2 Matrix Algebra

2.1 SOLUTIONS

Notes: The definition here of a matrix product AB gives the proper view of AB for nearly all matrix calculations. (The dual fact about the rows of A and the rows of AB is seldom needed, mainly because vectors here are usually written as columns.) I assign Exercise 13 and most of Exercises 17–22 to reinforce the definition of AB.

Exercises 23 and 24 are used in the proof of the Invertible Matrix Theorem, in Section 2.3. Exercises 23–25 are mentioned in a footnote in Section 2.2. A class discussion of the solutions of Exercises 23–25 can provide a transition to Section 2.2. Or, these exercises could be assigned after starting Section 2.2.

Exercises 27 and 28 are optional, but they are mentioned in Example 4 of Section 2.4. Outer products also appear in Exercises 31–34 of Section 4.6 and in the spectral decomposition of a symmetric matrix, in Section 7.1. Exercises 29–33 provide good training for mathematics majors.

1. $-2A = (-2)\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}$. Next, use $B - 2A = B + (-2A)$:

$$B - 2A = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}.$$

The product AC is not defined because the number of columns of A does not match the number of rows of C. $CD = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2(-1) & 1\cdot 5 + 2\cdot 4 \\ -2\cdot 3 + 1(-1) & -2\cdot 5 + 1\cdot 4 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$. For mental computation, the row-column rule is probably easier to use than the definition.

2. $A + 2B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + 2\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 2+14 & 0-10 & -1+2 \\ 4+2 & -5-8 & 2-6 \end{bmatrix} = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}$

The expression $3C - E$ is not defined because $3C$ has 2 columns and $-E$ has only 1 column.

$$CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1\cdot 7 + 2\cdot 1 & 1(-5) + 2(-4) & 1\cdot 1 + 2(-3) \\ -2\cdot 7 + 1\cdot 1 & -2(-5) + 1(-4) & -2\cdot 1 + 1(-3) \end{bmatrix} = \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$$

The product EB is not defined because the number of columns of E does not match the number of rows of B.

3. $3I_2 - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 3-4 & 0-(-1) \\ 0-5 & 3-(-2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$

$(3I_2)A = 3(I_2A) = 3\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}$, or

$(3I_2)A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 3\cdot 4+0 & 3(-1)+0 \\ 0+3\cdot 5 & 0+3(-2) \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}$

4. $A - 5I_3 = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$

$(5I_3)A = 5(I_3A) = 5A = 5\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}$, or

$(5I_3)A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 5\cdot 9+0+0 & 5(-1)+0+0 & 5\cdot 3+0+0 \\ 0+5(-8)+0 & 0+5\cdot 7+0 & 0+5(-6)+0 \\ 0+0+5(-4) & 0+0+5\cdot 1 & 0+0+5\cdot 8 \end{bmatrix}$

$= \begin{bmatrix} 45 & -5 & 15 \\ -45 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix}$

5. **a**. $A\mathbf{b}_1 = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ 12 \end{bmatrix}$, $A\mathbf{b}_2 = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}$

$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1\cdot 3+2(-2) & -1(-2)+2\cdot 1 \\ 5\cdot 3+4(-2) & 5(-2)+4\cdot 1 \\ 2\cdot 3-3(-2) & 2(-2)-3\cdot 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

6. **a**. $A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$, $A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$

$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 - 2\cdot 2 & 4\cdot 3 - 2(-1) \\ -3\cdot 1 + 0\cdot 2 & -3\cdot 3 + 0(-1) \\ 3\cdot 1 + 5\cdot 2 & 3\cdot 3 + 5(-1) \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$

7. Since A has 3 columns, B must match with 3 rows. Otherwise, AB is undefined. Since AB has 7 columns, so does B. Thus, B is 3×7.

8. The number of rows of B matches the number of rows of BC, so B has 3 rows.

9. $AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$, while $BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$.

Then $AB = BA$ if and only if $-10 + 5k = 15$ and $-9 = 6 - 3k$, which happens if and only if $k = 5$.

10. $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$, $AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

11. $AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$

$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$

Right-multiplication (that is, multiplication on the right) by the diagonal matrix D multiplies each *column* of A by the corresponding diagonal entry of D. Left-multiplication by D multiplies each *row* of A by the corresponding diagonal entry of D. To make $AB = BA$, one can take B to be a multiple of I_3. For instance, if $B = 4I_3$, then AB and BA are both the same as $4A$.

12. Consider $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$. To make $AB = 0$, one needs $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$. By inspection of A, a suitable $\mathbf{b}_1$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Example: $B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

13. Use the definition of AB written in reverse order: $[A\mathbf{b}_1 \ \cdots \ A\mathbf{b}_p] = A[\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$. Thus

$[Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_p] = QR$, when $R = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_p]$.

14. By definition, $UQ = U[\mathbf{q}_1 \ \cdots \ \mathbf{q}_4] = [U\mathbf{q}_1 \ \cdots \ U\mathbf{q}_4]$. From Example 6 of Section 1.8, the vector $U\mathbf{q}_1$ lists the total costs (material, labor, and overhead) corresponding to the amounts of products B and C specified in the vector $\mathbf{q}_1$. That is, the first column of UQ lists the total costs for materials, labor, and overhead used to manufacture products B and C during the first quarter of the year. Columns 2, 3, and 4 of UQ list the total amounts spent to manufacture B and C during the 2nd, 3rd, and 4th quarters, respectively.

15. **a**. False. See the definition of AB.

b. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.

c. True. See Theorem 2(b), read right to left.

d. True. See Theorem 3(b), read right to left.

e. False. The phrase "in the same order" should be "in the reverse order." See the box after Theorem 3.

16. **a**. False. AB must be a 3×3 matrix, but the formula for AB implies that it is 3×1. The plus signs should be just spaces (between columns). This is a common mistake.

b. True. See the box after Example 6.

c. False. The left-to-right order of B and C cannot be changed, in general.

d. False. See Theorem 3(d).

e. True. This general statement follows from Theorem 3(b).

17. Since $\begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} = AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$, the first column of B satisfies the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Row reduction: $\begin{bmatrix} A & A\mathbf{b}_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$. So $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$. Similarly, $\begin{bmatrix} A & A\mathbf{b}_2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$.

Note: An alternative solution of Exercise 17 is to row reduce $[A \quad A\mathbf{b}_1 \quad A\mathbf{b}_2]$ with one sequence of row operations. This observation can prepare the way for the inversion algorithm in Section 2.2.

18. The first two columns of AB are $A\mathbf{b}_1$ and $A\mathbf{b}_2$. They are equal since $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal.

