 u_2 = 0$?

2. Show that the projection of $\mathbf{v}$ onto $\mathbf{u}$ is always $\mathbf{0}$.

3. Show that $v_1 u_2 - v_2 u_1$ is never equal to zero.

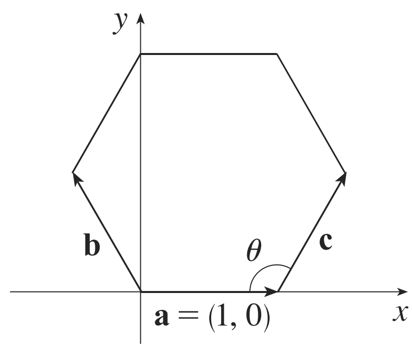
4. Show that any vector in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{u}$.

Group Work 3, Section 1.2
The Right Stuff

Consider the points (,), (,), and (,). These three points form a triangle. Is this triangle a right triangle? Justify your answer.

Group Work 4, Section 1.2
The Regular Hexagon

Consider the following regular hexagon:



1. Compute $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, and $\|\mathbf{c}\|$.

2. What is the angle θ?

3. What is $\mathbf{a} \cdot \mathbf{c}$?

4. What is $\mathbf{a} \cdot \mathbf{b}$?

5. Compute $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$ and $\operatorname{proj}_{\mathbf{b}} \mathbf{c}$.

6. Compute the x-component of $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

Group Work 5, Section 1.2
Find the Error (Part 1)

It is a beautiful spring morning. You are about to go to your 4 P.M. class, but have stopped at a convenience store to buy carrot sticks and bottled water for a healthy snack. As you wait in line to pay for your purchases, whistling to yourself, you notice a wild-eyed gentleman standing in line in front of you, buying a moon pie.

“Well aren’t you a merry grig?” he asks. You nod noncommittally, since you have no idea what a “grig” is. He takes your nod to mean that you would like further conversation, and asks, “Where are you off to now?”

“Why, I’m off to my linear algebra class, to learn some useful information about vectors.”

“Vectors, vectors,” he says, half to himself. “I remember learning about vectors... I remember learning... LIES!”

“What do you mean, ‘lies’?” you ask. “Everything we’ve learned about vectors is as true as it is useful.”

“Oh yes? You think you know it all, do you?” By this point, he has paid for his purchases. As you pay for yours, you notice him writing on his receipt:

Let $\mathbf{u}$ be a vector such that $\|\mathbf{u}\| = \sqrt{2}$. Choose a vector $\mathbf{v} \neq \mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 2$. Now we have

$$\begin{aligned}
2(\mathbf{u}\cdot\mathbf{u}) &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} \\
2(\mathbf{u}\cdot\mathbf{u}) - 2(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{2}(\mathbf{u}\cdot\mathbf{v}) \\
2(\mathbf{u}\cdot\mathbf{u}) - 2(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} \\
2\mathbf{u}\cdot(\mathbf{u}-\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}-\mathbf{v}) \\
2\mathbf{u} &= \mathbf{u} \\
2 &= 1
\end{aligned}$$

“I’ve seen someone try this before,” you say dismissively, “in college algebra. But you are not allowed to divide by zero.”

“Ah, but I am not dividing by zero! Since $\mathbf{u} \neq \mathbf{v}$, we know that $\mathbf{u} - \mathbf{v}$ cannot be zero! Now you go enjoy your class, while I go and enjoy my moony pie!” And the stranger leaves, singing a strange song to himself, and opening the wrapper to his moon pie.

Could Linear Algebra be flawed already? Two can’t equal one, can it? Find the error in the gentleman’s reasoning.

Group Work 6, Section 1.2
Find the Error (Part 2)

After determining the stranger's mistake, you go to your Linear Algebra class. Your teacher tells you to pay particular attention to page 18, so you take a scrap of paper to mark the page. You notice that you are using the gentleman's receipt, and that he has written something on the front of the receipt as well!

1 moon pie $1.50

Dear Merry Grig,

If I haven't already convinced you that your teacher is nothing but a purveyor of falsity, check this out:

Let $\mathbf{u}$ be a vector such that $\|\mathbf{u}\| = 1$. Choose a vector $\mathbf{v}$ such that $\mathbf{u}\cdot\mathbf{v} = 3$ and $\|\mathbf{v}\| = \sqrt{5}$. Now we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} - 2\,(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}\\ &= 0\end{aligned}$$

Hence $\mathbf{u} = \mathbf{v}$, since $\mathbf{u}-\mathbf{v} = \mathbf{0}$. But $\mathbf{u}$ and $\mathbf{v}$ have different lengths!

