Solutions for Chapter 1

Section 1.1

1. A is 2×2; B is 2×2; C is 2×2; D is 4×2; E is 4×2; F is 4×2; G is 2×3; H is 3×3; J is 1×5.

2. $a_{12}=2$ a_{31} does not exist
$b_{12}=6$ b_{31} does not exist
$c_{12}=0$ c_{31} does not exist
$d_{12}=1$ $d_{31}=3$
$e_{12}=2$ $e_{31}=5$
$f_{12}=1$ $f_{31}=0$
$g_{12}=\frac{1}{3}$ g_{31} does not exist
$h_{12}=\sqrt{3}$ $h_{31}=\sqrt{5}$
$j_{12}=0$ j_{31} does not exist

3. $a_{11}=1$; $a_{21}=3$; b_{32} does not exist; $d_{32}=-2$; $e_{22}=-2$; $g_{23}=-\frac{5}{6}$; $h_{33}=\sqrt{3}$; j_{21} does not exist.

4. $A(2\times 2)$; $B(2\times 2)$; $C(2\times 2)$; $H(3\times 3)$ are square matrices.

5. J is a row matrix; none are column matrices.

6. $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 \end{bmatrix}$$

8. $$\mathbf{A}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

9. $\mathbf{A}=\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$

10. $\mathbf{B}=\begin{bmatrix} n-2 & n-3 & \cdots & -1 \\ n-3 & n-4 & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & \cdots & -n \end{bmatrix}$

In the 3×3 case

$$\mathbf{B}=\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ -1 & -2 & -3 \end{bmatrix}$$

11. $\mathbf{C}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

12. $\mathbf{D}=\begin{bmatrix} 0 & -1 & -2 & -3 \\ 3 & 0 & -1 & -2 \\ 4 & 5 & 0 & -1 \end{bmatrix}$

13. $2\mathbf{A}=2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

14. $-5\mathbf{A}=-5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} -5 & -10 \\ -15 & -20 \end{bmatrix}$

15. $3\mathbf{D}=3\begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix}=\begin{bmatrix} 9 & 3 \\ -3 & 6 \\ 9 & -6 \\ 6 & 18 \end{bmatrix}$

16. $10\mathbf{E} = 10\begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} -20 & 20 \\ 0 & -20 \\ 50 & -30 \\ 50 & 10 \end{bmatrix}$

17. $-\mathbf{F} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ -2 & -2 \end{bmatrix}$

18. $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

19. $\mathbf{C} + \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 6 & 1 \end{bmatrix}$

20. $\mathbf{D} + \mathbf{E} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 8 & -5 \\ 7 & 7 \end{bmatrix}$

21. $\mathbf{D} + \mathbf{F} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 2 \\ 3 & -2 \\ 4 & 8 \end{bmatrix}$

22. $\mathbf{A} + \mathbf{D}$ - is undefined.

23. $\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$

24. $\mathbf{C} - \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & -7 \end{bmatrix}$

25. $\mathbf{D}-\mathbf{E}=\begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix}-\begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix}=\begin{bmatrix} 5 & -1 \\ -1 & 4 \\ -2 & 1 \\ -3 & 5 \end{bmatrix}$

26. $\mathbf{D}-\mathbf{F}=\begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 3 & -2 \\ 0 & 4 \end{bmatrix}$

27. $2\mathbf{A}+3\mathbf{B}=2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}+3\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}=\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}+\begin{bmatrix} 15 & 18 \\ 21 & 24 \end{bmatrix}=\begin{bmatrix} 17 & 22 \\ 27 & 32 \end{bmatrix}$

28. $3\mathbf{A}-2\mathbf{C}=3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}-2\begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix}=\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}+\begin{bmatrix} 2 & 0 \\ -6 & 6 \end{bmatrix}=\begin{bmatrix} 5 & 6 \\ 3 & 18 \end{bmatrix}$

29. $.1\mathbf{A}+.2\mathbf{C}=$

$$0.1\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}+0.2\begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix}=\begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}+\begin{bmatrix} -.2 & 0 \\ .6 & -.6 \end{bmatrix}=\begin{bmatrix} -0.1 & 0.2 \\ 0.9 & -0.2 \end{bmatrix}$$

30. $-2\mathbf{E}+\mathbf{F}=$

$$-2\begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix}+\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 4 & -4 \\ 0 & 4 \\ -10 & 6 \\ -10 & -2 \end{bmatrix}+\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 4 & -3 \\ -1 & 4 \\ -10 & 6 \\ -8 & 0 \end{bmatrix}$$

31. $\mathbf{A}+\mathbf{X}=\mathbf{B}$; $\mathbf{X}=\mathbf{B}-\mathbf{A}=\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}-\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

32. $2\mathbf{B}+\mathbf{Y}=\mathbf{C}$; $\mathbf{Y}=\mathbf{C}-2\mathbf{B}=$

$$\begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix}-2\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix}+\begin{bmatrix} -10 & -12 \\ -14 & -16 \end{bmatrix}=\begin{bmatrix} -11 & -12 \\ -11 & -19 \end{bmatrix}$$

33. $3\mathbf{D} - \mathbf{X} = \mathbf{E}$; $\mathbf{X} = 3\mathbf{D} - \mathbf{E} =$

$$3\begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ -3 & 6 \\ 9 & -6 \\ 6 & 18 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 0 & 2 \\ -5 & 3 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ -3 & 8 \\ 4 & -3 \\ 1 & 17 \end{bmatrix} =$$

34. $\mathbf{E} - 2\mathbf{Y} = \mathbf{F}$; $\mathbf{Y} = \frac{1}{2}[\mathbf{E} - \mathbf{F}] =$

$$\frac{1}{2}\left[\begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 5 & -3 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 5 & -3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \\ \frac{5}{2} & -\frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

35. $4\mathbf{A} + 5\mathbf{R} = 10\mathbf{C}$; $\mathbf{R} = \frac{1}{5}[10\mathbf{C} - 4\mathbf{A}] =$

$$\frac{1}{5}\left[10\begin{bmatrix} -1 & 0 \\ 3 & -3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right] = \frac{1}{5}\left[\begin{bmatrix} -10 & 0 \\ 30 & -30 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -12 & -16 \end{bmatrix}\right] = \frac{1}{5}\begin{bmatrix} -14 & -8 \\ 18 & -46 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{14}{5} & -\frac{8}{5} \\ \frac{18}{5} & -\frac{46}{5} \end{bmatrix}$$

36. $3\mathbf{F} - 2\mathbf{S} = \mathbf{D}$; $\mathbf{S} = \frac{1}{2}[3\mathbf{F} - \mathbf{D}] =$

$$\frac{1}{2}\left[3\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 3 & -2 \\ 2 & 6 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} -3 & 2 \\ -2 & -2 \\ -3 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 \\ -1 & -1 \\ -\frac{3}{2} & 1 \\ 2 & 0 \end{bmatrix}$$

37. $6\mathbf{A} - \theta\mathbf{B} =$

$$6\begin{bmatrix} \theta^2 & 2\theta - 1 \\ 4 & \dfrac{1}{\theta} \end{bmatrix} - \theta\begin{bmatrix} \theta^2 - 1 & 6 \\ \dfrac{3}{\theta} & \theta^2 + 2\theta + 1 \end{bmatrix} = \begin{bmatrix} 6\theta^2 & 12\theta - 6 \\ 24 & \dfrac{6}{\theta} \end{bmatrix} + \begin{bmatrix} -\theta^3 + 1 & -6\theta \\ -3 & -\theta^3 - 2\theta^2 - \theta \end{bmatrix} =$$

$$= \begin{bmatrix} -\theta^3 + 6\theta^2 + 1 & 6\theta - 6 \\ 21 & -\theta^3 - 2\theta^2 - \theta + \dfrac{6}{\theta} \end{bmatrix}$$

38. Prove: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

$$\left[a_{ij}\right] + \left[b_{ij}\right] = \left[a_{ij} + b_{ij}\right] = \left[b_{ij} + a_{ij}\right] = \left[b_{ij}\right] + \left[a_{ij}\right]$$

39. Prove: $\mathbf{A} + \mathbf{0} = \mathbf{A}$

$$\left[a_{ij}\right] + \left[0_{ij}\right] = \left[a_{ij} + 0_{ij}\right] = \left[a_{ij} + 0\right] = \left[a_{ij}\right]$$

40. Prove: $(\lambda_1 + \lambda_2)\mathbf{A} = \lambda_1\mathbf{A} + \lambda_2\mathbf{A}$

$$(\lambda_1 + \lambda_2)\left[a_{ij}\right] = \left[(\lambda_1 + \lambda_2)a_{ij}\right] = \left[\lambda_1 a_{ij} + \lambda_2 a_{ij}\right] = \left[\lambda_1 a_{ij}\right] + \left[\lambda_2 a_{ij}\right] = \lambda_1\left[a_{ij}\right] + \lambda_2\left[a_{ij}\right]$$

41. Prove: $(\lambda_1\lambda_2)\mathbf{A} = \lambda_1(\lambda_2\mathbf{A})$

$$(\lambda_1\lambda_2)\left[a_{ij}\right] = \left[(\lambda_1\lambda_2)a_{ij}\right] = \left[\lambda_1(\lambda_2 a_{ij})\right] = \lambda_1\left[\lambda_2 a_{ij}\right] = \lambda_1\left(\lambda_2\left[a_{ij}\right]\right)$$

42.

$$\begin{array}{cccc} \text{Refrigerators} & \text{Stoves} & \text{Washers} & \text{Dryers} \\ \end{array}$$
$$\begin{bmatrix} 3 & 5 & 3 & 4 \\ 0 & 2 & 9 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$$

43.

$$\begin{array}{l|ccc} & \text{Reg} & \text{Firm} & \text{Ex Firm} \\ \hline \text{Mich} & & & \\ \text{Tex} & & \left[\begin{matrix} 80 & 12 & 16 \\ 50 & 40 & 16 \\ 90 & 10 & 50 \end{matrix}\right] & \\ \text{Utah} & & & \end{array}$$

$$\begin{array}{lccc} & \text{Reg} & \text{Firm} & \text{Ex Firm} \\ \text{Mich} & 80 & 12 & 16 \\ \text{Tex} & 50 & 40 & 16 \\ \text{Utah} & 90 & 10 & 50 \end{array}$$

Damage matrix for next year (labeling the same for rows and columns):

$$\begin{bmatrix} 72 & 12 & 16 \\ 45 & 32 & 16 \\ 81 & 10 & 35 \end{bmatrix}$$

44.

$$\begin{array}{lcc} & \text{Purchase Price} & \text{Int. Rate} \\ \text{1st cert} & \$1000 & .07 \\ \text{2nd cert} & \$2000 & .075 \\ \text{3rd cert} & \$3000 & .0725 \end{array}$$

45. (a)

$$\begin{array}{cc} \text{IBM} & \text{ATT} \\ \end{array}$$
$$\begin{bmatrix} 200 & 150 \end{bmatrix}$$

(b)

$$\begin{array}{cc} \text{IBM} & \text{ATT} \\ \end{array}$$
$$\begin{bmatrix} 600 & 450 \end{bmatrix}$$

(c)

$$\begin{array}{cc} \text{IBM} & \text{ATT} \\ \end{array}$$
$$\begin{bmatrix} 550 & 350 \end{bmatrix}$$

46. (a) $\begin{bmatrix} 15 & 2 & 8 & 6 \end{bmatrix}$: 15 Tv sets, 2 Air Conditioners, 8 Refrigertors, 6 Dishwashers

(b)
$$\begin{bmatrix} 15 & 2 & 8 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 6 & 3 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 11 & 2 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 7 & 8 \end{bmatrix} =$$
$$= \begin{bmatrix} 9 & 4 & 10 & 8 \end{bmatrix}$$

47. (a) $\begin{bmatrix} 14,000 & 8,000 & 6,000 \end{bmatrix}$: 14,000 gals regular; 8,000 gals premium; 6,000 gals super

(b)
$$\begin{bmatrix} 14,000 & 8,000 & 6,000 \end{bmatrix} - \begin{bmatrix} 3,500 & 2,000 & 1,500 \end{bmatrix} =$$
$$= \begin{bmatrix} 10,500 & 6,000 & 4,500 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 10,500 & 6,000 & 4,500 \end{bmatrix} - \begin{bmatrix} 5,000 & 1,500 & 1,200 \end{bmatrix}$$
$$+ \begin{bmatrix} 30,000 & 10,000 & 0 \end{bmatrix} = \begin{bmatrix} 35,500 & 14,500 & 3,300 \end{bmatrix}$$

Section 1.2

1.

(a) Product **2**x4 & 4x**2**: order is 2x2.

(b) Product **4**x2& 2x**4**: order is 4x4.

(c) Product **2**x4 & 4x**1**: order is 2x1.

(d) Product **4**x1 & 2x**4**: is undefined.

1. (continued)

(e) Product **4**x1 & 1x**2**: order is 4x2.

(f) Product **2**x4 & 4x**4**: order is 2x4.

(g) Product **4**x4 & 4x**2**: order is 4x2.

(h) Product **4**x4 & 2x**2**: is undefined.

(i) Product **2**x4 & 4x2 & **4**x1: is undefined.

(j) Product **1**x2 & 2x4 & 4x**4**: order is 1x4.

(k) Product **4**x4 & 4x2 & 2x**4**: order is 4x4.

(l) Product **4**x4 & 4x4 & 4x1 & 1x**2**: order is 4x2.

2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

3.

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

4.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 3 \\ 9 & -8 & 7 \end{bmatrix}$$

5.

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -12 & 11 \\ 17 & -16 & 15 \end{bmatrix}$$

6. **CB** is undefined

7. $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -6 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -9 & -10 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix}$ undefined

11. $\begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 7 & -3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 7 & -4 & 1 \\ -8 & 4 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$ undefined

15. $\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix}$ undefined

16. $\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix}$ undefined

17. $$\begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 0 & -3 \\ 1 & 3 & 5 \end{bmatrix}$$

18. $$\begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

19. $$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 5 \end{bmatrix}$$

20. $$\begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

21. $$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -6 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

22.

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix}$$

Cancellation law is not valid for matrix multiplication.

23.

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 2 & 4 \end{bmatrix}$$

24. $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$$

25. $$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-z \\ 3x+y+z \\ x+3y \end{bmatrix}$$

26. $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x+a_{12}y \\ a_{21}x+a_{22}y \end{bmatrix}$$

27. $$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_{11}x+b_{12}y+b_{13}z \\ b_{21}x+b_{22}y+b_{23}z \end{bmatrix}$$

28. $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} =$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -16 & -12 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

29. $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 2\mathbf{I}) =$

$$\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 5 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 40 \\ -16 & 8 \end{bmatrix}$$

30. $(\mathbf{I}-\mathbf{A})(\mathbf{A}^2-\mathbf{I})$; $\mathbf{A}^2 = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} -1 & 1 & -1 \\ -3 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

31. Let **A** have order n x r and **B** have order r x p.
Show j^{th} column of (**AB**) = **A**$\times$ (j^{th} column of **B**).

Let **C** = j^{th} column of (**AB**). Then **C** = $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix}$ where

$$c_{11} = \sum_{k=1}^{r} a_{1k}b_{kj} \;;\; c_{21} = \sum_{k=1}^{r} a_{2k}b_{kj} \;;\; \cdots \;;\; c_{n1} = \sum_{k=1}^{r} a_{nk}b_{kj} \;.$$

Let **D** = **A**$\times$ (j^{th} column of **B**). Then **D** = $\begin{bmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{n1} \end{bmatrix}$ where

$d_{11} = \sum_{k=1}^{r} a_{1k}b_{kj} \;;\; d_{21} = \sum_{k=1}^{r} a_{2k}b_{kj} \;;\; \cdots \;;\; d_{n1} = \sum_{k=1}^{r} a_{nk}b_{kj}$. Hence **C** = **D**.

32. Let **A** have order n x r and **B** have order r x p.
Show i^{th} row of (**AB**) = (i^{th} row of **A**) $\times$ **B**.

32. (continued)

Let **C** = ith row of (**AB**). Then **C** = $\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \end{bmatrix}$ where

$$c_{11} = \sum_{k=1}^{r} a_{ik}b_{k1} \;;\; c_{12} = \sum_{k=1}^{r} a_{ik}b_{k2} \;;\; \cdots \;;\; c_{1p} = \sum_{k=1}^{r} a_{ik}b_{kp} \;.$$

Let **D** = (ith row of **A**) $\times$ **B**. Then **D** = $\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \end{bmatrix}$ where

$$d_{11} = \sum_{k=1}^{r} a_{ik}b_{k1} \;;\; d_{12} = \sum_{k=1}^{r} a_{ik}b_{k2} \;;\; \cdots \;;\; d_{1p} = \sum_{k=1}^{r} a_{ik}b_{kp}$$

. Hence **C** = **D**.

33. Let **A** have order n x r and **B** have order r x p. Let **C** = **AB**. Then $c_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}$. Suppose the ith row of **A** is a row of zeros, i.e., $a_{ik} = 0;\;\; k = 1,\cdots,r$, and $c_{ij} = \sum_{k=1}^{r}(0)b_{kj} = 0$ for every value of j. This means the ith row of **C** is a row of zeros, i.e., **AB** has a row of zeros.

34. Let **A** = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and **B** = $\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$. Then

AB = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 12 \end{bmatrix}$. I.e., **AB** need not have a row of zeros.

35. Let **A** have order n x r and **B** have order r x p. Let **C** = **AB**. Then $c_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}$. Suppose the jth column of **B** is a column of zeros, i.e., $b_{kj} = 0;\;\; k = 1,\cdots,r$, and $c_{ij} = \sum_{k=1}^{r} a_{ik}(0) = 0$ for every value of i. This means the jth column of **C** is a column of zeros, i.e., **AB** has a column of zeros.

36. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$. Then

$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$. I.e., **AB** need not have a column of zeros.

37. Prove: **A**(**BC**) = (**AB**)**C**
Let the orders of **A**, **B**, **C** be r x m, m x n, and n x s respectively. Let **D** = **BC** and **E** = **AB**. The orders of **D** and **E** are m x s and r x n respectively. Then $d_{ij} = \sum_{k=1}^{n} b_{ik}c_{kj}$ and $e_{ij} = \sum_{p=1}^{m} a_{ip}b_{pj}$. So we have

A(**BC**) = **AD** =

$$\left[\sum_{p=1}^{m} a_{ip}d_{pj}\right] = \left[\sum_{p=1}^{m} a_{ip}\left(\sum_{k=1}^{n} b_{pk}c_{kj}\right)\right] = \left[\sum_{p=1}^{m}\sum_{k=1}^{n} a_{ip}b_{pk}c_{kj}\right] = \left[\sum_{k=1}^{n}\left(\underbrace{\sum_{p=1}^{m} a_{ip}b_{pk}}_{e_{ik}}\right)c_{kj}\right] = \left[\sum_{k=1}^{n} e_{ik}c_{kj}\right] =$$

= **EC** = (**AB**)**C**.