19. (A solution is in the text). Write $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$. By definition, the third column of AB is $A\mathbf{b}_3$. By hypothesis, $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$. So $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$, by a property of matrix-vector multiplication. Thus, the third column of AB is the sum of the first two columns of AB.

20. The second column of AB is also all zeros because $A\mathbf{b}_2 = A\mathbf{0} = \mathbf{0}$.

21. Let $\mathbf{b}_p$ be the last column of B. By hypothesis, the last column of AB is zero. Thus, $A\mathbf{b}_p = \mathbf{0}$. However, $\mathbf{b}_p$ is not the zero vector, because B has no column of zeros. Thus, the equation $A\mathbf{b}_p = \mathbf{0}$ is a linear dependence relation among the columns of A, and so the columns of A are linearly dependent.

Note: The text answer for Exercise 21 is, "The columns of A are linearly dependent. Why?" The *Study Guide* supplies the argument above in case a student needs help.

22. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = A\mathbf{0}$ and $(AB)\mathbf{x} = \mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

23. If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and so $I_n\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. So every variable is a basic variable and every column of A is a pivot column. (A variation of this argument could be made using linear independence and Exercise 30 in Section 1.7.) Since each pivot is in a different row, A must have at least as many rows as columns.

24. Take any $\mathbf{b}$ in $\mathbb{R}^m$. By hypothesis, $AD\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Rewrite this equation as $A(D\mathbf{b}) = \mathbf{b}$. Thus, the vector $\mathbf{x} = D\mathbf{b}$ satisfies $A\mathbf{x} = \mathbf{b}$. This proves that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$. By Theorem 4 in Section 1.4, A has a pivot position in each row. Since each pivot is in a different column, A must have at least as many columns as rows.

25. By Exercise 23, the equation $CA = I_n$ implies that (number of rows in A) $\geq$ (number of columns), that is, $m \geq n$. By Exercise 24, the equation $AD = I_m$ implies that (number of rows in A) $\leq$ (number of columns), that is, $m \leq n$. Thus $m = n$. To prove the second statement, observe that $DAC = (DA)C = I_nC = C$, and also $DAC = D(AC) = DI_m = D$. Thus $C = D$. A shorter calculation is
$C = I_nC = (DA)C = D(AC) = DI_n = D$

26. Write $I_3 = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$ and $D = [\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3]$. By definition of AD, the equation $AD = I_3$ is equivalent |to the three equations $A\mathbf{d}_1 = \mathbf{e}_1$, $A\mathbf{d}_2 = \mathbf{e}_2$, and $A\mathbf{d}_3 = \mathbf{e}_3$. Each of these equations has at least one solution because the columns of A span $\mathbb{R}^3$. (See Theorem 4 in Section 1.4.) Select one solution of each equation and use them for the columns of D. Then $AD = I_3$.

27. The product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which usually is identified with a real number and is written without the matrix brackets.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a + 3b - 4c\,, \quad \mathbf{v}^T\mathbf{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = -2a + 3b - 4c$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

28. Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is,
$\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.

29. The (i, j)-entry of $A(B + C)$ equals the (i, j)-entry of $AB + AC$, because

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

The (i, j)-entry of $(B + C)A$ equals the (i, j)-entry of $BA + CA$, because

$$\sum_{k=1}^{n} (b_{ik} + c_{ik})a_{kj} = \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

30. The (i, j)))-entries of $r(AB)$, $(rA)B$, and $A(rB)$ are all equal, because $r\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} (ra_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(rb_{kj})$.

31. Use the definition of the product I_mA and the fact that $I_m\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^m$.

$I_mA = I_m[\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n] = [I_m\mathbf{a}_1 \quad \cdots \quad I_m\mathbf{a}_n] = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n] = A$

32. Let $\mathbf{e}_j$ and $\mathbf{a}_j$ denote the jth columns of I_n and A, respectively. By definition, the jth column of AI_n *is* $A\mathbf{e}_j$, which is simply $\mathbf{a}_j$ because $\mathbf{e}_j$ has 1 in the jth position and zeros elsewhere. Thus corresponding columns of AI_n and A are equal. Hence $AI_n = A$.

33. The (i, j)-entry of $(AB)^T$ is the (j, i)-entry of AB, which is $a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$

The entries in row i of B^T are $b_{1i}, \ldots, b_{ni}$, because they come from column i of B. Likewise, the entries in column j of A^T are $a_{j1}, \ldots, a_{jn}$, because they come from row j of A. Thus the (i, j)-entry in B^TA^T is $a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$, as above.

34. Use Theorem 3(d), treating $\mathbf{x}$ as an $n \times 1$ matrix: $(AB\mathbf{x})^T = \mathbf{x}^T(AB)^T = \mathbf{x}^TB^TA^T$.

35. [**M**] The answer here depends on the choice of matrix program. For MATLAB, use the **`help`** command to read about **`zeros`**, **`ones`**, **`eye`**, and **`diag`**. For other programs see the appendices in the *Study Guide*. (The TI calculators have fewer single commands that produce special matrices.)

36. [**M**] The answer depends on the choice of matrix program. In MATLAB, the command **`rand(6,4)`** creates a 6×4 matrix with random entries uniformly distributed between 0 and 1. The command **`round(19*(rand(6,4)-.5))`** creates a random 6×4 matrix with integer entries between –9 and 9. The same result is produced by the command **`randomint`** in the Laydata Toolbox on text website. For other matrix programs see the appendices in the *Study Guide*.

37. [**M**] $(A + I)(A - I) - (A^2 - I) = 0$ for all 4×4 matrices. However, $(A + B)(A - B) - A^2 - B^2$ is the zero matrix only in the special cases when $AB = BA$. In general,$(A + B)(A - B) = A(A - B) + B(A - B)$ $= AA - AB + BA - BB$.

38. [**M**] The equality $(AB)^T = A^TB^T$ is very likely to be false for 4×4 matrices selected at random.

39. [**M**] The matrix S "shifts" the entries in a vector (a, b, c, d, e) to yield $(b, c, d, e, 0)$. The entries in S^2 result from applying S to the columns of S, and similarly for S^3, and so on. This explains the patterns of entries in the powers of S:

$$S^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S^5 is the 5×5 zero matrix. S^6 is also the 5×5 zero matrix.

40. [**M**] $A^5 = \begin{bmatrix} .3318 & .3346 & .3336 \\ .3346 & .3323 & .3331 \\ .3336 & .3331 & .3333 \end{bmatrix}, A^{10} = \begin{bmatrix} .333337 & .333330 & .333333 \\ .333330 & .333336 & .333334 \\ .333333 & .333334 & .333333 \end{bmatrix}$

The entries in A^{20} all agree with .3333333333 to 9 or 10 decimal places. The entries in A^{30} all agree with .33333333333333 to at least 14 decimal places. The matrices appear to approach the matrix $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$. Further exploration of this behavior appears in Sections 4.9 and 5.2.