Well, gosh darn him anyway! How can two things be the same, and yet different?

Find the error.

1.3 Lines and Planes

Suggested Time and Emphasis

1 class. Recommended material.

Points to Stress

1. The definition and intuitive idea of "normal".
2. The normal, vector, parametric and general forms of the equation of a line.
3. The normal, vector, parametric and general forms of the equation of a plane.
4. The vector form of the equation of a line (plane) viewed as a translation of all linear combinations of one (two) fixed vector(s).

Drill Question

1. Consider the line with equation $2x + y = 0$. Sketch this line, and then write its equation in vector form, and then again in normal form.

   Answer $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$

2. Why do you even need the vector $\mathbf{p}$? Why can't you just specify a line with the direction vector alone?

   Answer Without the vector $\mathbf{p}$, the line would go through the origin.

Test Question

Do the following four points all lie on the same plane? Why or why not?

$$(3, 2, 1) \quad (3, 1, 2) \quad (5, 2, 3) \quad (2, 0, 2)$$

Answer Yes. An equation of the plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} t$$

Discussion Question

What would the equation of a line in 4-space look like? How about a a plane in 4-space? What would $ax + by + cz + dw = e$ represent?

Lecture Notes

- Students are being asked to do something that seems unusual to them. They are taking a concept that they presumably understand, finding the equation of a line, and asked to do it again, only this time in a more complicated way that they don't fully understand. One can motivate the students by discussing the problem of generalization. $ax + by = c$ (which simplifies to $y = mx + b$) is the equation of a line in 2-space. How do we get a line in 3-space? One guess would be $ax + by + cz = d$, but that turns out to be a plane, not a line. The problem with our standard way of writing the equation of a line is that it doesn't naturally extend to three dimensions. The techniques of this section not only generalize to three (or more) dimensions, but they do so in a simple, natural way.

- Example 1 is very dense, and provides the students with an excellent opportunity to learn how to read a mathematics textbook. Ask the students to reread Example 1, and give them time, waiting until almost all of the students are done before going on. And then go through it with them, sentence by sentence. Notice that almost every sentence requires a bit of thought. "The left-hand side of the equation is in the form of a dot product", "The vector $\mathbf{n}$ is perpendicular to the line", "The equation $\mathbf{n} \cdot \mathbf{x} = 0$ is the normal form of the equation of $\mathbf{l}$" all convey new concepts. Students may need to remind themselves that if the dot product of two vectors is zero, then the vectors are orthogonal. They may need to look up the definition of "orthogonal" if they've forgotten it. Students tend to try to read mathematics textbooks too quickly, and this example gives you an opportunity to demonstrate the process of truly understanding every sentence before going on, or at least making notes of what they need to ask questions about. (This isn't a bad thing to do two or three times throughout the semester.)

- Be sure to note the distinction between the point $(1, 2)$, the vector $[1, 2]$, and the vector $[1, 2]$ in standard position. When we discuss the normal and vector forms of the equation of a line, we are assuming that our vectors are in standard position. Perhaps discuss what happens if we remove this assumption. (The term for this hard-to-picture object is a *pencil*.)

- Students get direction vectors and normal vectors confused. Direction vectors tend to agree with the students' geometric intuition. They point along the line, and have a "rise" and "run" interpretation that harkens back to the concept of slope. The normal vector literally goes off at a right angle to their intuition. It is important to draw a picture such as Figure 5, and to show the students how the two different forms of the equation work.

- If you will be covering cross products (as done in the Exploration) this is a good time to foreshadow them: "Wouldn't it be useful if we had a good way of finding a vector perpendicular to two given vectors?"