38. Prove: (**B** + **C**)**A** = **BC** + **BA.**

(**B** + **C**)**A** =

$$\left(\left[b_{ij}\right]+\left[c_{ij}\right]\right)\left[a_{ij}\right] = \left[b_{ij}+c_{ij}\right]\left[a_{ij}\right] = \left[\sum_{k=1}^{n}\left(b_{ik}+c_{ik}\right)a_{kj}\right] = \left[\sum_{k=1}^{n}\left(b_{ik}a_{kj}+c_{ik}a_{kj}\right)\right] = \left[\sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}\right] =$$

$$= \left[\sum_{k=1}^{n} b_{ik}a_{kj}\right] + \left[\sum_{k=1}^{n} c_{ik}a_{kj}\right] = \left[b_{ij}\right]\left[a_{ij}\right] + \left[c_{ij}\right]\left[a_{ij}\right] =$$

BA + **CA**.

39. $\begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$

40. $\begin{bmatrix} 5 & 20 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 80 \\ -64 \end{bmatrix}$

41. $\begin{bmatrix} 3 & 3 \\ 6 & -8 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 100 \\ 300 \\ 500 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -2 & -6 \\ 4 & -9 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -8 \\ -5 \\ -3 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

44. $\begin{bmatrix} 2 & -1 & 0 \\ -4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \end{bmatrix}$

45. $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$

46. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

47. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

48. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 7 \\ 3 \end{bmatrix}$

49. $\begin{bmatrix} 5 & 3 & 2 & 4 \\ 1 & 1 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$

50. $\begin{bmatrix} 2 & -1 & 1 & -1 \\ 1 & 2 & -1 & 2 \\ 1 & -3 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

51. (a) **pn** = $\begin{bmatrix} 200 & 350 & 500 \end{bmatrix}\begin{bmatrix} 130 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 38{,}000 \end{bmatrix}$. This is the total revenue for the flight.

(b) **np** = $\begin{bmatrix} 130 \\ 20 \\ 10 \end{bmatrix}\begin{bmatrix} 200 & 350 & 500 \end{bmatrix} = \begin{bmatrix} 26{,}000 & 45{,}500 & 65{,}000 \\ 4{,}000 & 7{,}000 & 10{,}000 \\ 2{,}000 & 3{,}500 & 5{,}000 \end{bmatrix}$. This has no significance.

52. (a) **hP** =

$$\begin{bmatrix} 100 & 500 & 400 \end{bmatrix}\begin{bmatrix} 40 & 40\frac{1}{2} & 40\frac{7}{8} & 41 & 41 \\ 3\frac{1}{4} & 3\frac{5}{8} & 3\frac{1}{2} & 4 & 3\frac{7}{8} \\ 10 & 9\frac{3}{4} & 10\frac{1}{8} & 10 & 9\frac{5}{8} \end{bmatrix} = \begin{bmatrix} 9{,}625 & 9{,}762.5 & 9{,}887.5 & 10{,}100 & 9{,}887.5 \end{bmatrix}$$

This is the value of the portfolio at the end of each day.

52. (b) **Ph** = $\begin{bmatrix} 40 & 40\frac{1}{2} & 40\frac{7}{8} & 41 & 41 \\ 3\frac{1}{4} & 3\frac{5}{8} & 3\frac{1}{2} & 4 & 3\frac{7}{8} \\ 10 & 9\frac{3}{4} & 10\frac{1}{8} & 10 & 9\frac{5}{8} \end{bmatrix}\begin{bmatrix} 100 & 500 & 400 \end{bmatrix}$ This product is undefined so it has no significance.

53. $\mathbf{Tw} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 1.2 & 2.3 & 1.7 \\ 0.8 & 3.1 & 1.2 \end{bmatrix}\begin{bmatrix} 10.50 \\ 14.00 \\ 12.25 \end{bmatrix} = \begin{bmatrix} 14 \\ 65.625 \\ 66.5 \end{bmatrix}$. This gives the cost of producing each item.

54. $\mathbf{qTw} = \begin{bmatrix} 1{,}000 & 100 & 200 \end{bmatrix}\begin{bmatrix} 14 \\ 65.625 \\ 66.5 \end{bmatrix} = \begin{bmatrix} 33{,}862.50 \end{bmatrix}$. This gives the total cost of producing all items on order.

55. $\mathbf{FC} = \begin{bmatrix} 0.20 & 0.20 & 0.15 & 0.15 \\ 0.10 & 0.30 & 0.30 & 0.40 \\ 0.70 & 0.50 & 0.55 & 0.45 \end{bmatrix}\begin{bmatrix} 1050 & 950 \\ 1100 & 1050 \\ 360 & 500 \\ 860 & 1000 \end{bmatrix} = \begin{bmatrix} 613 & 625 \\ 887 & 960 \\ 1870 & 1915 \end{bmatrix}$. This gives the number of each gender in each state of illness.

Section 1.3

1. (a) $\mathbf{AB} = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 3 \\ -1 & 7 & 4 \end{bmatrix}$

$(\mathbf{AB})^{T} = \begin{bmatrix} -3 & -1 \\ 6 & 7 \\ 3 & 4 \end{bmatrix}$

1. (a) (continued)

$$\mathbf{A}^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{Product is undefined.}$$

$$\mathbf{B}^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 6 & 7 \\ 3 & 4 \end{bmatrix}$$

(b) $$\mathbf{AB} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 24 \\ 40 & 52 \end{bmatrix}$$

$$(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} 18 & 40 \\ 24 & 52 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 18 & 28 \\ 10 & 22 & 34 \\ 12 & 26 & 40 \end{bmatrix}$$

$$\mathbf{B}^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 48 \\ 24 & 52 \end{bmatrix}$$

(c) $$\mathbf{AB} = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 8 & -4 \\ 11 & -19 & 11 \\ 22 & 56 & -23 \end{bmatrix}$$

$$(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} 17 & 11 & 22 \\ 8 & -19 & 56 \\ -4 & 11 & -23 \end{bmatrix}$$

$$\mathbf{A}^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 7 \\ -1 & 3 & -8 \end{bmatrix}\begin{bmatrix} 6 & 2 & -1 \\ 1 & 0 & -7 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -15 \\ 52 & 3 & 2 \\ -27 & 6 & -36 \end{bmatrix}$$

1. (c) (continued)

$$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} 6 & 2 & -1 \\ 1 & 0 & -7 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 7 \\ -1 & 3 & -8 \end{bmatrix} = \begin{bmatrix} 17 & 11 & 22 \\ 8 & -19 & 56 \\ -4 & 11 & -23 \end{bmatrix}$$

2. $(\mathbf{A} + \mathbf{B})^T =$

$$\left(\begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 7 & 6 & 2 \\ 4 & 1 & 2 \\ -1 & 0 & -6 \end{bmatrix}^T = \begin{bmatrix} 7 & 4 & -1 \\ 6 & 1 & 0 \\ 2 & 2 & -6 \end{bmatrix}$$

$$\mathbf{A}^T + \mathbf{B}^T = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 7 \\ -1 & 3 & -8 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -1 \\ 1 & 0 & -7 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -1 \\ 6 & 1 & 0 \\ 2 & 2 & -6 \end{bmatrix}$$

3. $$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 29 \end{bmatrix}$$

$$\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ 6 & 9 & 12 \\ 8 & 12 & 16 \end{bmatrix}$$

4. (a) $(\mathbf{A}\mathbf{B}^T)^T = ((\mathbf{B}^T)^T\mathbf{A}^T) = \mathbf{B}\mathbf{A}^T$

(b) $(\mathbf{A} + \mathbf{B}^T)^T + \mathbf{A}^T = (\mathbf{A}^T + (\mathbf{B}^T)^T) + \mathbf{A}^T = 2\mathbf{A}^T + \mathbf{B}$

(c) $[\mathbf{A}^T(\mathbf{B} + \mathbf{C}^T)]^T = [\mathbf{A}^T\mathbf{B} + \mathbf{A}^T\mathbf{C}^T]^T = (\mathbf{A}^T\mathbf{B})^T + (\mathbf{A}^T\mathbf{C}^T)^T =$
$= \mathbf{B}^T(\mathbf{A}^T)^T + (\mathbf{C}^T)^T(\mathbf{A}^T)^T = \mathbf{B}^T\mathbf{A} + \mathbf{C}\mathbf{A} = (\mathbf{B}^T + \mathbf{C})\mathbf{A}$

(d) $[(\mathbf{A}\mathbf{B})^T + \mathbf{C}]^T = [((\mathbf{A}\mathbf{B})^T)^T + \mathbf{C}^T] = \mathbf{A}\mathbf{B} + \mathbf{C}^T$

(e) $[(\mathbf{A} + \mathbf{A}^T)(\mathbf{A} - \mathbf{A}^T)]^T = [\mathbf{A}^2 + \mathbf{A}^T\mathbf{A} - \mathbf{A}\mathbf{A}^T - (\mathbf{A}^T)^2]^T =$

$= (\mathbf{A}^2)^T + (\mathbf{A}^T\mathbf{A})^T - (\mathbf{A}\mathbf{A}^T)^T - ((\mathbf{A}^T)^2)^T = (\mathbf{A}^T)^2 + \mathbf{A}^T\mathbf{A} - \mathbf{A}\mathbf{A}^T - ((\mathbf{A}^2)^T)^T =$

$= (\mathbf{A}^T)^2 + \mathbf{A}^T\mathbf{A} - \mathbf{A}\mathbf{A}^T - \mathbf{A}^2$

5. (a) $\begin{bmatrix} \underline{1} & 2 & \underline{3} \\ 4 & 5 & 6 \\ \underline{7} & 8 & \underline{9} \end{bmatrix}$ hence a submatrix

(b) $\begin{bmatrix} \underline{1} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ hence a submatrix

(c) $\begin{bmatrix} \underline{1} & \underline{2} & 3 \\ 4 & 5 & 6 \\ 7 & \underline{8} & \underline{9} \end{bmatrix}$ hence not a submatrix

(d) $\begin{bmatrix} 1 & 2 & 3 \\ \underline{4} & 5 & \underline{6} \\ \underline{7} & 8 & \underline{9} \end{bmatrix}$ hence a submatrix

6. Submatrices of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \end{bmatrix}, \begin{bmatrix} c & d \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}, [a], [b], [c], [d]$$

7. $$\mathbf{A} = \left[\begin{array}{cc|cc} 4 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] = \begin{bmatrix} C & O \\ O & D \end{bmatrix};$$

$$\mathbf{A}^2 = \begin{bmatrix} C & O \\ O & D \end{bmatrix}\begin{bmatrix} C & O \\ O & D \end{bmatrix} = \begin{bmatrix} C^2 & O \\ O & D^2 \end{bmatrix};$$

7. (continued)

$$C^2 = \begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 12 & 6 \end{bmatrix}; \quad D^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{A}^2 = \left[\begin{array}{cc|cc} 18 & 6 & 0 & 0 \\ 12 & 6 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

8. $$\mathbf{B} = \left[\begin{array}{cc|cc} 3 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] = \begin{bmatrix} E & O \\ O & F \end{bmatrix}$$

$$\mathbf{B}^2 = \begin{bmatrix} E & O \\ O & F \end{bmatrix}\begin{bmatrix} E & O \\ O & F \end{bmatrix} = \begin{bmatrix} E^2 & O \\ O & F^2 \end{bmatrix};$$

$$E^2 = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -4 & -1 \end{bmatrix}; \quad F^2 = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{B}^2 = \left[\begin{array}{cc|cc} 7 & 8 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ \hline 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

9. $\mathbf{AB} = \begin{bmatrix} C & O \\ O & D \end{bmatrix}\begin{bmatrix} E & O \\ O & F \end{bmatrix} = \begin{bmatrix} CE & O \\ O & DF \end{bmatrix}$

$$CE = \begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ 4 & 6 \end{bmatrix}$$

$$DF = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$$

$$\mathbf{AB} = \left[\begin{array}{cc|cc} 11 & 9 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & -1 \end{array}\right]$$

10. $\mathbf{A} =$

$$\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} B & O \\ O & C \end{bmatrix}$$

10. (continued) $\mathbf{A}^2 = \begin{bmatrix} B^2 & O \\ O & C^2 \end{bmatrix} =$

$$= \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{A}^3 = \begin{bmatrix} B^3 & O \\ O & C^3 \end{bmatrix} =$

$$= \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

For $n > 3$:

$\mathbf{A}^n = \begin{bmatrix} B^n & O \\ O & C^n \end{bmatrix} =$

$$= \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

11. The matrices in row-reduced form are: **A**, **B**, **F**, **M**, **N**, **R**, **T**.

The matrices not in row-reduced form are: **C** (row 4), **D** (row 2), **E** (rows 1, 2, 3), **G** (row 3), **H** (row 1), **J** (row 2), **K** (row 2), **L** (rows 1, 2), **Q** (rows 1, 2), **S** (row 2).

12. The upper triangular matrices are: **E**, **F**, **K**, **L**, **M**, **N**, **R**, **T**.

13. Yes because of requirements (iii) and (iv) of definition 1.

14. No. See matrices **H** and **L** in problem 11.

15. Yes. See matrix **L** in problem 11.

16. $\mathbf{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

17. If **A** and **B** are diagonal matrices, then $a_{ij} = 0$ and $b_{ij} = 0$ for $i \neq j$.

$$\mathbf{AB} = \left[a_{ij}\right]\left[b_{ij}\right] = \left[\sum_{k=1}^{r} a_{ik}b_{kj}\right] = \begin{bmatrix} a_{11}b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}b_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & 0 & \cdots & 0 \\ 0 & b_{22}a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} =$$

$= \mathbf{BA}$

18. No. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$$

19.

$$\mathbf{AD} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 \\ 2 & 3 & -5 \\ 2 & 3 & -5 \end{bmatrix}$$

$$\mathbf{BD} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -10 \\ 6 & 12 & -25 \\ 12 & 21 & -40 \end{bmatrix}$$

Conclusion:

Let **D** = $\left[d_{ij}\right]$ be a diagonal matrix and **A** any square matrix for which the product **AD** is defined. Then the j^{th} column of **AD** is the j^{th} column of **A** multiplied by d_{jj}.

20.

$$\mathbf{DA} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ -5 & -5 & -5 \end{bmatrix}$$

$$\mathbf{DB} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 9 & 12 & 15 \\ -30 & -35 & -40 \end{bmatrix}$$

Conclusion:

Let **D** = $\left[d_{ij}\right]$ be a diagonal matrix and **A** any square matrix for which the product **DA** is defined. Then the i^{th} row of **DA** is the i^{th} row of **A** multiplied by d_{ii}.

21. Let **A** = $\left[a_{ij}\right]$. Then $(\mathbf{A}^{T})^{T} = \left(\left[a_{ij}\right]^{T}\right)^{T} = \left(\left[a_{ji}\right]\right)^{T} = \left[a_{ij}\right] = $ **A**.

22. Let **A** = $\left[a_{ij}\right]$. Then $(\lambda\mathbf{A})^{T} = \left[\lambda a_{ij}\right]^{T} = \left[\lambda a_{ji}\right] = \lambda\left[a_{ji}\right] = \lambda\mathbf{A}^{T}$.

23. $(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \left(\left[a_{ij}\right] + \left[b_{ij}\right]\right)^{T} = \left[a_{ij} + b_{ij}\right]^{T} = \left[a_{ji} + b_{ji}\right] = \left[a_{ji}\right] + \left[b_{ji}\right] = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$.

24. $(\mathbf{ABC})^{\mathrm{T}} = ((\mathbf{AB})\mathbf{C})^{\mathrm{T}} = (\mathbf{C}^{\mathrm{T}}(\mathbf{AB})^{\mathrm{T}}) = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

25. $\mathbf{B} = (\mathbf{A} + \mathbf{A}^{\mathrm{T}})/2$. $\quad \mathbf{B}^{\mathrm{T}} = ((\mathbf{A} + \mathbf{A}^{\mathrm{T}})/2)^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}} + \mathbf{A})/2 = \mathbf{B}$; i.e., $\mathbf{B}$ is a symmetric matrix.

26. $\mathbf{C} = (\mathbf{A} - \mathbf{A}^{\mathrm{T}})/2$. $\quad -\mathbf{C}^{\mathrm{T}} = -((\mathbf{A} - \mathbf{A}^{\mathrm{T}})/2)^{\mathrm{T}} = -(-\mathbf{A} + \mathbf{A}^{\mathrm{T}})/2 =$

$= (\mathbf{A} - \mathbf{A}^{\mathrm{T}})/2 = \mathbf{C}$; i.e., $\mathbf{C}$ is a skew-symmetric matrix.

27. $\mathbf{A} = (\mathbf{A} + \mathbf{A}^{\mathrm{T}})/2 + (\mathbf{A} - \mathbf{A}^{\mathrm{T}})/2$

28. From problem 1(c) $\mathbf{A} = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix}$. From problem 27:

$\mathbf{A} =$

$$\frac{1}{2}\left(\begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 7 \\ -1 & 3 & -8 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 1 & 5 & -1 \\ 2 & 1 & 3 \\ 0 & 7 & -8 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 7 \\ -1 & 3 & -8 \end{bmatrix}\right) =$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 7 & -1 \\ 7 & 2 & 10 \\ -1 & 10 & -16 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{7}{2} & -\frac{1}{2} \\ \frac{7}{2} & 1 & 5 \\ -\frac{1}{2} & 5 & -8 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{3}{2} & 0 & -2 \\ \frac{1}{2} & 2 & 0 \end{bmatrix}$$

29. From problem 1(c) $\mathbf{B} = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix}$. From problem 27:

$\mathbf{B} =$

$$\frac{1}{2}\left(\begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -1 \\ 1 & 0 & -7 \\ 3 & -1 & 2 \end{bmatrix}\right) + \frac{1}{2}\left(\begin{bmatrix} 6 & 1 & 3 \\ 2 & 0 & -1 \\ -1 & -7 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 2 & -1 \\ 1 & 0 & -7 \\ 3 & -1 & 2 \end{bmatrix}\right) =$$

$$= \frac{1}{2}\begin{bmatrix} 12 & 3 & 2 \\ 3 & 0 & -8 \\ 2 & -8 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 6 \\ -4 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 6 & \frac{3}{2} & 1 \\ \frac{3}{2} & 0 & -4 \\ 1 & -4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & 2 \\ \frac{1}{2} & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

30. $(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$ hence $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is symmetric for any matrix $\mathbf{A}$.

31. If $\mathbf{A}$ is skew-symmetric, then $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$.

The diagonal elements in $\mathbf{A}$ are : $a_{11},\ a_{22},\ \cdots,\ a_{nn}$.

The diagonal elements in $-\mathbf{A}^{\mathrm{T}}$ are : $-a_{11},\ -a_{22},\ \cdots,\ -a_{nn}$.

Since $a_{kk} = -a_{kk}$ for $k = 1, \cdots, n$, this means $a_{kk} = 0$ for all values of k. Hence the diagonal elements of a skew-symmetric matrix must be zero.

32. Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Suppose $\mathbf{A}$ commutes with every diagonal matrix. Then

$$\mathbf{AD} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$

32. (continued)

$$\mathbf{DA} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

Since **AD** = **DA**, we have $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$. This means $b = 0$ and $c = 0$. Hence **A** must be a diagonal matrix.

33. Let **A** = $\left[a_{ij}\right]$ be an n x n matrix and $\mathbf{D}_i$ an n x n diagonal matrix with $d_{kk} = 0$ for all $k \neq i$. Then

$$\mathbf{AD}_i = \begin{bmatrix} 0 & 0 & \cdots & a_{1i} & \cdots & 0 \\ 0 & 0 & \cdots & a_{2i} & \cdots & 0 \\ 0 & 0 & \cdots & a_{3i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{ni} & \cdots & 0 \end{bmatrix}$$

$$\mathbf{D}_i\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

If $\mathbf{AD}_i = \mathbf{D}_i\mathbf{A}$ for all i, then $a_{ij} = 0$ for all $i \neq j$, i.e., **A** is a diagonal matrix.

34. $$\mathbf{D}^2 = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} = \begin{bmatrix} d_{11}^2 & 0 & \cdots & 0 \\ 0 & d_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{nn}^2 \end{bmatrix}$$

34. (continued)
I.e., $\mathbf{D}^2 = \left[d_{ij}^2\right]$.

35. Let $\mathbf{A} = \left[a_{ij}\right]$ and $\mathbf{B} = \left[b_{ij}\right]$ be n x n upper triangular matrices.
Let $\mathbf{AB} = \mathbf{C} = \left[c_{ij}\right]$. Want to show that $c_{ij} = 0 \quad i > j$.

$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj}$. Since **A** and **B** are upper triangular

matrices, $a_{ik} = 0; \ i > k; \quad b_{kj} = 0; \ k > j$. Hence $\sum_{k=j+1}^{n} a_{ik}b_{kj} = \sum_{k=j+1}^{n} a_{ik}\left(0\right) = 0$.
Further, if $i > j \geq k$, then $\sum_{k=1}^{j} a_{ik}b_{kj} = \sum_{k=1}^{j} \left(0\right) b_{kj} = 0$. Hence $c_{ij} = 0 \quad i > j$.

Note the following notation will be used to indicate elementary row operations: cR_i - i^{th} row is multiplied by c; $cR_i + R_j$ – entries in the i^{th} row multiplied by c are added to the j^{th} row; $R_i \leftrightarrow R_j$ - entries in the i^{th} and j^{th} rows are interchanged.

Section 1.4

1.
(a) Substituting $x = 1; \ y = -3; \ z = 2$ into the given system of equations we have: $1 - 3 + 4 = 2; \ 1 + 3 - 4 = 0; \ 1 - 6 + 4 = -1$. Since the last equation is not satisfied by these values, the proposed values are not a solution to the given system.

(b) Substituting $x = 1; \ y = -1; \ z = 1$ into the given system of equations we have: $1 - 1 + 2 = 2; \ 1 + 1 - 2 = 0; \ 1 - 2 + 2 = 1$. Since all the equations are satisfied by these values, the proposed values are a solution to the given system.

2.

(a) Substituting $x_1 = 1;\ \ x_2 = 1;\ \ x_3 = 1$ into the given system of equations we have: $1+2+3=6;\ \ 1-3+2=0;\ \ 3-4+7=6.$ Since all the equations are satisfied by these values, the proposed values are a solution to the given system.

(b) Substituting $x_1 = 2;\ \ x_2 = 2;\ \ x_3 = 0$ into the given system of equations we have: $2+4+0=6;\ \ 1-3+0=-2.$ Since the second equation is not satisfied by these values, the proposed values are not a solution to the given system.

(c) Substituting $x_1 = 14;\ \ x_2 = 2;\ \ x_3 = -4$ into the given system of equations we have: $14+4-12=6;\ \ 14-6-8=0;\ \ 42-8-28=6.$ Since all the equations are satisfied by these values, the proposed values are a solution to the given system.

3. Substituting $x = 2;\ \ y = k$ into the given system of equations we have: $6+5k=11;\ \ 4-7k=-3.$ The solution to this system of equations is $k=1.$ So a solution of the given system of equations is $x=2;\ \ y=1.$

4. Substituting $x = 2k;\ \ y = -k;\ \ z = 0$ into the given system of equations we have:

$2k-2k+0=0;\ \ -4k+4k+0=0;\ \ 6k+6k+0=1.$ The solution to this system of equations is $k=\frac{1}{12}.$ So a solution of the system of equations is

$x=\frac{1}{6};\ \ y=-\frac{1}{12}\ \ z=0.$

5. Substituting $x = 2k;\ \ y = -k;\ \ z = 0$ into the given system of equations we have:

$2k-2k+0=0;\ \ -4k+4k+0=0;\ \ -6k+6k+0=0.$ This system is satisfied for all values of k. Hence any value for k will yield a solution for the system of equations .

6. The system of equations is

$$x + 2y = 5$$
$$y = 8$$

Substituting the value $y = 8$ into the first equation gives

$x + 16 = 5$ or $x = -11$. The solution is $x = -11;\ y = 8.$

7. The system of equations is

$$x - 2y + 3z = 10$$
$$y - 5z = -3$$
$$z = 4$$

Substituting the value $z = 4$ into the second equation gives $y - 20 = -3;\ y = 17$.

Substituting the values $z = 4;\ y = 17$ into the first equation gives

$x - 34 + 12 = 10;\ x = 32.$ The solution is $x = 32;\ y = 17;\ z = 4.$

8. The system of equations is

$$x_1 - 3x_2 + 12x_3 = 40$$
$$x_2 - 6x_3 = -200$$
$$x_3 = 25$$

Substituting the value $x_3 = 25$ into the second equation gives

$x_2 - 150 = -200;\ x_2 = -50$. Substituting the values $x_3 = 25;\ x_2 = -50$ into the first equation gives

$x_1 + 150 + 300 = 40;\ x_1 = -410.$ The solution is $x_1 = -410;\ x_2 = -50;\ x_3 = 25.$

9.
The system of equations is

$$x + 3y \quad = 10$$
$$y + 4z = 2$$

Substituting $y = 2 - 4z$ into the first equation gives

$x + 6 - 12z = -8;\ x = -14 + 12z$. The solution is $x = -14 + 12z;\ y = 2 - 4z,$
where z is arbitrary.

10. The system of equations is

$$\begin{aligned} x_1 - 7x_2 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Substituting $x_2 = x_3$ into the first equation gives $x_1 - 7x_3 + 2x_3 = 0;\ \ x_1 = 5x_3.$ The solution is $x_1 = 5x_3;\ \ x_2 = x_3;$ where x_3 is arbitrary.

11. The system of equations is

$$\begin{aligned} x_1 - x_2 \qquad &= 1 \\ x_2 - 2x_3 &= 2 \\ x_3 &= -3 \\ 0 &= 1 \end{aligned}$$

Since the last equation is invalid, the system has no solution.

12. $\left[\begin{array}{cc|c} 1 & -2 & 5 \\ -3 & 7 & 8 \end{array}\right] \xrightarrow{3R_1 + R_2} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & 23 \end{array}\right]$ The second row gives us $y = 23.$

Substituting $y = 23$ into the equation represented by the first row gives $x - 2(23) = 5;\ \ x = 51.$ The soution is $x = 51;\ \ y = 23.$

13.

$$\left[\begin{array}{cc|c} 4 & 24 & 20 \\ 2 & 11 & -8 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & 6 & 5 \\ 2 & 11 & -8 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 6 & 5 \\ 0 & -1 & -18 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 6 & 5 \\ 0 & 1 & 18 \end{array}\right]$$

Substituting $y = 18$ into the equation represented by the first row gives $x + 6(18) = 5;\ \ x = -103.$ The solution is $x = -103;\ \ y = 18.$

14.

$$\left[\begin{array}{cc|c} 0 & -1 & 6 \\ 2 & 7 & -5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & 7 & -5 \\ 0 & -1 & 6 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{7}{2} & -\frac{5}{2} \\ 0 & -1 & 6 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & \frac{7}{2} & -\frac{5}{2} \\ 0 & 1 & -6 \end{array}\right]$$

Substituting $y = -6$ into the equation represented by the first row gives $x + \frac{7}{2}(-6) = -\frac{5}{2}$; $x = 18.5$. The solution is $x = 18.5$; $y = -6$.

15.

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & 5 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 3 & 5 & 0 \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 14 & 0 \end{array}\right] \xrightarrow{\frac{1}{14}R_2} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Substituting $y = 0$ into the equation represented by the first row gives $x - 3(0) = 0$; $x = 0.$. The solution is $x = 0$; $y = 0$.

16. $$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 3 & -9 & 0 \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The first row gives $x - 3y = 0$; $x = 3y$. The solution is $x = 3y$ where y is arbitrary.

17.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ -1 & -1 & 2 & 3 \\ -2 & 3 & 0 & 0 \end{array}\right] \xrightarrow[2R_1 + R_3]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 7 & 6 & 8 \end{array}\right] \xrightarrow{-7R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -29 & -41 \end{array}\right] \xrightarrow{-\frac{1}{29}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & \frac{41}{29} \end{array}\right]$$

17. (continued)

Substituting $z=\frac{41}{29}$ into the equation represented by the second row gives $y+5\left(\frac{41}{29}\right)=7;\ \ y=-\frac{2}{29}$. Substituting these values for $z\,\&\,y$ into the equation represented by the first row gives $x+2\left(-\frac{2}{29}\right)+3\left(\frac{41}{29}\right)=4; x=-\frac{3}{29}$. The solution is $x=-\frac{3}{29};\ \ y=-\frac{2}{29};\ \ z=\frac{41}{29}$.

18.

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 4 \\ 1 & 3 & 2 & 1 \\ -2 & 3 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ -2 & 3 & 1 & 2 \end{array}\right] \xrightarrow{2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 9 & 5 & 4 \end{array}\right] \xrightarrow{-9R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 23 & -32 \end{array}\right] \xrightarrow{\frac{1}{23}R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -\frac{32}{23} \end{array}\right]$$

Substituting $z=-\frac{32}{23}$ into the equation represented by the second row gives $y-2\left(-\frac{32}{23}\right)=4;\ \ y=\frac{28}{23}$. Substituting these values for $z\,\&\,y$ into the equation represented by the first row gives $x+3\left(\frac{28}{23}\right)+2\left(-\frac{32}{23}\right)=1; x=\frac{3}{23}$. The solution is $x=\frac{3}{23};\ \ y=\frac{28}{23};\ \ z=-\frac{32}{23}$.

19.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ -1 & -4 & 3 & -1 \\ 2 & 0 & -1 & 3 \\ 2 & -1 & 4 & 2 \end{array}\right] \xrightarrow[-2R_1+R_4]{\substack{R_1+R_2 \\ -2R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -1 & 5 & -1 \\ 0 & -6 & -5 & 3 \\ 0 & -7 & 0 & 2 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & -6 & -5 & 3 \\ 0 & -7 & 0 & 2 \end{array}\right] \xrightarrow[7R_2+R_4]{6R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & -35 & 9 \\ 0 & 0 & -35 & 9 \end{array}\right] \xrightarrow{-\frac{1}{35}R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & -\frac{9}{35} \\ 0 & 0 & -35 & 9 \end{array}\right] \xrightarrow{35R_3+R_4} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & -\frac{9}{35} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Substituting $z = -\frac{9}{35}$ into the equation represented by the second row gives $y - 5\left(-\frac{9}{35}\right) = 1;\ y = -\frac{2}{7}$. Substituting the y & z values into the equation represented by the first row gives $x + 3\left(-\frac{2}{7}\right) + 2\left(-\frac{9}{35}\right) = 0;\ x = \frac{48}{35}.$

The solution is $x = \frac{48}{35};\ y = -\frac{2}{7};\ z = -\frac{9}{35}.$

20.

$$\left[\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ -4 & -8 & 2 & 0 \\ -2 & -4 & 1 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1/2 & 0 \\ -4 & -8 & 2 & 0 \\ -2 & -4 & 1 & -1 \end{array}\right] \xrightarrow[2R_1+R_3]{4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The last row represents the equation $0 = -1$ which is invalid. Hence there is no solution.

21.

$$\left[\begin{array}{ccc|c} -3 & 6 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 6 & -3 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first row represents the equation $x - 2y + z = 0$ or $x = 2y - z.$ The solution is $x = 2y - z$ with y and z arbitrary.

22.

$$\left[\begin{array}{ccc|c} -3 & -3 & -3 & 0 \\ 1 & -1 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ -3 & -3 & -3 & 0 \end{array}\right] \xrightarrow[3R_1+R_4]{\substack{-R_1+R_2 \\ -2R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{4R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third row represents the equation $z = 0$. Substituting $z = 0$ into the equation represented by the second row $y - \frac{1}{2}(0) = 0;\ \ y = 0$. Substituting the values for $y\,\&\,z$ into the equation represented by the first row $x + 0 + 0 = 0;\ \ x = 0$. The solution is $x = y = z = 0$.

23.

$$\left[\begin{array}{ccc|c} -3 & 6 & -3 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -3 & 6 & -3 & 0 \end{array}\right] \xrightarrow{3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The second row represents the equation $x_2 = 0$. Substituting $x_2 = 0$ into the equation represented by the first row $x_1 + 0 + x_3 = 0;\ \ x_1 = -x_3$. The solution is $x_1 = -x_3;\ \ x_2 = 0$ with x_3 arbitrary.

24. $$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first row represents the equation $x_1 - x_2 + 2x_3 = 0;\ \ x_1 = x_2 - 2x_3$. The solution is $x_1 = x_2 - 2x_3$ with $x_2\,\&\,x_3$ arbitrary.

25. $$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 3 & 1 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -5 & 10 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & -2 \end{array}\right]$$

25. (continued)
The second row represents the equation $x_2 = -2$. Substituting $x_2 = -2$ into the equation represented by the first row $x_1 + 2(-2) = -3;\ \ x_1 = 1$. The solution is $x_1 = 1;\ \ x_2 = -2$.

26. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & -3 & 2 & 4 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & -7 & 0 & 6 \end{array}\right] \xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -6/7 \end{array}\right]$

The second row represents the equation $x_2 = -6/7$. Substituting $x_2 = -6/7$ into the equation represented by the first row
$x_1 + 2(-6/7) + x_3 = -1;\ \ x_1 = 5/7 - x_3$. The solution is $x_1 = 5/7 - x_3;\ \ x_2 = -6/7$ with x_3 arbitrary.

27.
$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ -3 & 1 & 13 \\ 4 & 3 & 0 \end{array}\right] \xrightarrow[-4R_1 + R_3]{3R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 7 & 28 \\ 0 & -5 & -20 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & -5 & -20 \end{array}\right] \xrightarrow{5R_2 + R_3} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

The second row represents the equation $x_2 = 4$. Substituting $x_2 = 4$ into the equation represented by the first row $x_1 + 2x\ x_1 + 2(4) = 5;\ \ x_1 = -3$. The solution is $x_1 = -3;\ \ x_2 = 4$.

28.
$$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 2 & 1 & 8 \\ 5 & -3 & 7 & 15 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 8 \\ 5 & -3 & 7 & 15 \end{array}\right] \xrightarrow[-5R_1 + R_3]{-3R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -4 & 1 & 5 \\ 0 & -13 & 7 & 10 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/4 & -5/4 \\ 0 & -13 & 7 & 10 \end{array}\right] \xrightarrow{13R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/4 & -5/4 \\ 0 & 0 & 15/4 & -25/4 \end{array}\right] \xrightarrow{\frac{4}{15}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/4 & -5/4 \\ 0 & 0 & 1 & -5/3 \end{array}\right]$$
The third row represents the equation $x_3 = -5/3$. Substituting $x_3 = -5/3$ into the equation represented by the second row $x_2 - 1/4(-5/3) = -5/4;\ \ x_2 = -5/3$.

28. (continued)

Substituting $x_3 = -\frac{5}{3}$ and $x_2 = -\frac{5}{3}$ in the equation represented by the first row $x_1 + 2\left(-\frac{5}{3}\right) = 1$; $x_1 = \frac{13}{3}$. The solution is $x_1 = \frac{13}{3}$; $x_2 = -\frac{5}{3}$; $x_3 = -\frac{5}{3}$.

29.
$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 2 \\ 3 & -2 & 0 & -1 \\ 8 & -1 & -4 & 10 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 2 & 3 & -4 & 2 \\ 1 & -5 & 4 & -3 \\ 8 & -1 & -4 & 10 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 2 & 3 & -4 & 2 \\ 8 & -1 & -4 & 10 \end{array}\right] \xrightarrow[-8R_1+R_3]{-2R_1+R_2}$$
$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 13 & -12 & 8 \\ 0 & 39 & -36 & 34 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 13 & -12 & 8 \\ 0 & 0 & 0 & 10 \end{array}\right]$$
The last row represents the equation $0 = 10$ which is invalid. Hence there is no solution.

30. The graph of each equation is a plane. If the two planes do not intersect, there is no solution. If the planes intersect, their intersection is either a line or a plane. In either case there are infinitely many solutions.

31. Given: **Ax** = **b**; **Ay** = **b**; **Az** = **0**.
Then **Au** =**A**(**y** + **z**) = **Ay** + **Az** = **b** + **0** = **b**.

32. Given: **Ax** = **b**; **Ay** = **b**; **Az** = **0**.

(a) **Au** = **A**(**y** + α **z**) = **Ay** + **A**(α **z**) = **b** + α **Az** = **b** + **0** = **b**.
Hence **u** is a solution for all real values of α.

(b) **Au** = **A**(α **y** + **z**) = **A**(α **y**)+ **A**(**z**) = α **Ay** + **0** = α **b**. Hence **u** is a solution only for $\alpha = 1$.

33. 70,000 springs; 45,000 lbs. of stuffing; each regular mattress requires 50 springs and 30 lbs. of stuffing; each support mattress requires 60 springs and 40 lbs. of stuffing. Let r represent the number of regular mattresses produced daily, and s the number of support mattresses produced daily. The system of equations that models this problem is given by:
$$50r + 60s = 70,000$$
$$30r + 40s = 45,000$$

33. (continued)

This may be simplified to
$$\begin{aligned}5r + 6s &= 7,000\\ 3r + 4s &= 4,500\end{aligned}$$

To solve:

$$\left[\begin{array}{cc|c} 5 & 6 & 7,000 \\ 3 & 4 & 4,500 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|c} -1 & -2 & -2,000 \\ 3 & 4 & 4,500 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c} 1 & 2 & 2,000 \\ 3 & 4 & 4,500 \end{array}\right] \xrightarrow{-3R_1+R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 2,000 \\ 0 & -2 & -1,500 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 2 & 2,000 \\ 0 & 1 & 750 \end{array}\right]$$

The second row gives $s = 750.$ Substituting this value into the equation represented by the first row $r + 2(750) = 2,000;$ $r = 500.$ The solution is 500 regular mattresses and 750 support mattresses.

