1 Linear Equations in Linear Algebra

1.1 SOLUTIONS

Notes: The key exercises are 7 (or 11 or 12), 19–22, and 25. For brevity, the symbols R1, R2,…, stand for row 1 (or equation 1), row 2 (or equation 2), and so on. Additional notes are at the end of the section.

1. $\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$

Replace R2 by R2 + (2)R1 and obtain: $\quad \begin{aligned} x_1 + 5x_2 &= 7 \\ 3x_2 &= 9 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$

Scale R2 by 1/3: $\quad \begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$

Replace R1 by R1 + (–5)R2: $\quad \begin{aligned} x_1 &= -8 \\ x_2 &= 3 \end{aligned}$ $\quad \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$

The solution is $(x_1, x_2) = (-8, 3)$, or simply $(-8, 3)$.

2. $\begin{aligned} 3x_1 + 6x_2 &= -3 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$ $\quad \begin{bmatrix} 3 & 6 & -3 \\ 5 & 7 & 10 \end{bmatrix}$

Scale R1 by 1/3 and obtain: $\quad \begin{aligned} x_1 + 2x_2 &= -1 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -1 \\ 5 & 7 & 10 \end{bmatrix}$

Replace R2 by R2 + (–5)R1: $\quad \begin{aligned} x_1 + 2x_2 &= -1 \\ -3x_2 &= 15 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 15 \end{bmatrix}$

Scale R2 by –1/3: $\quad \begin{aligned} x_1 + 2x_2 &= -1 \\ x_2 &= -5 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix}$

Replace R1 by R1 + (–2)R2: $\quad \begin{aligned} x_1 &= 9 \\ x_2 &= -5 \end{aligned}$ $\quad \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -5 \end{bmatrix}$

The solution is $(x_1, x_2) = (9, -5)$, or simply $(9, -5)$.

3. The point of intersection satisfies the system of two linear equations:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$$

Replace R2 by R2 + (–1)R1 and obtain:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ -3x_2 &= -3 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \end{bmatrix}$$

Scale R2 by –1/3:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

Replace R1 by R1 + (–2)R2:

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (2, 1)$.

4. The point of intersection satisfies the system of two linear equations:

$$\begin{aligned} x_1 + 2x_2 &= -13 \\ 3x_1 - 2x_2 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -13 \\ 3 & -2 & 1 \end{bmatrix}$$

Replace R2 by R2 + (–3)R1 and obtain:

$$\begin{aligned} x_1 + 2x_2 &= -13 \\ -8x_2 &= 40 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -13 \\ 0 & -8 & 40 \end{bmatrix}$$

Scale R2 by –1/8:

$$\begin{aligned} x_1 + 2x_2 &= -13 \\ x_2 &= -5 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & -13 \\ 0 & 1 & -5 \end{bmatrix}$$

Replace R1 by R1 + (–2)R2:

$$\begin{aligned} x_1 &= -3 \\ x_2 &= -5 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (-3, -5)$.

5. The system is already in "triangular" form. The fourth equation is $x_4 = -5$, and the other equations do not contain the variable x_4. The next two steps should be to use the variable x_3 in the third equation to eliminate that variable from the first two equations. In matrix notation, that means to replace R2 by its sum with –4 times R3, and then replace R1 by its sum with 3 times R3.

6. One more step will put the system in triangular form. Replace R4 by its sum with –4 times R3, which produces $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -7 & 14 \end{bmatrix}$. After that, the next step is to scale the fourth row by –1/7.

7. Ordinarily, the next step would be to interchange R3 and R4, to put a 1 in the third row and third column. But in this case, the third row of the augmented matrix corresponds to the equation $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$, or simply, $0 = 1$. A system containing this condition has no solution. Further row operations are unnecessary once an equation such as $0 = 1$ is evident. The solution set is empty.

8. The standard row operations are:

$$\begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution set contains one solution: (0, 0, 0, 0).

9. The system has already been reduced to triangular form. Begin by replacing R3 by R3 + (3)R4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 1 & 0 & 14 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Next, replace R2 by R2 + (2)R3. Finally, replace R1 by R1 + R2:

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 21 \\ 0 & 0 & 1 & 0 & 14 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 16 \\ 0 & 1 & 0 & 0 & 21 \\ 0 & 0 & 1 & 0 & 14 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The solution set contains one solution: (16, 21, 14, 4).

10. The system has already been reduced to triangular form. Use the 1 in the fourth row to change the 3 and –2 above it to zeros. That is, replace R2 by R2 + (-3)R4 and replace R1 by R1 + (2)R4. For the final step, replace R1 by R1 + (-3)R2.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & -7 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & -11 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -47 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

The solution set contains one solution: (–47, 12, 2, –2).

11. First, swap R1 and R2. Then replace R3 by R3 + (–2)R1. Finally, replace R3 by R3 + (1)R2.

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = –2 if there were a solution. The solution set is empty.

12. Replace R2 by R2 + (–2)R1 and replace R3 by R3 + (2)R1. Finally, replace R3 by R3 + (3)R2.

$$\begin{bmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -9 & 15 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 5 if there were a solution. The solution set is empty.

13. $\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is (5, 3, –1).

14. $\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. The solution is (2, –1, 2).

15. First, replace R3 by R3 + (1)R1, then replace R4 by R4 + (1)R2, and finally replace R4 by R4 + (–1)R3.

$$\begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ -1 & 6 & 1 & 5 & 3 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & -1 & 5 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 & 5 \\ 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = –8 if there were a solution.

16. First replace R4 by R4 + (3/2)R1 and replace R4 by R4 + (–2/3)R2. (One could also scale R1 and R2 before adding to R4, but the arithmetic is rather easy keeping R1 and R2 unchanged.) Finally, replace R4 by R4 + (–1)R3.

$$\begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ -3 & 2 & 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 2 & 3 & -5 & -10 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -5 & -10 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -4 & -10 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & -9 & -9 \end{bmatrix}$$

The system is now in triangular form and has a solution. In fact, using the argument from Example 2, one can see that the solution is unique.

17. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 2 & 3 & -1 \\ 6 & 5 & 0 \\ 2 & -5 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & -8 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

The third equation, $0 = 2$, shows that the system is inconsistent, so the three lines have no point in common.

18. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{bmatrix}$$

The system is consistent, and using the argument from Example 2, there is only one solution. So the three planes have only one point in common.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$ Write c for $6 - 3h$. If $c = 0$, that is, if $h = 2$, then the system has no solution, because 0 cannot equal –4. Otherwise, when $h \neq 2$, the system has a solution.

20. $\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -5 \\ 0 & -8-2h & 16 \end{bmatrix}$ Write c for $-8 - 2h$. If $c = 0$, that is, if $h = –4$, then the system has no solution, because 0 cannot equal 16. Otherwise, when $h \neq –4$, the system has a solution.

21. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & h-12 & 0 \end{bmatrix}$ Write c for $h - 12$. Then the second equation $cx_2 = 0$ has a solution for every value of c. So the system is consistent for all h.

22. $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} -4 & 12 & h \\ 0 & 0 & -3+\frac{h}{2} \end{bmatrix}$ The system is consistent if and only if $-3+\frac{h}{2} = 0$, that is, if and only if $h = 6$.

23. **a**. True. See the remarks following the box titled *Elementary Row Operations.*

b. False. A 5×6 matrix has five rows.

c. False. The description applies to a single solution. The solution *set* consists of all possible solutions. Only in special cases does the solution set consist of exactly one solution. Mark a statement True only if the statement is *always* true.

d. True. See the box before Example 2.

24. **a**. False. The definition of *row equivalent* requires that there exist a sequence of row operations that transforms one matrix into the other.

b. True. See the box preceding the subsection titled *Existence and Uniqueness Questions.*

c. False. The definition of *equivalent systems* is in the second paragraph after equation (2).

d. True. By definition, a consistent system has *at least one* solution.

25. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$

Let b denote the number $k + 2g + h$. Then the third equation represented by the augmented matrix above is $0 = b$. This equation is possible if and only if b is zero. So the original system has a solution if and only if $k + 2g + h = 0$.

26. Row reduce the augmented matrix for the given system:

$$\begin{bmatrix} 2 & 4 & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & f/2 \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & f/2 \\ 0 & d-2c & g-c(f/2) \end{bmatrix}$$

This shows that $d - 2c$ must be nonzero, since f and g are arbitary. Otherwise, for some choices of f and g the second row would correspond to an equation of the form $0 = b$, where b is nonzero. Thus $d \neq 2c$.

27. Row reduce the augmented matrix for the given system. Scale the first row by $1/a$, which is possible since a is nonzero. Then replace R2 by R2 + $(-c)$R1.

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ 0 & d-c(b/a) & g-c(f/a) \end{bmatrix}$$

The quantity $d - c(b/a)$ must be nonzero, in order for the system to be consistent when the quantity $g - c(f/a)$ is nonzero (which can certainly happen). The condition that $d - c(b/a) \neq 0$ can also be written as $ad - bc \neq 0$, or $ad \neq bc$.

28. A basic principle of this section is that row operations do not affect the solution set of a linear system. Begin with a simple augmented matrix for which the solution is obviously (3, –2, –1), and then perform any elementary row operations to produce other augmented matrices. Here are three examples. The fact that they are all row equivalent proves that they all have the solution set (3, –2, –1).

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 4 \\ 2 & 0 & 1 & 5 \end{bmatrix}$$

29. Swap R1 and R3; swap R1 and R3.

30. Multiply R3 by –1/5; multiply R3 by –5.

31. Replace R3 by R3 + (–4)R1; replace R3 by R3 + (4)R1.

32. Replace R3 by R3 + (–4)R2; replace R3 by R3 + (4)R2.

33. The first equation was given. The others are:

$$T_2 = (T_1 + 20 + 40 + T_3)/4, \quad \text{or} \quad 4T_2 - T_1 - T_3 = 60$$

$$T_3 = (T_4 + T_2 + 40 + 30)/4, \quad \text{or} \quad 4T_3 - T_4 - T_2 = 70$$

$$T_4 = (10 + T_1 + T_3 + 30)/4, \quad \text{or} \quad 4T_4 - T_1 - T_3 = 40$$

Rearranging,

$$\begin{aligned} 4T_1 - T_2 - T_4 &= 30 \\ -T_1 + 4T_2 - T_3 &= 60 \\ -T_2 + 4T_3 - T_4 &= 70 \\ -T_1 - T_3 + 4T_4 &= 40 \end{aligned}$$

34. Begin by interchanging R1 and R4, then create zeros in the first column:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 4 & 40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 4 & 40 \\ 0 & 4 & 0 & -4 & 20 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix}$$

Scale R1 by –1 and R2 by 1/4, create zeros in the second column, and replace R4 by R4 + R3:

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & -4 & 14 & 195 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 12 & 270 \end{bmatrix}$$

Scale R4 by 1/12, use R4 to create zeros in column 4, and then scale R3 by 1/4:

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 4 & 0 & 120 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix}$$

The last step is to replace R1 by R1 + (–1)R3:

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 20.0 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30.0 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix}.$$ The solution is (20, 27.5, 30, 22.5).

Notes: The *Study Guide* includes a "Mathematical Note" about statements, "If … , then … ."

This early in the course, students typically use single row operations to reduce a matrix. As a result, even the small grid for Exercise 34 leads to about 80 multiplications or additions (not counting operations with zero). This exercise should give students an appreciation for matrix programs such as MATLAB. Exercise 14 in Section 1.10 returns to this problem and states the solution in case students have not already solved the system of equations. Exercise 31 in Section 2.5 uses this same type of problem in connection with an LU factorization.

For instructors who wish to use technology in the course, the *Study Guide* provides boxed MATLAB notes at the ends of many sections. Parallel notes for Maple, Mathematica, and the TI-83+/84+/89 calculators appear in separate appendices at the end of the *Study Guide*. The MATLAB box for Section 1.1 describes how to access the data that is available for all numerical exercises in the text. This feature has the ability to save students time if they regularly have their matrix program at hand when studying linear algebra. The MATLAB box also explains the basic commands **`replace`**, **`swap`**, and **`scale`**. These commands are included in the text data sets, available from the text web site, www.pearsonhighered.com/lay.

1.2 SOLUTIONS

Notes: The key exercises are 1–20 and 23–28. (Students should work at least four or five from Exercises 7–14, in preparation for Section 1.5.)

1. Reduced echelon form: a and b. Echelon form: d. Not echelon: c.

2. Reduced echelon form: a. Echelon form: b and d. Not echelon: c.

3. $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -3 & -12 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & 0 & -8 \\ 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Pivot cols 1 and 3. $\begin{bmatrix} \textcircled{1} & 2 & 4 & 8 \\ 2 & 4 & \textcircled{6} & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -18 \\ 0 & 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & -2 \\ 0 & 0 & \textcircled{1} & 2 \end{bmatrix}$. Pivot cols 1, 2, and 3 $\begin{bmatrix} \textcircled{1} & 2 & 4 & 5 \\ 2 & \textcircled{4} & 5 & 4 \\ 4 & 5 & \textcircled{4} & 2 \end{bmatrix}$

5. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & -5 \\ 0 & 0 & \textcircled{1} & 3 \end{bmatrix}$

Corresponding system of equations: $\begin{aligned} \textcircled{x_1} + 3x_2 &= -5 \\ \textcircled{x_3} &= 3 \end{aligned}$

The basic variables (corresponding to the pivot positions) are x_1 and x_3. The remaining variable x_2 is free. Solve for the basic variables in terms of the free variable. The general solution is

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Note: Exercise 7 is paired with Exercise 10.

8. $\begin{bmatrix} 1 & -3 & 0 & -5 \\ -3 & 7 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & -2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 4 \\ 0 & \textcircled{1} & 0 & 3 \end{bmatrix}$

Corresponding system of equations: $\begin{aligned} \textcircled{x_1} \quad\quad &= 4 \\ \textcircled{x_2} \quad\quad &= 3 \end{aligned}$

The basic variables (corresponding to the pivot positions) are x_1 and x_2. The remaining variable x_3 is free. Solve for the basic variables in terms of the free variable. In this particular problem, the basic variables do not depend on the value of the free variable.

General solution: $\begin{cases} x_1 = 4 \\ x_2 = 3 \\ x_3 \text{ is free} \end{cases}$

Note: A common error in Exercise 8 is to assume that x_3 is zero. To avoid this, identify the basic variables first. Any remaining variables are *free*. (This type of computation will arise in Chapter 5.)

9. $\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -6 \\ 0 & 1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -2 & 3 \\ 0 & \textcircled{1} & -2 & 3 \end{bmatrix}$

Corresponding system: $\begin{aligned} \textcircled{x_1} \quad &- 2x_3 = 3 \\ \textcircled{x_2} \quad &- 2x_3 = 3 \end{aligned}$

Basic variables: x_1, x_2; free variable: x_3. General solution: $\begin{cases} x_1 = 3 + 2x_3 \\ x_2 = 3 + 2x_3 \\ x_3 \text{ is free} \end{cases}$

10. $\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 0 & -7 & 14 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 0 & 2 \\ 0 & 0 & \textcircled{1} & -2 \end{bmatrix}$

Corresponding system: $\begin{aligned} \textcircled{x_1} \quad &- 2x_2 \quad\quad = 2 \\ \textcircled{x_3} \quad\quad\quad &= -2 \end{aligned}$

Basic variables: x_1, x_3; free variable: x_2. General solution: $\begin{cases} x_1 = 2 + 2x_2 \\ x_2 \text{ is free} \\ x_3 = -2 \end{cases}$

11. $\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2/3 & 4/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system: $\begin{aligned} \textcircled{x_1} \quad &- \frac{2}{3}x_2 + \frac{4}{3}x_3 = 0 \\ &0 = 0 \\ &0 = 0 \end{aligned}$

Basic variable: x_1; free variables x_2, x_3. General solution: $\begin{cases} x_1 = \frac{2}{3}x_2 - \frac{4}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

12. Since the bottom row of the matrix is equivalent to the equation 0 = 1, the system has no solutions.

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & -3 & 5 \\ 0 & \textcircled{1} & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{aligned} \textcircled{x_1} \quad\quad\quad - 3x_5 &= 5 \\ \textcircled{x_2} \quad\quad - 4x_5 &= 1 \\ \textcircled{x_4} + 9x_5 &= 4 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution: $\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

Note: The *Study Guide* discusses the common mistake $x_3 = 0$.

14. $\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{aligned} \textcircled{x_1} \quad - 5x_3 \quad\quad\quad &= 3 \\ \textcircled{x_2} + 4x_3 - x_4 \quad\quad &= 6 \\ \textcircled{x_5} &= 0 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1, x_2, x_5; free variables: x_3, x_4. General solution: $\begin{cases} x_1 = 3 + 5x_3 \\ x_2 = 6 - 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

15. **a**. The system is consistent. There are many solutions because x_3 is a free variable.

b. The system is consistent. There are many solutions because x_1 is a free variable.

16. **a**. The system is inconsistent. (The rightmost column of the augmented matrix is a pivot column).

b. The system is consistent. There are many solutions because x_2 is a free variable.

17. $\begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -1 & 4 \\ 0 & \textcircled{1} & h+8 \end{bmatrix}$ The system has a solution for all values of h since the augmented column cannot be a pivot column.

18. $\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 1 \\ 0 & 3h+6 & -h-2 \end{bmatrix}$ If $3h + 6$ is zero, that is, if $h = -2$, then the system has a solution, because 0 equals 0. When $h \neq -2$, the system has a solution since the augmented column cannot be a pivot column. Thus the system has a solution for all values of h.

19. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

a. When $h = 2$ and $k \neq 8$, the augmented column is a pivot column, and the system is inconsistent.

b. When $h \neq 2$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 2$ and $k = 8$, the system is consistent and has many solutions.

20. $\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix}$

a. When $h = -6$ and $k \neq 2$, the system is inconsistent, because the augmented column is a pivot column.

b. When $h \neq -6$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = -6$ and $k = 2$, the system is consistent and has many solutions.

21. **a**. False. See Theorem 1.

b. False. See the second paragraph of the section.

c. True. Basic variables are defined after equation (4).

d. True. This statement is at the beginning of *Parametric Descriptions of Solution Sets*.

e. False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.

22. **a**. True. See Theorem 1.

b. False. See Theorem 2.

c. False. See the beginning of the subsection *Pivot Positions*. The pivot positions in a matrix are determined completely by the positions of the leading entries in the nonzero rows of any echelon form obtained from the matrix.

d. True. See the paragraph just before Example 4.

e. False. The existence of at least one solution is not related to the presence or absence of free variables. If the system is inconsistent, the solution set is empty. See the solution of Practice Problem 2.

23. Since there are four pivots (one in each row), the augmented matrix must reduce to the form

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & a \\ 0 & \textcircled{1} & 0 & 0 & b \\ 0 & 0 & \textcircled{1} & 0 & c \\ 0 & 0 & 0 & \textcircled{1} & d \end{bmatrix} \quad \text{and so} \quad \begin{aligned} \textcircled{x_1} &= a \\ \textcircled{x_2} &= b \\ \textcircled{x_3} &= c \\ \textcircled{x_4} &= d \end{aligned}$$

No matter what the values of a, b, c, and d, the solution exists and is unique.

24. The system is consistent because there is not a pivot in column 5, which means that there is not a row of the form [0 0 0 0 1]. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has a solution.

25. If the coefficient matrix has a pivot position in every row, then there is a pivot position in the bottom row, and there is no room for a pivot in the augmented column. So, the system is consistent, by Theorem 2.

26. Since the coefficient matrix has three pivot columns, there is a pivot in each row of the coefficient matrix. Thus the augmented matrix will not have a row of the form [0 0 0 0 0 1], and the system is consistent.

27. *"If a linear system is consistent, then the solution is unique if and only if every column in the coefficient matrix is a pivot column; otherwise there are infinitely many solutions."*

This statement is true because the free variables correspond to *nonpivot* columns of the coefficient matrix. The columns are all pivot columns if and only if there are no free variables. And there are no free variables if and only if the solution is unique, by Theorem 2.

28. Every column in the augmented matrix *except the rightmost column* is a pivot column, and the rightmost column is *not* a pivot column.

29. An underdetermined system always has more variables than equations. There cannot be more basic variables than there are equations, so there must be at least one free variable. Such a variable may be assigned infinitely many different values. If the system is consistent, each different value of a free variable will produce a different solution, and the system will not have a unique solution. If the system is inconsistent, it will not have any solution.

30. Example:
$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 2x_3 &= 5 \end{aligned}$$

31. Yes, a system of linear equations with more equations than unknowns can be consistent.

Example (in which $x_1 = x_2 = 1$):
$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 0 \\ 3x_1 + 2x_2 &= 5 \end{aligned}$$

32. According to the numerical note in Section 1.2, when $n = 20$ the reduction to echelon form takes about $2(20)^3/3 \approx 5{,}333$ flops, while further reduction to reduced echelon form needs at most $(20)^2 = 400$ flops. Of the total flops, the "backward phase" is about 400/5733 = .07 or about 7%. When $n = 200$, the estimates are $2(200)^3/3 \approx 5{,}333{,}333$ flops for the reduction to echelon form and $(200)^2 = 40{,}000$ flops for the backward phase. The fraction associated with the backward phase is about $(4\times10^4)/(5.3\times10^6) = .007$, or about .7%.

33. For a quadratic polynomial $p(t) = a_0 + a_1t + a_2t^2$ to exactly fit the data (1, 6), (2, 15), and (3, 28), the coefficients a_0, a_1, a_2 must satisfy the systems of equations given in the text. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 28 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 2 & 8 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & 3 \\ 0 & 0 & \textcircled{1} & 2 \end{bmatrix}$$

The polynomial is $p(t) = 1 + 3t + 2t^2$.