- The following is a different approach to lines in $\mathbb{R}^2$ and planes in $\mathbb{R}^3$. It has the advantage of extending nicely to higher dimensions. Let $\mathbf{v} = [2, 4]$; this vector defines a (linear) function mapping $\mathbb{R}^2 \to \mathbb{R}$ by the rule $\mathbf{w} \mapsto \mathbf{w} \cdot \mathbf{v}$. Let $f_{\mathbf{v}}$ denote this function. A *level set* L_a of $f_{\mathbf{v}}$ is the set of all $\mathbf{w}$ such that $f_{\mathbf{v}}(\mathbf{w}) = a$. Sketch L_0. Now sketch L_1. Let's move up in dimension; let $\mathbf{v} = [0, 1, 0]$ and sketch (in $\mathbb{R}^3$) the level set L_0. What can say about these level sets in general? If you start with a plane in $\mathbb{R}^3$, can you express it as the level set of some $\mathbf{v}$? These level sets can obviously be defined in $\mathbb{R}^k$ for any integer k; these levels sets are called ***hyperplanes***.

Lecture Examples

- Various forms of the equation of the line that goes through the points $(1,3)$ and $(2,8)$ in $\mathbb{R}^2$:

$$y = 5x - 2$$

$$5x - y = 2$$

$$\begin{bmatrix} 5 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = 0$$

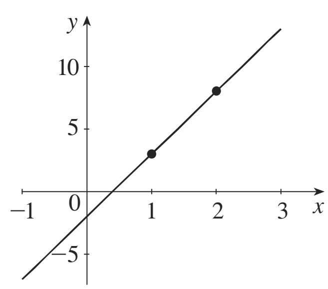
$$\begin{bmatrix} 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad (=2)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{cases} x = 1 + t \\ y = 3 + 5t \end{cases}$$

$$\begin{cases} x = 2 + t \\ y = 8 + 5t \end{cases}$$



- Various forms of the equation of the plane that goes through $(1, 1, 1), (2, 2, 1)$ and $(2, 1, 2)$. *Note:* We first observe that the vectors $[1, 1, 0]$ and $[1, 0, 1]$ are in the plane, and determine that $[1, -1, -1]$ is perpendicular to both by playing with dot-products.

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (= -1)$$

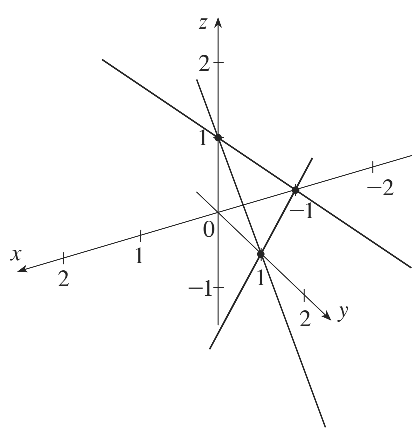
$$x - y - z = -1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} x = 1 + s + t \\ y = 1 + s \\ z = 1 + t \end{cases}$$

$$\begin{cases} x = 2 + s + t \\ y = 2 + s \\ z = 1 + t \end{cases}$$



- The distance between the point $(2, 1, 8)$ and the plane $2x - 3y + z = 5$ is $\frac{2}{7}\sqrt{14} \approx 1.069045$

Tech Tip

A CAS can draw a plane by expressing the general form of the plane as $z = f(x, y)$ and then using a 3D plot command. Use this approach to illustrate the different ways three planes can intersect (the empty set, a point, a line).

Group Work 1: The Match Game

This is a pandemonium-inducing game. Give each group four index cards. Each card contains an equation of a different line, and each card's equation is in a different form. Tell the students that their goal is to trade cards, and wind up with four different descriptions of the same line.

For the convenience of the teacher, each row below contains a winning combination. Make sure that each team starts with descriptions from different rows.

After the dust settles, lead the students in a discussion of optimal strategies for this game.