Note: The MATLAB box in the *Study Guide* introduces basic matrix notation and operations, including the commands that create special matrices needed in Exercises 35, 36 and elsewhere. The *Study Guide* appendices treat the corresponding information for the other matrix programs.

2.2 SOLUTIONS

Notes: The text includes the matrix inversion algorithm at the end of the section because this topic is popular. Students like it because it is a simple mechanical procedure. The final subsection is independent of the inversion algorithm and is needed for Exercises 35 and 36.

Key Exercises: 8, 11–24, 35. (Actually, Exercise 8 is only helpful for some exercises in this section. Section 2.3 has a stronger result.) Exercises 23 and 24 are used in the proof of the Invertible Matrix Theorem (IMT) in Section 2.3, along with Exercises 23 and 24 in Section 2.1. I recommend letting students work on two or more of these four exercises before proceeding to Section 2.3. In this way students *participate* in the proof of the IMT rather than simply watch an instructor carry out the proof. Also, this activity will help students understand *why* the theorem is true.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}^{-1} = \frac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}^{-1} = \frac{1}{12-14}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}^{-1} = \frac{1}{-40-(-35)}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ -1.4 & -1.6 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}^{-1} = \frac{1}{-24-(-28)}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the solution is

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$. Thus $x_1 = 7$ and $x_2 = -9$.

6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$, and the solution is $\mathbf{x} = A^{-1}\mathbf{b}$. To compute this by hand, the arithmetic is simplified by keeping the fraction 1/det(A) in front of the matrix for A^{-1}. (The *Study Guide* comments on this in its discussion of Exercise 7.) From Exercise 3,

$\mathbf{x} = A^{-1}\mathbf{b} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Thus $x_1 = 2$ and $x_2 = -5$.

7. **a**. $\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}^{-1} = \frac{1}{1\cdot 12 - 2\cdot 5}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$ or $\begin{bmatrix} 6 & -1 \\ -2.5 & .5 \end{bmatrix}$

$\mathbf{x} = A^{-1}\mathbf{b}_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. Similar calculations give

$A^{-1}\mathbf{b}_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$, $A^{-1}\mathbf{b}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $A^{-1}\mathbf{b}_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$.

b. $[A \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$

The solutions are $\begin{bmatrix} -9 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 11 \\ -5 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$, the same as in part (a).

Note: The *Study Guide* also discusses the number of arithmetic calculations for this Exercise 7, stating that when A is large, the method used in (b) is much faster than using A^{-1}.

8. Left-multiply each side of the equation $AD = I$ by A^{-1} to obtain

$A^{-1}AD = A^{-1}I$, $ID = A^{-1}$, and $D = A^{-1}$.

Parentheses are routinely suppressed because of the associative property of matrix multiplication.

9. **a**. True, by definition of *invertible.* **b**. False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

e. True, by the box just before Example 6.

10. **a**. False. The product matrix is invertible, but the product of inverses should be in the *reverse* order. See Theorem 6(b).

b. True, by Theorem 6(a). **c**. True, by Theorem 4.

d. True, by Theorem 7. **e**. False. The last part of Theorem 7 is misstated here.

11. (The proof can be modeled after the proof of Theorem 5.) The $n \times p$ matrix B is given (but is arbitrary). Since A is invertible, the matrix $A^{-1}B$ satisfies $AX = B$, because $A(A^{-1}B) = A\,A^{-1}B = IB = B$. To show this solution is unique, let X be any solution of $AX = B$. Then, left-multiplication of each side by A^{-1} shows that X must be $A^{-1}B$: Thus $A^{-1}(AX) = A^{-1}B$, so $IX = A^{-1}B$, and thus $X = A^{-1}B$.

12. If you assign this exercise, consider giving the following *Hint*: Use elementary matrices and imitate the proof of Theorem 7. The solution in the Instructor's Edition follows this hint. Here is another solution, based on the idea at the end of Section 2.2.

Write $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $X = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_p]$. By definition of matrix multiplication, $AX = [A\mathbf{u}_1 \ \cdots \ A\mathbf{u}_p]$. Thus, the equation $AX = B$ is equivalent to the p systems:

$A\mathbf{u}_1 = \mathbf{b}_1, \ \ \ldots \ A\mathbf{u}_p = \mathbf{b}_p$

Since A is the coefficient matrix in each system, these systems may be solved simultaneously, placing the augmented columns of these systems next to A to form $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p] = [A \ \ B]$. Since A is invertible, the solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are uniquely determined, and $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ must row reduce to $[I \ \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_p] = [I \ \ X]$. By Exercise 11, X is the unique solution $A^{-1}B$ of $AX = B$.

13. Left-multiply each side of the equation $AB = AC$ by A^{-1} to obtain $A^{-1}AB = A^{-1}AC$, so $IB = IC$, and $B = C$. This conclusion does not always follow when A is singular. Exercise 10 of Section 2.1 provides a counterexample.

14. Right-multiply each side of the equation $(B - C)D = 0$ by D^{-1} to obtain$(B - C)DD^{-1} = 0D^{-1}$, so $(B - C)I = 0$, thus $B - C = 0$, and $B = C$.

15. The box following Theorem 6 suggests what the inverse of ABC should be, namely, $C^{-1}B^{-1}A^{-1}$. To verify that this is correct, compute:
$(ABC)\ C^{-1}B^{-1}A^{-1} = ABCC^{-1}B^{-1}A^{-1} = ABIB^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$ and
$C^{-1}B^{-1}A^{-1}\ (ABC) = C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC = C^{-1}B^{-1}BC = C^{-1}IC = C^{-1}C = I$

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1}$, so $CB^{-1} = AI = A$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

Note: The *Study Guide* warns against using the formula $(AB)^{-1} = B^{-1}A^{-1}$ here, because this formula can be used only when both A and B are already known to be invertible.