34. **[M]** The system of equations to be solved is:

$$\begin{aligned}
a_0 + a_1\cdot 0 + a_2\cdot 0^2 + a_3\cdot 0^3 + a_4\cdot 0^4 + a_5\cdot 0^5 &= 0 \\
a_0 + a_1\cdot 2 + a_2\cdot 2^2 + a_3\cdot 2^3 + a_4\cdot 2^4 + a_5\cdot 2^5 &= 2.90 \\
a_0 + a_1\cdot 4 + a_2\cdot 4^2 + a_3\cdot 4^3 + a_4\cdot 4^4 + a_5\cdot 4^5 &= 14.8 \\
a_0 + a_1\cdot 6 + a_2\cdot 6^2 + a_3\cdot 6^3 + a_4\cdot 6^4 + a_5\cdot 6^5 &= 39.6 \\
a_0 + a_1\cdot 8 + a_2\cdot 8^2 + a_3\cdot 8^3 + a_4\cdot 8^4 + a_5\cdot 8^5 &= 74.3 \\
a_0 + a_1\cdot 10 + a_2\cdot 10^2 + a_3\cdot 10^3 + a_4\cdot 10^4 + a_5\cdot 10^5 &= 119
\end{aligned}$$

The unknowns are a_0, a_1, …, a_5. Use technology to compute the reduced echelon of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 1 & 4 & 16 & 64 & 256 & 1024 & 14.8 \\ 1 & 6 & 36 & 216 & 1296 & 7776 & 39.6 \\ 1 & 8 & 64 & 512 & 4096 & 32768 & 74.3 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 & 119 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 24 & 192 & 1248 & 7680 & 30.9 \\ 0 & 0 & 48 & 480 & 4032 & 32640 & 62.7 \\ 0 & 0 & 80 & 960 & 9920 & 99840 & 104.5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 192 & 2688 & 26880 & 8.7 \\ 0 & 0 & 0 & 480 & 7680 & 90240 & 14.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 0 & 384 & 7680 & -6.9 \\ 0 & 0 & 0 & 0 & 1920 & 42240 & -24.5 \end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0&0&0&0&0\\0&2&4&8&16&32&2.9\\0&0&8&48&224&960&9\\0&0&0&48&576&4800&3.9\\0&0&0&0&384&7680&-6.9\\0&0&0&0&0&3840&10\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&0&0&0\\0&2&4&8&16&32&2.9\\0&0&8&48&224&960&9\\0&0&0&48&576&4800&3.9\\0&0&0&0&384&7680&-6.9\\0&0&0&0&0&1&.0026\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0&0&0&0&0\\0&2&4&8&16&0&2.8167\\0&0&8&48&224&0&6.5000\\0&0&0&48&576&0&-8.6000\\0&0&0&0&384&0&-26.900\\0&0&0&0&0&1&.002604\end{bmatrix}\sim\cdots\sim\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&1.7125\\0&0&1&0&0&0&-1.1948\\0&0&0&1&0&0&.6615\\0&0&0&0&1&0&-.0701\\0&0&0&0&0&1&.0026\end{bmatrix}$$

Thus $p(t) = 1.7125t - 1.1948t^2 + .6615t^3 - .0701t^4 + .0026t^5$, and $p(7.5) = 64.6$ hundred lb.

Notes: In Exercise 34, if the coefficients are retained to higher accuracy than shown here, then $p(7.5) = 64.8$. If a polynomial of lower degree is used, the resulting system of equations is overdetermined. The augmented matrix for such a system is the same as the one used to find p, except that at least column 6 is missing. When the augmented matrix is row reduced, the sixth row of the augmented matrix will be entirely zero except for a nonzero entry in the augmented column, indicating that no solution exists.

Exercise 34 requires 25 row operations. It should give students an appreciation for higher-level commands such as **`gauss`** and **`bgauss`**, discussed in Section 1.4 of the *Study Guide*. The command **`ref`** (reduced echelon form) is available, but I recommend postponing that command until Chapter 2.

The *Study Guide* includes a "Mathematical Note" about the phrase, "If and only if," used in Theorem 2.

1.3 SOLUTIONS

Notes: The key exercises are 11–16, 19–22, 25, and 26. A discussion of Exercise 25 will help students understand the notation $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

1. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+(-3)\\2+(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$.

Using the definitions carefully,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+(-2)\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}(-2)(-3)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}, \text{ or, more quickly,}$$

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}. \text{ The intermediate step is often not written.}$$

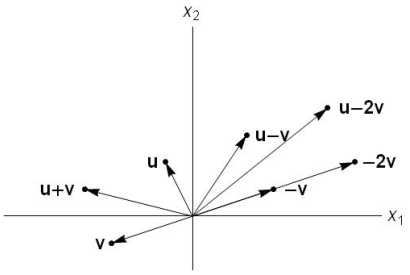
2. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3+2\\2-1\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$.

Using the definitions carefully,

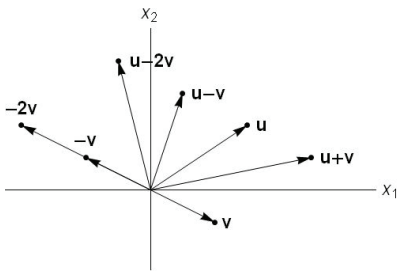
$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+(-2)\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}(-2)(2)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}, \text{ or, more quickly,}$$

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}-2\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}. \text{ The intermediate step is often not written.}$$

3.



4.



5. $x_1\begin{bmatrix}3\\-2\\8\end{bmatrix}+x_2\begin{bmatrix}5\\0\\-9\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix}$, $\begin{bmatrix}3x_1\\-2x_1\\8x_1\end{bmatrix}+\begin{bmatrix}5x_2\\0\\-9x_2\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix}$, $\begin{bmatrix}3x_1+5x_2\\-2x_1\\8x_1-9x_2\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix}$

$$\begin{aligned}3x_1 + 5x_2 &= 2\\ -2x_1 &= -3\\ 8x_1 - 9x_2 &= 8\end{aligned}$$

Usually the intermediate steps are not displayed.

6. $x_1\begin{bmatrix}3\\-2\end{bmatrix}+x_2\begin{bmatrix}7\\3\end{bmatrix}+x_3\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}3x_1\\-2x_1\end{bmatrix}+\begin{bmatrix}7x_2\\3x_2\end{bmatrix}+\begin{bmatrix}-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}3x_1+7x_2-2x_3\\-2x_1+3x_2+x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{aligned}3x_2 + 7x_2 - 2x_3 &= 0\\ -2x_1 + 3x_2 + x_3 &= 0\end{aligned}$$

Usually the intermediate steps are not displayed.

7. See the figure below. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

To write a vector **a** as a linear combination of **u** and **v**, imagine walking from the origin to **a** along the grid "streets" and keep track of how many "blocks" you travel in the **u**-direction and how many in the **v**-direction.

a. To reach **a** from the origin, you might travel 1 unit in the **u**-direction and –2 units in the **v**-direction (that is, 2 units in the negative **v**-direction). Hence $\mathbf{a} = \mathbf{u} - 2\mathbf{v}$.

b. To reach **b** from the origin, travel 2 units in the **u**-direction and –2 units in the **v**-direction. So $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$. Or, use the fact that **b** is 1 unit in the **u**-direction from **a**, so that

$$\mathbf{b} = \mathbf{a} + \mathbf{u} = (\mathbf{u} - 2\mathbf{v}) + \mathbf{u} = 2\mathbf{u} - 2\mathbf{v}$$

c. The vector **c** is –1.5 units from **b** in the **v**-direction, so

$$\mathbf{c} = \mathbf{b} - 1.5\mathbf{v} = (2\mathbf{u} - 2\mathbf{v}) - 1.5\mathbf{v} = 2\mathbf{u} - 3.5\mathbf{v}$$

d. The "map" suggests that you can reach **d** if you travel 3 units in the **u**-direction and –4 units in the **v**-direction. If you prefer to stay on the paths displayed on the map, you might travel from the origin to –3**v**, then move 3 units in the **u**-direction, and finally move –1 unit in the **v**-direction. So

$$\mathbf{d} = -3\mathbf{v} + 3\mathbf{u} - \mathbf{v} = 3\mathbf{u} - 4\mathbf{v}$$

Another solution is

$$\mathbf{d} = \mathbf{b} - 2\mathbf{v} + \mathbf{u} = (2\mathbf{u} - 2\mathbf{v}) - 2\mathbf{v} + \mathbf{u} = 3\mathbf{u} - 4\mathbf{v}$$

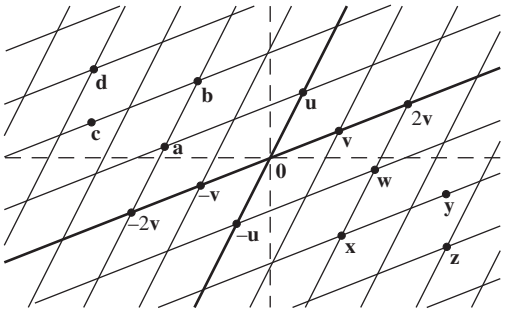


Figure for Exercises 7 and 8

8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, $\mathbf{w} = (-1)\mathbf{u} + 2\mathbf{v}$, or $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So $\mathbf{z} = 4\mathbf{v} - 3\mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$. If you prefer to stay on the paths displayed on the "map," you might travel from the origin to –2**u**, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$$

9.
$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0, \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

Note: The *Study Guide* says, "Check with your instructor whether you need to "show work" on a problem such as Exercise 9."

10.
$$\begin{array}{rcrcrcl} 3x_1 & - & 2x_2 & + & 4x_3 & = & 3 \\ -2x_1 & - & 7x_2 & + & 5x_3 & = & 1 \\ 5x_1 & + & 4x_2 & - & 3x_3 & = & 2 \end{array}, \qquad \begin{bmatrix} 3x_1 - 2x_2 + 4x_3 \\ -2x_1 - 7x_2 + 5x_3 \\ 5x_1 + 4x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 \\ -2x_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ -7x_2 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 5x_3 \\ -3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

11. The question

Is **b** a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ have a solution?

The equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \qquad (*)$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 5 & 2 \\ 0 & \textcircled{1} & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore **b** *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

12. The equation

$$x_1\begin{bmatrix}1\\0\\1\end{bmatrix}+x_2\begin{bmatrix}-2\\3\\-2\end{bmatrix}+x_3\begin{bmatrix}-6\\7\\5\end{bmatrix}=\begin{bmatrix}11\\-5\\9\end{bmatrix} \qquad (*)$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$

has the same solution set as the linear system whose augmented matrix is

$$M=\begin{bmatrix}1&-2&-6&11\\0&3&7&-5\\1&-2&5&9\end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M\sim\begin{bmatrix}\textcircled{1}&-2&-6&11\\0&\textcircled{3}&7&-5\\0&0&\textcircled{11}&-2\end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore $\mathbf{b}$ *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

13. Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. To determine if $\mathbf{b}$ is a linear combination of these columns, use the boxed fact in the subsection *Linear Combinations*. Row reduce the augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ until you reach echelon form:

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]=\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-4&2&3\\0&\textcircled{3}&5&-7\\0&0&0&\textcircled{3}\end{bmatrix}$$

The system for this augmented matrix is inconsistent, so $\mathbf{b}$ is *not* a linear combination of the columns of A.

14. Row reduce the augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ until you reach echelon form:

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]=\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&5&2\\0&\textcircled{1}&4&3\\0&0&0&0\end{bmatrix}.$$

The linear system corresponding to this matrix *has* a solution, so $\mathbf{b}$ is a linear combination of the columns of A.

15. $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}]=\begin{bmatrix}1&-5&3\\3&-8&-5\\-1&2&h\end{bmatrix}\sim\begin{bmatrix}1&-5&3\\0&7&-14\\0&-3&h+3\end{bmatrix}\sim\begin{bmatrix}1&-5&3\\0&1&-2\\0&-3&h+3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-5&3\\0&\textcircled{1}&-2\\0&0&h-3\end{bmatrix}$. The vector $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h-3$ is zero, that is, when $h=3$.

16. $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{y}] = \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & -5+2h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & h \\ 0 & \textcircled{1} & -3 \\ 0 & 0 & 4+2h \end{bmatrix}$. The vector $\mathbf{y}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ when $4 + 2h$ is zero, that is, when $h = -2$.

17. Noninteger weights are acceptable, of course, but some simple choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

18. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

19. By inspection, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is actually just a multiple of $\mathbf{v}_1$. For instance,

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(3/2)\mathbf{v}_1 = (a + 3b/2)\mathbf{v}_1$$

So $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and $\mathbf{0}$.

Note: Exercises 19 and 20 prepare the way for ideas in Sections 1.4 and 1.7.

20. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane in $\mathbf{R}^3$ through the origin, because neither vector in this problem is a multiple of the other.

21. Let $\mathbf{y} = \begin{bmatrix} h \\ k \end{bmatrix}$. Then $[\mathbf{u} \quad \mathbf{v} \quad \mathbf{y}] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & 2 & h \\ 0 & \textcircled{2} & k + h/2 \end{bmatrix}$. This augmented matrix corresponds to a consistent system for all h and k. So $\mathbf{y}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

23. **a**. False. The alternative notation for a (column) vector is (–4, 3), using parentheses and commas.

b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ were on the line through $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and the origin, then $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ would have to be a multiple of $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$, which is not the case.

c. True. See the line displayed just before Example 4.

d. True. See the box that discusses the matrix in (5).

e. False. The statement is often true, but Span{**u**, **v**} is not a plane when **v** is a multiple of **u**, or when **u** is the zero vector.

24. **a**. False. Span{**u**, **v**} can be a plane.

b. True. See the beginning of the subsection *Vectors in* $\mathbf{R}^n$.

c. True. See the comment following the definition of Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$}.

d. False. $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u} - \mathbf{v} + \mathbf{v} = \mathbf{u}$.

e. False. Setting all the weights equal to zero results in a legitimate linear combination of a set of vectors.

25. **a**. There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and **b** is not one of them.

b. There are infinitely many vectors in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if **b** is in W, use the method of Exercise 13.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -4 & 4 \\ 0 & \textcircled{3} & -2 & 1 \\ 0 & 0 & \textcircled{-1} & 2 \end{bmatrix}$$

The system for this augmented matrix is consistent, so **b** is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$}.

26. **a**. $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}] = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

No, **b** is not a linear combination of the columns of A, that is, **b** is not in W.

b. The second column of A is in W because $\mathbf{a}_2 = 0 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_2 + 0 \cdot \mathbf{a}_3$.

27. **a**. $5\mathbf{v}_1$ is the output of 5 days' operation of mine #1.

b. The total output is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, so x_1 and x_2 should satisfy $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 240 \\ 2824 \end{bmatrix}$.

c. **[M]** Reduce the augmented matrix $\begin{bmatrix} 30 & 40 & 240 \\ 600 & 380 & 2824 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.73 \\ 0 & 1 & 4.70 \end{bmatrix}$.

Operate mine #1 for 1.73 days and mine #2 for 4.70 days. (This is an approximate solution.)

28. **a**. The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1 \begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2 \begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1 \begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2 \begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix} = \begin{bmatrix} 162 \\ 23,610 \\ 1,623 \end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix} \sim \begin{bmatrix} 1.000 & 0 & 3.900 \\ 0 & 1.000 & 1.800 \\ 0 & 0 & 0 \end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

29. The total mass is 4 + 2 + 3 + 5 = 14. So $\mathbf{v} = (4\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 5\mathbf{v}_4)/14$. That is,

$$\mathbf{v} = \frac{1}{14}\left(4\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 8-8+12+5 \\ -8+4+0-30 \\ 16+6-6+0 \end{bmatrix} = \begin{bmatrix} 17/14 \\ -17/7 \\ 8/7 \end{bmatrix} \approx \begin{bmatrix} 1.214 \\ -2.429 \\ 1.143 \end{bmatrix}$$

30. Let m be the total mass of the system. By definition,

$$\mathbf{v} = \frac{1}{m}(m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k) = \frac{m_1}{m}\mathbf{v}_1 + \cdots + \frac{m_k}{m}\mathbf{v}_k$$

The second expression displays $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, which shows that $\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

31. **a**. The center of mass is $\frac{1}{3}\left(1\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} 8 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 10/3 \\ 2 \end{bmatrix}$.

b. The total mass of the new system is 9 grams. The three masses added, w_1, w_2, and w_3, satisfy the equation

$$\frac{1}{9}\left((w_1+1)\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (w_2+1)\cdot\begin{bmatrix} 8 \\ 1 \end{bmatrix} + (w_3+1)\cdot\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

which can be rearranged to

$$(w_1+1)\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (w_2+1)\cdot\begin{bmatrix} 8 \\ 1 \end{bmatrix} + (w_3+1)\cdot\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \end{bmatrix}$$

and

$$w_1\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2\cdot\begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3\cdot\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

The condition $w_1 + w_2 + w_3 = 6$ and the vector equation above combine to produce a system of three equations whose augmented matrix is shown below, along with a sequence of row operations:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & 2 & 8 \\ 1 & 1 & 4 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & 2 & 8 \\ 0 & 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3.5 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3.5 \\ 0 & 1 & 0 & .5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Answer: Add 3.5 g at (0, 1), add .5 g at (8, 1), and add 2 g at (2, 4).

Extra problem: Ignore the mass of the plate, and distribute 6 gm at the three vertices to make the center of mass at (2, 2). Answer: Place 3 g at (0, 1), 1 g at (8, 1), and 2 g at (2, 4).

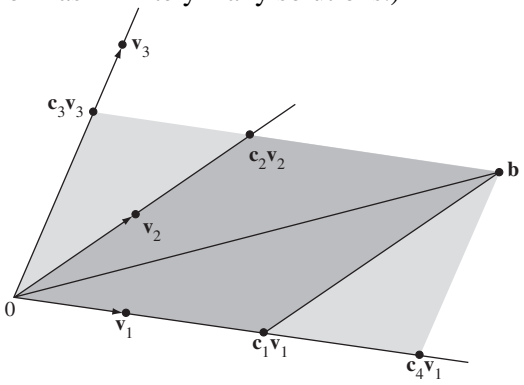
32. See the parallelograms drawn on the figure from the text that accompanies this exercise. Here c_1, c_2, c_3, and c_4 are suitable scalars. The darker parallelogram shows that **b** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\cdot\mathbf{v}_3 = \mathbf{b}$$

The larger parallelogram shows that **b** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, that is,

$$c_4\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

So the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ has at least two solutions, not just one solution. (In fact, the equation has infinitely many solutions.)



33. **a**. For $j = 1,\ldots, n$, the jth entry of $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is $(u_j + v_j) + w_j$. By associativity of addition in **R**, this entry equals $u_j + (v_j + w_j)$, which is the jth entry of $\mathbf{u} + (\mathbf{v} + \mathbf{w})$. By definition of equality of vectors, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

b. For any scalar c, the jth entry of $c(\mathbf{u} + \mathbf{v})$ is $c(u_j + v_j)$, and the jth entry of $c\mathbf{u} + c\mathbf{v}$ is $cu_j + cv_j$ (by definition of scalar multiplication and vector addition). These entries are equal, by a distributive law in **R**. So $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

34. **a**. For $j = 1,\ldots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of **R**. By vector equality,

$$\mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = \mathbf{0}.$$

b. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in **R** are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.

Note: When an exercise in this section involves a vector equation, the corresponding technology data (in the data files on the web) is usually presented as a set of (column) vectors. To use MATLAB or other technology, a student must first construct an augmented matrix from these vectors. The MATLAB note in the *Study Guide* describes how to do this. The appendices in the *Study Guide* give corresponding information about Maple, Mathematica, and the TI calculators.

1.4 SOLUTIONS

Notes: Key exercises are 1–20, 27, 28, 31 and 32. Exercises 29, 30, 33, and 34 are harder. Exercise 34 anticipates the Invertible Matrix Theorem but is not used in the proof of that theorem.

1. The matrix-vector $A\mathbf{x}$ is not defined because the number of columns (2) in the 3×2 matrix A does not match the number of entries (3) in the vector $\mathbf{x}$.

2. The matrix-vector $A\mathbf{x}$ is not defined because the number of columns (1) in the 3×1 matrix A does not match the number of entries (2) in the vector $\mathbf{x}$.

3. $A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot(-2)+2\cdot 3 \\ (-3)\cdot(-2)+1\cdot 3 \\ 1\cdot(-2)+6\cdot 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$$

4. $A\mathbf{x} = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\cdot\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1\cdot\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6-4 \\ 3+4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+3\cdot 2+(-4)\cdot 1 \\ 3\cdot 1+2\cdot 2+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

5. On the left side of the matrix equation, use the entries in the vector $\mathbf{x}$ as the weights in a linear combination of the columns of the matrix A:

$$2\cdot\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 1\cdot\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 1\cdot\begin{bmatrix} -3 \\ 1 \end{bmatrix} - 1\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

6. On the left side of the matrix equation, use the entries in the vector $\mathbf{x}$ as the weights in a linear combination of the columns of the matrix A:

$$-3\cdot\begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5\cdot\begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

7. The left side of the equation is a linear combination of three vectors. Write the matrix A whose columns are those three vectors, and create a variable vector $\mathbf{x}$ with three entries:

$$A = \begin{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} & \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} & \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \text{ Thus the equation } A\mathbf{x} = \mathbf{b} \text{ is}$$

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

For your information: The unique solution of this equation is (5, 7, 3). Finding the solution by hand would be time-consuming.

Note: The skill of writing a vector equation as a matrix equation will be important for both theory and application throughout the text. See also Exercises 27 and 28.

8. The left side of the equation is a linear combination of four vectors. Write the matrix A whose columns are those four vectors, and create a variable vector with four entries:

$$A = \begin{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} & \begin{bmatrix} -1 \\ 5 \end{bmatrix} & \begin{bmatrix} -4 \\ 3 \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}. \text{ Then the equation } A\mathbf{z} = \mathbf{b}$$

is $\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$.

For your information: One solution is (8, 7, 1, 3). The general solution is $z_1 = 37/6 + (17/6)z_3 - (1/3)z_4$, $z_2 = 22/3 + (5/3)z_3 - (2/3)z_4$, with z_3 and z_4 free.

9. The system has the same solution set as the vector equation

$$x_1 \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 5 & 1 & -3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

10. The system has the same solution set as the vector equation

$$x_1 \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 4 & -1 \\ 5 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$$

11. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & -11 \\ 0 & \textcircled{1} & 0 & 3 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$$

The solution is $\begin{cases} x_1 = -11 \\ x_2 = 3 \\ x_3 = 0 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$.

12. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -8 & 8 & -8 \\ 0 & 2 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & -4 \\ 0 & \textcircled{1} & 0 & 4 \\ 0 & 0 & \textcircled{1} & 3 \end{bmatrix}$$

The solution is $\begin{cases} x_1 = -4 \\ x_2 = 4 \\ x_3 = 3 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$.

13. The vector $\mathbf{u}$ is in the plane spanned by the columns of A if and only if $\mathbf{u}$ is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. (See the box preceding Example 3 in Section 1.4.) To study this equation, reduce the augmented matrix $[A \quad \mathbf{u}]$

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & 4 \\ 0 & \textcircled{8} & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so $\mathbf{u}$ is in the plane spanned by the columns of A.

For your information: The unique solution of $A\mathbf{x} = \mathbf{u}$ is (5/2, 3/2).

14. Reduce the augmented matrix $[A \quad \mathbf{u}]$ to echelon form:

$$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 2 & 5 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & 0 & 4 \\ 0 & \textcircled{1} & -1 & -1 \\ 0 & 0 & 0 & \textcircled{-3} \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has no solution, so $\mathbf{u}$ is not in the subset spanned by the columns of A.

15. The augmented matrix for $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} \textcircled{3} & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{bmatrix}$. This shows that the equation $A\mathbf{x} = \mathbf{b}$ is not consistent when $3b_1 + b_2$ is nonzero. The set of $\mathbf{b}$ for which the equation *is* consistent is a line through the origin–the set of all points (b_1, b_2) satisfying $b_2 = -3b_1$.

16. Row reduce the augmented matrix $[A \quad \mathbf{b}]$: $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 4b_1 + (7/2)(b_2 + 2b_1) \end{bmatrix} = \begin{bmatrix} \textcircled{1} & -2 & -1 & b_1 \\ 0 & \textcircled{-2} & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & 3b_1 + (7/2)b_2 + b_3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $3b_1 + (7/2)b_2 + b_3 = 0$, or $6b_1 + 7b_2 + 2b_3 = 0$. The set of such $\mathbf{b}$ is a plane through the origin in $\mathbf{R}^3$.