34. Desks: 5 hours for cutting, 10 hours for assembly
Bookcases: ¼ hour for cutting, 1 hour for assembly
Total times: 200 hours for cutting, 500 hours for assembly
Let d represent the number of desks and b the number of bookcases to be scheduled for completion in one day. The system of equations that models this problem is given by:

$$5d + \tfrac{1}{4}b = 200$$

$$10d + b = 500$$

To solve:

$$\left[\begin{array}{cc|c} 5 & \frac{1}{4} & 200 \\ 10 & 1 & 500 \end{array}\right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{20} & 40 \\ 10 & 1 & 500 \end{array}\right] \xrightarrow{-10R_1+R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{20} & 40 \\ 0 & \frac{1}{2} & 100 \end{array}\right] \xrightarrow{2R_2}$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{20} & 40 \\ 0 & 1 & 200 \end{array}\right]$$

34. (continued)
The second row gives $b = 200$. Substituting this value into the equation represented by the first row $d + \frac{1}{20}(200) = 40$; $d = 30$. The solution is 30 desks and 200 bookcases.

35. Must supply 70,000 tons low-grade (l-g) ore; 181,000 tons medium-grade (m-g) ore; 41,000 tons high-grade (h-g) ore. Daily production Mine A 8,000 tons l-g, 5,000 tons m-g, 1,000 tons h-g; Mine B 3,000 tons l-g, 12,000 tons m-g, 3,000 tons h-g; Mine C 1,000 tons l-g, 10,000 tons m-g, 2,000 tons h-g. Let A, B, C denote the number of days mines A,B,C must operate to meet demand. The system of equations that models this problem is given by

$$\begin{aligned} 8{,}000A + 3{,}000B + 1{,}000C &= 70{,}000 \\ 5{,}000A + 12{,}000B + 10{,}000C &= 181{,}000 \\ 1{,}000A + 3{,}000B + 2{,}000C &= 41{,}000 \end{aligned}$$

This may be simplified to

$$\begin{aligned} 8A + 3B + C &= 70 \\ 5A + 12B + 10C &= 181 \\ A + 3B + 2C &= 41 \end{aligned}$$

To solve:

$$\left[\begin{array}{ccc|c} 8 & 3 & 1 & 70 \\ 5 & 12 & 10 & 181 \\ 1 & 3 & 2 & 41 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 41 \\ 5 & 12 & 10 & 181 \\ 8 & 3 & 1 & 70 \end{array}\right] \xrightarrow[-8R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 41 \\ 0 & -3 & 0 & -24 \\ 0 & -21 & -15 & -258 \end{array}\right] \xrightarrow[-\frac{1}{3}R_3]{-\frac{1}{3}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 41 \\ 0 & 1 & 0 & 8 \\ 0 & 7 & 5 & 86 \end{array}\right] \xrightarrow{-7R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 41 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 5 & 30 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 41 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

The third row gives $C = 6$. The second row gives $B = 8$. Substituting these values into the equation represented by the first row $A + 3(8) + 2(6) = 41$; $A = 5$. The solution is Mine A – 5 days; Mine B – 8 days; Mine C – 6 days.

36. Bonus b is 5% of $400,000 minus 5% of the city and state taxes; city tax c is 2% of $400,000; state tax s is 3% of $400,000 minus 3% of the city tax. The system of equations that models this problem is given by

$$\begin{aligned} b + .05c + .05s &= 20,000 \\ c &= 8,000 \\ .03c + s &= 12,000 \end{aligned}$$

To solve:

$$\left[\begin{array}{ccc|c} 1 & .05 & .05 & 20,000 \\ 0 & 1 & 0 & 8,000 \\ 0 & .03 & 1 & 12,000 \end{array}\right] \xrightarrow{-3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & .05 & .05 & 20,000 \\ 0 & 1 & 0 & 8,000 \\ 0 & 0 & 1 & 11,760 \end{array}\right]$$

The third gives $s = 11,760$. The second row gives $c = 8,000.$ Substituting these values into the equation represented by the first row $b + .05(8,000) + .05(11,760) = 20,000;$ $b = 19,012$. The solution is the bonus is $19,012.

37. Fixed costs $800,000; variable cost $30 per barrel ($B$ - number of barrels); net sales (S) - $40 per barrel.

(a)

$$\begin{aligned} C &= 800,000 + 30B \\ S &= 40B \end{aligned}$$

(b) Adding the equation $C = S$ leads to the following system of equations that models this problem

$$\begin{aligned} -30B + C \quad &= 800,000 \\ -40B \quad + S &= 0 \\ C - S &= 0 \end{aligned}$$

37(b) (continued)

To solve

$$\left[\begin{array}{ccc|c} -30 & 1 & 0 & 800,000 \\ -40 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} -40 & 0 & 1 & 0 \\ -30 & 1 & 0 & 800,000 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{-\frac{1}{40}R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/40 & 0 \\ -30 & 0 & 1 & 800,000 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{30R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1/40 & 0 \\ 0 & 1 & -3/4 & 800,000 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{-R_2 + R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/40 & 0 \\ 0 & 1 & -3/4 & 800,000 \\ 0 & 0 & -1/4 & -800,000 \end{array}\right] \xrightarrow{-4R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1/40 & 0 \\ 0 & 1 & -3/4 & 800,000 \\ 0 & 0 & 1 & 3,200,000 \end{array}\right]$$

The third row gives $S = 3,200,000.$ Substituting this value into the equation represented by the second row $C - \frac{3}{4}(3,200,000) = 800,000;$ $C = 3,200,000.$ Substituting the values for $S \& C$ into the equation represented by the first row $B - \frac{1}{40}(3,200,000) = 0;$ $B = 80,000.$ The solution is 80,000 barrels must be produced.

38. The break-even equations are as follows

Farmer: $.40p_1 + .30p_2 + .50p_3 = p_1$

Carpenter: $.40p_1 + .25p_2 + .35p_3 = p_2$

Weaver: $.20p_1 + .45p_2 + .15p_3 = p_3$

38. (continued)
This may be rewritten as the following system of equations

$$-6p_1 + 3p_2 + 5p_3 = 0$$
$$8p_1 - 15p_2 + 7p_3 = 0$$
$$4p_1 + 9p_2 - 17p_3 = 0$$

To solve

$$\left[\begin{array}{ccc|c} -6 & 3 & 5 & 0 \\ 8 & -15 & 7 & 0 \\ 4 & 9 & -17 & 0 \end{array}\right] \xrightarrow{-\frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & -1/2 & -5/6 & 0 \\ 8 & -15 & 7 & 0 \\ 4 & 9 & -17 & 0 \end{array}\right] \xrightarrow[-4R_1+R_3]{-8R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1/2 & -5/6 & 0 \\ 0 & -11 & 41/3 & 0 \\ 0 & 11 & -41/3 & 0 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & -5/6 & 0 \\ 0 & -11 & 41/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{11}R_2} \left[\begin{array}{ccc|c} 1 & -1/2 & -5/6 & 0 \\ 0 & 1 & -41/33 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The second row gives $p_2 - \frac{41}{33}p_3 = 0$; $p_2 = \frac{41}{33}p_3$. Substituting this value into the equation represented by the first row $p_1 - \frac{1}{2}\left(\frac{41}{33}\right)p_3 - \frac{5}{6}p_3 = 0$; $p_1 = \frac{48}{33}p_3$.

The solution is $p_1 = \frac{48}{33}p_3$; $p_2 = \frac{41}{33}p_3$ where p_3 is arbitrary.

39. Let Paul, Jim, and Mary have wages p_1, p_2, p_3 repectively. The break-even equations are

Paul: $\frac{1}{2}p_1 + \frac{1}{3}p_2 + \frac{1}{6}p_3 = p_1$

Jim: $\frac{1}{4}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3 = p_2$

Mary: $\frac{1}{4}p_1 + \frac{1}{3}p_2 + \frac{1}{2}p_3 = p_3$

39. (continued)
This may be rewritten as the following system of equations.

$$\begin{aligned} 6p_1 - 4p_2 - 2p_3 &= 0 \\ 3p_1 - 8p_2 + 4p_3 &= 0 \\ 3p_1 + 4p_2 - 6p_3 &= 0 \end{aligned}$$

To solve

$$\left[\begin{array}{ccc|c} 6 & -4 & -2 & 0 \\ 3 & -8 & 4 & 0 \\ 3 & 4 & -6 & 0 \end{array}\right] \xrightarrow[-R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} 0 & 12 & -10 & 0 \\ 3 & -8 & 4 & 0 \\ 0 & 12 & -10 & 0 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|c} 0 & 12 & -10 & 0 \\ 3 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|c} 3 & -8 & 4 & 0 \\ 0 & 12 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & -8/3 & 4/3 & 0 \\ 0 & 12 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{12}R_2} \left[\begin{array}{ccc|c} 1 & -8/3 & 4/3 & 0 \\ 0 & 1 & -5/6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The second row gives $p_2 = \frac{5}{6}p_3$. Substituting this value into the equation represented by the first row $p_1 - \frac{8}{3}\left(\frac{5}{6}\right)p_3 + \frac{4}{3}p_3 = 0$; $p_1 = \frac{8}{9}p_3$. The solution is $p_1 = \frac{8}{9}p_3$; $p_2 = \frac{5}{6}p_3$ where p_3 is arbitrary.

40. Let $p_1,\ p_2,\ p_3,\ p_4$ denote the prices of the annual fruit harvests for countries A, B, C, D respectively. The break-even equations are

Country A: $.15p_1 + .10p_2 + 0p_3 + .15p_4 = p_1$

Country B: $.20p_1 + .40p_2 + \frac{1}{3}p_3 + .40p_4 = p_2$

Country C: $.30p_1 + .15p_2 + \frac{1}{3}p_3 + .45p_4 = p_3$

Country D: $.35p_1 + .35p_2 + \frac{1}{3}p_3 + 0p_4 = p_1$

40. (continued)

This gives the system of equations

$$-.85p_1 + .10p_2 + 0p_3 + .15p_4 = 0$$
$$.20p_1 - .60p_2 + \frac{1}{3}p_3 + .40p_4 = 0$$
$$.30p_1 + .15p_2 - \frac{2}{3}p_3 + .45p_4 = 0$$
$$.35p_1 + .35p_2 + \frac{1}{3}p_3 - p_4 = 0$$

This may be rewritten as

$$17p_1 - 2p_2 + 0p_3 - 3p_4 = 0$$
$$3p_1 - 9p_2 + 5p_3 + 6p_4 = 0$$
$$18p_1 + 9p_2 - 40p_3 + 27p_4 = 0$$
$$21p_1 + 21p_2 + 20p_3 - 60p_4 = 0$$

To solve

$$\left[\begin{array}{cccc|c} 17 & -2 & 0 & -3 & 0 \\ 3 & -9 & 5 & 6 & 0 \\ 18 & 9 & -40 & 27 & 0 \\ 21 & 21 & 20 & -60 & 0 \end{array}\right]$$

Since the computations fro this problem are tedious, the row-reduced from was obtained using computer software (MAPLE). The result is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -113/329 & 0 \\ 0 & 1 & 0 & -467/329 & 0 \\ 0 & 0 & 1 & -54/47 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is $p_1 = \frac{113}{329} \approx .34347$; $p_2 = \frac{467}{329} \approx 1.41945$; $p_3 = \frac{54}{47} \approx 1.14894$ where p_4 is arbitrary.

41. $\left[\begin{array}{cc|c} 1 & 3 & 35 \\ 4 & 8 & 15 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 4 & 8 & 15 \\ 1 & 3 & 35 \end{array}\right]$

The first pivot is 4.

42. $\left[\begin{array}{cc|c} 1 & -2 & -5 \\ 5 & 3 & 85 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 5 & 3 & 85 \\ 1 & -2 & -5 \end{array}\right]$

The first pivot is 5.

43. $\left[\begin{array}{ccc|c} -2 & 8 & -3 & 100 \\ 4 & 5 & 4 & 75 \\ -3 & -1 & 2 & 250 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 4 & 5 & 4 & 75 \\ -2 & 8 & -3 & 100 \\ -3 & -1 & 2 & 250 \end{array}\right]$

The first pivot is 4.

44. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \end{array}\right]$

The first pivot is 9.

45. $\left[\begin{array}{ccc|c} 1 & 8 & 8 & 400 \\ 0 & 1 & 7 & 800 \\ 0 & 3 & 9 & 600 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 8 & 8 & 400 \\ 0 & 3 & 9 & 600 \\ 0 & 1 & 7 & 800 \end{array}\right]$

The first pivot is 3 since the first column is in reduced form.

46. $\left[\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 0 \\ 1 & 0.4 & 0.8 & 0.1 & 90 \\ 4 & 10 & 1 & 8 & 40 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 4 & 10 & 1 & 8 & 40 \\ 1 & 0.4 & 0.8 & 0.1 & 90 \\ 0 & 2 & 3 & 4 & 0 \end{array}\right]$

The first pivot is 4.

Section 1.5

1. $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = (3)(6) - (4)(5) = -2$

2. $\begin{vmatrix} 3 & -4 \\ 5 & 6 \end{vmatrix} = (3)(6) - (-4)(5) = 38$

3. $\begin{vmatrix} 3 & 4 \\ -5 & 6 \end{vmatrix} = (3)(6) - (4)(-5) = 38$

4. $\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = (5)(8) - (6)(7) = -2$

5. $\begin{vmatrix} 5 & 6 \\ -7 & 8 \end{vmatrix} = (5)(8) - (6)(-7) = 82$

6. $\begin{vmatrix} 5 & 6 \\ 7 & -8 \end{vmatrix} = (5)(-8) - (6)(7) = -82$

7. $\begin{vmatrix} 1 & -1 \\ 2 & 7 \end{vmatrix} = (1)(7) - (-1)(2) = 9$

8. $\begin{vmatrix} -2 & -3 \\ -4 & 4 \end{vmatrix} = (-2)(4) - (-3)(-4) = -20$

9. $\begin{vmatrix} 3 & -1 \\ -3 & 8 \end{vmatrix} = (3)(8) - (-1)(-3) = 21$

10. $\begin{vmatrix} 1 & 2 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix} = (1)(2)(-3) = -6$ (Theorem 4)

11. (Using minors)

$$\begin{vmatrix} 3 & 2 & -2 \\ 1 & 0 & 4 \\ 2 & 0 & -3 \end{vmatrix} = -(2)\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} + (0)\begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix} - (0)\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = -2\big((1)(-3) - (4)(2)\big) = 22$$

12. $$\begin{vmatrix} 1 & -2 & -2 \\ 7 & 3 & -3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{(Theorem 5)}$$

Note the following notation is used in this section of problems.
cR_i - the ith is multiplied by c; $cR_i + R_j$ - entries in the ith row multiplied by c are added to the jth row; $R_i \leftrightarrow R_j$ - entries in the ith and jth rows are interchanged.

13.

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -3 \end{vmatrix} \xrightarrow{-2R_1 + R_3} \begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & -5 \end{vmatrix} = -(0)\begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} + (1)\begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} - (0)\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} =$$

$$= (1)\big((2)(-5) - (-1)(1)\big) = -9$$

14.

$$\begin{vmatrix} 3 & 5 & 2 \\ -1 & 0 & 4 \\ -2 & 2 & 7 \end{vmatrix} \xrightarrow[-2R_2 + R_3]{3R_2 + R_1} \begin{vmatrix} 0 & 5 & 14 \\ -1 & 0 & 4 \\ 0 & 2 & -1 \end{vmatrix} = (0)\begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 14 \\ 2 & -1 \end{vmatrix} + (0)\begin{vmatrix} 5 & 14 \\ 0 & 4 \end{vmatrix} =$$

$$= (5)(-1) - (14)(2) = -33$$

15.

$$\begin{vmatrix} 1 & -3 & -3 \\ 2 & 8 & 3 \\ 4 & 5 & 0 \end{vmatrix} \xrightarrow{R_1 + R_2} \begin{vmatrix} 1 & -3 & -3 \\ 3 & 5 & 0 \\ 4 & 5 & 0 \end{vmatrix} = (-3)\begin{vmatrix} 3 & 5 \\ 4 & 5 \end{vmatrix} - (0)\begin{vmatrix} 1 & -3 \\ 4 & 5 \end{vmatrix} + (0)\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} =$$

$$= (-3)\big((3)(5) - (5)(4)\big) = 15$$

16.

$$\begin{vmatrix} 2 & 1 & -9 \\ 3 & -1 & 1 \\ 3 & -1 & 2 \end{vmatrix} \xrightarrow[R_1+R_3]{R_1+R_2} \begin{vmatrix} 2 & 1 & -9 \\ 5 & 0 & -8 \\ 5 & 0 & -7 \end{vmatrix} = -(1)\begin{vmatrix} 5 & -8 \\ 5 & -7 \end{vmatrix} + (0)\begin{vmatrix} 2 & -9 \\ 5 & -7 \end{vmatrix} - (0)\begin{vmatrix} 2 & -9 \\ 5 & -8 \end{vmatrix} =$$

$$= -((5)(-7)-(-8)(5)) = -5$$

17. $$\begin{vmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{vmatrix} \xrightarrow[-R_1+R_3]{R_1+R_2} \begin{vmatrix} -1 & 3 & 3 \\ 0 & 4 & 7 \\ 0 & -2 & -1 \end{vmatrix} =$$

$$\begin{vmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{vmatrix} \xrightarrow[-R_1+R_3]{R_1+R_2} \begin{vmatrix} -1 & 3 & 3 \\ 0 & 4 & 7 \\ 0 & -2 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 4 & 7 \\ -2 & -1 \end{vmatrix} - (0)\begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix} + (0)\begin{vmatrix} 3 & 3 \\ 4 & 7 \end{vmatrix} =$$

$$= (-1)((4)(-1)-(7)(-2)) = -10$$

18. $$\begin{vmatrix} 1 & -3 & -3 \\ 2 & 8 & 4 \\ 3 & 5 & 1 \end{vmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & -3 & -3 \\ 0 & 14 & 10 \\ 0 & 14 & 10 \end{vmatrix} = 0$$ (Corollary 2)

19. $$\begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{vmatrix} \xrightarrow[-3R_1+R_3]{R_1+R_2} \begin{vmatrix} 2 & 1 & 3 \\ 5 & 0 & 5 \\ -4 & 0 & -4 \end{vmatrix} \xrightarrow[-\frac{1}{4}R_3]{\frac{1}{5}R_2} \left(\frac{1}{5}\right)\left(-\frac{1}{4}\right)\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

(Theorem 8 and Corollary 2)

20. $$\begin{vmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{vmatrix} = 0 \begin{vmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{vmatrix} = 0$$ (Corollary 2)

21.

$$\begin{vmatrix} 1 & 2 & -3 \\ 5 & 5 & 1 \\ 2 & -5 & -1 \end{vmatrix} \xrightarrow[-2R_1+R_3]{-5R_1+R_2} \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 16 \\ 0 & -9 & 5 \end{vmatrix} = (1)\begin{vmatrix} -5 & 16 \\ -9 & 5 \end{vmatrix} - (0)\begin{vmatrix} 2 & -3 \\ -9 & 5 \end{vmatrix} + (0)\begin{vmatrix} 2 & -3 \\ -5 & 16 \end{vmatrix} =$$

$$= (1)((-5)(5)-(16)(-9)) = 119$$

22. $\begin{vmatrix} -4 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & -2 \end{vmatrix} = (-4)(-1)(-2) = -8 \quad \text{(Theorem 4)}$

23.

$$\begin{vmatrix} 1 & 3 & 2 \\ -1 & 4 & 1 \\ 5 & 3 & 8 \end{vmatrix} \xrightarrow[-5R_1+R_3]{R_1+R_2} \begin{vmatrix} 1 & 3 & 2 \\ 0 & 7 & 3 \\ 0 & -12 & -2 \end{vmatrix} = (1)\begin{vmatrix} 7 & 3 \\ -12 & -2 \end{vmatrix} - (0)\begin{vmatrix} 3 & 2 \\ -12 & -2 \end{vmatrix} + (0)\begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix} =$$
$$= (1)((7)(-2) - (3)(-12)) = 22$$

24.