Category A	Category B	Category C	Category D
The line between $(0,0,1)$ and $(1,2,1)$	$[x,y,z]=[2,4,1]+t\,[1,2,0]$	$x=2t$ $y=4t$ $z=1$	A line through $(-4,-8,1)$ and parallel to $[3,6,0]$
The line between $(0,-3,3)$ and $(3,3,0)$	$[x,y,z]=[1,-1,2]+t\,[1,2,-1]$	$x=2+t$ $y=1+2t$ $z=1-t$	A line through $(4,5,-1)$ and parallel to $[2,4,-2]$
The line between $(1,3,2)$ and $(1,-1,6)$	$[x,y,z]=[1,2,3]+t\,[0,-1,1]$	$x=1$ $y=-t$ $z=5+t$	A line through $(1,0,5)$ and parallel to $[0,-\pi,\pi]$
The line between $(0,0,4)$ and $(12,8,8)$	$[x,y,z]=[9,6,7]+t\,[-3,-2,-1]$	$x=6-6t$ $y=4-4t$ $z=6-2t$	A line through $(-12,-8,0)$ and parallel to $[3,2,1]$
The line between $(5,0,7)$ and $(-2,-7,0)$	$[x,y,z]=[3,-2,5]+t\,[-1,-1,-1]$	$x=2-2t$ $y=-3-2t$ $z=4-2t$	A line through $(0,-5,2)$ and parallel to $[5,5,5]$
The line between $(-3,3,-9)$ and $(3,-3,9)$	$[x,y,z]=[0,0,0]+t\,[-1,1,-3]$	$x=-1+t$ $y=1-t$ $z=-3+3t$	A line through $(2,-2,6)$ and parallel to $[3,-3,9]$
The line between $(-4,2,1)$ and $(-11,1,-1)$	$[x,y,z]=[3,3,3]+t\,[7,1,2]$	$x=10+7t$ $y=4+t$ $z=5+2t$	A line through $(17,5,7)$ and parallel to $[-7,-1,-2]$

Group Work 2: Planes from Points

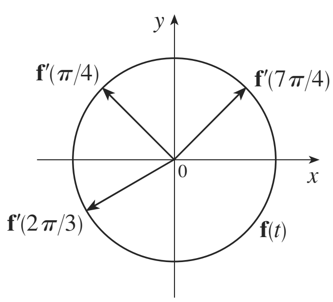
Give each group two sets of three points each, one non-collinear set and one collinear set. Ask the students to give a parametric equation of the unique plane containing the points. For the second set of points this is a trick question, since collinear points do not determine a plane. Sample sets of points are given below.

Non-collinear	Non-collinear	Non-collinear	Collinear	Collinear	Collinear
$(0,0,0)$	$(-1,4,2)$	$(0,-5,5)$	$(0,0,0)$	$(-1,4,2)$	$(0,-5,5)$
$(1,2,3)$	$(3,1,1)$	$(0,1,1)$	$(1,2,3)$	$(3,1,1)$	$(0,1,1)$
$(2,5,9)$	$(7,2,0)$	$(0,3,4)$	$\left(\frac{3}{2},3,\frac{9}{2}\right)$	$(7,-2,0)$	$(0,-2,3)$

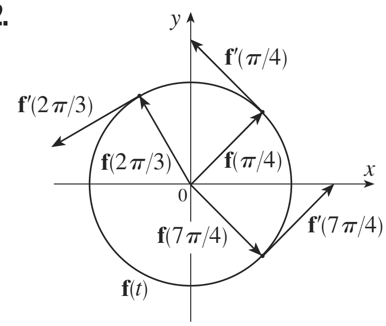
Group Work 3: Calculus and Linear Algebra

Answers

1.



2.



3. $\mathbf{x} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + s \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

4. They are orthogonal.

5. $\mathbf{x} = \mathbf{g}(t_0) + t\mathbf{g}'(t_0)$

Group Work 4: Plane to See

This activity is designed to give the students an opportunity to apply what they've learned about equations of planes.

Answers

1. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} s + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} t$

2. Answers will vary. Sample answers:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} s \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$$

3. $\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right) = 0$

4. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} s + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} t$

5. $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right) = 0$

Suggested Core Assignment

Exercises 2, 6, 9, 11, 13, 22, 24, 28, 32, 41$^{\mathrm{P}}$, 43, 48

Group Work 1, Section 1.3
The Match Game

Your teacher has just handed you four index cards, each with equations of four different lines. Your task is, by clever trading with other groups, to wind up with four different descriptions of the *same line*. The winning team gets a boffo prize, so go for it!

Group Work 2, Section 1.3
Planes from Points

1. Consider the following set of three points:

POINT 1: __________

POINT 2: __________

POINT 3: __________

Find a parametric equation of the unique plane containing these points.

2. Repeat Problem 1 using these points:

POINT 1: __________

POINT 2: __________

POINT 3: __________

Group Work 3, Section 1.3
Calculus and Linear Algebra

In beginning calculus, we learned to interpret the derivative of a real-valued function of one variable $f : \mathbb{R} \to \mathbb{R}$. A function of one variable that take on values in $\mathbb{R}^k$ is called a ***vector-valued function***. For example, if we let $\mathbf{f}(t) = \left[t, t^2, t^3\right]$, then $\mathbf{f}$ is a vector-valued function.