17. Row reduction shows that only three rows of A contain a pivot position:

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & 3 \\ 0 & \textcircled{2} & -1 & 4 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of A contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution for each $\mathbf{b}$ in $\mathbf{R}^4$.

18. Row reduction shows that only three rows of B contain a pivot position:

$$B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 4 & 1 & 2 \\ 0 & \textcircled{1} & 3 & -4 \\ 0 & 0 & 0 & \textcircled{15} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of B contains a pivot position, Theorem 4 in Section 1.4 shows that not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of B. The columns of B certainly do *not* span $\mathbf{R}^3$, because each column of B is in $\mathbf{R}^4$, not $\mathbf{R}^3$. (This question was asked to alert students to a fairly common misconception among students who are just learning about spanning.)

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do *not* span $\mathbf{R}^4$.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution for each $\mathbf{y}$ in $\mathbf{R}^4$, and the columns of A do *not* span $\mathbf{R}^4$.

21. Row reduce the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbf{R}^4$.

Note: Some students may realize that row operations are not needed, and thereby discover the principle covered in Exercises 31 and 32.

22. Row reduce the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & 9 & -6 \\ 0 & \textcircled{-3} & -2 \\ 0 & 0 & \textcircled{4} \end{bmatrix}$$

The matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbf{R}^3$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{R}^3$.

23. **a**. False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation*.

b. True. See the box before Example 3.

c. False. See the warning following Theorem 4.

d. True. See Example 4.

e. True. See parts (c) and (a) in Theorem 4.

f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

24. **a**. True. This statement is in Theorem 3. However, the statement is true without any "proof" because, by definition, $A\mathbf{x}$ is simply a notation for $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A.

b. True. See the box before Example 3.

c. True. See Example 2.

d. False. In Theorem 4, statement (d) is true if and only if statement (a) is true.

e. True. See Theorem 3.

f. False. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

26. The equation in x_1 and x_2 involves the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and it may be viewed as

$\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$. By definition of a matrix-vector product, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. The stated fact that $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $2\mathbf{u} - 3\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 2$, $x_2 = -3$.

27. The matrix equation can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$, where $c_1 = -3$, $c_2 = 1$, $c_3 = 2$, $c_4 = -1$, $c_5 = 2$, and

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$$

28. Place the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ into the columns of a matrix, say, Q and place the weights x_1, x_2, and x_3 into a vector, say, $\mathbf{x}$. Then the vector equation becomes

$$Q\mathbf{x} = \mathbf{v}, \text{ where } Q = [\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: If your answer is the equation $A\mathbf{x} = \mathbf{b}$, you need to specify what A and $\mathbf{b}$ are.

29. Start with any 3×3 matrix B in echelon form that has three pivot positions. Perform a row operation (a row interchange or a row replacement) that creates a matrix A that is *not* in echelon form. Then A has the desired property. The justification is given by row reducing A to B, in order to display the pivot positions. Since A has a pivot position in every row, the columns of A span $\mathbf{R}^3$, by Theorem 4.

30. Start with any nonzero 3×3 matrix B in echelon form that has fewer than three pivot positions. Perform a row operation that creates a matrix A that is *not* in echelon form. Then A has the desired property. Since A does not have a pivot position in every row, the columns of A do not span $\mathbf{R}^3$, by Theorem 4.

31. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$. Generally, if A is an $m\times n$ matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$.

32. A set of three vectors in $\mathbf{R}^4$ cannot span $\mathbf{R}^4$. Reason: the matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span $\mathbf{R}^4$, by Theorem 4. In general, a set of n vectors in $\mathbf{R}^m$ cannot span $\mathbf{R}^m$ when n is less than m.

33. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}$.

Note: Exercises 33 and 36 are difficult in the context of this section because the focus in Section 1.4 is on existence of solutions, not uniqueness. However, these exercises serve to review ideas from Section 1.2, and they anticipate ideas that will come later.

34. Given $A\mathbf{u}_1 = \mathbf{v}_1$ and $A\mathbf{u}_2 = \mathbf{v}_2$, you are asked to show that the equation $A\mathbf{x} = \mathbf{w}$ has a solution, where $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$. Observe that $\mathbf{w} = A\mathbf{u}_1 + A\mathbf{u}_2$ and use Theorem 5(a) with $\mathbf{u}_1$ and $\mathbf{u}_2$ in place of $\mathbf{u}$ and $\mathbf{v}$, respectively. That is, $\mathbf{w} = A\mathbf{u}_1 + A\mathbf{u}_2 = A(\mathbf{u}_1 + \mathbf{u}_2)$. So the vector $\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2$ is a solution of $\mathbf{w} = A\mathbf{x}$.

35. Suppose that $\mathbf{y}$ and $\mathbf{z}$ satisfy $A\mathbf{y} = \mathbf{z}$. Then $5\mathbf{z} = 5A\mathbf{y}$. By Theorem 5(b), $5A\mathbf{y} = A(5\mathbf{y})$. So $5\mathbf{z} = A(5\mathbf{y})$, which shows that $5\mathbf{y}$ is a solution of $A\mathbf{x} = 5\mathbf{z}$. Thus, the equation $A\mathbf{x} = 5\mathbf{z}$ is consistent.

36. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So the reduced echelon form of A must be $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Now it is clear that A has a pivot position in each *row*. By Theorem 4, the columns of A span $\mathbf{R}^4$.

37. **[M]** $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix} \sim \begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -11/7 & 3/7 & -23/7 \\ 0 & 58/7 & 16/7 & 1/7 \\ 0 & 11 & -3 & 23 \end{bmatrix} \sim \begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -11/7 & 3/7 & -23/7 \\ 0 & 0 & 50/11 & -189/11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

or, approximately $\begin{bmatrix} 7 & 2 & -5 & 8 \\ 0 & -1.57 & .429 & -3.29 \\ 0 & 0 & 4.55 & -17.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, to three significant figures. The original matrix does not have a pivot in every row, so its columns do not span $\mathbf{R}^4$, by Theorem 4.

38. **[M]** $\begin{bmatrix} 4 & -5 & -1 & 8 \\ 3 & -7 & -4 & 2 \\ 5 & -6 & -1 & 4 \\ 9 & 1 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 4 & -5 & -1 & 8 \\ 0 & -13/4 & -13/4 & -4 \\ 0 & 1/4 & 1/4 & -6 \\ 0 & 49/4 & 49/4 & -11 \end{bmatrix} \sim \begin{bmatrix} 4 & -5 & -1 & 8 \\ 0 & -13/4 & -13/4 & -4 \\ 0 & 0 & 0 & -82/13 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

With pivots only in the first three rows, the original matrix has columns that do not span $\mathbf{R}^4$, by Theorem 4.

39. **[M]** $\begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ -8 & 4 & -6 & -10 & -3 \\ -7 & 11 & -5 & -1 & -8 \\ 3 & -1 & 10 & 12 & 12 \end{bmatrix} \sim \begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ 0 & -8/5 & -26/5 & -34/5 & 9/5 \\ 0 & 61/10 & -43/10 & 9/5 & -19/5 \\ 0 & 11/10 & 97/10 & 54/5 & 51/5 \end{bmatrix}$

$$\sim \begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ 0 & -8/5 & -26/5 & -34/5 & 9/5 \\ 0 & 0 & -193/8 & -193/8 & 49/16 \\ 0 & 0 & 49/8 & 49/8 & 183/16 \end{bmatrix} \sim \begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ 0 & -8/5 & -26/5 & -34/5 & 9/5 \\ 0 & 0 & -193/8 & -193/8 & 49/16 \\ 0 & 0 & 0 & 0 & 4715/386 \end{bmatrix}$$

The original matrix has a pivot in every row, so its columns span $\mathbf{R}^4$, by Theorem 4.

40. **[M]** $\begin{bmatrix} 5 & 11 & -6 & -7 & 12 \\ -7 & -3 & -4 & 6 & -9 \\ 11 & 5 & 6 & -9 & -3 \\ -3 & 4 & -7 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 5 & 11 & -6 & -7 & 12 \\ 0 & 62/5 & -62/5 & -19/5 & 39/5 \\ 0 & -96/5 & 96/5 & 32/5 & -147/5 \\ 0 & 53/5 & -53/5 & -11/5 & 71/5 \end{bmatrix}$

$$\sim \begin{bmatrix} 5 & 11 & -6 & -7 & 12 \\ 0 & 62/5 & -62/5 & -19/5 & 39/5 \\ 0 & 0 & 0 & 16/31 & -537/31 \\ 0 & 0 & 0 & 65/62 & 467/62 \end{bmatrix} \sim \begin{bmatrix} \textcircled{5} & 11 & -6 & -7 & 12 \\ 0 & \textcircled{62/5} & -62/5 & -19/5 & 39/5 \\ 0 & 0 & 0 & \textcircled{16/31} & -537/31 \\ 0 & 0 & 0 & 0 & \textcircled{1367/62} \end{bmatrix}$$

The original matrix has a pivot in every row, so its columns span $\mathbf{R}^4$, by Theorem 4.

41. **[M]** Examine the calculations in Exercise 39. Notice that the fourth column of the original matrix, say A, is not a pivot column. Let A^o be the matrix formed by deleting column 4 of A, let B be the echelon form obtained from A, and let B^o be the matrix obtained by deleting column 4 of B. The sequence of row operations that reduces A to B also reduces A^o to B^o. Since B^o is in echelon form, it shows that A^o has a pivot position in each row. Therefore, the columns of A^o span $\mathbf{R}^4$.

It is possible to delete column 3 of A instead of column 4. In this case, the fourth column of A becomes a pivot column of A^o, as you can see by looking at what happens when column 3 of B is deleted. For later work, it is desirable to delete a nonpivot column.

Note: Exercises 41 and 42 help to prepare for later work on the column space of a matrix. (See Section 2.9 or 4.6.) The *Study Guide* points out that these exercises depend on the following idea, not explicitly mentioned in the text: when a row operation is performed on a matrix A, the calculations for each new entry depend only on the other entries in the *same column*. If a column of A is removed, forming a new matrix, the absence of this column has no affect on any row-operation calculations for entries in the other columns of A. (The absence of a column might affect the particular *choice* of row operations performed for some purpose, but that is not being considered here.)

42. **[M]** Examine the calculations in Exercise 40. The third column of the original matrix, say A, is not a pivot column. Let A^o be the matrix formed by deleting column 3 of A, let B be the echelon form obtained from A, and let B^o be the matrix obtained by deleting column 3 of B. The sequence of row operations that reduces A to B also reduces A^o to B^o. Since B^o is in echelon form, it shows that A^o has a pivot position in each row. Therefore, the columns of A^o span $\mathbf{R}^4$.

It is possible to delete column 2 of A instead of column 3. (See the remark for Exercise 41.) However, only *one* column can be deleted. If two or more columns were deleted from A, the resulting matrix would have fewer than four columns, so it would have fewer than four pivot positions. In such a case, not every row could contain a pivot position, and the columns of the matrix would not span $\mathbf{R}^4$, by Theorem 4.

Notes: At the end of Section 1.4, the *Study Guide* gives students a method for learning and mastering linear algebra concepts. Specific directions are given for constructing a review sheet that connects the basic definition of "span" with related ideas: equivalent descriptions, theorems, geometric interpretations, special cases, algorithms, and typical computations. I require my students to prepare such a sheet that reflects their choices of material connected with "span", and I make comments on their sheets to help them refine their review. Later, the students use these sheets when studying for exams.

The MATLAB box for Section 1.4 introduces two useful commands **`gauss`** and **`bgauss`** that allow a student to speed up row reduction while still visualizing all the steps involved. The command **`B = gauss(A,1)`** causes MATLAB to find the left-most nonzero entry in row 1 of matrix A, and use that entry as a pivot to create zeros in the entries below, using row replacement operations. The result is a matrix that a student might write next to A as the first stage of row reduction, since there is no need to write a new matrix after each separate row replacement. I use the **`gauss`** command frequently in lectures to obtain an echelon form that provides data for solving various problems. For instance, if a matrix has 5 rows, and if row swaps are not needed, the following commands produce an echelon form of A:

```
B = gauss(A,1),  B = gauss(B,2),  B = gauss(B,3),  B = gauss(B,4)
```

If an interchange is required, I can insert a command such as **`B = swap(B,2,5)`** . The command **`bgauss`** uses the left-most nonzero entry in a row to produce zeros *above* that entry. This command, together with **`scale`**, can change an echelon form into reduced echelon form.

The use of **`gauss`** and **`bgauss`** creates an environment in which students use their computer program the same way they work a problem by hand on an exam. Unless you are able to conduct your exams in a computer laboratory, it may be unwise to give students too early the power to obtain reduced echelon forms with one command—they may have difficulty performing row reduction by hand during an exam. Instructors whose students use a graphic calculator in class each day do not face this problem. In such a case, you may wish to introduce **`rref`** earlier in the course than Chapter 4 (or Section 2.8), which is where I finally allow students to use that command.

1.5 SOLUTIONS

Notes: The geometry helps students understand Span{**u**, **v**}, in preparation for later discussions of subspaces. The parametric vector form of a solution set will be used throughout the text. Figure 6 will appear again in Sections 2.9 and 4.8.

For solving homogeneous systems, the text recommends working with the augmented matrix, although no calculations take place in the augmented column. See the *Study Guide* comments on Exercise 7 that illustrate two common student errors.

All students need the practice of Exercises 1–14. (Assign all odd, all even, or a mixture. If you do not assign Exercise 7, be sure to assign both 8 and 10.) Otherwise, a few students may be unable later to find a basis for a null space or an eigenspace. Exercises 28–36 are important. Exercises 35 and 36 help students later understand how solutions of $A\mathbf{x} = \mathbf{0}$ encode linear dependence relations among the columns of A. Exercises 37–40 are more challenging. Exercise 37 will help students avoid the standard mistake of forgetting that Theorem 6 applies only to a *consistent* equation $A\mathbf{x} = \mathbf{b}$.

1. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -5 & 8 & 0 \\ 0 & \textcircled{-12} & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable x_3 is free, so the system has a nontrivial solution.

2. $$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 3 & 0 \\ 0 & \textcircled{-7} & 2 & 0 \\ 0 & 0 & \textcircled{3} & 0 \end{bmatrix}$$

There is no free variable; the system has only the trivial solution.

3. $\begin{bmatrix} -3 & 4 & -8 & 0 \\ -2 & 5 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & 4 & -8 & 0 \\ 0 & \textcircled{7/3} & 28/3 & 0 \end{bmatrix}$. The variable x_3 is free; the system has nontrivial solutions. An alert student will realize that row operations are unnecessary. With only two equations, there can be at most two basic variables. One variable *must* be free. Refer to Exercise 29 in Section 1.2.

4. $\begin{bmatrix} 5 & -3 & 2 & 0 \\ -3 & -4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & -3 & 2 & 0 \\ 0 & -29/5 & 16/5 & 0 \end{bmatrix}$. The variable x_3 is free; the system has nontrivial solutions. As in Exercise 3, row operations are unnecessary.

5. $\begin{bmatrix} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_3 = 0$
$x_2 + x_3 = 0$. The variable x_3 is free, $x_1 = -x_3$, and $x_2 = -x_3$.
$0 = 0$

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 - x_3 = 0$
$x_2 - x_3 = 0$. The variable x_3 is free, $x_1 = x_3$, and $x_2 = x_3$.
$0 = 0$

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$. $\begin{aligned} x_1 &+ 9x_3 - 8x_4 = 0 \\ x_2 &- 4x_3 + 5x_4 = 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

8. $\begin{bmatrix} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{bmatrix}$. $\begin{aligned} x_1 &- 2x_3 - 7x_4 = 0 \\ x_2 &+ 2x_3 - 4x_4 = 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = 2x_3 + 7x_4$ and $x_2 = -2x_3 + 4x_4$. The general solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 4x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

9. $\begin{bmatrix} 3 & -6 & 6 & 0 \\ -2 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ -2 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$

$$\begin{aligned} \textcircled{x_1} - 2x_2 \qquad &= 0 \\ \textcircled{x_3} &= 0 \end{aligned}.$$

The solution is $x_1 = 2x_2$, $x_3 = 0$, with x_2 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

10. $\begin{bmatrix} -1 & -4 & 0 & -4 & 0 \\ 2 & -8 & 0 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 4 & 0 \\ 2 & -8 & 0 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 4 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} \textcircled{x_1} \qquad + 4x_4 &= 0 \\ \textcircled{x_2} \qquad &= 0 \end{aligned}.$$

The basic variables are x_1 and x_2, so x_3 and x_4 are free. (Note that x_3 is not zero.) Also, $x_1 = -4x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} - 4x_2 \qquad + 5x_6 &= 0 \\ \textcircled{x_3} \qquad - x_6 &= 0 \\ \textcircled{x_5} - 4x_6 &= 0 \\ 0 &= 0 \end{aligned}.$$

The basic variables are x_1, x_3, and x_5. The remaining variables are free. In particular, x_4 is free (and not zero as some may assume). The solution is $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, $x_5 = 4x_6$, with x_2, x_4, and x_6 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_6 \\ 0 \\ x_6 \\ 0 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

Note: The *Study Guide* discusses two mistakes that students often make on this type of problem.

12.

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 3 & 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} - 2x_2 + 3x_3 \qquad + 29x_5 &= 0 \\ \textcircled{x_4} + 4x_5 &= 0 \\ \textcircled{x_6} &= 0 \\ 0 &= 0 \end{aligned}.$$

The basic variables are x_1, x_4, and x_6; the free variables are x_2, x_3, and x_5. The general solution is $x_1 = 2x_2 - 3x_3 - 29x_5$, $x_4 = -4x_5$, and $x_6 = 0$. In parametric vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_3 - 29x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -29x_5 \\ 0 \\ 0 \\ -4x_5 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -29 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

13. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 + 4x_3 \\ -2 - 7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -7x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = \mathbf{p} + x_3\mathbf{q}.$$

$$\uparrow \qquad \uparrow$$

$$\mathbf{p} \qquad \mathbf{q}$$

Geometrically, the solution set is the line through $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$ in the direction of $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$.

14. To write the general solution in parametric vector form, pull out the constant terms that do not involve the free variable:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_4 \\ 3-2x_4 \\ 2+5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 5x_4 \\ -2x_4 \\ 5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ -2 \\ 5 \\ 1 \end{bmatrix} = \mathbf{p} + x_4\mathbf{q}$$

$$\uparrow \qquad \uparrow$$

$$\mathbf{p} \qquad \mathbf{q}$$

The solution set is the line through $\mathbf{p}$ in the direction of $\mathbf{q}$.

15. Solve $x_1 + 5x_2 - 3x_3 = -2$ for the basic variable: $x_1 = -2 - 5x_2 + 3x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-5x_2+3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 + 5x_2 - 3x_3 = 0$ is $x_1 = -5x_2 + 3x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_2+3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

16. Solve $x_1 - 2x_2 + 3x_3 = 4$ for the basic variable: $x_1 = 4 + 2x_2 - 3x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4+2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 - 2x_2 + 3x_3 = 0$ is $x_1 = 2x_2 - 3x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

17. Row reduce the augmented matrix for the system:

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 4 & 8 \\ 0 & -3 & -3 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_3 = 8$. Thus $x_1 = 8 - x_3$, $x_2 = -4 - x_3$, and x_3 is free. In parametric vector form,

$x_2 + x_3 = -4$

$0 = 0$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 - x_3 \\ -4 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 5.

18. Row reduce the augmented matrix for the system:

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 1 & -3 & 13 \\ -1 & 1 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -3 & 3 & 3 \\ 0 & 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 7 \\ 0 & \textcircled{1} & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\textcircled{x_1} - x_3 = 7$. Thus $x_1 = 7 + x_3$, $x_2 = -1 + x_3$, and x_3 is free. In parametric vector form,

$\textcircled{x_2} - x_3 = -1$

$0 = 0$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 + x_3 \\ -1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 6.

19. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\,\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 3 \end{bmatrix}, \text{ or } \begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$$

20. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + t \begin{bmatrix} -7 \\ 6 \end{bmatrix}, \text{ or } \begin{cases} x_1 = 3 - 7t \\ x_2 = -2 + 6t \end{cases}$$

21. The line through **p** and **q** is parallel to **q – p**. So, given $\mathbf{p} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, form $\mathbf{q} - \mathbf{p} = \begin{bmatrix} 4-3 \\ 1-(-3) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and write the line as $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

22. The line through **p** and **q** is parallel to **q – p**. So, given $\mathbf{p} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, form $\mathbf{q} - \mathbf{p} = \begin{bmatrix} 0-(-3) \\ -3-2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and write the line as $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

Note: Exercises 21 and 22 prepare for Exercise 26 in Section 1.8.

23. **a**. True. See the first paragraph of the subsection titled *Homogeneous Linear Systems*.

b. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection entitled *Parametric Vector Form.*

c. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.

d. False. The line goes through **p** parallel to **v**. See the paragraph that precedes Fig. 5.

e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector **p** such that $A\mathbf{p} = \mathbf{b}$.

24. **a**. False. The trivial solution is always a solution to a homogeneous system of linear equations.

b. False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero **x** that satisfies the equation. See the sentence before Example 2.

c. True. See the paragraph following Example 3.

d. True. If the zero vector is a solution, then $\mathbf{b} = A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.

e. True. See Theorem 6.

25. Suppose **p** satisfies $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{p} = \mathbf{b}$. Theorem 6 says that the solution set of $A\mathbf{x} = \mathbf{b}$ equals the set $S = \{\mathbf{w} : \mathbf{w} = \mathbf{p} + \mathbf{v_h}$ for some $\mathbf{v}_h$ such that $A\mathbf{v_h} = \mathbf{0}\}$. There are two things to prove: (a) every vector in S satisfies $A\mathbf{x} = \mathbf{b}$, (b) every vector that satisfies $A\mathbf{x} = \mathbf{b}$ is in S.

a. Let **w** have the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $A\mathbf{v_h} = \mathbf{0}$. Then

$$A\mathbf{w} = A(\mathbf{p} + \mathbf{v_h}) = A\mathbf{p} + A\mathbf{v_h}. \text{ By Theorem 5(a) in section 1.4}$$
$$= \mathbf{b} + \mathbf{0} = \mathbf{b}$$

So every vector of the form $\mathbf{p} + \mathbf{v_h}$ satisfies $A\mathbf{x} = \mathbf{b}$.

b. Now let **w** be any solution of $A\mathbf{x} = \mathbf{b}$, and set $\mathbf{v_h} = \mathbf{w} - \mathbf{p}$. Then

$$A\mathbf{v_h} = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

So $\mathbf{v_h}$ satisfies $A\mathbf{x} = \mathbf{0}$. Thus every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$.