17. Right-multiply each side of $AB = BC$ by B^{-1}, thus $ABB^{-1} = BCB^{-1}$, so $AI = BCB^{-1}$, and $A = BCB^{-1}$.

18. Left-multiply each side of $A = PBP^{-1}$ by P^{-1}: thus $P^{-1}A = P^{-1}PBP^{-1}$, so $P^{-1}A = IBP^{-1}$, and $P^{-1}A = BP^{-1}$
Then right-multiply each side of the result by P: thus $P^{-1}AP = BP^{-1}P$, so $P^{-1}AP = BI$, and $P^{-1}AP = B$

19. Unlike Exercise 17, this exercise asks two things, "Does a solution exist and what is it?" First, find what the solution must be, if it exists. That is, suppose X satisfies the equation $C^{-1}(A + X)B^{-1} = I$. Left-multiply each side by C, and then right-multiply each side by B: thus $CC^{-1}(A + X)B^{-1} = CI$, so $I(A + X)B^{-1} = C$, thus $(A + X)B^{-1}B = CB$, and $(A + X)I = CB$

Expand the left side and then subtract A from both sides: thus $AI + XI = CB$, so $A + X = CB$, and $X = CB - A$

If a solution exists, it must be $CB - A$. To show that $CB - A$ really *is* a solution, substitute it for X:
$C^{-1}[A + (CB - A)]B^{-1} = C^{-1}[CB]B^{-1} = C^{-1}CBB^{-1} = II = I$.

Note: The *Study Guide* suggests that students ask their instructor about how many details to include in their proofs. After some practice with algebra, an expression such as $CC^{-1}(A + X)B^{-1}$ could be simplified directly to $(A + X)B^{-1}$ without first replacing CC^{-1} by I. However, you may wish this detail to be included in the homework for this section.

20. **a**. Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. Invert both sides of the original equation and use Theorem 6 about the inverse of a product (which applies because X^{-1} and B are invertible): $A - AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$

Then $A = AX + B^{-1}X = (A + B^{-1})X$. The product $(A + B^{-1})X$ is invertible because A is invertible. Since X is known to be invertible, so is the other factor, $A + B^{-1}$, by Exercise 16 or by an argument similar to part (a). Finally, $(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = X$

Note: This exercise is difficult. The algebra is not trivial, and at this point in the course, most students will not recognize the need to verify that a matrix is invertible.

21. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely, the zero solution. This means that the columns of A are linearly independent, by a remark in Section 1.7.

22. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (in fact, a unique solution) for each $\mathbf{b}$. By Theorem 4 in Section 1.4, the columns of A span $\mathbb{R}^n$.

23. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then there are no free variables in this equation, and so A has n pivot columns. Since A is *square* and the n pivot positions must be in different rows, the pivots in an echelon form of A must be on the main diagonal. Hence A is row equivalent to the $n \times n$ identity matrix.

24. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

25. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 0$. If $a = b = 0$, then examine $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ This has the solution $\mathbf{x}_1 = \begin{bmatrix} d \\ -c \end{bmatrix}$. This solution is nonzero, except when $a = b = c = d$. In that case, however, A is the zero matrix, and $A\mathbf{x} = \mathbf{0}$ for *every* vector $\mathbf{x}$. Finally, if a and b are not both zero, set $\mathbf{x}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$. Then $A\mathbf{x}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab + ba \\ -cb + da \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because $-cb + da = 0$. Thus, $\mathbf{x}_2$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. So, in all cases, the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. This is impossible when A is invertible (by Theorem 5), so A is *not* invertible.

26. $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} da - bc & 0 \\ 0 & -cb + ad \end{bmatrix}$. Divide both sides by $ad - bc$ to get $CA = I$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + da \end{bmatrix}$. Divide both sides by $ad - bc$. The right side is I. The left side is AC, because $\frac{1}{ad - bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = AC.$

27. **a**. Interchange A and B in equation (1) after Example 6 in Section 2.1: $\text{row}_i(BA) = \text{row}_i(B)\cdot A$. Then replace B by the identity matrix: $\text{row}_i(A) = \text{row}_i(IA) = \text{row}_i(I)\cdot A$.

b. Using part (a), when rows 1 and 2 of A are interchanged, write the result as

$$\begin{bmatrix} \text{row}_2(A) \\ \text{row}_1(A) \\ \text{row}_3(A) \end{bmatrix} = \begin{bmatrix} \text{row}_2(I)\cdot A \\ \text{row}_1(I)\cdot A \\ \text{row}_3(I)\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_2(I) \\ \text{row}_1(I) \\ \text{row}_3(I) \end{bmatrix} A = EA \qquad (*)$$

Here, E is obtained by interchanging rows 1 and 2 of I. The second equality in (*) is a consequence of the fact that $\text{row}_i(EA) = \text{row}_i(E)\cdot A$.

c. Using part (a), when row 3 of A is multiplied by 5, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ 5\cdot\text{row}_3(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ 5\cdot\text{row}_3(I)\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ 5\cdot\text{row}_3(I) \end{bmatrix} A = EA$$

Here, E is obtained by multiplying row 3 of I by 5.

28. When row 3 of A is replaced by $\text{row}_3(A) - 4\cdot\text{row}_1(A)$, write the result as

$$\begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) - 4\cdot\text{row}_1(A) \end{bmatrix} = \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ \text{row}_3(I)\cdot A - 4\cdot\text{row}_1(I)\cdot A \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ [\text{row}_3(I) - 4\cdot\text{row}_1(I)]\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ \text{row}_3(I) - 4\cdot\text{row}_1(I) \end{bmatrix} A = EA$$

Here, E is obtained by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\cdot\text{row}_1(I)$.

29. $$[A \;\; I] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

30. $$[A \;\; I] = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$$

31. $[A \quad I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

32. $[A \quad I] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}$. The matrix A is not invertible.

33. Let $B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$, and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$ (because $\mathbf{a}_j$ and $\mathbf{a}_{j+1}$ have the same entries except for the jth row), $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$ and $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$, $A\mathbf{b}_j = A(\mathbf{e}_j - \mathbf{e}_{j+1}) = A\mathbf{e}_j - A\mathbf{e}_{j+1} = \mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$ and $A\mathbf{b}_n = A\mathbf{e}_n = \mathbf{a}_n = \mathbf{e}_n$. Next, observe that $\mathbf{a}_j = \mathbf{e}_j + \cdots + \mathbf{e}_n$ for each j. Thus, $B\mathbf{a}_j = B(\mathbf{e}_j + \cdots + \mathbf{e}_n) = \mathbf{b}_j + \cdots + \mathbf{b}_n = (\mathbf{e}_j - \mathbf{e}_{j+1}) + (\mathbf{e}_{j+1} - \mathbf{e}_{j+2}) + \cdots + (\mathbf{e}_{n-1} - \mathbf{e}_n) + \mathbf{e}_n = \mathbf{e}_j$ This proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: Students who do this problem and then do the corresponding exercise in Section 2.4 will appreciate the Invertible Matrix Theorem, partitioned matrix notation, and the power of a proof by induction.