One important question is how we should interpret the derivative of a vector-valued function? Let's start with a definition. If $\mathbf{f}(t) = [f_1(t), \ldots, f_k(t)] : \mathbb{R} \to \mathbb{R}^k$, then $\mathbf{f}'(t) = [f_1'(t), \ldots, f_k'(t)]$. This should feel intuitive. If, for example, $\mathbf{f}(t) = \left[t, t^2, t^3\right]$, then we are defining $\mathbf{f}'(t)$ to be $\left[1, 2t, 3t^2\right]$. Notice that we interpret $\mathbf{f}'(t)$ not as a *point* in $\mathbb{R}^k$ but as a *vector* in $\mathbb{R}^k$.

1. Let $\mathbf{f}(t) = [\cos t, \sin t]$. Sketch the image curve $\mathbf{f}(t)$ in $\mathbb{R}^2$. Calculate $\mathbf{f}'(t)$ and draw a few $\mathbf{f}'(t)$ vectors in standard position on the same graph.

2. Notice that the $\mathbf{f}'(t)$ vectors, while quite beautiful, fail to reveal much about the curve $\mathbf{f}(t)$. For each vector you have drawn in standard position, redraw each one in a better position — a position that better captures the connection between $\mathbf{f}$ and $\mathbf{f}'$.

3. For each redrawn vector, give the equation of the corresponding tangent line using vector form.

4. Let $\mathbf{v}$ be the vector $[\cos t_0, \sin t_0]$ and let $\mathbf{u}$ be the derivative vector at t_0. What do you notice about $\mathbf{v}$ and $\mathbf{u}$?

5. In general, if $\mathbf{g}(t) = (g_1(t), \ldots, g_k(t))$, find the vector form of the tangent line to this curve at $t = t_0$.

Group Work 4, Section 1.3
Plane to See

We are going to explore how much information is needed to uniquely determine a plane in $\mathbb{R}^3$. Consider the three noncollinear points $P = (-1, 2, 3)$, $Q = (1, 2, 0)$, and $R = (1, 0, 1)$. For each of the following, either determine (in vector or normal form) the uniquely defined plane or exhibit two distinct planes satisfying the description.

1. A plane that contains P, Q and R.

2. A plane that contains the line determined by P and Q but does not contain R.

3. A plane that contains the vector $\overrightarrow{PQ}$ and is orthogonal to the vector $\overrightarrow{PR}$.

4. A plane that contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

5. A plane that contains P and is orthogonal to the vector $\overrightarrow{PQ} - \overrightarrow{PR}$.

1.4 Code Vectors and Modular Arithmetic

Suggested Time and Emphasis

$\frac{1}{2}$–1 class. Modular arithmetic recommended, error-correcting codes optional.

Points to Stress

1. Modular arithmetic, with an emphasis on binary arithmetic.
2. The concept of error-correcting codes.
3. Parity check digits.

Drill Question

Circle the binary vector which has a parity different from that of the others:

$$[1,0,0,1,1,1,0] \qquad [1,0,0,1,0,0,0] \qquad [1,1,1,1,1,0,1] \qquad [1,0,0,1,0,1,0]$$

Answer $[1,0,0,1,0,1,0]$

Discussion Questions

1. Why do we use error-correcting codes?
2. What would happen if every bit of a vector came across correctly *except* the parity bit?

Test Question

Is 0-534-34174-8 a valid ISBN? Why or why not?

Answer No. The parity bit is wrong.

Lecture Notes

- Point out that modular arithmetic comes up in other contexts besides this one. A good example to use is addition modulo 12. For example, if someone starts reading a book at 9 A.M., and reads for six hours, what time is it? If a student starts studying at 10 A.M., and studies for 23 hours, what time is it?
- One can introduce a little bit of symbolic logic as a way of justifying the importance of binary arithmetic. If the students look at the binary multiplication table, this corresponds to the idea of "and". The statement $ab = 1$ is true only if $a = 1$ *and* $b = 1$. Ask the students to think of a meaning for the addition table. It turns out to be *exclusive or*: $a + b = 1$ only if $a = 1$ *or* $b = 1$, but not both. (Exclusive or tends to come up in the English language, such as "You may have fries *or* cole slaw," or "Your money *or* your life".)
- Notice that there is a hierarchy of codes. Example 5 talks about a simple error-detecting code. We put a check bit at the end, and if there has been a single error, we know there is a problem. One drawback is that if there is an even number of errors, they are not detected. It is possible to fix this problem, but the price is that we need to add more than one extra bit. Another drawback is that we only know that an error has occurred, but we don't know which bit is faulty. It is possible to create error-correcting codes that tell you exactly what needs to be changed. But again, we pay the price of having to add extra bits. In general, the more sophisticated the code, the more bits that have to be added, and the more bits that we add, the slower the transmission rate. This trade-off is a special case of the age-old problem of speed versus accuracy.