26. When A is the 3×3 zero matrix, *every* **x** in $\mathbf{R}^3$ satisfies $A\mathbf{x} = \mathbf{0}$. So the solution set is all vectors in $\mathbf{R}^3$.

27. (*Geometric argument using Theorem* 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*Proof using free variables.*) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

28. **a**. When A is a 3×3 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence has no nontrivial solution.

b. With three pivot positions, A has a pivot position in each of its three rows. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$. The term "possible" in the exercise means that the only vectors considered in this case are those in $\mathbf{R}^3$, because A has three rows.

29. **a**. When A is a 4×4 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has three basic variables andone free variable. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With only three pivot positions, A cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^4$).

30. **a**. When A is a 2×5 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and three free variables. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b. With two pivot positions and only two rows, A has a pivot position in every row. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^2$).

31. **a**. When A is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbf{R}^3$), by Theorem 4 in Section 1.4.

32. No. If the solution set of $A\mathbf{x} = \mathbf{b}$ contained the origin, then $\mathbf{0}$ would satisfy $A\mathbf{0} = \mathbf{b}$, which is not true since $\mathbf{b}$ is not the zero vector.

33. Look for $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1 + 1\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3 = \mathbf{0}$. That is, construct A so that each row sum (the sum of the entries in a row) is zero.

34. Look for $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ such that $2\cdot\mathbf{a}_1 - 1\cdot\mathbf{a}_2 + 1\cdot\mathbf{a}_3 = \mathbf{0}$. That is, construct A so that subtracting the third column from the second column is twice the first column.

35. Look at $x_1\begin{bmatrix} -1 \\ 7 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 21 \\ -6 \end{bmatrix}$ and notice that the second column is 3 times the first. So suitable values for x_1 and x_2 would be 3 and –1 respectively. (Another pair would be 6 and –2, etc.) Thus $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

36. Inspect how the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are related. The second column is –2/3 times the first. Put another way, $2\mathbf{a}_1 + 3\mathbf{a}_2 = \mathbf{0}$. Thus $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

Note: Exercises 35 and 36 set the stage for the concept of linear dependence.

37. Since the solution set of $A\mathbf{x} = \mathbf{0}$ contains the point (4,1), the vector $\mathbf{x} = (4,1)$ satisfies $A\mathbf{x} = \mathbf{0}$. Write this equation as a vector equation, using $\mathbf{a}_1$ and $\mathbf{a}_2$ for the columns of A:

$$4\cdot\mathbf{a}_1 + 1\cdot\mathbf{a}_2 = \mathbf{0}$$

Then $\mathbf{a}_2 = -4\mathbf{a}_1$. So choose any nonzero vector for the first column of A and multiply that column by -4 to get the second column of A. For example, set $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$.

Finally, the only way the solution set of $A\mathbf{x} = \mathbf{b}$ could *not* be parallel to the line through (1,4) and the origin is for the solution set of $A\mathbf{x} = \mathbf{b}$ to be *empty*. This does not contradict Theorem 6, because that theorem applies only to the case when the equation $A\mathbf{x} = \mathbf{b}$ has a nonempty solution set. For $\mathbf{b}$, take any vector that is *not* a multiple of the columns of A.

Note: In the *Study Guide*, a "Checkpoint" for Section 1.5 will help students with Exercise 37.

38. If $\mathbf{w}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $A\mathbf{w} = \mathbf{0}$. For any scalar c, Theorem 5(b) in Section 1.4 shows that $A(c\mathbf{w}) = cA\mathbf{w} = c\mathbf{0} = \mathbf{0}$.

39. Suppose $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. Then, since $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$ by Theorem 5(a) in Section 1.4,

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Now, let c and d be scalars. Using both parts of Theorem 5,

$$A(c\mathbf{v} + d\mathbf{w}) = A(c\mathbf{v}) + A(d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\mathbf{0} + d\mathbf{0} = \mathbf{0}.$$

40. No. If $A\mathbf{x} = \mathbf{b}$ has no solution, then A cannot have a pivot in each row. Since A is 3×3, it has at most two pivot positions. So the equation $A\mathbf{x} = \mathbf{y}$ for any $\mathbf{y}$ has at most two basic variables and at least one free variable. Thus, the solution set for $A\mathbf{x} = \mathbf{y}$ is either empty or has infinitely many elements.

Note: The MATLAB box in the *Study Guide* introduces the **`zeros`** command, in order to augment a matrix with a column of zeros.

1.6 SOLUTIONS

1. Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

	Distribution of Output From:			
	Goods	Services		Purchased by:
output	↓	↓	input	
	.2	.7	→	Goods
	.8	.3	→	Services

Denote the total annual output (in dollars) of the sectors by p_G and p_S. From the first row, the total input to the Goods sector is $.2\,p_G + .7\,p_S$. The Goods sector must pay for that. So the equilibrium prices must satisfy

income		expenses
p_G	=	$.2p_G + .7\,p_S$

From the second row, the input (that is, the expense) of the Services sector is $.8\,p_G + .3\,p_S$. The equilibrium equation for the Services sector is

$$\begin{array}{cc} \text{income} & \text{expenses} \\ p_S & = .8p_G + .3p_S \end{array}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .8p_G - .7p_S &= 0 \\ -.8p_G + .7p_S &= 0 \end{aligned}$$

Row reduce the augmented matrix:

$$\begin{bmatrix} .8 & -.7 & 0 \\ -.8 & .7 & 0 \end{bmatrix} \sim \begin{bmatrix} .8 & -.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -.875 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_G = .875\,p_S$, with p_S free. One equilibrium solution is $p_S = 1000$ and $p_G = 875$. If one uses fractions instead of decimals in the calculations, the general solution would be written $p_G = (7/8)\,p_S$, and a natural choice of prices might be $p_S = 80$ and $p_G = 70$. Only the *ratio* of the prices is important: $p_G = .875\,p_S$. The economic equilibrium is unaffected by a proportional change in prices.

2. Take some other value for p_S, say 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices, because the solution set of the system of equations consists of all multiples of one vector. Changing the unit of measurement to another currency such as Japanese yen has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

3. **a**. Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

	Distribution of Output From:				Purchased by :
	Fuels and Power	Manufacturing	Services		
output	↓	↓	↓	input	
	.10	.10	.20	→	Fuels and Power
	.80	.10	.40	→	Manufacturing
	.10	.80	.40	→	Services

b. Denote the total annual output (in dollars) of the sectors by p_F, p_M, and p_S. From the first row of the table, the total input to the Fuels & Power sector is $.1p_F + .1p_M + .2p_S$. So the equilibrium prices must satisfy

$$\begin{array}{cc} \text{income} & \text{expenses} \\ p_F & = .1p_F + .1p_M + .2p_S \end{array}$$

From the second and third rows of the table, the income/expense requirements for the Manufacturing sector and the Services sector are, respectively,

$$\begin{aligned} p_M &= .8p_F + .1p_M + .4p_S \\ p_S &= .1p_F + .8p_M + .4p_S \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .9p_F - .1p_M - .2p_S &= 0 \\ -.8p_F + .9p_M - .4p_S &= 0 \\ -.1p_F - .8p_M + .6p_S &= 0 \end{aligned} \qquad \begin{bmatrix} .9 & -.1 & -.2 & 0 \\ -.8 & .9 & -.4 & 0 \\ -.1 & -.8 & .6 & 0 \end{bmatrix}$$

c. **[M]** You can obtain the reduced echelon form with a matrix program.

$$\begin{bmatrix} .9 & -.1 & -.2 & 0 \\ -.8 & .9 & -.4 & 0 \\ -.1 & -.8 & .6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -.301 & 0 \\ 0 & \textcircled{1} & -.712 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of decimal places displayed is somewhat arbitrary.

The general solution is $p_F = .301\ p_S$, $p_M = .712\ p_S$, with p_S free. If p_S is assigned the value of 100, then $p_F = 30.1$ and $p_M = 71.2$. Note that only the *ratios* of the prices are determined. This makes sense, for if they were converted from, say, dollars to yen or Euros, the inputs and outputs of each sector would still balance. The economic equilibrium is not affected by a proportional change in prices.

4. **a**. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

	Distribution of Output From:					
	Mining	Lumber	Energy	Transportation		Purchased by:
output	↓	↓	↓	↓	input	
	.30	.15	.20	.20	→	Mining
	.10	.15	.15	.10	→	Lumber
	.60	.50	.45	.50	→	Energy
	0	.20	.20	.20	→	Transportation

b. **[M]** Denote the total annual output of the sectors by p_M, p_L, p_E, and p_T, respectively. From the first row of the table, the total input to Agriculture is $.30p_M + .15p_L + .20p_E + .20\ p_T$. So the equilibrium prices must satisfy

income expenses

$$p_M = .30p_M + .15p_L + .20p_E + .20p_T$$

From the second, third, and fourth rows of the table, the equilibrium equations are

$$\begin{aligned} p_L &= .10p_M + .15p_L + .15p_E + .10p_T \\ p_E &= .60p_M + .50p_L + .45p_E + .50p_T \\ p_T &= .20p_L + .20p_E + .20p_{\mathrm{T}} \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .70p_M - .15p_L - .20p_E - .20p_T &= 0 \\ -.10p_M + .85p_L - .15p_E - .10p_T &= 0 \\ -.60p_M - .50p_L + .55p_E - .50p_T &= 0 \\ -.20p_L - .20p_E + .80p_T &= 0 \end{aligned}$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} .70 & -.15 & -.20 & -.20 & 0 \\ -.10 & .85 & -.15 & -.10 & 0 \\ -.60 & -.50 & .55 & -.50 & 0 \\ 0 & -.20 & -.20 & .80 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1.37 & 0 \\ 0 & 1 & 0 & -.84 & 0 \\ 0 & 0 & 1 & -3.16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for the basic variables in terms of the free variable p_T, and obtain $p_M = 1.37p_T$, $p_L = .84p_T$, and $p_E = 3.16p_T$. The data probably justifies at most two significant figures, so take $p_T = 100$ and round off the other prices to $p_M = 137$, $p_L = 84$, and $p_E = 316$.

5. **a**. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

	Distribution of Output From:					
	Agriculture	Manufacturing	Services	Transportation		Purchased by:
output	↓	↓	↓	↓	input	
	.20	.35	.10	.20	→	Agriculture
	.20	.10	.20	.30	→	Manufacturing
	.30	.35	.50	.20	→	Services
	.30	.20	.20	.30	→	Transportation

b. **[M]** Denote the total annual output of the sectors by p_A, p_M, p_S, and p_T, respectively. The equilibrium equations are

$$\begin{aligned} p_A &= .20p_A + .35p_M + .10p_S + .20p_T \\ p_M &= .20p_A + .10p_M + .20p_S + .30p_T \\ p_S &= .30p_A + .35p_M + .50p_S + .20p_T \\ p_T &= .30p_A + .20p_M + .20p_S + .30p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .80p_A - .35p_M - .10p_S - .20p_T &= 0 \\ -.20p_A + .90p_M - .20p_S - .30p_T &= 0 \\ -.30p_A - .35p_M + .50p_S - .20p_T &= 0 \\ -.30p_A - .20p_M - .20p_S + .70p_T &= 0 \end{aligned}$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} .80 & -.35 & -.10 & -.20 & 0 \\ -.20 & .90 & -.20 & -.30 & 0 \\ -.30 & -.35 & .50 & -.20 & 0 \\ -.30 & -.20 & -.20 & .70 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -.799 & 0 \\ 0 & 1 & 0 & -.836 & 0 \\ 0 & 0 & 1 & -1.465 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for the basic variables in terms of the free variable p_T, and obtain $p_A = .799p_T$, $p_M = .836p_T$, and $p_S = 1.465p_T$. Take $p_T = \$10.00$ and round off the other prices to $p_A = \$7.99$, $p_M = \$8.36$, and $p_S = \$14.65$ per unit.

c. Construct the new exchange table one column at a time. The entries in each column must sum to 1.

	Distribution of Output From:					
	Agriculture	Manufacturing	Services	Transportation		Purchased by:
output	↓	↓	↓	↓	input	
	.20	.35	.10	.20	→	Agriculture
	.10	.10	.20	.30	→	Manufacturing
	.40	.35	.50	.20	→	Services
	.30	.20	.20	.30	→	Transportation

d. **[M]** The new equilibrium equations are

$$\begin{aligned} p_A &= .20p_A + .35p_M + .10p_S + .20p_T \\ p_M &= .10p_A + .10p_M + .20p_S + .30p_T \\ p_S &= .40p_A + .35p_M + .50p_S + .20p_T \\ p_T &= .30p_A + .20p_M + .20p_S + .30p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .80p_A - .35p_M - .10p_S - .20p_T &= 0 \\ -.10p_A + .90p_M - .20p_S - .30p_T &= 0 \\ -.40p_A - .35p_M + .50p_S - .20p_T &= 0 \\ -.30p_A - .20p_M - .20p_S + .70p_T &= 0 \end{aligned}$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} .80 & -.35 & -.10 & -.20 & 0 \\ -.10 & .90 & -.20 & -.30 & 0 \\ -.40 & -.35 & .50 & -.20 & 0 \\ -.30 & -.20 & -.20 & .70 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & -.781 & 0 \\ 0 & \textcircled{1} & 0 & -.767 & 0 \\ 0 & 0 & \textcircled{1} & -1.562 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for the basic variables in terms of the free variable p_T, and obtain $p_A = .781p_T$, $p_M = .767p_T$, and $p_S = 1.562p_T$. Take $p_T = \$10.00$ and round off the other prices to $p_A = \$7.81$, $p_M = \$7.67$, and $p_S = \$15.62$ per unit. The campaign has caused unit prices for the Agriculture and Manufacturing sectirs to go down slightly, while increasing the unit price for the Services sector to increase by \$.10 per unit. The campaign has benefited the Services sector the most.

6. The following vectors list the numbers of atoms of aluminum (Al), oxygen (O), and carbon (C):

$$\underset{\text{aluminum}}{\begin{matrix}\text{Al}_2\text{O}_3: \\ 2\end{matrix}} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \underset{\text{oxygen}}{\begin{matrix}\text{C}: \\ 0\end{matrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underset{\text{carbon}}{\begin{matrix}\text{Al}: \\ 0\end{matrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix}\text{CO}_2: \\ 1\end{matrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The coefficients in the equation $x_1 \cdot \text{Al}_2\text{O}_3 + x_2 \cdot \text{C} \rightarrow x_3 \cdot \text{Al} + x_4 \cdot \text{CO}_2$ satisfy

$$x_1\begin{bmatrix}2\\3\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\1\end{bmatrix}=x_3\begin{bmatrix}1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\2\\1\end{bmatrix}$$

Move the right terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix}2&0&-1&0&0\\3&0&0&-2&0\\0&1&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/2&0&0\\3&0&0&-2&0\\0&1&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/2&0&0\\0&0&3/2&-2&0\\0&1&0&-1&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&-1/2&0&0\\0&1&0&-1&0\\0&0&3/2&-2&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/2&0&0\\0&1&0&-1&0\\0&0&1&-4/3&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-2/3&0\\0&1&0&-1&0\\0&0&1&-4/3&0\end{bmatrix}$$

The general solution is $x_1 = (2/3)x_4$, $x_2 = x_4$, $x_3 = (4/3)x_4$, with x_4 free. Take $x_4 = 3$. Then $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$. The balanced equation is

$$2Al_2O_3 + 3C \rightarrow 4Al + 3CO_2$$

7. The following vectors list the numbers of atoms of sodium (Na), hydrogen (H), carbon (C), and oxygen (O):

$$NaHCO_3:\begin{bmatrix}1\\1\\1\\3\end{bmatrix},\ H_3C_6H_5O_7:\begin{bmatrix}0\\8\\6\\7\end{bmatrix},\ Na_3C_6H_5O_7:\begin{bmatrix}3\\5\\6\\7\end{bmatrix},\ H_2O:\begin{bmatrix}0\\2\\0\\1\end{bmatrix},\ CO_2:\begin{bmatrix}0\\0\\1\\2\end{bmatrix}\begin{matrix}\text{sodium}\\\text{hydrogen}\\\text{carbon}\\\text{oxygen}\end{matrix}$$

The order of the various atoms is not important. The list here was selected by writing the elements in the order in which they first appear in the chemical equation, reading left to right:

$$x_1\cdot NaHCO_3 + x_2\cdot H_3C_6H_5O_7 \rightarrow x_3\cdot Na_3C_6H_5O_7 + x_4\cdot H_2O + x_5\cdot CO_2.$$

The coefficients $x_1, \ldots, x_5$ satisfy the vector equation

$$x_1\begin{bmatrix}1\\1\\1\\3\end{bmatrix}+x_2\begin{bmatrix}0\\8\\6\\7\end{bmatrix}=x_3\begin{bmatrix}3\\5\\6\\7\end{bmatrix}+x_4\begin{bmatrix}0\\2\\0\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\1\\2\end{bmatrix}$$

Move all the terms to the left side (changing the sign of each entry in the third, fourth, and fifth vectors) and reduce the augmented matrix:

$$\begin{bmatrix}1&0&-3&0&0&0\\1&8&-5&-2&0&0\\1&6&-6&0&-1&0\\3&7&-7&-1&-2&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&-1&0\\0&1&0&0&-1/3&0\\0&0&1&0&-1/3&0\\0&0&0&1&-1&0\end{bmatrix}$$

The general solution is $x_1 = x_5$, $x_2 = (1/3)x_5$, $x_3 = (1/3)x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 3$. Then $x_1 = x_4 = 3$, and $x_2 = x_3 = 1$. The balanced equation is

$$3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

8. The following vectors list the numbers of atoms of hydrogen (H), oxygen (O), calcium (Ca), and carbon (C):

$$H_3O:\begin{bmatrix}3\\1\\0\\0\end{bmatrix},\ CaCO_3:\begin{bmatrix}0\\3\\1\\1\end{bmatrix},\ H_2O:\begin{bmatrix}2\\1\\0\\0\end{bmatrix},\ Ca:\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ CO_2:\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\begin{matrix}\text{hydrogen}\\\text{oxygen}\\\text{calcium}\\\text{carbon}\end{matrix}$$

The coefficients in the chemical equation

$$x_1\cdot H_3O + x_2\cdot CaCO_3 \ \rightarrow\ x_3\cdot H_2O + x_4\cdot Ca + x_5\cdot CO_2$$

satisfy the vector equation

$$x_1\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\3\\1\\1\end{bmatrix}=x_3\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}0\\2\\0\\1\end{bmatrix}$$

Move the terms to the left side (changing the sign of each entry in the last three vectors) and reduce the augmented matrix:

$$\begin{bmatrix}3&0&-2&0&0&0\\1&3&-1&0&-2&0\\0&1&0&-1&0&0\\0&1&0&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&-2&0\\0&1&0&0&-1&0\\0&0&1&0&-3&0\\0&0&0&1&-1&0\end{bmatrix}$$

The general solution is $x_1 = 2x_5$, $x_2 = x_5$, $x_3 = 3x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 1$. Then $x_1 = 2$, and $x_2 = x_4 = 1$, and $x_3 = 3$. The balanced equation is

$$2H_3O + CaCO_3 \ \rightarrow\ 3H_2O + Ca + CO_2$$

9.The following vectors list the numbers of atoms of boron (B), sulfur (S), hydrogen (H), and oxygen (O):

$$B_2S_3:\begin{bmatrix}2\\3\\0\\0\end{bmatrix},\ H_2O:\begin{bmatrix}0\\0\\2\\1\end{bmatrix},\ H_3BO_3:\begin{bmatrix}1\\0\\3\\3\end{bmatrix},\ H_2S:\begin{bmatrix}0\\1\\2\\0\end{bmatrix}\begin{matrix}\text{boron}\\\text{sulfur}\\\text{hydrogen}\\\text{oxygen}\end{matrix}$$

The coefficients in the equation $x_1\cdot B_2S_3 + x_2\cdot H_2O \ \rightarrow\ x_3\cdot H_3BO_3 + x_4\cdot H_2S$ satisfy

$$x_1\begin{bmatrix}2\\3\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\\1\end{bmatrix}=x_3\begin{bmatrix}1\\0\\3\\3\end{bmatrix}+x_4\begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

Move the terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix}2&0&-1&0&0\\3&0&0&-1&0\\0&2&-3&-2&0\\0&1&-3&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1/3&0\\0&1&0&-2&0\\0&0&1&-2/3&0\\0&0&0&0&0\end{bmatrix}$$

The general solution is $x_1 = (1/3)\,x_4$, $x_2 = 2x_4$, $x_3 = (2/3)\,x_4$, with x_4 free. Take $x_4 = 3$. Then $x_1 = 1$, $x_2 = 6$, and $x_3 = 2$. The balanced equation is

$$B_2S_3 + 6H_2O \rightarrow 2H_3BO_3 + 3H_2S$$

10. **[M]** Set up vectors that list the atoms per molecule. Using the order lead (Pb), nitrogen (N), chromium (Cr), manganese (Mn), and oxygen (O), the vector equation to be solved is

$$x_1\begin{bmatrix}1\\6\\0\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\1\\2\\8\end{bmatrix} = x_3\begin{bmatrix}3\\0\\0\\0\\4\end{bmatrix} + x_4\begin{bmatrix}0\\0\\2\\0\\3\end{bmatrix} + x_5\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix} + x_6\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} \quad \begin{matrix}\text{lead}\\\text{nitrogen}\\\text{chromium}\\\text{manganese}\\\text{oxygen}\end{matrix}$$

The general solution is $x_1 = (1/6)x_6$, $x_2 = (22/45)x_6$, $x_3 = (1/18)x_6$, $x_4 = (11/45)x_6$, $x_5 = (44/45)x_6$, and x_6 is free. Take $x_6 = 90$. Then $x_1 = 15$, $x_2 = 44$, $x_3 = 5$, $x_4 = 22$, and $x_5 = 88$. The balanced equation is

$$15PbN_6 + 44CrMn_2O_8 \rightarrow 5Pb_3O_4 + 22Cr_2O_3 + 88MnO_2 + 90NO$$

11. **[M]** Set up vectors that list the atoms per molecule. Using the order manganese (Mn), sulfur (S), arsenic (As), chromium (Cr), oxygen (O), and hydrogen (H), the vector equation to be solved is

$$x_1\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\2\\10\\35\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1\\0\\0\\4\\2\end{bmatrix} = x_4\begin{bmatrix}1\\0\\0\\0\\4\\1\end{bmatrix} + x_5\begin{bmatrix}0\\0\\1\\0\\0\\3\end{bmatrix} + x_6\begin{bmatrix}0\\3\\0\\1\\12\\0\end{bmatrix} + x_7\begin{bmatrix}0\\0\\0\\0\\1\\2\end{bmatrix} \quad \begin{matrix}\text{manganese}\\\text{sulfur}\\\text{arsenic}\\\text{chromium}\\\text{oxygen}\\\text{hydrogen}\end{matrix}$$

In rational format, the general solution is $x_1 = (16/327)x_7$, $x_2 = (13/327)x_7$, $x_3 = (374/327)x_7$, $x_4 = (16/327)x_7$, $x_5 = (26/327)x_7$, $x_6 = (130/327)x_7$, and x_7 is free. Take $x_7 = 327$ to make the other variables whole numbers. The balanced equation is

$$16MnS + 13As_2Cr_{10}O_{35} + 374H_2SO_4 \rightarrow 16HMnO_4 + 26AsH_3 + 130CrS_3O_{12} + 327H_2O$$

Note that some students may use decimal calculation and simply "round off" the fractions that relate $x_1, \ldots, x_6$ to x_7. The equations they construct may balance most of the elements but miss an atom or two. Here is a solution submitted by two of my students:

$$5MnS + 4As_2Cr_{10}O_{35} + 115H_2SO_4 \rightarrow 5HMnO_4 + 8AsH_3 + 40CrS_3O_{12} + 100H_2O$$

Everything balances except the hydrogen. The right side is short 1 hydrogen atom. Perhaps the students thought that it escaped!

12. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	$x_1 + x_4$	=	x_2
B	x_2	=	$x_3 + 100$
C	$x_3 + 80$	=	x_4

Rearrange the equations:

$$\begin{aligned} x_1 - x_2 \quad\quad + x_4 &= 0 \\ x_2 - x_3 \quad\quad &= 100 \\ x_3 - x_4 &= -80 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -80 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 20 \\ 0 & \textcircled{1} & 0 & -1 & 20 \\ 0 & 0 & \textcircled{1} & -1 & -80 \end{bmatrix}$$

The general solution (written in the style of Section 1.2) is

$$\begin{cases} x_1 = 20 \\ x_2 = 20 + x_4 \\ x_3 = -80 + x_4 \\ x_4 \text{ is free} \end{cases}$$

Since x_3 cannot be negative, the minimum value of x_4 is 80.

13. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	$x_2 + 30$	=	$x_1 + 80$
B	$x_3 + x_5$	=	$x_2 + x_4$
C	$x_6 + 100$	=	$x_5 + 40$
D	$x_4 + 40$	=	$x_6 + 90$
E	$x_1 + 60$	=	$x_3 + 20$

Rearrange the equations:

$$\begin{aligned} x_1 - x_2 \quad\quad\quad\quad\quad\quad\quad\quad &= -50 \\ x_2 - x_3 + x_4 - x_5 \quad\quad &= 0 \\ x_5 - x_6 &= 60 \\ x_4 \quad\quad - x_6 &= 50 \\ x_1 \quad\quad - x_3 \quad\quad &= -40 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & \textcircled{1} & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & \textcircled{1} & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$

b. To find minimum flows, note that since x_1 cannot be negative, $x_3 \geq 40$. This implies that $x_2 \geq 50$. Also, since x_6 cannot be negative, $x_4 \geq 50$ and $x_5 \geq 60$. The minimum flows are $x_2 = 50$, $x_3 = 40$, $x_4 = 50$, $x_5 = 60$ (when $x_1 = 0$ and $x_6 = 0$).

14. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	80	$=$	$x_1 + x_5$
B	$x_1 + x_2 + 100$	$=$	x_4
C	x_3	$=$	$x_2 + 90$
D	$x_4 + x_5$	$=$	$x_3 + 90$

Rearrange the equations:

$$\begin{array}{rcrcrcrcr} x_1 & & & & & & + & x_5 & = & 80 \\ x_1 & + & x_2 & & & - & x_4 & & = & -100 \\ & & x_2 & - & x_3 & & & & = & -90 \\ & & & & x_3 & - & x_4 & - & x_5 & = & -90 \end{array}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 80 \\ 1 & 1 & 0 & -1 & 0 & -100 \\ 0 & 1 & -1 & 0 & 0 & -90 \\ 0 & 0 & 1 & -1 & -1 & -90 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 1 & 80 \\ 0 & \textcircled{1} & 0 & -1 & -1 & -180 \\ 0 & 0 & \textcircled{1} & -1 & -1 & -90 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = 80 - x_5 \\ x_2 = x_4 + x_5 - 180 \\ x_3 = x_4 + x_5 - 90 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$

b. If $x_5 = 0$, then the general solution is $\begin{cases} x_1 = 80 \\ x_2 = x_4 - 180 \\ x_3 = x_4 - 90 \\ x_4 \text{ is free} \end{cases}$

c. Since x_2 cannot be negative, the minimum value of x_4 when $x_5 = 0$ is 180.

15. Write the equations for each intersection.

Intersection	Flow in		Flow out
A	$x_6 + 60$	=	x_1
B	x_1	=	$x_2 + 70$
C	$x_2 + 100$	=	x_3
D	x_3	=	$x_4 + 90$
E	$x_4 + 80$	=	x_5
F	x_5	=	$x_6 + 80$

Rearrange the equations:

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad\qquad\qquad - x_6 &= 60\\
x_1 - x_2 \qquad\qquad\qquad\qquad\qquad &= 70\\
x_2 - x_3 \qquad\qquad\qquad\qquad &= -100\\
x_3 - x_4 \qquad\qquad\qquad &= 90\\
x_4 - x_5 \qquad\qquad &= -80\\
x_5 - x_6 &= 80
\end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 60 \\ 1 & -1 & 0 & 0 & 0 & 0 & 70 \\ 0 & 1 & -1 & 0 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 0 & 90 \\ 0 & 0 & 0 & 1 & -1 & 0 & -80 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 60 \\ 0 & 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 1 & 0 & 0 & -1 & 90 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $\begin{cases} x_1 = 60 + x_6 \\ x_2 = -10 + x_6 \\ x_3 = 90 + x_6 \\ x_4 = x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}$.

Since x_2 cannot be negative, the minimum value of x_6 is 10.

Note: The MATLAB box in the *Study Guide* discusses rational calculations, needed for balancing the chemical equations in Exercises 10 and 11. As usual, the appendices cover this material for Maple, Mathematica, and the TI calculators.

1.7 SOLUTIONS

Note: Key exercises are 9–20 and 23–30. Exercise 30 states a result that could be a theorem in the text. There is a danger, however, that students will memorize the result without understanding the proof, and then later mix up the words row and column. Exercises 37 and 38 anticipate the discussion in Section 1.9 of one-to-one transformations. Exercise 44 is fairly difficult for my students.

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{5} & 7 & 9 & 0 \\ 0 & \textcircled{2} & 4 & 0 \\ 0 & 0 & \textcircled{4} & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

2. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 3 & 0 \\ 3 & -8 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & 0 & 3 & 0 \\ 0 & \textcircled{-8} & -7/2 & 0 \\ 0 & 0 & \textcircled{-1} & 0 \end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

3. Use the method of Example 3 (or the box following the example). By comparing entries of the vectors, one sees that the second vector is –2 times the first vector. Thus, the two vectors are linearly dependent.

4. From the first entries in the vectors, it seems that the second vector of the pair $\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \end{bmatrix}$ may be 3 times the first vector. But there is a sign problem with the second entries. So neither of the vectors is a multiple of the other. The vectors are linearly independent.

5. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & -3 & 9 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -5 & 0 \\ 1 & -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 \\ 2 & 1 & -7 & 0 \\ -1 & 4 & -5 & 0 \\ 0 & -3 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 \\ 0 & 9 & -3 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & -3 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 \\ 0 & 9 & -3 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -4 & -2 & 0 \\ 0 & \textcircled{9} & -3 & 0 \\ 0 & 0 & \textcircled{-7} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

6. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & -1 & 5 & 0 \\ -4 & -3 & 0 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 1 & -20 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & -15 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 1 & -5 & 0 \\ 0 & \textcircled{-1} & 5 & 0 \\ 0 & 0 & \textcircled{-15} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so the columns of A are linearly independent.

7. Study the equation $A\mathbf{x} = \mathbf{0}$. Some people may start with the method of Example 2:

$$\begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 4 & -3 & 0 & 0 \\ 0 & \textcircled{1} & -1 & 1 & 0 \\ 0 & 0 & \textcircled{6} & -6 & 0 \end{bmatrix}$$

But this is a waste of time. There are only 3 rows, so there are at most three pivot positions. Hence, at least one of the four variables must be free. So the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

8. Same situation as with Exercise 7. The (unnecessary) row operations are

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 0 \\ -2 & 4 & -6 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 2 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 3 & 2 & 0 \\ 0 & \textcircled{1} & -1 & 3 & 0 \\ 0 & 0 & 0 & \textcircled{6} & 0 \end{bmatrix}$$

Again, because there are at most three pivot positions yet there are four variables, the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and the columns of A are linearly dependent.

9. **a**. The vector $\mathbf{v}_3$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 5 \\ 0 & 0 & \textcircled{8} \\ 0 & 0 & h-10 \end{bmatrix}$$

At this point, the equation $0 = 8$ shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 5 & 0 \\ 0 & 0 & \textcircled{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

10. **a**. The vector $\mathbf{v}_3$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -5 \\ -5 & 15 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & h+10 \end{bmatrix}$$

At this point, the equation $0 = 1$ shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -3 & 9 & -5 & 0 \\ -5 & 15 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

11. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ -2 & -6 & 2 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & h+4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & 4 & -2 & 0 \\ 0 & \textcircled{-2} & 0 & 0 \\ 0 & 0 & h+4 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 4 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -4$.

12. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{3} & -6 & 9 & 0 \\ 0 & \textcircled{-1} & 0 & 0 \\ 0 & 0 & h+18 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 18 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -18$.

13. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 3 & 0 \\ 0 & \textcircled{1} & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

14. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 7 & 1 & 0 \\ -4 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -6 & h+8 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & 0 \\ 0 & \textcircled{1} & 5 & 0 \\ 0 & 0 & h+38 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 38 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -38$.

15. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

16. The set is linearly dependent because the second vector is –3/2 times the first vector.

17. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

18. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are – 4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

20. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

21. **a**. False. A homogeneous system *always* has the trivial solution. See the box before Example 2.

b. False. See the warning after Theorem 7.

c. True. See Fig. 3, after Theorem 8.

d. True. See the remark following Example 4.

22. **a**. True. See Theorem 7.

b. True. See Example 4.

c. False. For instance, the set consisting of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is linearly dependent. See the warning after Theorem 8.

d. False. See Example 3(a).

23. $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

24. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

25. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

26. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. The columns must be linearly independent, by Theorem 7, because the first column is not zero, the second column is not a multiple of the first, and the third column is not a linear combination of the preceding two columns (because $\mathbf{a}_3$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$).

27. All four columns of the 6×4 matrix A must be pivot columns. Otherwise, the equation $A\mathbf{x} = \mathbf{0}$ would have a free variable, in which case the columns of A would be linearly dependent.

28. If the columns of a 4×6 matrix A span $\mathbf{R}^4$, then A has a pivot in each row, by Theorem 4. Since each pivot position is in a different column, A has four pivot columns.

29. *A*: any 3×2 matrix with one column a multiple of the other.
B: any 3×2 matrix with two nonzero columns such that neither column is a multiple of the other. In this case the columns are linearly independent and so the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. a. n

b. The columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x} = \mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

31. Think of $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$. The text points out that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$. Rewrite this as $\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 1, -1)$.

32. Think of $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$. The text points out that $\mathbf{a}_1 - 3\mathbf{a}_2 = \mathbf{a}_3$. Rewrite this as $\mathbf{a}_1 - 3\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, -3, -1)$.

33. True, by Theorem 7. (The *Study Guide* adds another justification.)

34. False. The vector $\mathbf{v}_1$ could be the zero vector.

35. True, by Theorem 9.

36. False. Counterexample: Take $\mathbf{v}_1$ and $\mathbf{v}_2$ to be multiples of one vector. Take $\mathbf{v}_3$ to be *not* a multiple of that vector. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

37. True. A linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ may be extended to a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ by placing a zero weight on $\mathbf{v}_4$.

38. True. If the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a nontrivial solution (with at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\cdot\mathbf{v}_4 = \mathbf{0}$. But that cannot happen because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 37, if you know that the statement there is true.

39. If for all $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, then take $\mathbf{b} = \mathbf{0}$, and conclude that the equation $A\mathbf{x} = \mathbf{0}$ has at most one solution. Then the trivial solution is the only solution, and so the columns of A are linearly independent.

40. An $m\times n$ matrix with n pivot columns has a pivot in each column. So the equation $A\mathbf{x} = \mathbf{b}$ has no free variables. If there is a solution, it must be unique.

41. **[M]** $A = \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ -5 & -3 & -7 & -11 & 15 \\ 4 & 3 & 5 & 2 & 1 \\ 8 & -7 & 23 & 4 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ 0 & -29/3 & 29/3 & 2/3 & 25/3 \\ 0 & 25/3 & -25/3 & -22/3 & 19/3 \\ 0 & 11/3 & -11/3 & -44/3 & 77/3 \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ 0 & -29/3 & 29/3 & 2/3 & 25/3 \\ 0 & 0 & 0 & -196/29 & 392/29 \\ 0 & 0 & 0 & -418/29 & 836/29 \end{bmatrix} \sim \begin{bmatrix} \textcircled{3} & -4 & 10 & 7 & -4 \\ 0 & \textcircled{-29/3} & 29/3 & 2/3 & 25/3 \\ 0 & 0 & 0 & \textcircled{-196/29} & 392/29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Use the pivot columns of A to form $B = \begin{bmatrix} 3 & -4 & 7 \\ -5 & -3 & -11 \\ 4 & 3 & 2 \\ 8 & -7 & 4 \end{bmatrix}$. Other choices are possible.

42. **[M]** $\begin{bmatrix} 12 & 10 & -6 & 8 & 4 & -14 \\ -7 & -6 & 4 & -5 & -7 & 9 \\ 9 & 9 & -9 & 9 & 9 & -18 \\ -4 & -3 & -1 & 0 & -8 & 1 \\ 8 & 7 & -5 & 6 & 1 & -11 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{12} & 10 & -6 & 8 & 4 & -14 \\ 0 & \textcircled{-1/6} & 1/2 & -1/3 & -14/3 & 5/6 \\ 0 & 0 & \textcircled{-2} & 2 & -16 & -2 \\ 0 & 0 & 0 & 0 & \textcircled{-36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Use the pivot columns of A to form $B = \begin{bmatrix} 12 & 10 & -6 & 4 \\ -7 & -6 & 4 & -7 \\ 9 & 9 & -9 & 9 \\ -4 & -3 & -1 & -8 \\ 8 & 7 & -5 & 1 \end{bmatrix}$. Other choices are possible.

43. **[M]** Make **v** any one of the columns of A that is not in B and row reduce the augmented matrix $[B \quad \mathbf{v}]$. The calculations will show that the equation $B\mathbf{x} = \mathbf{v}$ is consistent, which means that **v** is a linear combination of the columns of B. Thus, each column of A that is not a column of B is in the set spanned by the columns of B.

44. **[M]** Calculations made as for Exercise 43 will show that each column of A that is not a column of B is in the set spanned by the columns of B. *Reason*: The original matrix A has only four pivot columns. If one or more columns of A are removed, the resulting matrix will have at most four pivot columns. (Use exactly the same row operations on the new matrix that were used to reduce A to echelon form.) If **v** is a column of A that is not in B, then row reduction of the augmented matrix $[B \quad \mathbf{v}]$ will display at most four pivot columns. Since B itself was constructed to have four pivot columns, adjoining **v** cannot produce a fifth pivot column. Thus the first four columns of $[B \quad \mathbf{v}]$ are the pivot columns. This implies that the equation $B\mathbf{x} = \mathbf{v}$ has a solution.

Note: At the end of Section 1.7, the *Study Guide* has another note to students about "Mastering Linear Algebra Concepts." The note describes how to organize a review sheet that will help students form a mental image of linear independence. The note also lists typical misuses of terminology, in which an adjective is applied to an inappropriate noun. (This is a major problem for my students.) I require my students to prepare a review sheet as described in the *Study Guide*, and I try to make helpful comments on their sheets. I am convinced, through personal observation and student surveys, that the students who

prepare many of these review sheets consistently perform better than other students. Hopefully, these students will remember important concepts for some time beyond the final exam.

1.8 SOLUTIONS

Notes: The key exercises are 17–20, 25 and 31. Exercise 20 is worth assigning even if you normally assign only odd exercises. Exercise 25 (and 26) can be used to make a few comments about computer graphics, even if you do not plan to cover Section 2.6. For Exercise 31, the *Study Guide* encourages students *not* to look at the proof before trying hard to construct it. Then the *Guide* explains how to create the proof.

Exercises 19 and 20 provide a natural segue into Section 1.9. I arrange to discuss the homework on these exercises when I am ready to begin Section 1.9. The definition of the standard matrix in Section 1.9 follows naturally from the homework, and so I've covered the first page of Section 1.9 before students realize we are working on new material.

The text does not provide much practice determining whether a transformation is linear, because the time needed to develop this skill would have to be taken away from some other topic. If you want your students to be able to do this, you may need to supplement Exercises 23, 24, 32 and 33.

If you skip the concepts of one-to-one and "onto" in Section 1.9, you can use the result of Exercise 31 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.)

1. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$

2. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a/3 \\ b/3 \\ c/3 \end{bmatrix}$

3. $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & -2 \\ -3 & 1 & 6 & 3 \\ 2 & -2 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & -2 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} 7 \\ 6 \\ 1 \end{bmatrix}$, unique solution

4. $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & -1 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -3 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} -17 \\ -7 \\ -1 \end{bmatrix}$, unique solution

5. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 3 \\ 0 & \textcircled{1} & 2 & 1 \end{bmatrix}$

Note that a solution is *not* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To avoid this common error, write the equations:

$$\begin{aligned} \textcircled{x_1} \quad & + \quad 3x_3 = 3 \\ \textcircled{x_2} \quad & + \quad 2x_3 = 1 \end{aligned}$$ and solve for the basic variables: $\begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_3 \text{ is free} \end{cases}$

The general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. For a particular solution, one might choose $x_3 = 0$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

6. $[A \quad \mathbf{b}] = \begin{bmatrix} 1 & -3 & 2 & 1 \\ 3 & -8 & 8 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 8 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 8 & 10 \\ 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Write the equations:

$$\begin{aligned} \textcircled{x_1} \quad & + \quad 8x_3 = 10 \\ \textcircled{x_2} \quad & + \quad 2x_3 = 3 \end{aligned}$$ and solve for the basic variables: $\begin{cases} x_1 = 10 - 8x_3 \\ x_2 = 3 - 2x_3 \\ x_3 \text{ is free} \end{cases}$

The general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 8x_3 \\ 3 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -8 \\ -2 \\ 1 \end{bmatrix}$. For a particular solution, one might choose $x_3 = 0$ and $\mathbf{x} = \begin{bmatrix} 10 \\ 3 \\ 0 \end{bmatrix}$.

7. The value of a is 5. The domain of T is $\mathbf{R}^5$, because a 6×5 matrix has 5 columns and for $A\mathbf{x}$ to be defined, $\mathbf{x}$ must be in $\mathbf{R}^5$. The value of b is 6. The codomain of T is $\mathbf{R}^6$, because $A\mathbf{x}$ is a linear combination of the columns of A, and each column of A is in $\mathbf{R}^6$.

8. The matrix A must have 7 rows and 5 columns. For the domain of T to be $\mathbf{R}^5$, A must have five columns so that $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbf{R}^5$. For the codomain of T to be $\mathbf{R}^7$, the columns of A must have seven entries (in which case A must have seven rows), because $A\mathbf{x}$ is a linear combination of the columns of A.

9. Solve $A\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 2 & -4 & 4 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 2 & -6 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & -5 & 0 \\ 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}$

$$\sim\begin{bmatrix}\textcircled{1} & 0 & -4 & 0 & 0\\ 0 & \textcircled{1} & -3 & 0 & 0\\ 0 & 0 & 0 & \textcircled{1} & 0\end{bmatrix}\begin{matrix}\textcircled{x_1} & & -4x_3 & & = 0\\ & \textcircled{x_2} & -3x_3 & & = 0\\ & & & \textcircled{x_4} & = 0\end{matrix},\quad\begin{cases}x_1 = 4x_3\\ x_2 = 3x_3\\ x_3 \text{ is free}\\ x_4 = 0\end{cases}$$

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}4x_3\\ 3x_3\\ x_3\\ 0\end{bmatrix} = x_3\begin{bmatrix}4\\ 3\\ 1\\ 0\end{bmatrix}$$

10. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix}3 & 2 & 10 & -6 & 0\\ 1 & 0 & 2 & -4 & 0\\ 0 & 1 & 2 & 3 & 0\\ 1 & 4 & 10 & 8 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -4 & 0\\ 3 & 2 & 10 & -6 & 0\\ 0 & 1 & 2 & 3 & 0\\ 1 & 4 & 10 & 8 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -4 & 0\\ 0 & 2 & 4 & 6 & 0\\ 0 & 1 & 2 & 3 & 0\\ 0 & 4 & 8 & 12 & 0\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 0 & 2 & -4 & 0\\ 0 & 1 & 2 & 3 & 0\\ 0 & 2 & 4 & 6 & 0\\ 0 & 4 & 8 & 12 & 0\end{bmatrix} \sim \begin{bmatrix}\textcircled{1} & 0 & 2 & -4 & 0\\ 0 & \textcircled{1} & 2 & 3 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{matrix}\textcircled{x_1} & & +2x_3 & - & 4x_4 & = 0\\ & \textcircled{x_2} & +2x_3 & + & 3x_4 & = 0\end{matrix}\quad\begin{cases}x_1 = -2x_3 + 4x_4\\ x_2 = -2x_3 - 3x_4\\ x_3 \text{ is free}\\ x_4 \text{ is free}\end{cases}\quad \mathbf{x} = \begin{bmatrix}-2x_3\\ -2x_3\\ x_3\\ 0\end{bmatrix} + \begin{bmatrix}4x_4\\ -3x_4\\ 0\\ x_4\end{bmatrix} = x_3\begin{bmatrix}-2\\ -2\\ 1\\ 0\end{bmatrix} + x_4\begin{bmatrix}4\\ -3\\ 0\\ 1\end{bmatrix}$$

11. Is the system represented by $[A \quad \mathbf{b}]$ consistent? Yes, as the following calculation shows.

$$\begin{bmatrix}1 & -3 & 5 & -5 & -1\\ 0 & 1 & -3 & 5 & 1\\ 2 & -4 & 4 & -4 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -3 & 5 & -5 & -1\\ 0 & 1 & -3 & 5 & 1\\ 0 & 2 & -6 & 6 & 2\end{bmatrix} \sim \begin{bmatrix}\textcircled{1} & -3 & 5 & -5 & -1\\ 0 & \textcircled{1} & -3 & 5 & 1\\ 0 & 0 & 0 & \textcircled{-4} & 0\end{bmatrix}$$

The system is consistent, so $\mathbf{b}$ is in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

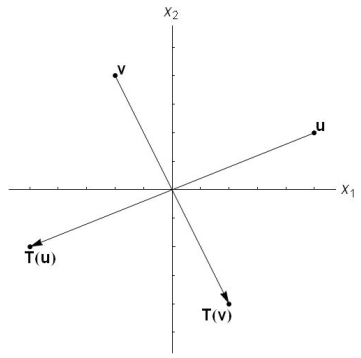
12. Is the system represented by $[A \quad \mathbf{b}]$ consistent?

$$\begin{bmatrix}3 & 2 & 10 & -6 & -1\\ 1 & 0 & 2 & -4 & 3\\ 0 & 1 & 2 & 3 & -1\\ 1 & 4 & 10 & 8 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -4 & 3\\ 3 & 2 & 10 & -6 & -1\\ 0 & 1 & 2 & 3 & -1\\ 1 & 4 & 10 & 8 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -4 & 3\\ 0 & 2 & 4 & 6 & -10\\ 0 & 1 & 2 & 3 & -1\\ 0 & 4 & 8 & 12 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 2 & -4 & 3\\ 0 & 1 & 2 & 3 & -1\\ 0 & 2 & 4 & 6 & -10\\ 0 & 4 & 8 & 12 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -4 & 3\\ 0 & 1 & 2 & 3 & -1\\ 0 & 0 & 0 & 0 & -8\\ 0 & 0 & 0 & 0 & 5\end{bmatrix}$$

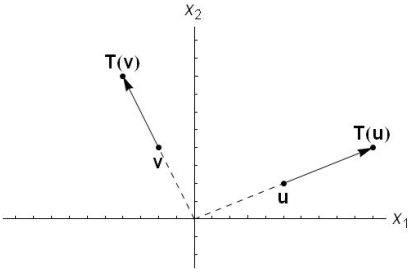
The system is inconsistent, so **b** is not in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

13.