34. Let $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & & \\ 0 & -1/3 & 1/3 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & -1/n & 1/n \end{bmatrix}$

and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j = j(\mathbf{e}_j + \cdots + \mathbf{e}_n)$, $\mathbf{b}_j = \dfrac{1}{j}\mathbf{e}_j - \dfrac{1}{j+1}\mathbf{e}_{j+1}$, and $\mathbf{b}_n = \dfrac{1}{n}\mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$,

$$A\mathbf{b}_j = A\left(\frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}\right) = \frac{1}{j}\mathbf{a}_j - \frac{1}{j+1}\mathbf{a}_{j+1} = (\mathbf{e}_j + \cdots + \mathbf{e}_n) - (\mathbf{e}_{j+1} + \cdots + \mathbf{e}_n) = \mathbf{e}_j$$

Also, $A\mathbf{b}_n = A\left(\frac{1}{n}\mathbf{e}_n\right) = \frac{1}{n}\mathbf{a}_n = \mathbf{e}_n$. Finally, for $j = 1, \ldots, n$, the sum $\mathbf{b}_j + \cdots + \mathbf{b}_n$ is a "telescoping sum" whose value is $\frac{1}{j}\mathbf{e}_j$. Thus, $B\mathbf{a}_j = j(B\mathbf{e}_j + \cdots + B\mathbf{e}_n) = j(\mathbf{b}_j + \cdots + \mathbf{b}_n) = j\left(\frac{1}{j}\mathbf{e}_j\right) = \mathbf{e}_j$

which proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: If you assign Exercise 34, you may wish to supply a hint using the notation from Exercise 33: Express each column of A in terms of the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the identity matrix. Do the same for B.

35. Row reduce $[A \quad \mathbf{e}_3]$:

$$\begin{bmatrix} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Answer: The third column of A^{-1} is $\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$.

36. **[M]** Write $B = [A \quad F]$, where F consists of the last two columns of I_3, and row reduce:

$$B = \begin{bmatrix} -25 & -9 & -27 & 0 & 0 \\ 546 & 180 & 537 & 1 & 0 \\ 154 & 50 & 149 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/2 & -9/2 \\ 0 & 1 & 0 & -433/6 & 439/2 \\ 0 & 0 & 1 & 68/3 & -69 \end{bmatrix}$$

The last two columns of A^{-1} are $\begin{bmatrix} 1.5000 & -4.5000 \\ -72.1667 & 219.5000 \\ 22.6667 & -69.0000 \end{bmatrix}$

37. There are many possibilities for C, but $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ is the only one whose entries are 1, –1, and 0.

With only three possibilities for each entry, the construction of C can be done by trial and error. This is probably faster than setting up a system of 4 equations in 6 unknowns. The fact that A cannot be invertible follows from Exercise 25 in Section 2.1, because A is not square.

38. Write $AD = A[\mathbf{d}_1 \quad \mathbf{d}_2] = [A\mathbf{d}_1 \quad A\mathbf{d}_2]$. The structure of A shows that $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

[There are 25 possibilities for D if entries of D are allowed to be 1, –1, and 0.] There is *no* 4×2 matrix C such that $CA = I_4$. If this were true, then $CA\mathbf{x}$ would equal $\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^4$. This cannot happen because

the columns of A are linearly dependent and so $A\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x}$. For such an $\mathbf{x}$, $CA\mathbf{x} = C(\mathbf{0}) = \mathbf{0}$. An alternate justification would be to cite Exercise 23 or 25 in Section 2.1.

39. $\mathbf{y} = D\mathbf{f} = \begin{bmatrix} .005 & .002 & .001 \\ .002 & .004 & .002 \\ .001 & .002 & .005 \end{bmatrix} \begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix} = \begin{bmatrix} .27 \\ .30 \\ .23 \end{bmatrix}$. The deflections are .27 in., .30 in., and .23 in. at points 1, 2, and 3, respectively.

40. **[M]** The *stiffness matrix* is D^{-1}. Use an "inverse" command to produce $D^{-1} = 125\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

To find the forces (in pounds) required to produce a deflection of .04 cm at point 3, most students will use technology to solve $D\mathbf{f} = (0, 0, .04)$ and obtain $(0, -5, 10)$.

Here is another method, based on the idea suggested in Exercise 42. The first column of D^{-1} lists the forces required to produce a deflection of 1 in. at point 1 (with zero deflection at the other points). Since the transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, the forces required to produce a deflection of .04 cm at point 3 is given by .04 times the third column of D^{-1}, namely (.04)(125) times $(0, -1, 2)$, or $(0, -5, 10)$ pounds.

41. To determine the forces that produce a deflections of .08, .12, .16, and .12 cm at the four points on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (.08, .12, .16, .12)$. The forces at the four points are 12, 1.5, 21.5, and 12 newtons, respectively.

42. **[M]** To determine the forces that produce a deflection of .24 cm at the second point on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (0, .24, 0, 0)$. The forces at the four points are –104, 167, –113, and 56.0 newtons, respectively. These forces are .24 times the entries in the second column of D^{-1}.

Reason: The transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, so the forces required to produce a deflection of .24 cm at the second point are .24 times the forces required to produce a deflection of 1 cm at the second point. These forces are listed in the second column of D^{-1}.

Another possible discussion: The solution of $D\mathbf{x} = (0, 1, 0, 0)$ is the second column of D^{-1}. Multiply both sides of this equation by .24 to obtain $D(.24\mathbf{x}) = (0, .24, 0, 0)$. So $.24\mathbf{x}$ is the solution of $D\mathbf{f} = (0, .24, 0, 0)$. (The argument uses linearity, but students may not mention this.)

Note: The *Study Guide* suggests using **`gauss`**, **`swap`**, **`bgauss`**, and **`scale`** to reduce $[A \quad I]$, because I prefer to postpone the use of **`ref`** (or **`rref`**) until later. If you wish to introduce **`ref`** now, see the *Study Guide*'s technology notes for Sections 2.8 or 4.3. (Recall that Sections 2.8 and 2.9 are only covered when an instructor plans to skip Chapter 4 and get quickly to eigenvalues.)

2.3 SOLUTIONS

Notes: This section ties together most of the concepts studied thus far. With strong encouragement from an instructor, most students can use this opportunity to review and reflect upon what they have learned, and form a solid foundation for future work. Students who fail to do this now usually struggle throughout the rest of the course. Section 2.3 can be used in at least three different ways.

(1) Stop after Example 1 and assign exercises only from among the Practice Problems and Exercises 1 to 28. I do this when teaching "Course 3" described in the text's "Notes to the Instructor. " If you did not cover Theorem 12 in Section 1.9, omit statements (f) and (i) from the Invertible Matrix Theorem.

(2) Include the subsection "Invertible Linear Transformations" in Section 2.3, if you covered Section 1.9. I do this when teaching "Course 1" because our mathematics and computer science majors take this class. Exercises 29–40 support this material.