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 1 & 2 \\ -3 & 4 & 1 \end{vmatrix} \xrightarrow{-2R_3+R_2} \begin{vmatrix} 3 & -2 & 0 \\ 7 & -7 & 0 \\ -3 & 4 & 1 \end{vmatrix} = (0)\begin{vmatrix} 7 & -7 \\ -3 & 4 \end{vmatrix} - (0)\begin{vmatrix} 3 & -2 \\ -3 & 4 \end{vmatrix} + (1)\begin{vmatrix} 3 & -2 \\ 7 & -7 \end{vmatrix} =$$
$$= (1)((3)(-7) - (-2)(7)) = -7$$

25. $\begin{vmatrix} -4 & 0 & 0 & 0 \\ 1 & -5 & 0 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & -2 & 1 \end{vmatrix} = (-4)(-5)(-2)(1) = -40 \quad \text{(Theorem 4)}$

26.

$$\begin{vmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{vmatrix} \xrightarrow{-R_1+R_3} \begin{vmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 3 & 0 & -2 & -1 \\ 2 & 0 & -3 & 2 \end{vmatrix} = -(2)\begin{vmatrix} 1 & 3 & -1 \\ 3 & -2 & -1 \\ 2 & -3 & 2 \end{vmatrix} \xrightarrow[2R_1+R_3]{-R_1+R_2} (-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -5 & 0 \\ 4 & 3 & 0 \end{vmatrix} =$$
$$= (-2)(-1)\begin{vmatrix} 2 & -5 \\ 4 & 3 \end{vmatrix} = (2)((2)(3) - (-5)(4)) = 52$$

27.

$$\begin{vmatrix} 1 & 1 & 2 & -2 \\ 1 & 5 & 2 & -1 \\ -2 & -2 & 1 & 3 \\ -3 & 4 & -1 & 8 \end{vmatrix} \xrightarrow[3R_1+R_4]{\substack{-R_1+R_2 \\ 2R_1+R_3}} \begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 7 & 5 & 2 \end{vmatrix} = (1)\begin{vmatrix} 4 & 0 & 1 \\ 0 & 5 & -1 \\ 7 & 5 & 2 \end{vmatrix} \xrightarrow[-2R_1+R_3]{R_1+R_2} (1)\begin{vmatrix} 4 & 0 & 1 \\ 4 & 5 & 0 \\ -1 & 5 & 0 \end{vmatrix} =$$

$$= (1)(1)\begin{vmatrix} 4 & 5 \\ -1 & 5 \end{vmatrix} = \big((4)(5)-(5)(-1)\big) = 25$$

28.

$$\begin{vmatrix} -1 & 3 & 2 & -2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{vmatrix} \xrightarrow[3R_1+R_4]{\substack{R_1+R_2 \\ 3R_1+R_3}} \begin{vmatrix} -1 & 3 & 2 & -2 \\ 0 & -2 & -2 & 4 \\ 0 & 3 & 7 & -5 \\ 0 & 5 & 9 & -9 \end{vmatrix} = (-1)\begin{vmatrix} -2 & -2 & 4 \\ 3 & 7 & -5 \\ 5 & 9 & -9 \end{vmatrix} \xrightarrow{R_1+R_2}$$

$$(-1)\begin{vmatrix} -2 & -2 & 4 \\ 1 & 5 & -1 \\ 5 & 9 & -9 \end{vmatrix} \xrightarrow[-5R_2+R_3]{2R_2+R_1} (-1)\begin{vmatrix} 0 & 8 & 2 \\ 1 & 5 & -1 \\ 0 & -16 & -4 \end{vmatrix} \xrightarrow{2R_1+R_3} (-1)\begin{vmatrix} 0 & 8 & 2 \\ 1 & 5 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(Theorem 5)

29. $\begin{vmatrix} 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ -3 & 4 & 0 & 8 \end{vmatrix} = 0$ (Theorem 5)

30.

$$\begin{vmatrix} 1 & 2 & 1 & -1 \\ 4 & 0 & 3 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & -2 & 1 & 1 \end{vmatrix} \xrightarrow[-2R_1+R_4]{\substack{-4R_1+R_2 \\ -R_1+R_3}} \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & -8 & -1 & 4 \\ 0 & -1 & -1 & 6 \\ 0 & -6 & -1 & 3 \end{vmatrix} = (1)\begin{vmatrix} -8 & -1 & 4 \\ -1 & -1 & 6 \\ -6 & -1 & 3 \end{vmatrix} \xrightarrow[-6R_2+R_3]{-8R_2+R_1} (1)\begin{vmatrix} 0 & 7 & -44 \\ -1 & -1 & 6 \\ 0 & 5 & -33 \end{vmatrix} =$$

$$= (1)\big((-)(-1)\big)\begin{vmatrix} 7 & -44 \\ 5 & -33 \end{vmatrix} = \big((7)(-33)-(-44)(5)\big) = -11$$

31. $\begin{vmatrix} 11 & 1 & 0 & 9 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{vmatrix} = 0$ (Theorem 5)

32. $\begin{vmatrix} t & 2t \\ 1 & t \end{vmatrix} = t^2 - 2t = t(t-2) = 0; \quad t = 0, 2$

33. $\begin{vmatrix} t-2 & t \\ 3 & t+2 \end{vmatrix} = (t-2)(t+2) - 3t = t^2 - 3t - 4 = (t-4)(t+1) = 0; \quad t = 4, -1$

34. $\begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0 \quad \lambda = 2, 3$

35. $\begin{vmatrix} 1-\lambda & 5 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 5 = \lambda^2 - 6 = 0; \quad \lambda = \pm\sqrt{6}$

36. $\begin{vmatrix} 3-\lambda & 4 \\ 5 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) - 20 = \lambda^2 - 9\lambda - 2$

37. $\begin{vmatrix} 3-\lambda & -4 \\ 5 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) + 20 = \lambda^2 - 9\lambda + 38$

38. $\begin{vmatrix} 5-\lambda & 6 \\ 7 & 8-\lambda \end{vmatrix} = (5-\lambda)(8-\lambda) - 42 = \lambda^2 - 13\lambda - 2$

39. $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) + 2 = \lambda^2 - 8\lambda + 9$

40.

$$\begin{vmatrix} 3-\lambda & 2 & -2 \\ 1 & -\lambda & 4 \\ 2 & 0 & -3-\lambda \end{vmatrix} \xrightarrow[-2R_2+R_3]{(-3+\lambda)R_2+R_1} \begin{vmatrix} 0 & 2+3\lambda-\lambda^2 & -14+4\lambda \\ 1 & -\lambda & 4 \\ 0 & 2\lambda & -11-\lambda \end{vmatrix} = -(1)\begin{vmatrix} 2+3\lambda-\lambda^2 & -14+4\lambda \\ 2\lambda & -11-\lambda \end{vmatrix} =$$

$$= (-1)\left((2+3\lambda-\lambda^2)(-11-\lambda) - (-14+4\lambda)(2\lambda)\right) = -\lambda^3 + 7\lambda + 22$$

41.

$$\begin{vmatrix} 1-\lambda & -2 & -2 \\ 7 & 3-\lambda & -3 \\ 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 7 & 3-\lambda \end{vmatrix} = (-\lambda)\left((1-\lambda)(3-\lambda)+14\right) = -\lambda^3 + 4\lambda^2 - 17\lambda$$

42.

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 1 & 1-\lambda & 1 \\ 3 & 2 & -3-\lambda \end{vmatrix} \xrightarrow[(-3-\lambda)R_1+R_3]{R_1+R_2} \begin{vmatrix} 2-\lambda & 0 & -1 \\ 3-\lambda & 1-\lambda & 0 \\ \lambda^2+\lambda-3 & 2 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 3-\lambda & 1-\lambda \\ \lambda^2+\lambda-3 & 2 \end{vmatrix} =$$

$$= (-1)\Big((3-\lambda)(2)-(1-\lambda)\left(\lambda^2+\lambda-3\right)\Big) = -\lambda^3+6\lambda-9$$

43.

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ -1 & -\lambda & 4 \\ -2 & 2 & 7-\lambda \end{vmatrix} \xrightarrow[-2R_2+R_3]{(3-\lambda)R_2+R_1} \begin{vmatrix} 0 & \lambda^2-3\lambda+5 & 14-4\lambda \\ -1 & -\lambda & 4 \\ 0 & 2+2\lambda & -1-\lambda \end{vmatrix} = -(-1)\begin{vmatrix} \lambda^2-3\lambda+5 & 14-4\lambda \\ 2+2\lambda & -1-\lambda \end{vmatrix} =$$

$$= (1)\Big(\left(\lambda^2-3\lambda+5\right)(-1-\lambda)-(14-4\lambda)(2+2\lambda)\Big) = -\lambda^3+10\lambda^2-22\lambda-33$$

44. $\mathbf{AB} = \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & -5 \\ 7 & 1 \end{bmatrix}$; $\det(\mathbf{A}) = \begin{vmatrix} 6 & 1 \\ 1 & 2 \end{vmatrix} = 12-1 = 11$;

$\det(\mathbf{B}) = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 3+2=5$; $\det(\mathbf{AB}) = \begin{vmatrix} 20 & -5 \\ 7 & 1 \end{vmatrix} 20+35=55$; $(11)(5)=55$.

I.e., $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

45. Area $= \left\| \begin{matrix} -1 & 2 \\ 3 & -3 \end{matrix} \right\| = |3-6| = 3$.

46. Area $= \left\| \begin{matrix} 1 & -4 \\ -5 & -4 \end{matrix} \right\| = |-4-20| = 24$.

47. Area $= \left\| \begin{matrix} 2 & 3 \\ 4 & -8 \end{matrix} \right\| = |-16-12| = 28$.

48.

$$\begin{vmatrix} 1 & 2 & -2 \\ 1 & 3 & 3 \\ 2 & 5 & 0 \end{vmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 5 \\ 0 & 1 & 4 \end{vmatrix} \xrightarrow{-R_2+R_3} \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{vmatrix} \xrightarrow{-R_3} (-1)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

49. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow[-7R_1+R_3]{-4R_1+R_2} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} \xrightarrow{-2R_2+R_3} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

50.

$$\begin{vmatrix} 3 & -4 & 2 \\ -1 & 5 & 7 \\ 1 & 9 & -6 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} (-1)\begin{vmatrix} 1 & 9 & -6 \\ -1 & 5 & 7 \\ 3 & -4 & 2 \end{vmatrix} \xrightarrow[-3R_1+R_3]{R_1+R_2} (-1)\begin{vmatrix} 1 & 9 & -6 \\ 0 & 14 & 1 \\ 0 & -31 & 20 \end{vmatrix} \xrightarrow{\frac{1}{14}R_2}$$

$$(-1)(14)\begin{vmatrix} 1 & 9 & -6 \\ 0 & 1 & 1/14 \\ 0 & -31 & 20 \end{vmatrix} \xrightarrow{31R_2+R_3} (-14)\begin{vmatrix} 1 & 9 & -6 \\ 0 & 1 & 1/14 \\ 0 & 0 & 311/14 \end{vmatrix} \xrightarrow{\frac{14}{311}R_3} (-14)\left(\frac{311}{14}\right)\begin{vmatrix} 1 & 9 & -6 \\ 0 & 1 & 1/14 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= (-14)\left(\frac{311}{14}\right) = -311$$

51.

$$\begin{vmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} (-1)\begin{vmatrix} 1 & 1 & 4 \\ -1 & 3 & 3 \\ -1 & 1 & 2 \end{vmatrix} \xrightarrow[R_1+R_3]{R_1+R_2} (-1)\begin{vmatrix} 1 & 1 & 4 \\ 0 & 4 & 7 \\ 0 & 2 & 6 \end{vmatrix} \xrightarrow{\frac{1}{4}R_2}$$

$$(-1)(4)\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 7/4 \\ 0 & 2 & 6 \end{vmatrix} \xrightarrow{-2R_2+R_3} (-4)\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 7/4 \\ 0 & 0 & 5/2 \end{vmatrix} \xrightarrow{\frac{2}{5}R_3} (-4)\left(\frac{5}{2}\right)\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & 7/4 \\ 0 & 0 & 1 \end{vmatrix} = (-4)\left(\frac{5}{2}\right) = -10$$

52. $\begin{vmatrix} 1 & -3 & -3 \\ 2 & 8 & 4 \\ 3 & 5 & 1 \end{vmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & -3 & -3 \\ 0 & 14 & 10 \\ 0 & 14 & 10 \end{vmatrix} = 0$ (Corollary 2)

53.

$$\begin{vmatrix} 2 & 1 & -9 \\ 3 & -1 & 1 \\ 3 & -1 & 2 \end{vmatrix} \xrightarrow[-R_2+R_3]{-R_2+R_1} \begin{vmatrix} -1 & 2 & -10 \\ 3 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow{3R_1+R_2} \begin{vmatrix} -1 & 2 & -10 \\ 0 & 5 & -29 \\ 0 & 0 & 1 \end{vmatrix} \xrightarrow[\frac{1}{5}R_2]{-R_1}$$

$$(-1)(5)\begin{vmatrix} 1 & -2 & 10 \\ 0 & 1 & -29/5 \\ 0 & 0 & 1 \end{vmatrix} = -5$$

54.

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{vmatrix} \xrightarrow{-R_2+R_1} \begin{vmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 3 & 5 \end{vmatrix} \xrightarrow[2R_1+R_3]{3R_1+R_2} \begin{vmatrix} -1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 7 & 7 \end{vmatrix} \xrightarrow{\frac{5}{7}R_3} \begin{vmatrix} -1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{vmatrix} = 0$$

(Corollary 2)

55. $\begin{vmatrix} -1 & 3 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 3 \end{vmatrix} = 0$ (Corollary 2)

56.

$$\begin{vmatrix} 1 & 2 & -3 \\ 5 & 5 & 1 \\ 2 & -5 & -1 \end{vmatrix} \xrightarrow[-2R_1+R_3]{-5R_1+R_2} \begin{vmatrix} 1 & 2 & -3 \\ 0 & -5 & 16 \\ 0 & -9 & 5 \end{vmatrix} \xrightarrow{-\frac{1}{5}R_2} (-5)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -16/5 \\ 0 & -9 & 5 \end{vmatrix} \xrightarrow{9R_2+R_3}$$

$$(-5)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -16/5 \\ 0 & 0 & -119/5 \end{vmatrix} \xrightarrow{-\frac{5}{119}R_3} (-5)\left(-\frac{119}{5}\right)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -16/5 \\ 0 & 0 & 1 \end{vmatrix} = (-5)\left(-\frac{119}{5}\right) = 119$$

57.

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 2 & -3 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} (-1)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 3 & 2 & -3 \end{vmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & -1 & -6 \end{vmatrix} \xrightarrow{-\frac{1}{2}R_2}$$

$$(-1)(-2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & -1 & -6 \end{vmatrix} \xrightarrow{R_2+R_3} (-1)(-2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & -9/2 \end{vmatrix} \xrightarrow{-2/9 R_3} (-1)(-2)\left(-\frac{9}{2}\right)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= (-1)(-2)\left(-\frac{9}{2}\right) = -9$$

58.

$$\begin{vmatrix} 3 & 5 & 2 \\ -1 & 0 & 4 \\ -2 & 2 & 7 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_2} (-1)\begin{vmatrix} -1 & 0 & 4 \\ 3 & 5 & 2 \\ -2 & 2 & 7 \end{vmatrix} \xrightarrow[-2R_1+R_3]{3R_1+R_2} (-1)\begin{vmatrix} -1 & 0 & 4 \\ 0 & 5 & 14 \\ 0 & 2 & -1 \end{vmatrix} \xrightarrow[\frac{1}{5}R_2]{-R_1}$$

$$(-1)(-1)(5)\begin{vmatrix} 1 & 0 & -4 \\ 0 & 1 & 14/5 \\ 0 & 2 & -1 \end{vmatrix} \xrightarrow{-2R_2+R_3} (5)\begin{vmatrix} 1 & 0 & -4 \\ 0 & 1 & 14/5 \\ 0 & 0 & -33/5 \end{vmatrix} \xrightarrow{-\frac{5}{33}R_3} (5)\left(-\frac{33}{5}\right)\begin{vmatrix} 1 & 0 & -4 \\ 0 & 1 & 14/5 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= (5)\left(-\frac{33}{5}\right) = -33$$

59.

$$\begin{vmatrix} 1 & -3 & -3 \\ 2 & 8 & 3 \\ 4 & 5 & 0 \end{vmatrix} \xrightarrow[-4R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & -3 & -3 \\ 0 & 14 & 9 \\ 0 & 17 & 12 \end{vmatrix} \xrightarrow{\frac{1}{14}R_2} (14)\begin{vmatrix} 1 & -3 & -3 \\ 0 & 1 & 9/14 \\ 0 & 17 & 12 \end{vmatrix} \xrightarrow{-17R_2+R_3}$$

$$(14)\begin{vmatrix} 1 & -3 & -3 \\ 0 & 1 & 9/14 \\ 0 & 0 & 15/14 \end{vmatrix} \xrightarrow{\frac{14}{15}R_3} (14)\left(\frac{15}{14}\right)\begin{vmatrix} 1 & -3 & -3 \\ 0 & 1 & 9/14 \\ 0 & 0 & 1 \end{vmatrix} = (14)\left(\frac{15}{14}\right) = 15$$

60.