A reflection through the origin.

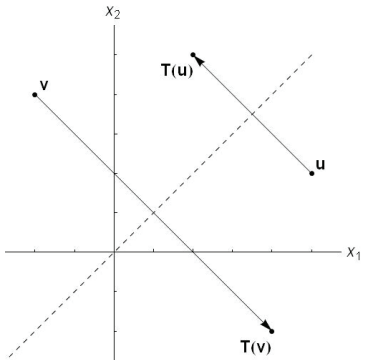
14.



A scaling by the factor 2.

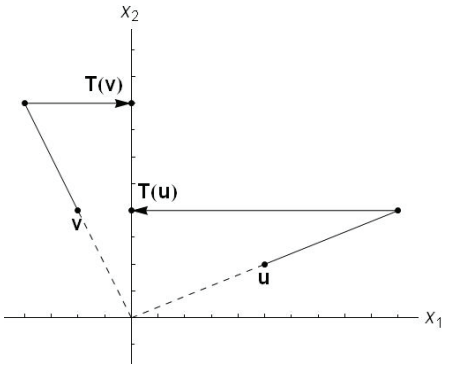
The transformation in Exercise 13 may also be described as a rotation of π radians about the origin or a rotation of $-\pi$ radians about the origin.

15.



A reflection through the line $x_2 = x_1$.

16.

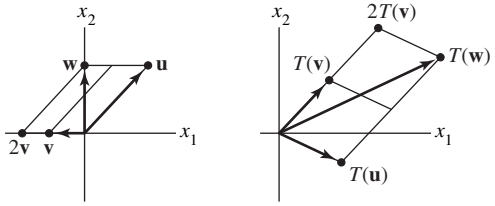


A scaling by a factor of 2 and a projection onto the x_2 axis.

17. $T(2\mathbf{u}) = 2T(\mathbf{u}) = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, $T(3\mathbf{v}) = 3T(\mathbf{v}) = 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$, and

$$T(2\mathbf{u} + 3\mathbf{v}) = 2T(\mathbf{u}) + 3T(\mathbf{v}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}.$$

18. Draw a line through **w** parallel to **v**, and draw a line through **w** parallel to **u**. See the left part of the figure below. From this, estimate that $\mathbf{w} = \mathbf{u} + 2\mathbf{v}$. Since T is linear, $T(\mathbf{w}) = T(\mathbf{u}) + 2T(\mathbf{v})$. Locate $T(\mathbf{u})$ and $2T(\mathbf{v})$ as in the right part of the figure and form the associated parallelogram to locate $T(\mathbf{w})$.



19. All we know are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the fact that T is linear. The key idea is to write $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$. Then, from the linearity of T, write

$$T(\mathbf{x}) = T(5\mathbf{e}_1 - 3\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}.$$

To find the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, observe that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

20. Use the basic definition of $A\mathbf{x}$ to construct A. Write

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ 5 & -2 \end{bmatrix}\mathbf{x}, \quad A = \begin{bmatrix} -3 & 7 \\ 5 & -2 \end{bmatrix}$$

21. **a**. True. Functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ are defined before Fig. 2. A linear transformation is a function with certain properties.

b. False. The domain is $\mathbf{R}^5$. See the paragraph before Example 1.

c. False. The range is the set of all linear combinations of the columns of A. See the paragraph before Example 1.

d. False. See the paragraph after the definition of a linear transformation.

e. True. See the paragraph following the box that contains equation (4).

22. **a**. True. See the subsection on *Matrix Transformations*.

b. True. See the subsection on *Linear Transformations*.

c. False. The question is an existence question. See the remark about Example 1(d), following the solution of Example 1.

d. True. See the discussion following the definition of a linear transformation.

e. True. $T(\mathbf{0}) = \mathbf{0}$. See the box after the definition of a linear transformation.

23. **a**. When $b = 0$, $f(x) = mx$. In this case, for all x,y in $\mathbf{R}$ and all scalars c and d,

$$f(cx + dy) = m(cx + dy) = mcx + mdy = c(mx) + d(my) = c\cdot f(x) + d\cdot f(y)$$

This shows that f is linear.

b. When $f(x) = mx + b$, with b nonzero, $f(0) = m(0) = b = b \neq 0$. This shows that f is not linear, because every linear transformation maps the zero vector in its domain into the zero vector in the codomain. (In this case, both zero vectors are just the number 0.) Another argument, for instance, would be to calculate $f(2x) = m(2x) + b$ and $2f(x) = 2mx + 2b$. If b is nonzero, then $f(2x)$ is not equal to $2f(x)$ and so f is not a linear transformation.

c. In calculus, f is called a "linear function" because the graph of f is a line.

24. Let $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for $\mathbf{x}$ in $\mathbf{R}^n$. If $\mathbf{b}$ is not zero, $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$. Actually, T fails both properties
of a linear transformation. For instance, $T(2\mathbf{x}) = A(2\mathbf{x}) + \mathbf{b} = 2A\mathbf{x} + \mathbf{b}$, which is not the same as $2T(\mathbf{x}) = 2(A\mathbf{x} + \mathbf{b}) = 2A\mathbf{x} + 2\mathbf{b}$. Also,

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) + \mathbf{b} = A\mathbf{x} + A\mathbf{y} + \mathbf{b}$$

which is not the same as

$$T(\mathbf{x}) + T(\mathbf{y}) = A\mathbf{x} + \mathbf{b} + A\mathbf{y} + \mathbf{b}$$

25. Any point $\mathbf{x}$ on the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ satisfies the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ for some value of t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v}) \qquad (*)$$

If $T(\mathbf{v}) = \mathbf{0}$, then $T(\mathbf{x}) = T(\mathbf{p})$ for all values of t, and the image of the original line is just a single point. Otherwise, (*) is the parametric equation of a line through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$.

26. **a**. From the figure following Exercise 22 in Section 1.5, the line through $\mathbf{p}$ and $\mathbf{q}$ is in the direction of $\mathbf{q} - \mathbf{p}$, and so the equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t\mathbf{q} - t\mathbf{p} = (1 - t)\mathbf{p} + t\mathbf{q}$.

b. Consider $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$ for t such that $0 \le t \le 1$. Then, by linearity of T,

$$T(\mathbf{x}) = T((1 - t)\mathbf{p} + t\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{q}) \qquad 0 \le t \le 1 \qquad (*)$$

If $T(\mathbf{p})$ and $T(\mathbf{q})$ are distinct, then (*) is the equation for the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$, as shown in part (a) Otherwise, the set of images is just the single point $T(\mathbf{p})$, because

$$(1 - t)T(\mathbf{p}) + tT(\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$$

27. Any point $\mathbf{x}$ on the plane P satisfies the parametric equation $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ for some values of s and t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v}) \qquad (s, t \text{ in } \mathbf{R})$$

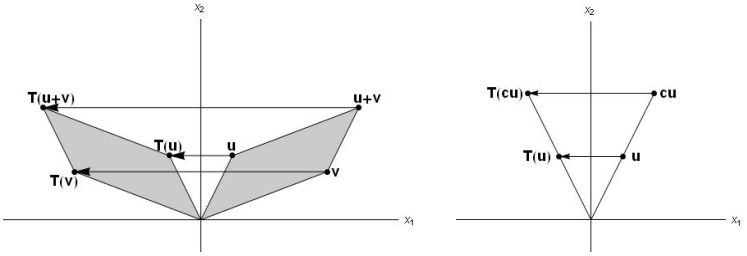
The set of images is just Span$\{T(\mathbf{u}), T(\mathbf{v})\}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent and not both zero, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

28. Consider a point $\mathbf{x}$ in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ for $0 \le a \le 1$, $0 \le b \le 1$. By linearity of T, the image of $\mathbf{x}$ is

$$T(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}), \text{ for } 0 \le a \le 1,\ 0 \le b \le 1$$

This image point lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$. Special "degenerate" cases arise when $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. If one of the images is not zero, then the "parallelogram" is actually the line segment from $\mathbf{0}$ to $T(\mathbf{u}) + T(\mathbf{v})$. If both $T(\mathbf{u})$ and $T(\mathbf{v})$ are zero, then the parallelogram is just $\{\mathbf{0}\}$. Another possibility is that even $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, in which case the original parallelogram is degenerate (either a line segment or the zero vector). In this case, the set of images must be degenerate, too.

29.



30. Given any $\mathbf{x}$ in $\mathbf{R}^n$, there are constants $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$, because $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbf{R}^n$. Then, from property (5) of a linear transformation,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = c_1\mathbf{0} + \cdot\cdot + c_p\mathbf{0} = \mathbf{0}$$

31. (The *Study Guide* has a more detailed discussion of the proof.) Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

Then $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0}) = \mathbf{0}$. Since T is linear,

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$$

Since not all the weights are zero, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is a linearly dependent set.

32. Take any vector (x_1, x_2) with $x_2 \neq 0$, and use a negative scalar. For instance, $T(0, 1) = (-2, -4)$, but $T(-1\cdot(0, 1)) = T(0, -1) = (-2, 4) \neq (-1)\cdot T(0, 1)$.

33. One possibility is to show that T does not map the zero vector into the zero vector, something that every linear transformation *does* do. $T(0, 0) = (0, -3, 0)$.

34. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, cu_2 + dv_2, -(cu_3 + dv_3)) = (cu_1 + dv_1, cu_2 + dv_2,\ cu_3 - dv_3) \\ &= (cu_1, cu_2, -cu_3) + (dv_1, dv_2, -dv_3) = c(u_1, u_2, -u_3) + d(v_1, v_2, -v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

35. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, 0, cu_3 + dv_3) = (cu_1, 0, cu_3) + (dv_1, 0, dv_3) \\ &= c(u_1, 0, u_3) + d(v_1, 0, v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

36. Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in $\mathbf{R}^n$ and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then there exist weights c_1, c_2, not both zero, such that $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$. Because T is linear, $T(c_1\mathbf{u} + c_2\mathbf{v}) = \mathbf{0}$. That is, the vector $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$ satisfies $T(\mathbf{x}) = \mathbf{0}$. Furthermore, $\mathbf{x}$ cannot be the zero vector, since that would mean that a nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$ is zero, which is impossible because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus, the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

37. **[M]** $\begin{bmatrix} 2 & 3 & 5 & -5 & 0 \\ -7 & 7 & 0 & 0 & 0 \\ -3 & 4 & 1 & 3 & 0 \\ -9 & 3 & -6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$ $\mathbf{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

38. **[M]** $\begin{bmatrix} 3 & 4 & -7 & 0 & 0 \\ 5 & -8 & 7 & 4 & 0 \\ 6 & -8 & 6 & 4 & 0 \\ 9 & -7 & -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = -x_4 \\ x_4 \text{ is free} \end{cases}$ $\mathbf{x} = x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

39. **[M]** $\begin{bmatrix} 2 & 3 & 5 & -5 & 8 \\ -7 & 7 & 0 & 0 & 7 \\ -3 & 4 & 1 & 3 & 5 \\ -9 & 3 & -6 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 & 0 & 1 \\ 0 & \textcircled{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation, because the augmented matrix shows a consistent system. In fact, the general solution is $\begin{cases} x_1 = 1 - x_3 \\ x_2 = 2 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$; when $x_3 = 0$ a solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

40. **[M]** $\begin{bmatrix} 3 & 4 & -7 & 0 & 4 \\ 5 & -8 & 7 & 4 & -4 \\ 6 & -8 & 6 & 4 & -4 \\ 9 & -7 & -2 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 1 \\ 0 & \textcircled{1} & 0 & 1 & 2 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation, because the augmented matrix shows a consistent system. In fact, the general solution is $\begin{cases} x_1 = 1 - x_4 \\ x_2 = 2 - x_4 \\ x_3 = 1 - x_4 \\ x_4 \text{ is free} \end{cases}$; when $x_4 = 0$ a solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Notes: At the end of Section 1.8, the *Study Guide* provides a list of equations, figures, examples, and connections with concepts that will strengthen a student's understanding of linear transformations. I encourage my students to continue the construction of review sheets similar to those for "span" and "linear independence," but I refrain from collecting these sheets. At some point the students have to assume the responsibility for mastering this material.

If your students are using MATLAB or another matrix program, you might insert the definition of matrix multiplication after this section, and then assign a project that uses random matrices to explore properties of matrix multiplication. See Exercises 34–36 in Section 2.1. Meanwhile, in class you can continue with your plans for finishing Chapter 1. When you get to Section 2.1, you won't have much to do. The Study Guide's MATLAB note for Section 2.1 contains the matrix notation students will need for a project on matrix multiplication. The appendices in the Study Guide have the corresponding material for Mathematica, Maple, and the TI-83+/84+/89 calculators.

.

1.9 SOLUTIONS

Notes: This section is optional if you plan to treat linear transformations only lightly, but many instructors will want to cover at least Theorem 10 and a few geometric examples. Exercises 15 and 16 illustrate a fast way to solve Exercises 17–22 without explicitly computing the images of the standard basis.

The purpose of introducing *one-to-one* and *onto* is to prepare for the term *isomorphism* (in Section 4.4) and to acquaint math majors with these terms. Mastery of these concepts would require a substantial digression, and some instructors prefer to omit these topics (and Exercises 25–40). In this case, you can use the result of Exercise 31 in Section 1.8 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.) The notions of one-to-one and onto appear in the Invertible Matrix Theorem (Section 2.3), but can be omitted there if desired

Exercises 25–28 and 31–36 offer fairly easy writing practice. Exercises 31, 32, and 35 provide important links to earlier material.

1. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

2. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$

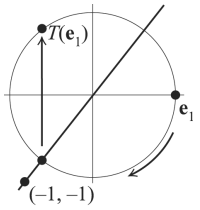
3. $T(\mathbf{e}_1) = \mathbf{e}_1 - 3\mathbf{e}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $T(\mathbf{e}_2) = \mathbf{e}_2$, $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

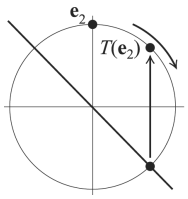
4. $T(\mathbf{e}_1) = \mathbf{e}_1$, $T(\mathbf{e}_2) = \mathbf{e}_2 + 2\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

5. $T(\mathbf{e}_1) = \mathbf{e}_2$, $T(\mathbf{e}_2) = -\mathbf{e}_1$. $A = \begin{bmatrix} \mathbf{e}_2 & -\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

6. $T(\mathbf{e}_1) = \mathbf{e}_2$, $T(\mathbf{e}_2) = -\mathbf{e}_1$. $A = \begin{bmatrix} \mathbf{e}_2 & -\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

7. Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. Since $\mathbf{e}_1$ is on the unit circle in the plane, it rotates through $-3\pi/4$ radians into a point on the unit circle that lies in the third quadrant and on the line $x_2 = x_1$ (that is, $y = x$ in more familiar notation). The point (–1,–1) is on the ine $x_2 = x_1$, but its distance from the origin is $\sqrt{2}$. So the rotational image of $\mathbf{e}_1$ is $(-1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2}, 1/\sqrt{2})$. Similarly, $\mathbf{e}_2$ rotates into a point on the unit circle that lies in the second quadrant and on the





line $x_2 = -x_1$, namely, $(1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(1/\sqrt{2}, 1/\sqrt{2})$. When the two calculations described above are written in vertical vector notation, the transformation's standard matrix $[T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$ is easily seen:

$$\mathbf{e}_1 \to \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{e}_2 \to \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

8. The horizontal shear maps $\mathbf{e}_1$ into $\mathbf{e}_1$, and then the reflection in the line $x_2 = -x_1$ maps $\mathbf{e}_1$ into $-\mathbf{e}_2$. (See Table 1.) The horizontal shear maps $\mathbf{e}_2$ into $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$. To find the image of $\mathbf{e}_2 + 2\mathbf{e}_1$ when it is reflected in the line $x_2 = -x_1$, use the fact that such a reflection is a linear transformation. So, the image of $\mathbf{e}_2 + 2\mathbf{e}_1$ is the same linear combination of the images of $\mathbf{e}_2$ and $\mathbf{e}_1$, namely, $-\mathbf{e}_1 + 2(-\mathbf{e}_2) = -\mathbf{e}_1 - 2\mathbf{e}_2$. To summarize,

$$\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_2 \text{ and } \mathbf{e}_2 \to \mathbf{e}_2 + 2\mathbf{e}_1 \to -\mathbf{e}_1 - 2\mathbf{e}_2, \text{ so } A = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

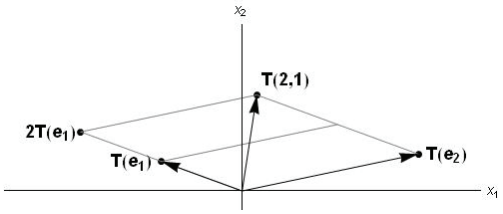
9. $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_2$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$, so $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

10. $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$, so $A = \begin{bmatrix} \mathbf{e}_2 & -\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

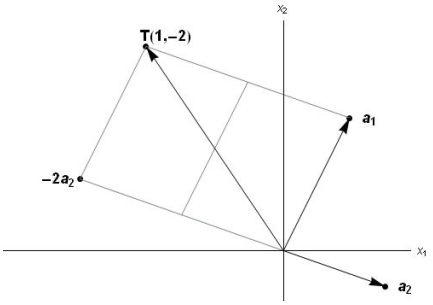
11. The transformation T described maps $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_1$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_2$. A rotation through π radians also maps $\mathbf{e}_1$ into $-\mathbf{e}_1$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_2$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

12. The transformation T in Exercise 10 maps $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$. A rotation about the origin through $\pi/2$ radians also maps $\mathbf{e}_1$ into $\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_1$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

13. Since (2, 1)=2 $\mathbf{e}_1$+ $\mathbf{e}_2$, the image of (2, 1) under T is $2T(\mathbf{e}_1) + T(\mathbf{e}_2)$, by linearity of T. On the figure in the exercise, locate $2T(\mathbf{e}_1)$ and use it with $T(\mathbf{e}_2)$ to form the parallelogram shown below.



14. Since $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{a}_1 - 2\mathbf{a}_2$, when $\mathbf{x} = (1, -2)$, the image of $\mathbf{x}$ is located by forming the parallelogram shown below.



15. By inspection, $\begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$

16. By inspection, $\begin{bmatrix} 3 & -2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ x_1 + 4x_2 \\ x_2 \end{bmatrix}$

17. To express $T(\mathbf{x})$ as $A\mathbf{x}$, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors, and then fill in the entries in A by inspection, as done in Exercises 15 and 16. Note that since $T(\mathbf{x})$ and $\mathbf{x}$ have four entries, A must be a 4×4 matrix.

$$T(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

18. As in Exercise 17, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors. Since $\mathbf{x}$ has 2 entries, A has 2 columns. Since $T(\mathbf{x})$ has 4 entries, A has 4 rows.

$$\begin{bmatrix} x_1 + 4x_2 \\ 0 \\ x_1 - 3x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

19. Since $T(\mathbf{x})$ has 2 entries, A has 2 rows. Since $\mathbf{x}$ has 3 entries, A has 3 columns.

$$\begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

20. Since $T(\mathbf{x})$ has 1 entry, A has 1 row. Since $\mathbf{x}$ has 4 entries, A has 4 columns.

$$[3x_1 + 4x_3 - 2x_4] = [\quad A \quad]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [3 \quad 0 \quad 4 \quad -2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

21. $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, row reduce the augmented matrix: $\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 7 \\ 0 & \textcircled{1} & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

22. $T(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$, row reduce the augmented matrix:

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

23. **a**. True. See Theorem 10.

b. True. See Example 3.

c. False. See the paragraph before Table 1.

d. False. See the definition of *onto*. *Any* function from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps each vector onto another vector.

e. False. See Example 5.

24. **a**. False. See Theorem 12.

b. True. See Theorem 10.

c. True. See Theorem 10.

d. False. See the definition of one-to-one. Any *function* from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps a vector onto a single (unique) vector.

e. False. See Table 3.

25. A row interchange and a row replacement on the standard matrix A of the transformation T in Exercise 17 produce $\begin{bmatrix} \textcircled{1} & 2 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & 0 & \textcircled{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This matrix shows that A has only three pivot positions, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. Also, since A does not have a pivot in each row, the columns of A do not span $\mathbf{R}^4$. By Theorem 12, T does *not* map $\mathbf{R}^4$ onto $\mathbf{R}^4$.

26. The standard matrix A of the transformation T in Exercise 2 is 2×3. Its columns are linearly dependent because A has more columns than rows. So T is *not* one-to-one, by Theorem 12. Also, A is row equivalent to $\begin{bmatrix} \textcircled{1} & -2 & 3 \\ 0 & \textcircled{17} & -20 \end{bmatrix}$, which shows that the rows of A span $\mathbf{R}^2$. By Theorem 12, T maps $\mathbf{R}^3$onto $\mathbf{R}^2$.

27. The standard matrix A of the transformation T in Exercise 19 is $\begin{bmatrix} \textcircled{1} & -5 & 4 \\ 0 & \textcircled{1} & -6 \end{bmatrix}$. The columns of A are linearly dependent because A has more columns than rows. So T is *not* one-to-one, by Theorem 12. Also, A has a pivot in each row, so the rows of A span $\mathbf{R}^2$. By Theorem 12, T maps $\mathbf{R}^3$ onto $\mathbf{R}^2$.