(3) Skip the linear transformation material here, but discuss the **condition number** and the Numerical Notes. Assign exercises from among 1–28 and 41–45, and perhaps add a computer project on the condition number. (See the projects on our web site.) I do this when teaching "Course 2" for our engineers.

The abbreviation IMT (here and in the *Study Guide*) denotes the Invertible Matrix Theorem (Theorem 8).

1. The columns of the matrix $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ are not multiples, so they are linearly independent. By (e) in the IMT, the matrix is invertible. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

2. The fact that the columns of $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$ are multiples is not so obvious. The fastest check in this case may be the determinant, which is easily seen to be zero. By Theorem 4 in Section 2.2, the matrix is not invertible.

3. Row reduction to echelon form is trivial because there is really no need for arithmetic calculations:
$\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ The 3×3 matrix has 3 pivot positions and hence is invertible, by (c) of the IMT. [Another explanation could be given using the transposed matrix. But see the note below that follows the solution of Exercise 14.]

4. The matrix $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

5. $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ -4 & -9 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & -9 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is not invertible because it is not row equivalent to the identity matrix.

6. $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & -9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix is not invertible because it is not row equivalent to the identity matrix.

7. $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The 4×4 matrix has four pivot positions and so is invertible by (c) of the IMT.

8. The 4×4 matrix $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ is invertible because it has four pivot positions, by (c) of the IMT.

9. **[M]** $\begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ 4 & 0 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & -11 & -7 & 15 \\ 0 & 9 & 31 & 12 \\ 0 & 8 & 5 & -11 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 9 & 31 & 12 \\ 0 & -11 & -7 & 15 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 25.375 & 24.375 \\ 0 & 0 & -.1250 & -.1250 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 25.375 & 24.375 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 25.375 & 24.375 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 2 & 3 & -1 \\ 0 & 8 & 5 & -11 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

The 4×4 matrix is invertible because it has four pivot positions, by (c) of the IMT.

10. **[M]** $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & .8 & 1.6 & .2 & -3.6 \\ 0 & .6 & 2.2 & -21.6 & -21.2 \\ 0 & .2 & .4 & -.2 & -10.4 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & 0 & 0 & 1 & 34 \\ 0 & 0 & 1 & -21 & 7 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & 0 & 1 & -21 & 7 \\ 0 & 0 & 0 & 1 & 34 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

The 5×5 matrix is invertible because it has five pivot positions, by (c) of the IMT.

11. **a**. True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).

b. True. If statement (h) of the IMT is true, then so is statement (e).

c. False. Statement (g) of the IMT is true only for invertible matrices.

d. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.

e. True, by the IMT. If A^T is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

12. **a**. True. If statement (k) of the IMT is true, then so is statement (j).

b. True. If statement (e) of the IMT is true, then so is statement (h).

c. True. See the remark immediately following the proof of the IMT.

d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n \times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, yet not every such matrix has n pivot positions.

e. True, by the IMT. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

Note: The solutions below for Exercises 13–30 refer mostly to the IMT. In many cases, however, part or all of an acceptable solution could also be based on various results that were used to establish the IMT.

13. If a square upper triangular $n \times n$ matrix has nonzero diagonal entries, then because it is already in echelon form, the matrix is row equivalent to I_n and hence is invertible, by the IMT. Conversely, if the matrix is invertible, it has n pivots on the diagonal and hence the diagonal entries are nonzero.

14. If A is lower triangular with nonzero entries on the diagonal, then these n diagonal entries can be used as pivots to produce zeros below the diagonal. Thus A has n pivots and so is invertible, by the IMT. If one of the diagonal entries in A is zero, A will have fewer than n pivots and hence be singular.

Notes: For Exercise 14, another correct analysis of the case when A has nonzero diagonal entries is to apply the IMT (or Exercise 13) to A^T. Then use Theorem 6 in Section 2.2 to conclude that since A^T is invertible so is its transpose, A. You might mention this idea in class, but I recommend that you not spend much time discussing A^T and problems related to it, in order to keep from making this section too lengthy. (The transpose is treated infrequently in the text until Chapter 6.)

If you do plan to ask a test question that involves A^T and the IMT, then you should give the students some extra homework that develops skill using A^T. For instance, in Exercise 14 replace "columns" by "rows." Also, you could ask students to explain why an $n \times n$ matrix with linearly independent columns must also have linearly independent rows.

15. If A has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that A cannot be invertible.

16. Part (h) of the IMT shows that a 5×5 matrix cannot be invertible when its columns do not span $\mathbb{R}^5$.

17. If A is invertible, so is A^{-1}, by Theorem 6 in Section 2.2. By (e) of the IMT applied to A^{-1}, the columns of A^{-1} are linearly independent.

18. By (g) of the IMT, C is invertible. Hence, each equation $C\mathbf{x} = \mathbf{v}$ has a unique solution, by Theorem 5 in Section 2.2. This fact was pointed out in the paragraph following the proof of the IMT.

19. By (e) of the IMT, D is invertible. Thus the equation $D\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^7$, by (g) of the IMT. Even better, the equation $D\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^7$, by Theorem 5 in Section 2.2. (See the paragraph following the proof of the IMT.)

20. By the box following the IMT, E and F are invertible and are inverses. So $FE = I = EF$, and so E and F commute.

21. The matrix G cannot be invertible, by Theorem 5 in Section 2.2 or by the box following the IMT. So (g), and hence (h), of the IMT are false and the columns of G do not span $\mathbb{R}^n$.

22. Statement (g) of the IMT is false for H, so statement (d) is false, too. That is, the equation $H\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

23. Statement (b) of the IMT is false for K, so statements (e) and (h) are also false. That is, the columns of K are linearly *de*pendent and the columns do *not* span $\mathbb{R}^n$.

24. No conclusion about the columns of L may be drawn, because no information about L has been given. The equation $L\mathbf{x} = \mathbf{0}$ *always* has the trivial solution.

25. Suppose that A is square and $AB = I$. Then A is invertible, by the (k) of the IMT. Left-multiplying each side of the equation $AB = I$ by A^{-1}, one has $A^{-1}AB = A^{-1}I$, $IB = A^{-1}$, and $B = A^{-1}$.

By Theorem 6 in Section 2.2, the matrix B (which is A^{-1}) is invertible, and its inverse is $(A^{-1})^{-1}$, which is A.

26. If the columns of A are linearly independent, then since A is square, A is invertible, by the IMT. So A^2, which is the product of invertible matrices, is invertible. By the IMT, the columns of A^2 span $\mathbb{R}^n$.