- The textbook gives two examples of error-correcting codes that are used in common products, the UPC and the ISBN. Students could be asked to verify the check digit on their *Linear Algebra* textbook for both the ISBN and the UPC. Discuss why the check vector used with UPCs will detect all single errors.

- Discuss self-correcting codes. One simple self-correcting code just repeats each word three times. For example, to encode 1011 we would send the string 101110111011. Now, if there is a single error, we can deduce where the error was and correct it. For example, if you received the transmission 101011101010, you would write

$$\begin{matrix}1010\\1110\\1010\end{matrix}$$

  You would know that there was an error, but that the intended message was 1010. The disadvantage to this method is that it takes $3k$ bits to send a k-bit message. There are much more efficient self-correcting codes created by some very smart people.

Lecture Examples

- A binary vector with a check digit, where a transmission error has occurred:

$$[1, 0, 0, 1, 0, 0, 1, 0, 0, 0]$$

- A valid UPC:

7 81376 84047 5

- An invalid UPC:

0 51125 66206 8

- An invalid UPC that would go undetected:

8 2697-90760-2 6 — Error undetected

8 2796-90760-2 6 — Original

Tech Tip

Have the students design a function that returns 1 if an given UPC is valid and 0 if it is invalid.

Group Work 1: Find the Counterfeit

This activity is a direct application of the UPC described in the textbook.

Answer The UPC of "The Starry Nights" is invalid.

Group Work 2: Beyond Parity

You may want to do Part 1 with the students, or a different example of your own devising.

Answers

1. $[1, 0, 1, 1, 1, 1, 0]$
2. $[1, 1, 1, 0, 1, 0, 1]$
3. If the intended message was $[1, 1, 0, 1]$ then the transmitted word should be $[1, 1, 0, 1, 1, 0, 1]$. We didn't get that, so there is an error.

   Now, since we are assuming there was only one error, we can try flipping each bit of the received word, in turn:

$$\begin{aligned}
&[0, 1, 0, 1, 0, 1, 1] \text{ is valid}\\
&[1, 0, 0, 1, 0, 1, 1] \text{ is valid}\\
&[1, 1, 1, 1, 0, 1, 1] \text{ is invalid}\\
&[1, 1, 0, 0, 0, 1, 1] \text{ is invalid}\\
&[1, 1, 0, 1, 1, 1, 1] \text{ is invalid}\\
&[1, 1, 0, 1, 0, 0, 1] \text{ is invalid}\\
&[1, 1, 0, 1, 0, 1, 0] \text{ is invalid}
\end{aligned}$$

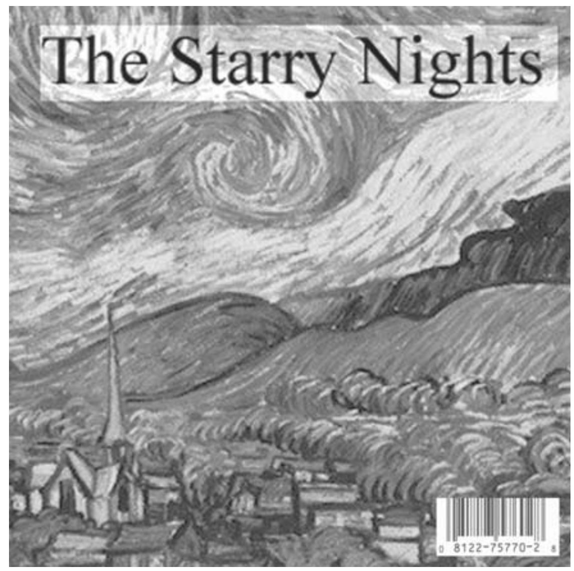
   So we know that the first or second bit is faulty, but we don't know which.
4. This particular system allows us to narrow down where there error is, but does not allow us to find it exactly. It is an improvement over a single parity bit, because it allows us to localize the error. It has the disadvantage of taking 7 bits of bandwidth to send a 4-bit message. There are actually more complex codes that allow us to find the error precisely.