28. The standard matrix A of the transformation T in Exercise 14 has linearly independent columns, because the figure in that exercise shows that $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiples. So T is one-to-one, by Theorem 12. Also, A must have a pivot in each column because the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. Thus, the echelon form of A is $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$. Since A has a pivot in each row, the columns of A span $\mathbf{R}^2$. So T maps $\mathbf{R}^2$ onto $\mathbf{R}^2$. An alternate argument for the second part is to observe directly from the figure in Exercise 14 that $\mathbf{a}_1$ and $\mathbf{a}_2$ span $\mathbf{R}^2$. This is more or less evident, based on experience with grids such as those in Figure 8 and Exercise 7 of Section 1.3.

29. By Theorem 12, the columns of the standard matrix A must be linearly independent and hence the equation $A\boldsymbol{x} = \mathbf{0}$ has no free variables. So each column of A must be a pivot column:

$A \sim \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. Note that T cannot be onto because of the shape of A.

30. By Theorem 12, the columns of the standard matrix A must span $\mathbb{R}^3$. By Theorem 4, the matrix must have a pivot in each row. There are four possibilities for the echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Note that T cannot be one-to-one because of the shape of A.

31. "T is one-to-one if and only if A has n pivot columns." By Theorem 12(b), T is one-to-one if and only if the columns of A are linearly independent. And from the statement in Exercise 30 in Section 1.7, the columns of A are linearly independent if and only if A has n pivot columns.

32. The transformation T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$, by Theorem 12. This happens if and only if A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A has m rows, this happens if and only if A has m pivot columns. Thus, "T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only A has m pivot columns."

33. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B, and let A be the standard matrix for T. By definition, $A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)]$, where $\mathbf{e}_j$ is the jth column of I_n. However, by matrix-vector multiplication, $T(\mathbf{e}_j) = B\mathbf{e}_j = \mathbf{b}_j$, the jth column of B. So $A = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n] = B$.

34. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^p$ and let c and d be scalars. Then

$T(S(c\mathbf{u} + d\mathbf{v})) = T(c \cdot S(\mathbf{u}) + d \cdot S(\mathbf{v}))$ because S is linear

$= c \cdot T(S(\mathbf{u})) + d \cdot T(S(\mathbf{v}))$ because T is linear

This calculation shows that the mapping $\mathbf{x} \to T(S(\mathbf{x}))$ is linear. See equation (4) in Section 1.8.

35. If $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then its standard matrix A has a pivot in each row, by Theorem 12 and by Theorem 4 in Section 1.4. So A must have at least as many columns as rows. That is, $m \le n$. When T is one-to-one, A must have a pivot in each column, by Theorem 12, so $m \ge n$.

36. The transformation T maps $\mathbf{R}^n$ *onto* $\mathbf{R}^m$ if and only if for each $\mathbf{y}$ in $\mathbf{R}^m$ *there exists* an $\mathbf{x}$ in $\mathbf{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.

37. **[M]** $\begin{bmatrix} -5 & 6 & -5 & -6 \\ 8 & 3 & -3 & 8 \\ 2 & 9 & 5 & -12 \\ -3 & 2 & 7 & -12 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There is no pivot in the fourth column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. (For a shorter argument, use the result of Exercise 31.)

38. **[M]** $\begin{bmatrix} 7 & 5 & 9 & -9 \\ 5 & 6 & 4 & -4 \\ 4 & 8 & 0 & 7 \\ -6 & -6 & 6 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$. Yes. There is a pivot in every column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. By Theorem 11, the transformation T is one-to-one. (For a shorter argument, use the result of Exercise 31.)

39. **[M]** $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 5 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There is not a pivot in every row, so the columns of the standard matrix do not span $\mathbf{R}^5$. By Theorem 12, the transformation T does *not* map $\mathbf{R}^5$ onto $\mathbf{R}^5$.

40. **[M]** $\begin{bmatrix} 9 & 43 & 5 & 6 & -1 \\ 14 & 15 & -7 & -5 & 4 \\ -8 & -6 & 12 & -5 & -9 \\ -5 & -6 & -4 & 9 & 8 \\ 13 & 14 & 15 & 3 & 11 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$. There is a pivot in every row, so the columns of the standard matrix span $\mathbf{R}^5$. By Theorem 12, the transformation T maps $\mathbf{R}^5$ onto $\mathbf{R}^5$.

1.10 SOLUTIONS

1. **a**. If x_1 is the number of servings of Cheerios and x_2 is the number of servings of 100% Natural Cereal, then x_1 and x_2 should satisfy

$$x_1 \begin{bmatrix} \text{nutrients} \\ \text{per serving} \\ \text{of Cheerios} \end{bmatrix} + x_2 \begin{bmatrix} \text{nutrients} \\ \text{per serving of} \\ \text{100\% Natural} \end{bmatrix} = \begin{bmatrix} \text{quantities} \\ \text{of nutrients} \\ \text{required} \end{bmatrix}$$

That is,

$$x_1\begin{bmatrix}110\\4\\20\\2\end{bmatrix}+x_2\begin{bmatrix}130\\3\\18\\5\end{bmatrix}=\begin{bmatrix}295\\9\\48\\8\end{bmatrix}$$

b. The equivalent matrix equation is $\begin{bmatrix}110 & 130\\4 & 3\\20 & 18\\2 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}295\\9\\48\\8\end{bmatrix}$. To solve this, row reduce the augmented matrix for this equation.

$$\begin{bmatrix}110 & 130 & 295\\4 & 3 & 9\\20 & 18 & 48\\2 & 5 & 8\end{bmatrix}\sim\begin{bmatrix}2 & 5 & 8\\4 & 3 & 9\\20 & 18 & 48\\110 & 130 & 295\end{bmatrix}\sim\begin{bmatrix}1 & 2.5 & 4\\4 & 3 & 9\\10 & 9 & 24\\110 & 130 & 295\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2.5 & 4\\0 & -7 & -7\\0 & -16 & -16\\0 & -145 & -145\end{bmatrix}\sim\begin{bmatrix}1 & 2.5 & 4\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 1.5\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

The desired nutrients are provided by 1.5 servings of Cheerios together with 1 serving of 100% Natural Cereal.

2. Set up nutrient vectors for one serving of Shredded Wheat (SW) and Kellogg's Crispix (Crp):

Nutrients:	SW	Crp
calories	160	110
protein	5	2
fiber	6	.1
fat	1	.4

a. Let $B=\begin{bmatrix}\text{SW} & \text{Crp}\end{bmatrix}=\begin{bmatrix}160 & 110\\5 & 2\\6 & .1\\1 & .4\end{bmatrix}$, $\mathbf{u}=\begin{bmatrix}3\\2\end{bmatrix}$.

Then $B\mathbf{u}$ lists the amounts of calories, protein, carbohydrate, and fat in a mixture of three servings of Shredded Wheat and two servings of Crispix.

b. **[M]** Let u_1 and u_2 be the number of servings of Shredded Wheat and Crispix, respectively. Can these numbers satisfy the equation $B\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}120\\3.2\\2.46\\.64\end{bmatrix}$? To find out, row reduce the augmented matrix

$$\begin{bmatrix} 160 & 110 & 130 \\ 5 & 2 & 3.2 \\ 6 & .1 & 2.46 \\ 1 & .4 & .64 \end{bmatrix} \sim \begin{bmatrix} 1 & .4 & .64 \\ 0 & 0 & 0 \\ 0 & -2.3 & -1.38 \\ 0 & 46 & 27.6 \end{bmatrix} \sim \begin{bmatrix} 1 & .4 & .64 \\ 0 & 46 & 27.6 \\ 0 & -2.3 & -1.38 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & .4 & .64 \\ 0 & 1 & .6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & .4 \\ 0 & 1 & .6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the system is consistent, it is possible for a mixture of the two creals to provide the desired nutrients. The mixture is .4 servings of Shredded Wheat and .6 servings of Crispix.

3. **a**. **[M]** Let x_1, x_2, and x_3 be the number of servings ofAnnies's Mac and Cheese, broccoli, and chicken, respectively, needed for the lunch. The values of x_1, x_2, and x_3 should satisfy

$$x_1 \begin{bmatrix} \text{nutrients} \\ \text{per serving} \\ \text{of Mac and Cheese} \end{bmatrix} + x_2 \begin{bmatrix} \text{nutrients} \\ \text{per serving} \\ \text{of broccoli} \end{bmatrix} + x_3 \begin{bmatrix} \text{nutrients} \\ \text{per serving} \\ \text{of chicken} \end{bmatrix} = \begin{bmatrix} \text{quantities} \\ \text{of nutrients} \\ \text{required} \end{bmatrix}$$

From the given data,

$$x_1 \begin{bmatrix} 270 \\ 10 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 51 \\ 5.4 \\ 5.2 \end{bmatrix} + x_3 \begin{bmatrix} 70 \\ 15 \\ 0 \end{bmatrix} = \begin{bmatrix} 400 \\ 30 \\ 10 \end{bmatrix}$$

To solve, row reduce the corresponding augmented matrix:

$$\begin{bmatrix} 270 & 51 & 70 & 400 \\ 10 & 5.4 & 15 & 30 \\ 2 & 5.2 & 0 & 10 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & .99 \\ 0 & 1 & 0 & 1.54 \\ 0 & 0 & 1 & .79 \end{bmatrix}$$

$$\mathbf{x} \approx \begin{bmatrix} .99 \\ 1.54 \\ .74 \end{bmatrix} = \begin{bmatrix} \text{servings of Mac and Cheese} \\ \text{servings of broccoli} \\ \text{servings of chicken} \end{bmatrix}$$

b. **[M]** Changing from Annie's Mac and Cheese to Annie's Whole Wheat Shells and White Cheddar changes the vector equation to

$$x_1 \begin{bmatrix} 260 \\ 9 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 51 \\ 5.4 \\ 5.2 \end{bmatrix} + x_3 \begin{bmatrix} 70 \\ 15 \\ 0 \end{bmatrix} = \begin{bmatrix} 400 \\ 30 \\ 10 \end{bmatrix}$$

To solve, row reduce the corresponding augmented matrix:

$$\begin{bmatrix} 260 & 51 & 70 & 400 \\ 9 & 5.4 & 15 & 30 \\ 5 & 5.2 & 0 & 10 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 1.09 \\ 0 & 1 & 0 & .88 \\ 0 & 0 & 1 & 1.03 \end{bmatrix}$$

$$\mathbf{x} \approx \begin{bmatrix} 1.09 \\ .88 \\ 1.03 \end{bmatrix} = \begin{bmatrix} \text{servings of Shells} \\ \text{servings of broccoli} \\ \text{servings of chicken} \end{bmatrix}$$

Notice that the number of servings of broccoli has decreased as was desired.

4. Here are the data, assembled from Table 1 and Exercise 4:

Nutrient	Mg of Nutrients/Unit				Nutrients Required (milligrams)
	milk	soy flour	whey	soy prot.	
protein	36	51	13	80	33
carboh.	52	34	74	0	45
fat	0	7	1.1	3.4	3
calcium	1.26	.19	.8	.18	.8

a. Let x_1, x_2, x_3, x_4 represent the number of units of nonfat milk, soy flour, whey, and isolated soy protein, respectively. These amounts must satisfy the following matrix equation

$$\begin{bmatrix} 36 & 51 & 13 & 80 \\ 52 & 34 & 74 & 0 \\ 0 & 7 & 1.1 & 3.4 \\ 1.26 & .19 & .8 & .18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 33 \\ 45 \\ 3 \\ .8 \end{bmatrix}$$

b. **[M]**

$$\begin{bmatrix} 36 & 51 & 13 & 80 & 33 \\ 52 & 34 & 74 & 0 & 45 \\ 0 & 7 & 1.1 & 3.4 & 3 \\ 1.26 & .19 & .8 & .18 & .8 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & .64 \\ 0 & 1 & 0 & 0 & .54 \\ 0 & 0 & 1 & 0 & -.09 \\ 0 & 0 & 0 & 1 & -.21 \end{bmatrix}$$

The "solution" is $x_1 = .64$, $x_2 = .54$, $x_3 = -.09$, $x_4 = -.21$. This solution is not feasible, because the mixture cannot include negative amounts of whey and isolated soy protein. Although the coefficients of these two ingredients are fairly small, they cannot be ignored. The mixture of .64 units of nonfat milk and .54 units of soy flour provide 50.6 g of protein, 51.6 g of carbohydrate, 3.8 g of fat, and .9 g of calcium. Some of these nutrients are nowhere close to the desired amounts.

5. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 11 \\ -5 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \text{Total of RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Current } I_4 \text{ does not flow in loop 1} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -5 \\ 10 \\ -1 \\ 0 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 10 \end{bmatrix}$, and $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{r}_4] = \begin{bmatrix} 11 & -5 & 0 & 0 \\ -5 & 10 & -1 & 0 \\ 0 & -1 & 9 & -2 \\ 0 & 0 & -2 & 10 \end{bmatrix}$.

Notice that each off-diagonal entry of R is negative (or zero). This happens because the loop current directions are all chosen in the same direction on the figure. (For each loop j, this choice forces the currents in other loops adjacent to loop j to flow in the direction opposite to current I_j.)

Next, set $\mathbf{v} = \begin{bmatrix} 50 \\ -40 \\ 30 \\ -30 \end{bmatrix}$. The voltages in loops 2 and 4 are negative because the battery orientation in each loop is opposite to the direction chosen for positive current flow. Thus, the equation $R\mathbf{i} = \mathbf{v}$ becomes

$$\begin{bmatrix} 11 & -5 & 0 & 0 \\ -5 & 10 & -1 & 0 \\ 0 & -1 & 9 & -2 \\ 0 & 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 50 \\ -40 \\ 30 \\ -30 \end{bmatrix}. \quad \textbf{[M]}: \text{ The solution is } \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 3.68 \\ -1.90 \\ 2.57 \\ -2.49 \end{bmatrix}.$$

6. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 6 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \text{Total of RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Current } I_4 \text{ does not flow in loop 1} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -1 \\ 9 \\ -4 \\ 0 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -4 \\ 7 \\ -2 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 7 \end{bmatrix}$, and $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{r}_4] = \begin{bmatrix} 6 & -1 & 0 & 0 \\ -1 & 9 & -4 & 0 \\ 0 & -4 & 7 & -2 \\ 0 & 0 & -2 & 7 \end{bmatrix}$. Set $\mathbf{v} = \begin{bmatrix} 30 \\ 20 \\ 40 \\ 10 \end{bmatrix}$. Then $R\mathbf{i} = \mathbf{v}$ becomes

$$\begin{bmatrix} 6 & -1 & 0 & 0 \\ -1 & 9 & -4 & 0 \\ 0 & -4 & 7 & -2 \\ 0 & 0 & -2 & 7 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 30 \\ 20 \\ 40 \\ 10 \end{bmatrix}. \quad \textbf{[M]}: \text{ The solution is } \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 6.36 \\ 8.14 \\ 11.73 \\ 4.78 \end{bmatrix}.$$

7. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 12 \\ -7 \\ 0 \\ -4 \end{bmatrix} \begin{array}{l} \text{Total of RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Voltage drop for } I_4 \text{ is negative; } I_4 \text{ flows in opposite direction} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -7 \\ 15 \\ -6 \\ 0 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -6 \\ 14 \\ -5 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} -4 \\ 0 \\ -5 \\ 13 \end{bmatrix}$, and $R = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{r}_4] = \begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix}$.

Notice that each off-diagonal entry of R is negative (or zero). This happens because the loop current directions are all chosen in the same direction on the figure. (For each loop j, this choice forces the currents in other loops adjacent to loop j to flow in the direction opposite to current I_j.)

Next, set $\mathbf{v} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$. Note the negative voltage in loop 4. The current direction chosen in loop 4 is opposed by the orientation of the voltage source in that loop. Thus $R\mathbf{i} = \mathbf{v}$ becomes

$$\begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$$

. **[M]**: The solution is $\mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 11.43 \\ 10.55 \\ 8.04 \\ 5.84 \end{bmatrix}$.

8. Loop 1: The resistance vector is

$$\mathbf{r}_1 = \begin{bmatrix} 9 \\ -1 \\ 0 \\ -1 \\ -4 \end{bmatrix} \begin{array}{l} \text{Total of RI voltage drops for current } I_1 \\ \text{Voltage drop for } I_2 \text{ is negative; } I_2 \text{ flows in opposite direction} \\ \text{Current } I_3 \text{ does not flow in loop 1} \\ \text{Voltage drop for } I_4 \text{ is negative; } I_4 \text{ flows in opposite direction} \\ \text{Voltage drop for } I_5 \text{ is negative; } I_5 \text{ flows in opposite direction} \end{array}$$

Loop 2: The resistance vector is

$$\mathbf{r}_2 = \begin{bmatrix} -1 \\ 7 \\ -2 \\ 0 \\ -3 \end{bmatrix} \begin{array}{l} \text{Voltage drop for } I_1 \text{ is negative; } I_1 \text{ flows in opposite direction} \\ \text{Total of RI voltage drops for current } I_2 \\ \text{Voltage drop for } I_3 \text{ is negative; } I_3 \text{ flows in opposite direction} \\ \text{Current } I_4 \text{ does not flow in loop 2} \\ \text{Voltage drop for } I_5 \text{ is negative; } I_5 \text{ flows in opposite direction} \end{array}$$

Also, $\mathbf{r}_3 = \begin{bmatrix} 0 \\ -2 \\ 10 \\ -3 \\ -3 \end{bmatrix}$, $\mathbf{r}_4 = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 7 \\ -2 \end{bmatrix}$, $\mathbf{r}_5 = \begin{bmatrix} -4 \\ -3 \\ -3 \\ -2 \\ 12 \end{bmatrix}$, and $R = \begin{bmatrix} 9 & -1 & 0 & -1 & -4 \\ -1 & 7 & -2 & 0 & -3 \\ 0 & -2 & 10 & -3 & -3 \\ -1 & 0 & -3 & 7 & -2 \\ -4 & -3 & -3 & -2 & 12 \end{bmatrix}$. Set $\mathbf{v} = \begin{bmatrix} 50 \\ -30 \\ 20 \\ -40 \\ 0 \end{bmatrix}$. Note the negative voltages for loops where the chosen current direction is opposed by the orientation of the voltage source in that loop. Thus $R\mathbf{i} = \mathbf{v}$ becomes:

$$\begin{bmatrix} 9 & -1 & 0 & -1 & -4 \\ -1 & 7 & -2 & 0 & -3 \\ 0 & -2 & 10 & -3 & -3 \\ -1 & 0 & -3 & 7 & -2 \\ -4 & -3 & -3 & -2 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 50 \\ -30 \\ 20 \\ -40 \\ 0 \end{bmatrix}.$$ **[M]** The solution is $\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 4.00 \\ -4.38 \\ -.90 \\ -5.80 \\ -.96 \end{bmatrix}$.

9. The population movement problems in this section assume that the total population is constant, with no migration or immigration. The statement that "about 7% of the city's population moves to the suburbs" means also that the rest of the city's population (93%) remain in the city. This determines the entries in the first column of the migration matrix (which concerns movement *from* the city).

From: City	Suburbs	To:
.93		City
.07		Suburbs

Likewise, if 5% of the suburban population moves to the city, then the other 95% remain in the suburbs. This determines the second column of the migration matrix:, $M = \begin{bmatrix} .93 & .05 \\ .07 & .95 \end{bmatrix}$. The difference equation is $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$. Also, $\mathbf{x}_0 = \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix}$

The population in 2011 (when $k = 1$) is $\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .93 & .05 \\ .07 & .95 \end{bmatrix} \begin{bmatrix} 800{,}000 \\ 500{,}000 \end{bmatrix} = \begin{bmatrix} 769{,}000 \\ 531{,}000 \end{bmatrix}$

The population in 2012 (when $k = 2$) is $\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .93 & .05 \\ .07 & .95 \end{bmatrix} \begin{bmatrix} 769{,}000 \\ 531{,}000 \end{bmatrix} = \begin{bmatrix} 741{,}720 \\ 558{,}280 \end{bmatrix}$

10. The data in the first sentence implies that the migration matrix has the form:

	From:	
City	Suburbs	To:
$\begin{bmatrix} \quad & .04 \\ .06 & \quad \end{bmatrix}$		City Suburbs

The remaining entries are determined by the fact that the numbers in each column must sum to 1. (For instance, if 6% of the city people move to the suburbs, then the rest, or 94%, remain in the city.) So the migration matrix is $M = \begin{bmatrix} .94 & .04 \\ .06 & .96 \end{bmatrix}$. The initial population is $\mathbf{x}_0 = \begin{bmatrix} 10,000,000 \\ 800,000 \end{bmatrix}$.

The population in 2011 (when $k = 1$) is $\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .94 & .04 \\ .06 & .96 \end{bmatrix}\begin{bmatrix} 10,000,000 \\ 800,000 \end{bmatrix} = \begin{bmatrix} 9,432,000 \\ 1,368,000 \end{bmatrix}$

The population in 2012 (when $k = 2$) is $\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .94 & .04 \\ .06 & .96 \end{bmatrix}\begin{bmatrix} 9,432,000 \\ 1,368,000 \end{bmatrix} = \begin{bmatrix} 8,920,800 \\ 1,879,200 \end{bmatrix}$

11. The problem concerns two groups of people–those living in California and those living outside California (and in the United States). It is reasonable, but not essential, to consider the people living inside California first. That is, the first entry in a column or row of a vector will concern the people living in California. With this choice, the migration matrix has the form:

	From:	
Calif.	Outside	To:
$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$		Calif. Outside

a. For the first column of the migration matrix M, compute

$$\frac{\begin{Bmatrix}\text{Calif. persons}\\ \text{who moved}\end{Bmatrix}}{\{\text{Total Calif. pop.}\}} = \frac{516,100}{31,524,000} = .016372$$

The other entry in the first column is 1 – .016372 = .983628. The exercise requests that 5 decimal places be used. So this number should be rounded to .98363. Whatever number of decimal places is used, it is important that the two entries sum to 1. So, for the first fraction, use .01637.

For the second column of M, compute $\dfrac{\begin{Bmatrix}\text{outside persons}\\ \text{who moved}\end{Bmatrix}}{\{\text{Total outside pop.}\}} = \dfrac{381,262}{228,680,000} = .00167$. The other entry is 1 – .00167 = .99833. Thus, the migration matrix is

	From:	
Calif.	Outside	To:
$\begin{bmatrix} .98363 & .00167 \\ .01637 & .99833 \end{bmatrix}$		Calif. Outside

b. **[M]** The initial vector is $\mathbf{x}_0 = (31.524, 228.680)$, with data in millions of persons. Since $\mathbf{x}_0$ describes the population in 1994, and $\mathbf{x}_1$ describes the population in 1995, the vector $\mathbf{x}_6$ describes the projected population for the year 2000, assuming that the migration rates remain constant and

there are no deaths, births, or migration. Here are the vectors $\mathbf{x}_0$ through $\mathbf{x}_6$ with the first 5 figures displayed. Numbers are in millions of persons:

$$\mathbf{x}_0 = \begin{bmatrix} 31.524 \\ 228.68 \end{bmatrix}, \begin{bmatrix} 31.390 \\ 228.82 \end{bmatrix}, \begin{bmatrix} 31.258 \\ 228.95 \end{bmatrix}, \begin{bmatrix} 31.129 \\ 229.08 \end{bmatrix}, \begin{bmatrix} 31.002 \\ 229.20 \end{bmatrix}, \begin{bmatrix} 30.877 \\ 229.33 \end{bmatrix}, \begin{bmatrix} 30.755 \\ 229.45 \end{bmatrix} = \mathbf{x}_6.$$

12. Set $M = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix}$. Then $\mathbf{x}_1 = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix} \begin{bmatrix} 295 \\ 55 \\ 150 \end{bmatrix} \approx \begin{bmatrix} 304 \\ 57 \\ 139 \end{bmatrix}$, and

$\mathbf{x}_2 = \begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix} \begin{bmatrix} 304 \\ 57 \\ 139 \end{bmatrix} \approx \begin{bmatrix} 312 \\ 58 \\ 130 \end{bmatrix}$. The entries in $\mathbf{x}_2$ give the approximate distribution of cars on Wednesday, two days after Monday.