27. Let W be the inverse of AB. Then $ABW = I$ and $A(BW) = I$. Since A is square, A is invertible, by (k) of the IMT.

Note: The *Study Guide* for Exercise 27 emphasizes here that the equation $A(BW) = I$, *by itself*, does not show that A is invertible. Students are referred to Exercise 38 in Section 2.2 for a counterexample. Although there is an overall assumption that matrices in this section are square, I insist that my students mention this fact when using the IMT. Even so, at the end of the course, I still sometimes find a student who thinks that an equation $AB = I$ implies that A is invertible.

28. Let W be the inverse of AB. Then $WAB = I$ and $(WA)B = I$. By (j) of the IMT applied to B in place of A, the matrix B is invertible.

29. Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one, statement (f) of the IMT is false. Then (i) is also false and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ does not map $\mathbb{R}^n$ onto $\mathbb{R}^n$. Also, A is not invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not invertible, by Theorem 9.

30. Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, statement (f) of the IMT is true. Then (i) is also true and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Also, A is invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible, by Theorem 9.

31. Since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$, the matrix A has a pivot in each row (Theorem 4 in Section 1.4). Since A is square, A has a pivot in each column, and so there are no free variables in the equation $A\mathbf{x} = \mathbf{b}$, which shows that the solution is unique.

Note: The preceding argument shows that the (square) shape of A plays a crucial role. A less revealing proof is to use the "pivot in each row" and the IMT to conclude that A is invertible. Then Theorem 5 in Section 2.2 shows that the solution of $A\mathbf{x} = \mathbf{b}$ is unique.

32. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A must have a pivot in each of its n columns. Since A is square (and this is the key point), there must be a pivot in each *row* of A. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

Another argument: Statement (d) of the IMT is true, so A is invertible. By Theorem 5 in Section 2.2, the equation $A\mathbf{x} = \mathbf{b}$ has a (unique) solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

33. (Solution in *Study Guide*) The standard matrix of T is $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$, which is invertible because $\det A \neq 0$. By Theorem 9, the transformation T is invertible and the standard matrix of T^{-1} is A^{-1}. From the formula for a 2×2 inverse, $A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$. So

$$T^{-1}(x_1, x_2) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (7x_1 + 9x_2, 4x_1 + 5x_2)$$

34. The standard matrix of T is $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, which is invertible because $\det A = 2 \neq 0$. By Theorem 9, T is invertible, and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$. Thus

$$T^{-1}(x_1, x_2) = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2 \right)$$

35. (Solution in *Study Guide*) To show that T is one-to-one, suppose that $T(\mathbf{u}) = T(\mathbf{v})$ for some vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Then $S(T(\mathbf{u})) = S(T(\mathbf{v}))$, where S is the inverse of T. By Equation (1), $\mathbf{u} = S(T(\mathbf{u}))$ and $S(T(\mathbf{v})) = \mathbf{v}$, so $\mathbf{u} = \mathbf{v}$. Thus T is one-to-one. To show that T is onto, suppose $\mathbf{y}$ represents an arbitrary vector in $\mathbb{R}^n$ and define $\mathbf{x} = S(\mathbf{y})$. Then, using Equation (2), $T(\mathbf{x}) = T(S(\mathbf{y})) = \mathbf{y}$, which shows that T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ Second proof: By Theorem 9, the standard matrix A of T is invertible. By the IMT, the columns of A are linearly independent and span $\mathbb{R}^n$. By Theorem 12 in Section 1.9, T is one-to-one and maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

36. If T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, then the columns of its standard matrix A span $\mathbb{R}^n$ by Theorem 12 in Section 1.9. By the IMT, A is invertible. Hence, by Theorem 9 in Section 2.3, T is invertible, and A^{-1} is the standard matrix of T^{-1}. Since A^{-1} is also invertible, by the IMT, its columns are linearly independent and span $\mathbb{R}^n$. Applying Theorem 12 in Section 1.9 to the transformation T^{-1}, we conclude that T^{-1} is a one-to-one mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

37. Let A and B be the standard matrices of T and U, respectively. Then AB is the standard matrix of the mapping $\mathbf{x} \mapsto T(U(\mathbf{x}))$, because of the way matrix multiplication is defined (in Section 2.1). By hypothesis, this mapping is the identity mapping, so $AB = I$. Since A and B are square, they are invertible, by the IMT, and $B = A^{-1}$. Thus, $BA = I$. This means that the mapping $\mathbf{x} \mapsto U(T(\mathbf{x}))$ is the identity mapping, i.e., $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

38. Let A be the standard matrix of T. By hypothesis, T is not a one-to-one mapping. So, by Theorem 12 in Section 1.9, the standard matrix A of T has linearly dependent columns. Since A is square, the columns of A do not span $\mathbb{R}^n$. By Theorem 12, again, T cannot map $\mathbb{R}^n$ onto $\mathbb{R}^n$

39. Given any $\mathbf{v}$ in $\mathbb{R}^n$, we may write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$, because T is an onto mapping. Then, the assumed properties of S and U show that $S(\mathbf{v}) = S(T(\mathbf{x})) = \mathbf{x}$ and $U(\mathbf{v}) = U(T(\mathbf{x})) = \mathbf{x}$. So $S(\mathbf{v})$ and $U(\mathbf{v})$ are equal for each $\mathbf{v}$. That is, S and U are the same function from $\mathbb{R}^n$ into $\mathbb{R}^n$

40. Given **u**, **v** in $\mathbb{R}^n$ let $\mathbf{x} = S(\mathbf{u})$ and $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x})=T(S(\mathbf{u})) = \mathbf{u}$ and $T(\mathbf{y}) = T(S(\mathbf{v})) = \mathbf{v}$, by equation (2). Hence

$$\begin{aligned} S(\mathbf{u}+\mathbf{v}) &= S(T(\mathbf{x})+T(\mathbf{y})) \\ &= S(T(\mathbf{x}+\mathbf{y})) && \text{Because } T \text{ is linear} \\ &= \mathbf{x}+\mathbf{y} && \text{By equation (1)} \\ &= S(\mathbf{u})+S(\mathbf{v}) \end{aligned}$$

So, S preserves sums. For any scalar r,

$$\begin{aligned} S(r\mathbf{u}) &= S(rT(\mathbf{x})) = S(T(r\mathbf{x})) && \text{Because } T \text{ is linear} \\ &= r\mathbf{x} && \text{By equation (1)} \\ &= rS(\mathbf{u}) \end{aligned}$$

So S preserves scalar multiples. Thus S ia a linear transformation.