Suggested Core Assignment

Exercises 3, 14, 16, 22, 23, 24, 36, 37, 46, 53$^{\text{P}}$, 54

Group Work 1, Section 1.4
Find the Counterfeit

Three of the following CDs were bought at my local music store. One of them was bought from an Evil Counterfeiter, who will soon be brought to justice. Which one is the pirated CD?



Group Work 2, Section 1.4
Beyond Parity

We wish to transmit binary vectors of length 4 such as $[1, 0, 1, 1]$ and $[1, 1, 1, 0]$. Because we are afraid of transmission error, we are going to add some bits to make an error-detecting code. Let $\mathbf{v}$ be the 4-bit vector we want to transmit. Bit number 5 will be $\mathbf{v} \cdot [1, 1, 1, 1]$, bit number 6 will be $\mathbf{v} \cdot [1, 1, 0, 0]$, and bit number 7 will be $\mathbf{v} \cdot [0, 0, 1, 1]$. All addition is modulo two. For example, $[1, 1, 0, 1, 1] \cdot [1, 1, 1, 1, 0] = 1$.

1. If $\mathbf{v} = [1, 0, 1, 1]$, what vector will we transmit?

2. If $\mathbf{v} = [1, 1, 1, 0]$, what vector will we transmit?

3. If you receive the vector $[1, 1, 0, 1, 0, 1, 1]$, was there an error? If you know for sure there was exactly one error, can you determine which bit is faulty?

4. What are the advantages of this system over the single-parity-bit system? What are the disadvantages?

2 Linear Equations

2.1 Introduction to Systems of Linear Equations

Suggested Time and Emphasis

$\frac{1}{2}$–1 class. Essential material.

Points to Stress

1. Basic definitions including linear equations, systems of linear equations and augmented matrices.
2. Geometric interpretations of the solution set of a system of linear equations, including viewing them as sets of vectors.
3. Dependent, independent, and inconsistent systems: systems with no solution, systems with a single solution, and systems with infinitely many solutions. (The details of solving these systems is given in the next section.)

Drill Question

Consider the system of equations

$$\begin{aligned} x + y &= -1 \\ 2x - 3y &= 8 \end{aligned}$$

(a) Find the augmented matrix corresponding to this system.

(b) Put the matrix you obtained in part (a) into upper triangular form.

Answer (a) $\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & -3 & 8 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -5 & 10 \end{array}\right]$

Discussion Question

Can a system with 2 equations and 3 unknowns be inconsistent?

Answer Yes

Sample Test Question:

Consider the system

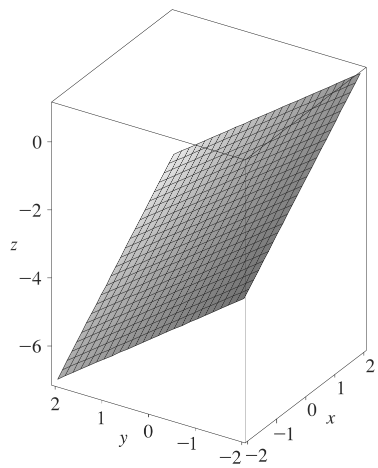
$$\begin{aligned} ax + 3y + 2z &= 5 \\ bx + cy + 4z &= 9 \\ 5x + by + cz &= 16 \end{aligned}$$

If $\{x = 1, y = 2, z = 3\}$ is a solution to this system, then find a, b and c.

Answer $a = -7, b = -37, c = 17$

Lecture Notes

- After showing that every linear system has an associated augmented matrix, ask the question, "Does every matrix have an associated linear system?" The students may not think to look at $n \times 1$ matrices.
- Ask the question, "Why isn't there any other possibility for the number of solutions to a linear system besides 0, 1, and infinitely many?" Consider a linear system of m equations in n variables. We will show that if there is more than one solution, then there must be infinitely many solutions. Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are solutions to the system. Let λ be a number between 0 and 1. Let $\mathbf{w} = \lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$. Now $\mathbf{w}$ solves the system for every λ.
- Inconsistent systems can be illustrated by starting with the equation $x - y - z = 3$ and sketching the plane of solutions for this equation.



We can then add a second equation, such as $x - y - z = 4$, such that the two equations form an inconsistent system, and sketch the two parallel planes, noting they do not intersect.