13. **[M]** The order of entries in a column of a migration matrix must match the order of the columns. For instance, if the first column concerns the population in the city, then the first entry in *each* column must be the fraction of the population that moves to (or remains in) the city. In this case, the data in the exercise leads to $M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$

a. Some of the population vectors are

$$\mathbf{x}_5 = \begin{bmatrix} 523{,}293 \\ 476{,}707 \end{bmatrix}, \ \mathbf{x}_{10} = \begin{bmatrix} 472{,}737 \\ 527{,}263 \end{bmatrix}, \ \mathbf{x}_{15} = \begin{bmatrix} 439{,}417 \\ 560{,}583 \end{bmatrix}, \ \mathbf{x}_{20} = \begin{bmatrix} 417{,}456 \\ 582{,}544 \end{bmatrix}$$

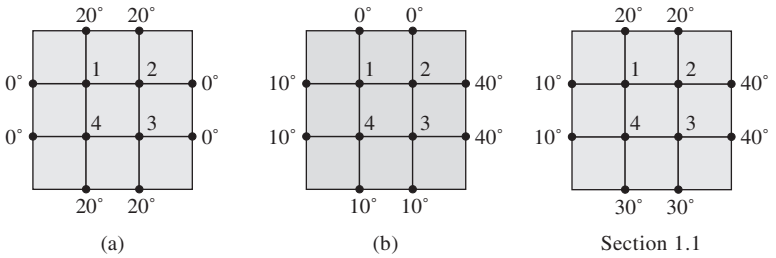
The data here shows that the city population is declining and the suburban population is increasing, but the changes in population each year seem to grow smaller.

b. When $\mathbf{x}_0 = \begin{bmatrix} 350{,}000 \\ 650{,}000 \end{bmatrix}$, the situation is different. Now

$$\mathbf{x}_5 = \begin{bmatrix} 358{,}523 \\ 641{,}477 \end{bmatrix}, \ \mathbf{x}_{10} = \begin{bmatrix} 364{,}140 \\ 635{,}860 \end{bmatrix}, \ \mathbf{x}_{15} = \begin{bmatrix} 367{,}843 \\ 632{,}157 \end{bmatrix}, \ \mathbf{x}_{20} = \begin{bmatrix} 370{,}283 \\ 629{,}717 \end{bmatrix}$$

The city population is increasing slowly and the suburban population is decreasing. No other conclusions are expected. (This example will be analyzed in greater detail later in the text.)

14. Here are Figs. (a) and (b) for Exercise 13, followed by the figure for Exercise 34 in Section 1.1:



(a) (b) Section 1.1

For Fig. (a), the equations are

$$\begin{aligned} 4T_1 &= 0 + 20 + T_2 + T_4 \\ 4T_2 &= T_1 + 20 + 0 + T_3 \\ 4T_3 &= T_4 + T_2 + 0 + 20 \\ 4T_4 &= 0 + T_1 + T_3 + 20 \end{aligned}$$

To solve the system, rearrange the equations and row reduce the augmented matrix. Interchanging rows 1 and 4 speeds up the calculations. The first five steps are shown in detail.

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 20 \\ -1 & 4 & -1 & 0 & 20 \\ 0 & -1 & 4 & -1 & 20 \\ -1 & 0 & -1 & 4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ -1 & 4 & -1 & 0 & 20 \\ 0 & -1 & 4 & -1 & 20 \\ 4 & -1 & 0 & -1 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & -1 & 4 & -1 & 20 \\ 0 & -1 & -4 & 15 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 4 & -1 & 20 \\ 0 & -1 & -4 & 15 & 100 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -2 & 20 \\ 0 & 0 & -4 & 14 & 100 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -20 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -2 & 20 \\ 0 & 0 & 0 & 12 & 120 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

For Fig (b), the equations are

$$\begin{aligned} 4T_1 &= 10 + 0 + T_2 + T_4 \\ 4T_2 &= T_1 + 0 + 40 + T_3 \\ 4T_3 &= T_4 + T_2 + 40 + 10 \\ 4T_4 &= 10 + T_1 + T_3 + 10 \end{aligned}$$

Rearrange the equations and row reduce the augmented matrix:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 10 \\ -1 & 4 & -1 & 0 & 40 \\ 0 & -1 & 4 & -1 & 50 \\ -1 & 0 & -1 & 4 & 20 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 17.5 \\ 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & 12.5 \end{bmatrix}$$

a. Here are the solution temperatures for the three problems studied:

Fig. (a) in Exercise 14 of Section 1.10: (10, 10, 10, 10)

Fig. (b) in Exercise 14 of Section 1.10: (10, 17.5, 20, 12.5)

Figure for Exercises 34 in Section 1.1 (20, 27.5, 30, 22.5)

When the solutions are arranged this way, it is evident that the third solution is the sum of the first two solutions. What might not be so evident is that list of boundary temperatures of the third problem is the sum of the lists of boundary temperatures of the first two problems. (The temperatures are listed clockwise, starting at the left of T_1.)

Fig. (a): (0, 20, 20, 0, 0, 20, 20, 0)

Fig. (b): (10, 0, 0, 40, 40, 10, 10, 10)

Fig. from Section 1.1: (10, 20, 20, 40, 40, 30, 30, 10)

b. When the boundary temperatures in Fig. (a) are multiplied by 3, the new interior temperatures are also multiplied by 3.

c. The correspondence from the list of eight boundary temperatures to the list of four interior temperatures is a linear transformation. A verification of this statement is not expected. However, it can be shown that the solutions of the steady-state temperature problem here satisfy a superposition principle. The system of equations that approximate the interior temperatures can

be written in the form $A\mathbf{x} = \mathbf{b}$, where A is determined by the arrangement of the four interior points on the plate and $\mathbf{b}$ is a vector in $\mathbf{R}^4$ determined by the boundary temperatures.

Note: The MATLAB box in the *Study Guide* for Section 1.10 discusses scientific notation and shows how to generate a matrix whose columns list the vectors $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, …, determined by an equation $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $k = 0, 1, \ldots$.

Chapter 1 SUPPLEMENTARY EXERCISES

1. **a**. False. (The word "reduced" is missing.) Counterexample:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The matrix A is row equivalent to matrices B and C, both in echelon form.

b. False. Counterexample: Let A be any $n \times n$ matrix with fewer than n pivot columns. Then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. (Theorem 2 in Section 1.2 says that a system has either zero, one, or infinitely many solutions, but it does not say that a system with infinitely many solutions exists. Some counterexample is needed.)

c. True. If a linear system has more than one solution, it is a consistent system and has a free variable. By the Existence and Uniqueness Theorem in Section 1.2, the system has infinitely many solutions.

d. False. Counterexample: The following system has no free variables and no solution:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 5 \\ x_1 + x_2 &= 2 \end{aligned}$$

e. True. See the box after the definition of elementary row operations, in Section 1.1. If $[A \quad \mathbf{b}]$ is transformed into $[C \quad \mathbf{d}]$ by elementary row operations, then the two augmented matrices are row equivalent.

f. True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the nonhomogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

g. False. For the columns of A to span $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ must be consistent for *all* $\mathbf{b}$ in $\mathbf{R}^m$, not for just one vector $\mathbf{b}$ in $\mathbf{R}^m$.

h. False. *Any* matrix can be transformed by elementary row operations into reduced echelon form, but not every matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.

i. True. If A is row equivalent to B, then A can be transformed by elementary row operations first into B and then further transformed into the reduced echelon form U of B. Since the reduced echelon form of A is unique, it must be U.

j. False. Every equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution whether or not some variables are free.

k. True, by Theorem 4 in Section 1.4. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbf{R}^m$, then A must have a pivot position in every one of its m rows. If A has m pivot positions, then A has m pivot columns, each containing one pivot position.

l. False. The word "unique" should be deleted. Let A be any matrix with m pivot columns but more than m columns altogether. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has m basic variables and at least one free variable. Thus the equation does not does not have a unique solution.

m. True. If A has n pivot positions, it has a pivot in each of its n columns and in each of its n rows. The reduced echelon form has a 1 in each pivot position, so the reduced echelon form is the $n \times n$ identity matrix.

n. True. Both matrices A and B can be row reduced to the 3×3 identity matrix, as discussed in the previous question. Since the row operations that transform B into I_3 are reversible, A can be transformed first into I_3 and then into B.

o. True. The reason is essentially the same as that given for question f.

p. True. If the columns of A span $\mathbf{R}^m$, then the reduced echelon form of A is a matrix U with a pivot in each row, by Theorem 4 in Section 1.4. Since B is row equivalent to A, B can be transformed by row operations first into A and then further transformed into U. Since U has a pivot in each row, so does B. By Theorem 4, the columns of B span $\mathbf{R}^m$.

q. False. See Example 5 in Section 1.7.

r. True. Any set of three vectors in $\mathbf{R}^2$ would have to be linearly dependent, by Theorem 8 in Section 1.7.

s. False. If a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ were to span $\mathbf{R}^5$, then the matrix $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$ would have a pivot position in each of its five rows, which is impossible since A has only four columns.

t. True. The vector $-\mathbf{u}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, namely, $-\mathbf{u} = (-1)\mathbf{u} + 0\mathbf{v}$.

u. False. If $\mathbf{u}$ and $\mathbf{v}$ are multiples, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a line, and $\mathbf{w}$ need not be on that line.

v. False. Let $\mathbf{u}$ and $\mathbf{v}$ be any linearly independent pair of vectors and let $\mathbf{w} = 2\mathbf{v}$. Then $\mathbf{w} = 0\mathbf{u} + 2\mathbf{v}$, so $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. However, $\mathbf{u}$ cannot be a linear combination of $\mathbf{v}$ and $\mathbf{w}$ because if it were, $\mathbf{u}$ would be a multiple of $\mathbf{v}$. That is not possible since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

w. False. The statement would be true if the condition $\mathbf{v}_1$ is not zero were present. See Theorem 7 in Section 1.7. However, if $\mathbf{v}_1 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, no matter what else might be true about $\mathbf{v}_2$ and $\mathbf{v}_3$.

x. True. "Function" is another word used for "transformation" (as mentioned in the definition of "transformation" in Section 1.8), and a linear transformation is a special type of transformation.

y. True. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to map $\mathbf{R}^5$ onto $\mathbf{R}^6$, the matrix A would have to have a pivot in every row and hence have six pivot columns. This is impossible because A has only five columns.

z. False. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one, A must have a pivot in each column. Since A has n columns and m pivots, m might be less than n.

2. If $a \neq 0$, then $x = b/a$; the solution is unique. If $a = 0$, and $b \neq 0$, the solution set is empty, because $0x = 0 \neq b$. If $a = 0$ and $b = 0$, the equation $0x = 0$ has infinitely many solutions.

3. **a**. Any consistent linear system whose echelon form is

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Any consistent linear system whose coefficient matrix has reduced echelon form I_3.

c. Any inconsistent linear system of three equations in three variables.

4. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$. A solution of $A\mathbf{x} = \mathbf{b}$ exists for all $\mathbf{b}$ because there is a pivot in each row of A. Each solution is unique because there are no free variables.

5. a. $\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}$. If $h = 12$ and $k \neq 2$, the second row of the augmented matrix indicates an inconsistent system of the form $0x_2 = b$, with b nonzero. If $h = 12$, and $k = 2$, there is only one nonzero equation, and the system has infinitely many solutions. Finally, if $h \neq 12$, the coefficient matrix has two pivots and the system has a unique solution.

b. $\begin{bmatrix} -2 & h & 1 \\ 6 & k & -2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-2} & h & 1 \\ 0 & k+3h & 1 \end{bmatrix}$. If $k + 3h = 0$, the system is inconsistent. Otherwise, the coefficient matrix has two pivots and the system has a unique solution.

6. a. Set $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$." Or, "Determine if $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$." To do this, compute $\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{4} & -2 & 7 & -5 \\ 0 & \textcircled{1} & -4 & 7 \end{bmatrix}$. The system is consistent, so $\mathbf{b}$ *is* in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. Set $A = \begin{bmatrix} 4 & -2 & 7 \\ 8 & -3 & 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. "Determine if $\mathbf{b}$ is a linear combination of the columns of A."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if $\mathbf{b}$ is in the range of T."

7. a. Set $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. "Determine if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbf{R}^3$." To do this, row reduce $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$:

$\begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 \\ 0 & -9 & -4 \\ 0 & 9 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -4 & -2 \\ 0 & \textcircled{-9} & -4 \\ 0 & 0 & 0 \end{bmatrix}$. The matrix does not have a pivot in each row, so its columns do not span $\mathbf{R}^3$, by Theorem 4 in Section 1.4.

b. Set $A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}$. "Determine if the columns of A span $\mathbf{R}^3$."

c. Define $T(\mathbf{x}) = A\mathbf{x}$. "Determine if T maps $\mathbf{R}^3$ onto $\mathbf{R}^3$."

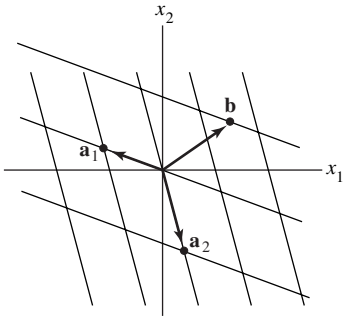
8. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$ **b.** $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

9. The first line is the line spanned by $\begin{bmatrix}1\\2\end{bmatrix}$. The second line is spanned by $\begin{bmatrix}2\\1\end{bmatrix}$. So the problem is to write $\begin{bmatrix}5\\6\end{bmatrix}$ as the sum of a multiple of $\begin{bmatrix}1\\2\end{bmatrix}$ and a multiple of $\begin{bmatrix}2\\1\end{bmatrix}$. That is, find x_1 and x_2 such that $x_1\begin{bmatrix}2\\1\end{bmatrix}+x_2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$. Reduce the augmented matrix for this equation:

$$\begin{bmatrix}2&1&5\\1&2&6\end{bmatrix}\sim\begin{bmatrix}1&2&6\\2&1&5\end{bmatrix}\sim\begin{bmatrix}1&2&6\\0&-3&-7\end{bmatrix}\sim\begin{bmatrix}1&2&6\\0&1&7/3\end{bmatrix}\sim\begin{bmatrix}1&0&4/3\\0&1&7/3\end{bmatrix}$$

Thus, $\begin{bmatrix}5\\6\end{bmatrix}=\frac{4}{3}\begin{bmatrix}2\\1\end{bmatrix}+\frac{7}{3}\begin{bmatrix}1\\2\end{bmatrix}$ or $\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}8/3\\4/3\end{bmatrix}+\begin{bmatrix}7/3\\14/3\end{bmatrix}$.

10. The line through $\mathbf{a}_1$ and the origin and the line through $\mathbf{a}_2$ and the origin determine a "grid" on the x_1x_2-plane as shown below. Every point in $\mathbf{R}^2$ can be described uniquely in terms of this grid. Thus, $\mathbf{b}$ can be reached from the origin by traveling a certain number of units in the $\mathbf{a}_1$-direction and a certain number of units in the $\mathbf{a}_2$-direction.



11. A solution set is a line when the system has one free variable. If the coefficient matrix is 2×3, then two of the columns should be pivot columns. For instance, take $\begin{bmatrix}1&2&*\\0&3&*\end{bmatrix}$. Put anything in column 3. The resulting matrix will be in echelon form. Make one row replacement operation on the second row to create a matrix *not* in echelon form, such as $\begin{bmatrix}1&2&1\\0&3&1\end{bmatrix}\sim\begin{bmatrix}1&2&1\\1&5&2\end{bmatrix}$

12. A solution set is a plane where there are two free variables. If the coefficient matrix is 2×3, then only one column can be a pivot column. The echelon form will have all zeros in the second row. Use a row replacement to create a matrix not in echelon form. For instance, let $A=\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}$.

13. The reduced echelon form of A looks like $E=\begin{bmatrix}1&0&*\\0&1&*\\0&0&0\end{bmatrix}$. Since E is row equivalent to A, the equation $E\mathbf{x}=\mathbf{0}$ has the same solutions as $A\mathbf{x}=\mathbf{0}$. Thus $\begin{bmatrix}1&0&*\\0&1&*\\0&0&0\end{bmatrix}\begin{bmatrix}3\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

By inspection, $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

14. Row reduce the augmented matrix for $x_1\begin{bmatrix} 1 \\ a \end{bmatrix} + x_2\begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (*).

$$\begin{bmatrix} 1 & a & 0 \\ a & a+2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & a & 0 \\ 0 & a+2-a^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & (2-a)(1+a) & 0 \end{bmatrix}$$

The equation (*) has a nontrivial solution only when $(2 - a)(1 + a) = 0$. So the vectors are linearly independent for all a except $a = 2$ and $a = -1$.

15. **a**. If the three vectors are linearly independent, then a, c, and f must all be nonzero. (The converse is true, too.) Let A be the matrix whose columns are the three linearly independent vectors. Then A must have three pivot columns. (See Exercise 30 in Section 1.7, or realize that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so there can be no free variables in the system of equations.) Since A is 3×3, the pivot positions are exactly where a, c, and f are located.

b. The numbers $a, \ldots, f$ can have any values. Here's why. Denote the columns by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Observe that v_1 is not the zero vector. Next, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ because the third entry of $\mathbf{v}_2$ is nonzero. Finally, $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because the fourth entry of $\mathbf{v}_3$ is nonzero. By Theorem 7 in Section 1.7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

16. Denote the columns from right to left by $\mathbf{v}_1, \ldots, \mathbf{v}_4$. The "first" vector $\mathbf{v}_1$ is nonzero, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ (because the third entry of $\mathbf{v}_2$ is nonzero), and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (because the second entry of $\mathbf{v}_3$ is nonzero). Finally, by looking at first entries in the vectors, $\mathbf{v}_4$ cannot be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. By Theorem 7 in Section 1.7, the columns are linearly independent.

17. Here are two arguments. The first is a "direct" proof. The second is called a "proof by contradiction."

i. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, $\mathbf{v}_1 \neq \mathbf{0}$. Also, Theorem 7 shows that $\mathbf{v}_2$ cannot be a multiple of $\mathbf{v}_1$, and $\mathbf{v}_3$ cannot be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. By hypothesis, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Thus, by Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ cannot be a linearly dependent set and so must be linearly independent.

ii. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Then by Theorem 7, one of the vectors in the set is a linear combination of the preceding vectors. This vector cannot be $\mathbf{v}_4$ because $\mathbf{v}_4$ is *not* in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, none of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combinations of the preceding vectors, by Theorem 7. So the linear dependence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is impossible. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

18. Suppose that c_1 and c_2 are constants such that

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0} \qquad (*)$$

Then $(c_1 + c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, both $c_1 + c_2 = 0$ and $c_2 = 0$. It follows that both c_1 and c_2 in (*) must be zero, which shows that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

19. Let M be the line through the origin that is parallel to the line through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are both on M. So one of these two vectors is a multiple of the other, say $\mathbf{v}_2 - \mathbf{v}_1 = k(\mathbf{v}_3 - \mathbf{v}_1)$. This equation produces a linear dependence relation $(k - 1)\mathbf{v}_1 + \mathbf{v}_2 - k\mathbf{v}_3 = \mathbf{0}$.

A second solution: A parametric equation of the line is $\mathbf{x} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1)$. Since $\mathbf{v}_3$ is on the line, there is some t_0 such that $\mathbf{v}_3 = \mathbf{v}_1 + t_0(\mathbf{v}_2 - \mathbf{v}_1) = (1 - t_0)\mathbf{v}_1 + t_0\mathbf{v}_2$. So $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

20. If $T(\mathbf{u}) = \mathbf{v}$, then since T is linear,

$$T(-\mathbf{u}) = T((-1)\mathbf{u}) = (-1)T(\mathbf{u}) = -\mathbf{v}.$$

21. Either compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$ to make the columns of A, or write the vectors vertically in the definition of T and fill in the entries of A by inspection:

$$A\mathbf{x} = \begin{bmatrix} ? & ? & ? \\ ? & A & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22. By Theorem 12 in Section 1.9, the columns of A span $\mathbf{R}^3$. By Theorem 4 in Section 1.4, A has a pivot in each of its three rows. Since A has three columns, each column must be a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables, and the columns of A are linearly independent. By Theorem 12 in Section 1.9, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

23. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ implies that $\begin{array}{rcrcl} 4a & - & 3b & = & 5 \\ 3a & + & 4b & = & 0 \end{array}$. Solve:

$$\begin{bmatrix} 4 & -3 & 5 \\ 3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 25/4 & -15/4 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 16/5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \end{bmatrix}$$

Thus $a = 4/5$ and $b = -3/5$.

24. The matrix equation displayed gives the information $2a - 4b = 2\sqrt{5}$ and $4a + 2b = 0$. Solve for a and b:

$$\begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 0 & 10 & -4\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & \sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/\sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix}$$

So $a = 1/\sqrt{5},\ b = -2/\sqrt{5}$.

25. **a**. The vector lists the number of three-, two-, and one-bedroom apartments provided when x_1 floors of plan A are constructed.

b. $x_1\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}$

c. **[M]** Solve $x_1\begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 136 \end{bmatrix}$

$$\begin{bmatrix} 3 & 4 & 5 & 66 \\ 7 & 4 & 3 & 74 \\ 8 & 8 & 9 & 136 \end{bmatrix} \sim \cdots \begin{bmatrix} 1 & 0 & -1/2 & 2 \\ 0 & 1 & 13/8 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{rcrcl} x_1 & & - & (1/2)x_3 & = & 2 \\ & x_2 & + & (13/8)x_3 & = & 15 \\ & & & 0 & = & 0 \end{array}$$

The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + (1/2)x_3 \\ 15 - (13/8)x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ -13/8 \\ 1 \end{bmatrix}$$

However, the only feasible solutions must have whole numbers of floors for each plan. Thus, x_3 must be a multiple of 8, to avoid fractions. One solution, for $x_3 = 0$, is to use 2 floors of plan A and 15 floors of plan B. Another solution, for $x_3 = 8$, is to use 6 floors of plan A , 2 floors of plan B, and 8 floors of plan C. These are the only feasible solutions. A larger positive multiple of 8 for x_3 makes x_2 negative. A negative value for x_3, of course, is not feasible either.