$$\begin{vmatrix} 3 & 5 & 4 & 6 \\ -2 & 1 & 0 & 7 \\ -5 & 4 & 7 & 2 \\ 8 & -3 & 1 & 1 \end{vmatrix} \xrightarrow{R_2+R_1} \begin{vmatrix} 1 & 6 & 4 & 13 \\ -2 & 1 & 0 & 7 \\ -5 & 4 & 7 & 2 \\ 8 & -3 & 1 & 1 \end{vmatrix} \xrightarrow[-8R_1+R_4]{\substack{2R_1+R_2 \\ 5R_1+R_3}} \begin{vmatrix} 1 & 6 & 4 & 13 \\ 0 & 13 & 8 & 33 \\ 0 & 34 & 27 & 67 \\ 0 & -51 & -31 & -103 \end{vmatrix} \xrightarrow{\frac{1}{13}R_2}$$

$$(13)\begin{vmatrix} 1 & 6 & 4 & 13 \\ 0 & 1 & 8/13 & 33/13 \\ 0 & 34 & 27 & 67 \\ 0 & -51 & -31 & -103 \end{vmatrix} \xrightarrow[51R_2+R_4]{-34R_2+R_3} (13)\begin{vmatrix} 1 & 6 & 4 & 13 \\ 0 & 1 & 8/13 & 33/13 \\ 0 & 0 & 79/13 & -251/13 \\ 0 & 0 & 5/13 & 344/13 \end{vmatrix} \xrightarrow{\frac{13}{79}R_3}$$

$$(13)\left(\frac{79}{13}\right)\begin{vmatrix} 1 & 6 & 4 & 13 \\ 0 & 1 & 8/13 & 33/13 \\ 0 & 0 & 1 & -251/79 \\ 0 & 0 & 5/13 & 344/13 \end{vmatrix} \xrightarrow{-\frac{5}{13}R_3+R_4} (13)\left(\frac{79}{13}\right)\begin{vmatrix} 1 & 6 & 4 & 13 \\ 0 & 1 & 8/13 & 33/13 \\ 0 & 0 & 1 & -251/79 \\ 0 & 0 & 0 & 28,431/1,027 \end{vmatrix} \xrightarrow{\frac{1,027}{28,431}R_4}$$

$$(13)\left(\frac{79}{13}\right)\left(\frac{28,431}{1,027}\right)\begin{vmatrix} 1 & 6 & 4 & 13 \\ 0 & 1 & 8/13 & 33/13 \\ 0 & 0 & 1 & -251/79 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (13)\left(\frac{79}{13}\right)\left(\frac{28,431}{1,027}\right) = 2187$$

61.

$$\begin{vmatrix} -1 & 2 & 1 & 2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{vmatrix} \xrightarrow{-R_1} (-1)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 1 & 0 & 3 & -1 \\ 2 & 2 & -1 & 1 \\ 2 & 0 & -3 & 2 \end{vmatrix} \xrightarrow[-2R_1+R_4]{\substack{-R_1+R_2 \\ -2R_1+R_3}}$$

$$(-1)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 0 & 2 & 4 & 1 \\ 0 & 6 & 1 & 5 \\ 0 & 4 & -1 & 6 \end{vmatrix} \xrightarrow{\frac{1}{2}R_2} (-1)(2)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & 1/2 \\ 0 & 6 & 1 & 5 \\ 0 & 4 & -1 & 6 \end{vmatrix} \xrightarrow[-4R_2+R_4]{-6R_2+R_3}$$

61. (continued)

$$(-1)(2)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & -11 & 2 \\ 0 & 0 & -9 & 4 \end{vmatrix} \xrightarrow{-\frac{1}{11}R_3} (-1)(2)(-11)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{11} \\ 0 & 0 & -9 & 4 \end{vmatrix} \xrightarrow{9R_3+R_4}$$

$$(22)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{11} \\ 0 & 0 & 0 & \frac{26}{11} \end{vmatrix} \xrightarrow{\frac{11}{26}R_4} (22)\left(\frac{26}{11}\right)\begin{vmatrix} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{11} \\ 0 & 0 & 0 & 1 \end{vmatrix} = (22)\left(\frac{26}{11}\right) = 52$$

62.

$$\begin{vmatrix} 1 & 1 & 2 & -2 \\ 1 & 5 & 2 & -1 \\ -2 & -2 & 1 & 3 \\ -3 & 4 & -1 & 8 \end{vmatrix} \xrightarrow[\substack{2R_1+R_3 \\ 3R_1+R_4}]{-R_1+R_2} \begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 5 & -1 \\ 0 & 7 & 5 & 2 \end{vmatrix} \xrightarrow{\frac{1}{4}R_2} (4)\begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 5 & -1 \\ 0 & 7 & 5 & 2 \end{vmatrix} \xrightarrow{-7R_2+R_3}$$

$$(4)\begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 5 & \frac{1}{4} \end{vmatrix} \xrightarrow{\frac{1}{5}R_3} (4)(5)\begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 5 & \frac{1}{4} \end{vmatrix} \xrightarrow{-5R_3+R_4} (4)(5)\begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & \frac{5}{4} \end{vmatrix} \xrightarrow{\frac{4}{5}R_4}$$

$$(20)\left(\frac{5}{4}\right)\begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 & 1 \end{vmatrix} = (20)\left(\frac{5}{4}\right) = 25$$

63.

$$\begin{vmatrix} -1 & 3 & 2 & -2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{vmatrix} \xrightarrow{-R_1} (-1)\begin{vmatrix} 1 & -3 & -2 & 2 \\ 1 & -5 & -4 & 6 \\ 3 & -6 & 1 & 1 \\ 3 & -4 & 3 & -3 \end{vmatrix} \xrightarrow[-3R_1+R_4]{\substack{-R_1+R_2 \\ -3R_1+R_3}} (-1)\begin{vmatrix} 1 & -3 & -2 & 2 \\ 0 & -2 & -2 & 4 \\ 0 & 3 & 7 & -5 \\ 0 & 5 & 9 & -9 \end{vmatrix} \xrightarrow{-\frac{1}{2}R_2}$$

$$(-1)(-2)\begin{vmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & 7 & -5 \\ 0 & 5 & 9 & -9 \end{vmatrix} \xrightarrow[-5R_2+R_4]{-3R_2+R_3} (2)\begin{vmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 4 & 1 \end{vmatrix} = 0$$

(Corollary 2)

64. $\begin{vmatrix} 1 & 1 & 0 & -2 \\ 1 & 5 & 0 & -1 \\ -2 & -2 & 0 & 3 \\ -3 & 4 & 0 & 8 \end{vmatrix} = 0$ (Theorem 5)

65.

$$\begin{vmatrix} -2 & 0 & 1 & 3 \\ 4 & 0 & 2 & -2 \\ -3 & 1 & 0 & 1 \\ 5 & 4 & 1 & 7 \end{vmatrix} \xrightarrow{C_3 \leftrightarrow C_1} (-1)\begin{vmatrix} 1 & 0 & -2 & 3 \\ 2 & 0 & 4 & -2 \\ 0 & 1 & -3 & 1 \\ 1 & 4 & 5 & 7 \end{vmatrix} \xrightarrow[-R_1+R_4]{-2R_1+R_2} (-1)\begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 0 & 8 & -8 \\ 0 & 1 & -3 & 1 \\ 0 & 4 & 7 & 4 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$(-1)(-1)\begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 8 & -8 \\ 0 & 4 & 7 & 4 \end{vmatrix} \xrightarrow{\frac{1}{8}R_3} (-1)(-1)(8)\begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 4 & 7 & 4 \end{vmatrix} \xrightarrow{-4R_2+R_4}$$

65. (continued)

$$(8)\begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 19 & 0 \end{vmatrix} \xrightarrow{-19R_3+R_4} (8)\begin{vmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 19 \end{vmatrix} \xrightarrow{\frac{1}{19}R_4}$$

$$(8)(19)\begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (8)(19) = 152$$

66.

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} \xrightarrow[-R_2]{2R_1} (-1)\left(\frac{1}{2}\right)\begin{vmatrix} 2a & 2b & 2c \\ -r & -s & -t \\ x & y & z \end{vmatrix} \xrightarrow{2C_2} (-1)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\begin{vmatrix} 2a & 4b & 2c \\ -r & -2s & -t \\ x & 2y & z \end{vmatrix} =$$

$$= \left(-\frac{1}{4}\right)\begin{vmatrix} 2a & 4b & 2c \\ -r & -2s & -t \\ x & 2y & z \end{vmatrix}$$

67.

$$\begin{vmatrix} a-3x & b-3y & c-3z \\ a+5x & b+5y & c+5z \\ x & y & z \end{vmatrix} \xrightarrow{-R_1+R_2} \begin{vmatrix} a-3x & b-3y & c-3z \\ 8x & 8y & 8z \\ x & y & z \end{vmatrix} \xrightarrow{\frac{1}{8}R_2} (8)\begin{vmatrix} a-3x & b-3y & c-3z \\ x & y & z \\ x & y & z \end{vmatrix} = 0$$

(Corollary 2)

68.

$$\begin{vmatrix} 2a & 3a & c \\ 2r & 3r & t \\ 2x & 3x & z \end{vmatrix} \xrightarrow[\frac{1}{3}C_2]{\frac{1}{2}C_1} (2)(3)\begin{vmatrix} a & a & c \\ r & r & t \\ x & x & z \end{vmatrix} = 0$$ (Corollary 2 for columns)

69. $\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix} \xrightarrow{C_2 \leftrightarrow C_3} (-1)\begin{vmatrix} a & x & r \\ b & y & s \\ c & z & t \end{vmatrix}$

70. $\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix} \xrightarrow[-C_3+C_2]{C_3+C_1} \begin{vmatrix} a+x & r-x & x \\ b+y & s+y & y \\ c+z & t-z & z \end{vmatrix}$

71.
$$(-12)\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix} \xrightarrow[3C_2]{2C_1} (-12)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\begin{vmatrix} 2a & 3r & x \\ 2b & 3s & y \\ 2c & 3t & z \end{vmatrix} \xrightarrow[-R_3]{2R_2} (-2)\left(\frac{1}{2}\right)(-1)\begin{vmatrix} 2a & 3r & x \\ 4b & 6s & 2y \\ -2c & -3t & -z \end{vmatrix} =$$
$$= \begin{vmatrix} 2a & 3r & x \\ 4b & 6s & 2y \\ -2c & -3t & -z \end{vmatrix}$$

72.
$$(5)\begin{vmatrix} a & r & x \\ b & s & y \\ c & t & z \end{vmatrix} \xrightarrow{-3R_2+R_1} (5)\begin{vmatrix} a-3b & r-3s & x-3y \\ b & s & y \\ c & t & z \end{vmatrix} \xrightarrow{-2R_3+R_2} (5)\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ c & t & z \end{vmatrix} \xrightarrow{5R_3}$$
$$(5)\left(\frac{1}{5}\right)\begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix} = \begin{vmatrix} a-3b & r-3s & x-3y \\ b-2c & s-2t & y-2z \\ 5c & 5t & 5z \end{vmatrix}$$

73. From problem 48 $\begin{vmatrix} 1 & 2 & -2 \\ 1 & 3 & 3 \\ 2 & 5 & 0 \end{vmatrix} = -1$. Determinant of the transpose:

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ -2 & 3 & 0 \end{vmatrix} \xrightarrow[2R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 4 \end{vmatrix} = (1)\begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} = 4-5=-1$$

From problem 49 $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$. Determinant of the transpose:

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{vmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = (1)\begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 36-36=0$$

From problem 50 $\begin{vmatrix} 3 & -4 & 2 \\ -1 & 5 & 7 \\ 1 & 9 & -6 \end{vmatrix} = -311$. Determinant of the transpose:

$$\begin{vmatrix} 3 & -1 & 1 \\ -4 & 5 & 9 \\ 2 & 7 & -6 \end{vmatrix} \xrightarrow[-R_3+R_1]{2R_3+R_2} \begin{vmatrix} 1 & -8 & 7 \\ 0 & 19 & -3 \\ 2 & 7 & -6 \end{vmatrix} \xrightarrow{-2R_1+R_3} \begin{vmatrix} 1 & -8 & 7 \\ 0 & 19 & -3 \\ 0 & 23 & -20 \end{vmatrix} =$$

$$= (1)\begin{vmatrix} 19 & -3 \\ 23 & -20 \end{vmatrix} = -380+69=-311$$

From problem 51 $\begin{vmatrix} -1 & 3 & 3 \\ 1 & 1 & 4 \\ -1 & 1 & 2 \end{vmatrix} = -10$. Determinant of the transpose:

$$\begin{vmatrix} -1 & 1 & -1 \\ 3 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} \xrightarrow[3R_1+R_3]{3R_1+R_2} \begin{vmatrix} -1 & 1 & -1 \\ 0 & 4 & -2 \\ 0 & 7 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 4 & -2 \\ 7 & -1 \end{vmatrix} = (-1)(-4+14) = -10$$

74. For this example $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -3 & 4 \end{bmatrix}$; $\lambda = 3$; $n = 2$. $\lambda\mathbf{A} = \begin{bmatrix} 3 & 9 \\ -9 & 12 \end{bmatrix}$

$$\begin{vmatrix} 3 & 9 \\ -9 & 12 \end{vmatrix} = 36 + 81 = 117$$

$$\begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 4 + 9 = 13; \quad \lambda^2 = 9; \quad (9)(13) = 117.$$

75. For this example $\mathbf{A} = \begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix}$; $\lambda = -2$; $n = 2$. $\lambda\mathbf{A} = \begin{bmatrix} -4 & -6 \\ 6 & 4 \end{bmatrix}$

$$\begin{vmatrix} -4 & -6 \\ 6 & 4 \end{vmatrix} = -16 + 30 = 20$$

$$\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} = -4 + 9 = 5; \quad \lambda^2 = 4; \quad (4)(5) = 20.$$

76. For this example $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$; $\lambda = -1$; $n = 2$. $\lambda\mathbf{A} = \begin{bmatrix} -3 & -4 \\ -5 & -6 \end{bmatrix}$

$$\begin{vmatrix} -3 & -4 \\ -5 & -6 \end{vmatrix} = 18 - 20 = -2$$

$$\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2; \quad \lambda^2 = 1; \quad (1)(-2) = -2.$$

Section 1.6

1. (a) $\begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 1 & 1/3 \\ 1/2 & 1/9 \end{bmatrix} = \begin{bmatrix} 5/2 & \\ & \end{bmatrix}$ Not an inverse for **A**.

(b) $\begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ -2 & -9 \end{bmatrix} = \begin{bmatrix} -7 & \\ & \end{bmatrix}$ Not an inverse for **A**.

(c) $\begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ It is an inverse for **A**.

(d) $\begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & \\ & \end{bmatrix}$ Not an inverse for **A**.

2. (a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \\ & \end{bmatrix}$ Not an inverse for **A**.

(b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \\ & \end{bmatrix}$ Not an inverse for **A**.

(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \\ & \end{bmatrix}$ Not an inverse for **A**.

(d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & \end{bmatrix}$ Not an inverse for **A**.

In problems 3 – 12 the answer is the matrix on the left as demonstrated by the multiplication.

3. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

4. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ c & d \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -5c & -5d \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ -5d & -5e & -5f \\ g & h & k \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 3a+c & 3b+d \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ c & d \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ d+3g & e+3h & f+3k \\ g & h & k \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5a+g & 5b+h & 5c+k \end{bmatrix}$

11.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

12. $\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} a & b & c & d & e \\ f & g & h & k & r \end{bmatrix} = \begin{bmatrix} a & b & c & d & e \\ 7f & 7g & 7h & 7k & 7r \end{bmatrix}$

13. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

23. $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 4 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right]$

The inverse is $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$.

24.

$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2}$

$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right]$

The inverse is $\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$.

25. $\left[\begin{array}{cc|cc} 4 & 4 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|cc} 4 & 4 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$ It has no inverse.

26.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right] \xrightarrow{-R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

The inverse is $\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$.

27.

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

The inverse is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

28.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|ccc} -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & -5 & 3 & 0 & -2 \end{array}\right] \xrightarrow{2R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 3 & 2 & -2 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 2 \end{array}\right] \xrightarrow{-R_2+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 2 \end{array}\right] \xrightarrow{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 6 & 5 & -4 \\ 0 & 0 & 1 & -3 & -2 & 2 \end{array}\right]$$

The inverse is $\begin{bmatrix} -1 & -1 & 1 \\ 6 & 5 & -4 \\ -3 & -2 & 2 \end{bmatrix}$.

29.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-7R_1+R_3]{-4R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

It has no inverse.

30.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4R_1+R_3]{-5R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -5/2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -5/2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1 \end{array}\right]$$

The inverse is $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$.

31.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 5 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-\frac{5}{2}R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1/2 & 0 & -5/4 \\ 0 & 1 & 0 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/6 & -4/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

The inverse is $\frac{1}{6}\begin{bmatrix} 3 & -1 & -8 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

32.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 \\ 3 & 9 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 1 & 0 \\ 3 & 9 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \\ 3 & 9 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 8 & 1 & 4 & -3 & 0 \\ 0 & 15 & 2 & 3 & -3 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 8 & 1 & 4 & -3 & 0 \\ 0 & -1 & 0 & -5 & 3 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -5 & 3 & 1 \\ 0 & 8 & 1 & 4 & -3 & 0 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 8 & 1 & 4 & -3 & 0 \end{array}\right] \xrightarrow[2R_2+R_1]{-8R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -5 & -2 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -36 & 21 & 8 \end{array}\right]$$

The inverse is $\begin{bmatrix} 9 & -5 & -2 \\ 5 & -3 & -1 \\ -36 & 21 & 8 \end{bmatrix}$.

33.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -4 & 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -4 & 3 & -2 & 1 & 0 \\ 0 & -1 & 5 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 5 & -1 & 1 & 1 \\ 0 & -4 & 3 & -2 & 1 & 0 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -5 & 1 & -1 & -1 \\ 0 & -4 & 3 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-2R_2+R_1]{4R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & -1 & 2 & 2 \\ 0 & 1 & -5 & 1 & -1 & -1 \\ 0 & 0 & -17 & 2 & -3 & -4 \end{array}\right] \xrightarrow{-\frac{1}{17}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & -1 & 2 & 2 \\ 0 & 1 & -5 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2/17 & 3/17 & 4/17 \end{array}\right] \xrightarrow[-9R_3+R_1]{5R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/17 & 7/17 & -2/17 \\ 0 & 1 & 0 & 7/17 & -2/17 & 3/17 \\ 0 & 0 & 1 & -2/17 & 3/17 & 4/17 \end{array}\right]$$

33. (continued)

The inverse is $\frac{1}{17}\begin{bmatrix} 1 & 7 & -2 \\ 7 & -2 & 3 \\ -2 & 3 & 4 \end{bmatrix}$.

34.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & -2 & -4 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & -7 & -3 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \\ 0 & -8 & -7 & -3 & 1 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & -8 & -7 & -3 & 1 & 0 \end{array}\right] \xrightarrow{8R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 17 & 13 & 1 & -8 \end{array}\right] \xrightarrow{\frac{1}{17}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 1 & 13/17 & 1/17 & -8/17 \end{array}\right] \xrightarrow[-3R_3+R_2]{-R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4/17 & -1/17 & 8/17 \\ 0 & 1 & 0 & -5/17 & -3/17 & 7/17 \\ 0 & 0 & 1 & 13/17 & 1/17 & -8/17 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14/17 & 5/17 & -6/17 \\ 0 & 1 & 0 & -5/17 & -3/17 & 7/17 \\ 0 & 0 & 1 & 13/17 & 1/17 & -8/17 \end{array}\right]$$

The inverse is $\frac{1}{17}\begin{bmatrix} 14 & 5 & -6 \\ -5 & -3 & 7 \\ 13 & 1 & -8 \end{bmatrix}$.

35.

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 3 & -4 & -4 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 5 & 0 & -1 & 1 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_3}$$

$\left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 5 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$ It has no inverse.

36.

$$\left[\begin{array}{ccc|ccc} 5 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -5 & 1 & -2 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-2R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -5 & 1 & -2 & 0 \\ 0 & -5 & 12 & -2 & 5 & 0 \\ 0 & -1 & 9 & -2 & 4 & 1 \end{array}\right] \xrightarrow{-5R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -5 & 1 & -2 & 0 \\ 0 & 0 & -33 & 8 & -15 & -5 \\ 0 & -1 & 9 & -2 & 4 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -5 & 1 & -2 & 0 \\ 0 & -1 & 9 & -2 & 4 & 1 \\ 0 & 0 & -33 & 8 & -15 & -5 \end{array}\right] \xrightarrow[-\frac{1}{33}R_3]{-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -5 & 1 & -2 & 0 \\ 0 & 1 & -9 & 2 & -4 & -1 \\ 0 & 0 & 1 & -8/33 & 15/33 & 5/33 \end{array}\right] \xrightarrow[5R_3+R_1]{9R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -7/33 & 9/33 & 25/33 \\ 0 & 1 & 0 & -6/33 & 3/33 & 12/33 \\ 0 & 0 & 1 & -8/33 & 15/33 & 5/33 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/33 & 3/33 & 1/33 \\ 0 & 1 & 0 & -6/33 & 3/33 & 12/33 \\ 0 & 0 & 1 & -8/33 & 15/33 & 5/33 \end{array}\right]$$

The inverse is $\frac{1}{33}\begin{bmatrix} 5 & 3 & 1 \\ -6 & 3 & 12 \\ -8 & 15 & 5 \end{bmatrix}$.

37.

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 0 & -8 & 4 & 1 & -3 & 0 \\ 0 & -3 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-\frac{1}{8}R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & -1/8 & 3/8 & 0 \\ 0 & -3 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow[-3R_2+R_1]{3R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 3/8 & -1/8 & 0 \\ 0 & 1 & -1/2 & -1/8 & 3/8 & 0 \\ 0 & 0 & -1/2 & -3/8 & -7/8 & 1 \end{array}\right] \xrightarrow[-R_3+R_2]{R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2/8 & 10/8 & -1 \\ 0 & 0 & -1/2 & -3/8 & -7/8 & 1 \end{array}\right] \xrightarrow{-2R_3}$$

37. (continued)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1/4 & 5/4 & -1 \\ 0 & 0 & 0 & 3/4 & 7/4 & -2 \end{array}\right]$$ The inverse is $\frac{1}{4}\begin{bmatrix} 0 & -4 & 4 \\ 1 & 5 & -4 \\ 3 & 7 & -8 \end{bmatrix}$.

38.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_4]{\frac{1}{2}R_3}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right] \xrightarrow[\substack{-R_4+R_2 \\ -R_4+R_1}]{-\frac{3}{2}R_4+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right] \xrightarrow[-2R_3+R_1]{R_3+R_2}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1/2 & 5/4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 3/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right]$$

The inverse is $\frac{1}{4}\begin{bmatrix} 4 & -4 & -4 & -4 \\ 0 & 4 & 2 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

39.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 6 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 4 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_4]{\substack{-2R_1+R_2 \\ -4R_1+R_3}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 4 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 6 & 2 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 4 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_2+R_4]{-6R_2+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -16 & 6 & 1 & 0 \\ 0 & 0 & 4 & -1 & -7 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -8 & 3 & 1/2 & 0 \\ 0 & 0 & 4 & -1 & -7 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-4R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -8 & 3 & 1/2 & 0 \\ 0 & 0 & 0 & -1 & 25 & -10 & -2 & 1 \end{array}\right] \xrightarrow{-R_4}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -8 & 3 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -25 & 10 & 2 & -1 \end{array}\right] \text{ The inverse is } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -8 & 3 & 1/2 & 0 \\ -25 & 10 & 2 & -1 \end{bmatrix}.$$

40. $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad-bc}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore\ \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

41. (a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$; $\mathbf{A}^{-1} = \dfrac{1}{4-3}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$; $\mathbf{A}^{-1} = \dfrac{1}{\frac{1}{3}-\frac{1}{4}}\begin{bmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$

42. The system of equations is given in matrix form by

$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. To solve using matrix inversion:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The solution is $x = 1$; $y = -2$.

43. The system of equations is given in matrix form by

$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$. To solve using matrix inversion:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 13 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 13 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -21 \\ 28 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

The solution is $a = -3$; $b = 4$.

44. The system of equations is given in matrix form by

$\begin{bmatrix} 4 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$. To solve using matrix inversion:

44. (continued)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = -\frac{1}{16}\begin{bmatrix} -3 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = -\frac{1}{16}\begin{bmatrix} -20 \\ -8 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/2 \end{bmatrix}$$

The solution is $x = \frac{5}{4}$; $y = \frac{1}{2}$.

45. The system of equations is given in matrix form by

$\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} l \\ p \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. To solve using matrix inversion:

$$\begin{bmatrix} l \\ p \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The solution is $l = 1$; $p = 3$.

46. The system of equations is given in matrix form by

$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 24 \end{bmatrix}$. This cannot be solved using matrix inversion since $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ does not have an inverse.

47. The system of equations is given in matrix form by

$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$. To solve using matrix inversion:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 & -1 & -7 \\ -2 & 1 & 4 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -24 \\ 15 \\ 9 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \\ 3 \end{bmatrix}$$

The solution is $x = -8$; $y = 5$; $z = 3$.

48. The system of equations is given in matrix form by

$$\begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}.$$ To solve using matrix inversion:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ 3 & -1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 7 & 11 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The solution is $x = y = z = 1$.

49. The system of equations (after simplifying) is given in matrix form by

$$\begin{bmatrix} 6 & 3 & 2 \\ 6 & 4 & 3 \\ 20 & 15 & 12 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -10 \end{bmatrix}.$$ To solve using matrix inversion:

$$\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 6 & 3 & 2 \\ 6 & 4 & 3 \\ 20 & 15 & 12 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -2 \\ -10 \end{bmatrix} = \begin{bmatrix} 3/2 & -3 & 1/2 \\ -6 & 16 & -3 \\ 5 & -15 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

The solution is $l = 1;\ \ m = -2;\ \ n = 0.$

50. The system of equations is given in matrix form by

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & -2 & 0 \\ 8 & -1 & -4 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \\ 10 \end{bmatrix}.$$ This cannot be solved using matrix inversion

since $\begin{bmatrix} 2 & 3 & -4 \\ 3 & -2 & 0 \\ 8 & -1 & -4 \end{bmatrix}$ does not have an inverse.

51. The system of equations is given in matrix form by

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}.$$ To solve using matrix inversion:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 2 & 0 & -4 \\ -1 & 3 & 5 \\ -3 & 3 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The solution is $x = y = 1; \;\; z = 2.$

52. $\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}^{-1}=\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$

(a) $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix}=\begin{bmatrix} 70 \\ -40 \end{bmatrix}$ The solution is $x = 70; \;\; y = -40.$

(b) $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -8 \\ 22 \end{bmatrix}=\begin{bmatrix} 134 \\ -82 \end{bmatrix}$ The solution is $x = 134; \;\; y = -82.$

(c) $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}=\begin{bmatrix} 1.9 \\ -1.1 \end{bmatrix}$ The solution is $x = 1.9; \;\; y = -1.1.$

(d) $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix}=\begin{bmatrix} 25 \\ -15 \end{bmatrix}$ The solution is $x = 25; \;\; y = -15.$

53. $\begin{bmatrix} 2 & 4 & 0 \\ 3 & 2 & 1 \\ 5 & -3 & 7 \end{bmatrix}^{-1}=\frac{1}{30}\begin{bmatrix} -17 & 28 & -4 \\ 16 & -14 & 2 \\ 19 & -26 & 8 \end{bmatrix}$

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{30}\begin{bmatrix} -17 & 28 & -4 \\ 16 & -14 & 2 \\ 19 & -26 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 15 \end{bmatrix}=\frac{1}{30}\begin{bmatrix} 130 \\ -50 \\ -50 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 13 \\ -5 \\ -5 \end{bmatrix}$

53. (a) (continued)

The solution is $x = \frac{13}{3}$; $y = -\frac{5}{3}$; $z = -\frac{5}{3}$.

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -17 & 28 & -4 \\ 16 & -14 & 2 \\ 19 & -26 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \\ 15 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 113 \\ -34 \\ -34 \end{bmatrix}$

The solution is $x = \frac{113}{30}$; $y = -\frac{34}{30}$ $z = -\frac{31}{30}$.

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -17 & 28 & -4 \\ 16 & -14 & 2 \\ 19 & -26 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \\ 15 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 158 \\ -64 \\ -76 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 79 \\ -32 \\ -38 \end{bmatrix}$

The solution is $x = \frac{79}{15}$ $y = -\frac{32}{15}$; $z = -\frac{38}{15}$.

(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -17 & 28 & -4 \\ 16 & -14 & 2 \\ 19 & -26 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 14 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 123 \\ -54 \\ -51 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 41 \\ -18 \\ -17 \end{bmatrix}$

The solution is $x = \frac{41}{10}$; $y = -\frac{18}{10}$; $z = -\frac{17}{10}$.

54. (a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$;

$\mathbf{A}^{-2} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -4 \\ -8 & 3 \end{bmatrix}$;

$\mathbf{A}^{-3} = \mathbf{A}^{-2}\,\mathbf{A}^{-1} = \begin{bmatrix} 11 & -4 \\ -8 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -15 \\ -30 & 11 \end{bmatrix}$.

54. (continued)

(b) $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$;

$$\mathbf{A}^{-2} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -20 \\ -4 & 9 \end{bmatrix};$$

$$\mathbf{A}^{-3} = \mathbf{A}^{-2}\,\mathbf{A}^{-1} = \begin{bmatrix} 9 & -20 \\ -4 & 9 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -38 & 85 \\ 17 & -38 \end{bmatrix}.$$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$;

$$\mathbf{A}^{-2} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 19 & -5 \\ -15 & 4 \end{bmatrix};$$

$$\mathbf{A}^{-3} = \mathbf{A}^{-2}\,\mathbf{A}^{-1} = \begin{bmatrix} 19 & -5 \\ -15 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 91 & -24 \\ -71 & 19 \end{bmatrix};$$

(d) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\mathbf{A}^{-2} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{A}^{-3} = \mathbf{A}^{-2}\,\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

54. (continued)

(e) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\mathbf{A}^{-2} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{A}^{-3} = \mathbf{A}^{-2}\,\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -6 & -9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

55. Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.
But $\mathbf{AB} = \mathbf{0}$ for every matrix $\mathbf{B}$. Hence $\mathbf{A}^{-1}$ does not exist.

56. Using the result of problem 20 in section 1.3: If $\mathbf{D}$ is a diagonal matrix, then for any square matrix $\mathbf{A}$ for which the product is defined, the ith row of $\mathbf{DA}$ is the ith row of $\mathbf{A}$ multiplied by d_{ii}.
Let $\mathbf{D}$ be a diagonal matrix in which $d_{ii} = 0$ for some i. If $\mathbf{D}^{-1}$ exists, then $\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$. But the ith row of $\mathbf{D}\mathbf{D}^{-1}$ is a row of zeros (from problem 20, section 1.3). Hence $\mathbf{D}^{-1}$ does not exist.

57. If $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{AA} = \mathbf{I}$ and by definition of the inverse, $\mathbf{A} = \mathbf{A}^{-1}$.

58. Since $\mathbf{A}$ is symmetric, $\mathbf{A} = \mathbf{A}^T$.
$(\mathbf{B}\mathbf{A}^{-1})^T(\mathbf{A}^{-1}\mathbf{B}^T)^{-1} = [(\mathbf{A}^{-1})^T\mathbf{B}^T][(\mathbf{B}^T)^{-1}(\mathbf{A}^{-1})^{-1}] = [(\mathbf{A}^T)^{-1}\mathbf{B}^T][(\mathbf{B}^T)^{-1}(\mathbf{A})] =$

$= [\mathbf{A}^{-1}\mathbf{B}^T][(\mathbf{B}^T)^{-1}(\mathbf{A})] = [\mathbf{A}^{-1}(\mathbf{B}^T(\mathbf{B}^T)^{-1}\mathbf{A}] = \mathbf{A}^{-1}\mathbf{I}\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

59. Let $\mathbf{B} = (\mathbf{A}^{-1})^{-1}$. Then $(\mathbf{A}^{-1})\mathbf{B} = \mathbf{I}$ and $\mathbf{A}[(\mathbf{A}^{-1})\mathbf{B}] = \mathbf{AI} = \mathbf{A}$.
But $\mathbf{A}[(\mathbf{A}^{-1})\mathbf{B}] = (\mathbf{A}\mathbf{A}^{-1})\mathbf{B} = \mathbf{IB} = \mathbf{B}$. Hence $\mathbf{B} = \mathbf{A}$, i.e., $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

60. $((1/\lambda)\mathbf{A}^{-1})(\lambda\mathbf{A}) = (1/\lambda \cdot \lambda)(\mathbf{A}^{-1})(\mathbf{A}) = \mathbf{I}$. Similarly, $(\lambda\mathbf{A})((1/\lambda)\mathbf{A}^{-1}) = \mathbf{I}$.

Hence $(\lambda\mathbf{A})^{-1} = ((1/\lambda)\mathbf{A}^{-1})$.

61. $(\mathbf{ABC})^{-1} = ((\mathbf{AB})(\mathbf{C}))^{-1} = \mathbf{C}^{-1}(\mathbf{AB})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

62. Let **A** be an upper triangular matrix which is non-singular. (Note: this means all diagonal entries are non-zero.) To show $\mathbf{A}^{-1}$ is upper triangular let $\mathbf{B} = \mathbf{A}^{-1}$. Then $\mathbf{AB} = \mathbf{C} = \mathbf{I}$. Consider b_{21}. Want to show $b_{21} = 0$. Since **A** is upper triangular, $a_{k1} = 0$ for $k = 2, \cdots, n$. Since $\mathbf{C} = \mathbf{I}$, $c_{21} = 0$. Further, $0 = c_{21} = b_{21}a_{11} + \underbrace{b_{22}a_{21} + b_{23}a_{31} + \cdots + b_{2n}a_{n1}}_{0}$. And since $a_{11} \neq 0$, this means $b_{21} = 0$.

In a similar way one can show that $b_{i1} = 0$ for $i = 3, \cdots, n$. Now consider b_{32}.

$0 = c_{32} = \underbrace{b_{31}a_{12}}_{0} + b_{32}a_{22} + \underbrace{b_{33}a_{32} + b_{34}a_{42} + \cdots + b_{3n}a_{n2}}_{0}$. Since $a_{22} \neq 0$, this means $b_{32} = 0$. Continuing in this way it can be seen that $b_{ij} = 0$ for $i > j$. Hence $\mathbf{B} = \mathbf{A}^{-1}$ is an upper triangular matrix.

63. Proof of Theorem 1 (c):

Let **E** be an elementary matrix that adds to the cth row a constant k times the rth row. Let **F** be the elementary matrix obtained by replacing k by $-k$ in the matrix **E**. An example is given by the following matrices:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & -k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}.$$

63. (continued)

Note that $e_{cc} = 1;\ \ e_{cr} = k$ and $f_{cc} = 1;\ \ f_{cr} = -k.$ Let **B** = **EF**, i.e., $b_{ij} = \sum_{k=1}^{n} e_{ik} f_{kj}$.

For $i = j$: $b_{ii} = \underbrace{e_{i1} f_{1i} + e_{i2} f_{2i} + \cdots +}_{0} \underbrace{e_{ii} f_{ii}}_{1} + \underbrace{\cdots + e_{in} f_{ni}}_{0} = 1$.

For $i \neq j;\ \ i \neq c;\ \ j \neq r$: $b_{ij} = \underbrace{e_{i1} f_{1j} + e_{i2} f_{2j} + \cdots +}_{0} \underbrace{e_{ii} f_{ij}}_{1\bullet 0} + \underbrace{\cdots}_{0} + \underbrace{e_{ij} f_{jj}}_{0\bullet 1} + \underbrace{\cdots + e_{in} f_{nj}}_{0} = 0$.

$$b_{cr} = \underbrace{e_{c1} f_{1r} + e_{c2} f_{2r} + \cdots +}_{0} \underbrace{e_{cc} f_{cr}}_{k} + \underbrace{\cdots}_{0} + \underbrace{e_{cr} f_{rr}}_{-k} + \underbrace{\cdots + e_{cn} f_{nr}}_{0} = 0 .$$

Hence **B** = **I** so that **EF** = **I**, i.e., **F** = $\mathbf{E}^{-1}$.

64.

$$\begin{bmatrix} A_1 & & & & O \\ & A_2 & & & \\ & & A_3 & & \\ & & & \ddots & \\ O & & & & A_k \end{bmatrix} \begin{bmatrix} A_1^{-1} & & & & \\ & A_2^{-1} & & & \\ & & A_3^{-1} & & \\ & & & \ddots & \\ & & & & A_n^{-1} \end{bmatrix} = \begin{bmatrix} I & & & & O \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ O & & & & I \end{bmatrix} = \mathbf{I}$$

Section 1.7

1. $\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$;

$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$;

$\mathbf{Ly} = \mathbf{b}$; $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$; $\alpha = 1,\ \ 3\alpha + \beta = -6,\ \ \beta = -9.$

$\mathbf{Ux} = \mathbf{y}$; $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \end{bmatrix}$; $x + y = 1,\ \ y = -9;\ \ x = 10$. $\mathbf{x} = \begin{bmatrix} 10 \\ -9 \end{bmatrix}$.

2. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 1 \\ 0 & 3/2 \end{bmatrix}$; $\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 0 & 3/2 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$;

$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3/2 \end{bmatrix}$;

$\mathbf{Ly} = \mathbf{b}$; $\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$; $\alpha = 11,\ \ 1/2\,\alpha + \beta = -2,\ \ \beta = -\frac{15}{2}$

$\mathbf{Ux} = \mathbf{y}$; $\begin{bmatrix} 2 & 1 \\ 0 & 3/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -15/2 \end{bmatrix}$; $2x + y = 11,\ \ \frac{3}{2}y = -\frac{15}{2},\ \ y = -5,\ \ x = 8.$

$\mathbf{x} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

3. $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \xrightarrow{-\frac{5}{8}R_1 + R_2} \begin{bmatrix} 8 & 3 \\ 0 & 1/8 \end{bmatrix}$; $\mathbf{U} = \begin{bmatrix} 8 & 3 \\ 0 & 1/8 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 5/8 & 1 \end{bmatrix}$;

$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} 1 & 0 \\ 5/8 & 1 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 0 & 1/8 \end{bmatrix}$;

$\mathbf{Ly} = \mathbf{b}$; $\begin{bmatrix} 1 & 0 \\ 5/8 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 625 \\ 550 \end{bmatrix}$; $\alpha = 625,\ \ \frac{5}{8}\alpha + \beta = 550,\ \ \beta = \frac{1275}{8}$

3. (continued)

Ux = **y**;

$$\begin{bmatrix} 8 & 3 \\ 0 & 1/8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 625 \\ 1275/8 \end{bmatrix}; \;\; 8x + 3y = 625, \;\; \frac{1}{8}y = \frac{1275}{8}, y = 1275, \;\; x = -400 .$$

$$\mathbf{x} = \begin{bmatrix} -400 \\ 1275 \end{bmatrix}.$$

4. $$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}; \;\; \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \;\; \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix};$$

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}; \;\; \alpha = 4, \;\; \alpha + \beta = 1, \;\; -\beta + \gamma = -1, \;\; \beta = -3, \;\; \gamma = -4$$

Ux = **y**;

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -4 \end{bmatrix}; \;\; x + y = 4, \;\; -y + z = -3, \;\; 2z = -4; \;\; z = -2, \;\; y = 1, \;\; x = 3$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$$

5. $\begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 1 \\ 2 & -2 & 3 \end{bmatrix} \xrightarrow[2R_1+R_3]{R_1+R_2} \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

$\mathbf{U} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$; $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$;

$\mathbf{Ly} = \mathbf{b}$;

$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$; $\alpha = -1,\ -\alpha + \beta = -2,\ -2\alpha - 2\beta + \gamma = 3;\ \alpha = -1,\ \beta = -3,\ \gamma = -5$

$\mathbf{Ux} = \mathbf{y}$;

$\begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix}$; $-x + 2y = -1,\ -y + z = -3,\ 5z = -5;\ z = -1,\ y = 2,\ x = 5$

$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.

6. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ -2 & -1 & -2 \end{bmatrix} \xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{U} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$;

6. (continued)

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 10 \\ -40 \\ 0 \end{bmatrix}; \;\; \alpha = 10, \;\; 2\alpha + \beta = -40, \;\; -\alpha + \gamma = 0; \;\; \alpha = 10, \;\; \beta = -60, \;\; \gamma = 10$$

Ux = **y**;

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -60 \\ 10 \end{bmatrix}; \;\; 2x + y + 3z = 10, \;\; -y - 6z = -60, \;\; z = 10; \;\; x = -10, \;\; y = 0, \;\; z = 10$$

$$\mathbf{x} = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}.$$

7. $$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 3 & 9 & 2 \end{bmatrix} \xrightarrow[-R_1+R_3]{-\frac{4}{3}R_1+R_2} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -8/3 & -1/3 \\ 0 & 7 & 1 \end{bmatrix} \xrightarrow{\frac{21}{8}R_2+R_3} \begin{bmatrix} 3 & 2 & 1 \\ 0 & -8/3 & -1/3 \\ 0 & 0 & 1/8 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -8/3 & -1/3 \\ 0 & 0 & 1/8 \end{bmatrix}; \;\; \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ 1 & -21/8 & 1 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ 1 & -21/8 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 0 & -8/3 & -1/3 \\ 0 & 0 & 1/8 \end{bmatrix}$$

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ 1 & -21/8 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 50 \\ 80 \\ 20 \end{bmatrix}; \;\; \alpha = 50, \;\; \frac{4}{3}\alpha + \beta = 80, \;\; \alpha - \frac{21}{8}\beta + \gamma = 20; \;\; \alpha = 50, \;\; \beta = \frac{40}{3}, \;\; \gamma = 5$$

7. (continued)

Ux = **y;**

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -8/3 & -1/3 \\ 0 & 0 & 1/8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 50 \\ 40/3 \\ 5 \end{bmatrix}; \ 3x+2y+z=50, \ -\frac{8}{3}y-\frac{1}{3}z=\frac{40}{3}, \ \frac{1}{8}z=5;$$

$x=10, \ \ y=-10, \ \ z=40.$

$$\mathbf{x} = \begin{bmatrix} 10 \\ -10 \\ 40 \end{bmatrix}.$$

8. $$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow[R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{\frac{3}{4}R_2+R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 17/4 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 17/4 \end{bmatrix}; \ \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/4 & 1 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 17/4 \end{bmatrix}$$

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/4 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 80 \\ 159 \\ -75 \end{bmatrix}; \ \alpha=80, \ 2\alpha+\beta=159, \ \alpha-\frac{3}{4}\beta+\gamma=-75;$$

$\alpha=80, \ \ \beta=-1, \ \ \gamma=\frac{17}{4}.$

8. (continued)

Ux = **y**;

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 17/4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 80 \\ -1 \\ 17/4 \end{bmatrix}; \quad x + 2y - z = 80, \ -4y + 3z = -1, \ \frac{17}{4}z = \frac{17}{4}; \quad x = 79, \quad y = 1, \quad z = 1.$$

$$\mathbf{x} = \begin{bmatrix} 79 \\ 1 \\ 1 \end{bmatrix}.$$

9. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{U} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{Ly} = \mathbf{b}; \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}; \quad \alpha = 8, \quad \beta = -1, \quad \gamma = 5.$$

Ux = **y**;

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}; \quad x + 2y - z = 8, \quad 2y + z = -1, \quad z = 5; \quad x = 19, \quad y = -3, \quad z = 5.$$

$$\mathbf{x} = \begin{bmatrix} 19 \\ -3 \\ 5 \end{bmatrix}.$$

10. $$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[-R_1 + R_2]{-3R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

10. (continued)

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \tfrac{1}{2} & 1 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \tfrac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

$\mathbf{Ly} = \mathbf{b}$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \tfrac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}; \quad \alpha = 2, \quad 3\alpha + \beta = 4, \quad \alpha + \frac{1}{2}\beta + \gamma = 2; \quad \alpha = 2, \quad \beta = -2, \quad \gamma = 1.$$

$\mathbf{Ux} = \mathbf{y}$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}; \quad x = 2, \quad 2y = -2, \quad 2z = 1; \quad x = 2, \quad y = -1, \quad z = \frac{1}{2}.$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ \tfrac{1}{2} \end{bmatrix}.$$

11.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[-R_2+R_4]{-R_2+R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{-2R_3+R_4}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix};$$

11. (continued)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix};$$

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -2 \\ -2 \end{bmatrix}; \quad \alpha = 4, \quad \alpha + \beta = -3, \quad \alpha + \beta + \gamma = -2, \quad \beta + 2\gamma + \delta = -2;$$

$\alpha = 4, \quad \beta = -7, \quad \gamma = 1, \quad \delta = 3.$

Ux = **y**;

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \\ 3 \end{bmatrix}; \quad x + z + w = 4, \quad y - z = -7, \quad z - w = 1, \quad 3w = 3;$$

$x = 1, \quad y = -5, \quad z = 2, \quad w = 1.$

$$\mathbf{x} = \begin{bmatrix} 1 \\ -5 \\ 2 \\ 1 \end{bmatrix}.$$

12.

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 4 & 2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1+R_2} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{2}{7}R_2+R_4} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5/7 & 8/7 \end{bmatrix} \xrightarrow{-\frac{5}{7}R_3+R_4}$$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3/7 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3/7 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2/7 & 5/7 & 1 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2/7 & 5/7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3/7 \end{bmatrix};$$

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2/7 & 5/7 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1,000 \\ 200 \\ 100 \\ 100 \end{bmatrix}; \quad \alpha = 1,000, \quad \frac{1}{2}\alpha + \beta = 200, \quad \gamma = 100, \quad \frac{2}{7}\beta + \frac{5}{7}\gamma + \delta = 100;$$

$$\alpha = 1,000, \quad \beta = -300, \quad \gamma = 100, \quad \delta = \frac{800}{7}.$$

12. (continued)

Ux = **y**;

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3/7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1{,}000 \\ -300 \\ 100 \\ 800/7 \end{bmatrix}; \quad 2x + y - z + 3w = 1{,}000, \quad \frac{7}{2}y + \frac{5}{2}z - \frac{1}{2}w = -300, \quad -z + w = 1$$

$$x = \frac{800}{3}, \quad y = -\frac{500}{3}, \quad z = \frac{500}{3}, \quad w = \frac{800}{3}.$$

$$\mathbf{x} = \begin{bmatrix} 800/3 \\ -500/3 \\ 500/3 \\ 800/3 \end{bmatrix}$$

13.

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_1+R_4]{-R_1+R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{2R_3+R_4} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix};$$

13. (continued)

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 10 \\ 10 \end{bmatrix}; \quad \alpha = 30, \quad \alpha + \beta = 30, \quad \alpha + \beta + \gamma = 10, \quad -\beta - 2\gamma + \delta = 10;$$

$\alpha = 30, \quad \beta = 0, \quad \gamma = -20, \quad \delta - 30.$

Ux = **y**;

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ -20 \\ -30 \end{bmatrix}; \quad x + 2y + z + w = 30, \quad -y + z = 0, \quad -z + w = -20, \quad 3w = -30;$$

$x = 10, \quad y = 10, \quad z = 10, \quad w = -10.$

$$\mathbf{x} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ -10 \end{bmatrix}.$$

14.

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 6 \\ -4 & 3 & 1 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix} \xrightarrow[-\frac{1}{2}R_1 + R_4]{\substack{-R_1 + R_2 \\ 2R_1 + R_3}} \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 3 & 5 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{-\frac{3}{2}R_2 + R_3} \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_3 + R_4}$$

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

14. (continued)

$$\mathbf{U} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 3 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 3/2 & 1 & 0 \\ 1/2 & 0 & 1/4 & 1 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 3/2 & 1 & 0 \\ 1/2 & 0 & 1/4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 3 \end{bmatrix};$$

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 3/2 & 1 & 0 \\ 1/2 & 0 & 1/4 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 9 \\ 4 \end{bmatrix}; \quad \alpha = -2, \quad \alpha + \beta = 4, \quad -2\alpha + \frac{3}{2}\beta + \gamma = 9, \quad \frac{1}{2}\alpha + \frac{1}{4}\gamma + \delta = 4;$$

$\alpha = -2, \quad \beta = 6, \quad \gamma = -4, \quad \delta = 6.$

Ux = **y**;

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -4 \\ 6 \end{bmatrix}; \quad 2x + 2z = -2, \quad 2y - 2z + 6w = 6, \quad 8z - 8w = -4, \quad 3w = 6;$$

$x = -\frac{5}{2}, \quad y = -\frac{3}{2}, \quad z = \frac{3}{2}, \quad w = 2.$

$$\mathbf{x} = \begin{bmatrix} -5/2 \\ -3/2 \\ 3/2 \\ 2 \end{bmatrix}.$$

15. (a) $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \xrightarrow{2R_1+R_2} \begin{bmatrix} -1 & 2 \\ 0 & 7 \end{bmatrix}$; $\mathbf{U} = \begin{bmatrix} -1 & 2 \\ 0 & 7 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$;

$\mathbf{Ly} = \mathbf{b}$; $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$; $\alpha = -9,\ -2\alpha + \beta = 4$; $\alpha = -9,\ \ \beta = -14.$

$\mathbf{Ux} = \mathbf{y}$; $\begin{bmatrix} -1 & 2 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ -14 \end{bmatrix}$; $-x + 2y = -9,\ \ 7y = -14$; $x = 5,\ \ y = -2.$

The solution is $x = 5,\ \ y = -2.$

(b) $\mathbf{Ly} = \mathbf{b}$; $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\alpha = 1,\ -2\alpha + \beta = -1$; $\alpha = 1,\ \ \beta = 1.$

$\mathbf{Ux} = \mathbf{y}$; $\begin{bmatrix} -1 & 2 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $-x + 2y = 1,\ \ 7y = 1$; $x = -\frac{5}{7},\ \ y = \frac{1}{7}.$

The solution is $x = -\frac{5}{7},\ \ y = \frac{1}{7}.$

16. (a) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \\ 2 & 7 & -4 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$;

$\mathbf{U} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$;

16. (a) (continued)

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -6 \end{bmatrix};\ \alpha = -1,\ \ 2\alpha + \beta = 4,\ \ 2\alpha - \beta + \gamma = -6;\ \ \alpha = -1,\ \ \beta = 6,\ \ \gamma = 2.$$

Ux = **y**; $$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix};\ x + 3y - z = -1;\ -y + 3z = 6,\ \ z = 2;$$

$$x = 1,\ \ y = 0,\ \ z = 2.$$

The solution is $x = 1,\ \ y = 0,\ \ z = 2.$

(b) **Ly** = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix};\ \alpha = 10,\ \ 2\alpha + \beta = 10,\ \ 2\alpha - \beta + \gamma = 10;\ \ \alpha = 10,\ \ \beta = -10,\ \ \gamma = -20.$$

Ux = **y**; $$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ -20 \end{bmatrix};\ x + 3y - z = 10;\ -y + 3z = -10,\ \ z = -20;$$

$$x = 140,\ \ y = -50,\ \ z = -20.$$

The solution is $x = 140,\ \ y = -50,\ \ z = -20.$

17. From problem 4:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix};\quad \mathbf{U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix};$$

17. (continued)
(a)

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -4 \end{bmatrix};\ \alpha = 5,\ \ \alpha+\beta = 7,\ \ -\beta+\gamma = -4;\ \ \alpha = 5,\ \ \beta = 2,\ \ \gamma = -2.$$

Ux = **y**;

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix};\ x+y = 5,\ \ -y+z = 2,\ \ 2z = -2;\ \ x = 8,\ \ y = -3,\ \ z = -1.$$

The solution is $\mathbf{x} = \begin{bmatrix} 8 \\ -3 \\ -1 \end{bmatrix}$.

(b) **Ly** = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix};\ \alpha = 2,\ \ \alpha+\beta = 2,\ \ -\beta+\gamma = 0;\ \ \alpha = 2,\ \ \beta = 0,\ \ \gamma = 0.$$

Ux = **y**;

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix};\ x+y = 2,\ \ -y+z = 0,\ \ 2z = 0;\ \ x = 2,\ \ y = 0,\ \ z = 0.$$

The solution is $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

17. (continued)

(c) **Ly** = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 40 \\ 50 \\ 20 \end{bmatrix}; \quad \alpha = 40, \;\; \alpha + \beta = 50, \;\; -\beta + \gamma = 20; \;\; \alpha = 40, \;\; \beta = 10, \;\; \gamma = 30.$$

Ux = **y**;

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \\ 30 \end{bmatrix}; \quad x + y = 40, \;\; -y + z = 10, \;\; 2z = 30; \;\; x = 35, \;\; y = 5, \;\; z = 15.$$

The solution is $\mathbf{x} = \begin{bmatrix} 35 \\ 5 \\ 15 \end{bmatrix}$.

(d) **Ly** = **b**;

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}; \quad \alpha = 1, \;\; \alpha + \beta = 1, \;\; -\beta + \gamma = 3; \;\; \alpha = 1, \;\; \beta = 0, \;\; \gamma = 3.$$

Ux = **y**;

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}; \quad x + y = 1, \;\; -y + z = 0, \;\; 2z = 3; \;\; x = -\frac{1}{2}, \;\; y = \frac{3}{2}, \;\; z = \frac{3}{2}.$$

The solution is $\mathbf{x} = \begin{bmatrix} -0.5 \\ 1.5 \\ 1.5 \end{bmatrix}$.

18. From problem 13:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}; \quad \mathbf{U} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix};$$

(a) $\mathbf{Ly} = \mathbf{b}$;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \quad \alpha = -1, \quad \alpha + \beta = 1, \quad \alpha + \beta + \gamma = 1, \quad -\beta - 2\gamma + \delta = 1;$$

$\alpha = -1, \quad \beta = 2, \quad \gamma = 0, \quad \delta = 3.$

$\mathbf{Ux} = \mathbf{y}$;

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}; \quad x + 2y + z + w = -1, \quad -y + z = 2, \quad -z + w = 0, \quad 3w = 3;$$

$x = -1, \quad y = -1, \quad z = 1, \quad w = 1.$

The solution is $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

(b) $\mathbf{Ly} = \mathbf{b}$;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \alpha = 0, \quad \alpha + \beta = 0, \quad \alpha + \beta + \gamma = 0, \quad -\beta - 2\gamma + \delta = 0;$$

$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0.$

18. (b)(continued)

Ux = **y**;

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad x + 2y + z + w = 0, \; -y + z = 0, \; -z + w = 0, \; 3w = 0;$$

$x = 0, \; y = 0, \; z = 0, \; w = 0.$

The solution is $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) **Ly** = **b**;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 190 \\ 130 \\ 160 \\ 60 \end{bmatrix}; \quad \alpha = 190, \; \alpha + \beta = 130, \; \alpha + \beta + \gamma = 160, \; -\beta - 2\gamma + \delta = 60;$$

$\alpha = 190, \; \beta = -60, \; \gamma = 30, \; \delta = 60.$

Ux = **y**;

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 190 \\ -60 \\ 30 \\ 60 \end{bmatrix}; \quad x + 2y + z + w = 190, \; -y + z = -60, \; -z + w = 30, \; 3w = 60;$$

$x = 80, \; y = 50, \; z = -10, \; w = 20.$

The solution is $\mathbf{x} = \begin{bmatrix} 80 \\ 50 \\ -10 \\ 20 \end{bmatrix}$.

18. (continued)
(d)

Ly = **b**;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \quad \alpha = 1, \quad \alpha + \beta = 1, \quad \alpha + \beta + \gamma = 1, \quad -\beta - 2\gamma + \delta = 1;$$

$\alpha = 1, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 1.$

Ux = **y**;

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad x + 2y + z + w = 1, \quad -y + z = 0, \quad -z + w = 0, \quad 3w = 1;$$

$x = -\frac{1}{3}, \quad y = \frac{1}{3}, \quad z = \frac{1}{3}, \quad w = \frac{1}{3}.$

The solution is $\mathbf{x} = \begin{bmatrix} -1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

19. $\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix}$. The **LU** decomposition procedure cannot be used since there is a 0 in the 1-1 position. Interchanging the first and second equations (rows):

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 2 & -1 & -1 \end{bmatrix} \xrightarrow{-2R_1 + R_3} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & -3 & -7 \end{bmatrix} \xrightarrow{\frac{3}{2}R_2 + R_3} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -11/2 \end{bmatrix}$$

19. (continued)

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -11/2 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -3/2 & 1 \end{bmatrix}.$$

20. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -3 \\ 0 & -1 & 1 \end{bmatrix}$ The **LU** decomposition procedure cannot be used since there is a 0 in the 2-2 position. Interchanging the first and third equations (rows):

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2+R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & 3/2 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}.$$

21. (a) $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 9 \end{bmatrix}$ The **LU** decomposition procedure cannot be used since there is a 0 in the 1-1 position. The same problem occurs even if the rows are interchanged.

21. (continued)

(b) $\mathbf{LU} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 9 \end{bmatrix} = \mathbf{A}$

(c) $\mathbf{LU} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 9 \end{bmatrix} = \mathbf{A}$

(d) Because **A** is singular.