41. **[M]**

a. The exact solution of (3) is $x_1 = 3.94$ and $x_2 = .49$. The exact solution of (4) is $x_1 = 2.90$ and $x_2 = 2.00$.

b. When the solution of (4) is used as an approximation for the solution in (3) , the error in using the value of 2.90 for x_1 is about 26%, and the error in using 2.0 for x_2 is about 308%.

c. The condition number of the coefficient matrix is 3363. The percentage change in the solution from (3) to (4) is about 7700 times the percentage change in the right side of the equation. This is the same order of magnitude as the condition number. The condition number gives a rough measure of how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ can be to changes in **b**. Further information about the condition number is given at the end of Chapter 6 and in Chapter 7.

Note: See the *Study Guide*'s MATLAB box, or a technology appendix, for information on condition number. Only the TI-83+ and TI-89 lack a command for this.

42. **[M]** MATLAB gives cond(A) = 23683, which is approximately 10^4. If you make several trials with MATLAB, which records 16 digits accurately, you should find that **x** and $\mathbf{x}_1$ agree to at least 12 or 13 significant digits. So about 4 significant digits are lost. Here is the result of one experiment. The vectors were all computed to the maximum 16 decimal places but are here displayed with only four decimal places:

$$\mathbf{x} = \text{rand}(4,1) = \begin{bmatrix} .9501 \\ .21311 \\ .6068 \\ .4860 \end{bmatrix}, \quad \mathbf{b} = A\mathbf{x} = \begin{bmatrix} -3.8493 \\ 5.5795 \\ 20.7973 \\ .8467 \end{bmatrix}. \text{ The MATLAB solution is } \mathbf{x}_1 = A\backslash\mathbf{b} = \begin{bmatrix} .9501 \\ .2311 \\ .6068 \\ .4860 \end{bmatrix}.$$

However, $\mathbf{x} - \mathbf{x}_1 = \begin{bmatrix} .0171 \\ .4858 \\ -.2360 \\ .2456 \end{bmatrix} \times 10^{-12}$. The computed solution $\mathbf{x}_1$ is accurate to about 12 decimal places.

43. **[M]** MATLAB gives cond(A) = 68,622. Since this has magnitude between 10^4 and 10^5, the estimated accuracy of a solution of $A\mathbf{x} = \mathbf{b}$ should be to about four or five decimal places *less* than the 16 decimal places that MATLAB usually computes accurately. That is, one should expect the solution to be accurate to only about 11 or 12 decimal places. Here is the result of one experiment. The vectors were all computed to the maximum 16 decimal places but are here displayed with only four decimal places:

$\mathbf{x}$ = rand(5,1) = $\begin{bmatrix} .2190 \\ .0470 \\ .6789 \\ .6793 \\ .9347 \end{bmatrix}$, $\mathbf{b} = A\mathbf{x} = \begin{bmatrix} 15.0821 \\ .8165 \\ 19.0097 \\ -5.8188 \\ 14.5557 \end{bmatrix}$. The MATLAB solution is $\mathbf{x}_1 = A\backslash\mathbf{b} = \begin{bmatrix} .2190 \\ .0470 \\ .6789 \\ .6793 \\ .9347 \end{bmatrix}$.

However, $\mathbf{x} - \mathbf{x}_1 = \begin{bmatrix} .3165 \\ -.6743 \\ .3343 \\ .0158 \\ -.0005 \end{bmatrix} \times 10^{-11}$. The computed solution $\mathbf{x}_1$ is accurate to about 11 decimal places.

44. **[M]** Solve $A\mathbf{x} = (0, 0, 0, 0, 1)$. MATLAB shows that $\text{cond}(A) \approx 4.8 \times 10^5$. Since MATLAB computes numbers accurately to 16 decimal places, the entries in the computed value of $\mathbf{x}$ should be accurate to at least 11 digits. The exact solution is (630, –12600, 56700, –88200, 44100).

45. **[M]** Some versions of MATLAB issue a warning when asked to invert a Hilbert matrix of order 12 or larger using floating-point arithmetic. The product AA^{-1} should have several off-diagonal entries that are far from being zero. If not, try a larger matrix.

Note: All matrix programs supported by the *Study Guide* have data for Exercise 45, but only MATLAB and Maple have a single command to create a Hilbert matrix. The HP-48G data for Exercise 45 contain a program that can be edited to create other Hilbert matrices.

Notes: The *Study Guide* for Section 2.3 organizes the statements of the Invertible Matrix Theorem in a table that imbeds these ideas in a broader discussion of rectangular matrices. The statements are arranged in three columns: statements that are logically equivalent for any $m \times n$ matrix and are related to existence concepts, those that are equivalent only for any $n \times n$ matrix, and those that are equivalent for any $n \times p$ matrix and are related to uniqueness concepts. Four statements are included that are not in the text's official list of statements, to give more symmetry to the three columns. You may or may not wish to comment on them.

I believe that students cannot fully understand the concepts in the IMT if they do not know the correct wording of each statement. (Of course, this knowledge is not sufficient for understanding.) The *Study Guide*'s Section 2.3 has an example of the type of question I often put on an exam at this point in the course. The section concludes with a discussion of reviewing and reflecting, as important steps to a mastery of linear algebra.

2.4 SOLUTIONS

Notes: Partitioned matrices arise in theoretical discussions in essentially every field that makes use of matrices. The *Study Guide* mentions some examples (with references).

Every student should be exposed to some of the ideas in this section. If time is short, you might omit Example 4 and Theorem 10, and replace Example 5 by a problem similar to one in Exercises 1–10. (A sample

replacement is given at the end of these solutions.) Then select homework from Exercises 1–13, 15, and 21–24.

The exercises just mentioned provide a good environment for practicing matrix manipulation. Also, students will be reminded that an equation of the form $AB = I$ does not by itself make A or B invertible. (The matrices must be square and the IMT is required.)

1. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left*. $\begin{bmatrix} I & 0 \\ E & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} IA+0C & IB+0D \\ EA+IC & EB+ID \end{bmatrix} = \begin{bmatrix} A & B \\ EA+C & EB+D \end{bmatrix}$

2. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left*. $\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} EA+0C & EB+0D \\ 0A+FC & 0B+FD \end{bmatrix} = \begin{bmatrix} EA & EB \\ FC & FD \end{bmatrix}$

3. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left*. $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} 0W+IY & 0X+IZ \\ IW+0Y & IX+0Z \end{bmatrix} = \begin{bmatrix} Y & Z \\ W & X \end{bmatrix}$

4. Apply the row-column rule as if the matrix entries were numbers, but for each product always write the entry of the left block-matrix on the *left*.

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} IA+0C & IB+0D \\ -XA+IC & -XB+ID \end{bmatrix} = \begin{bmatrix} A & B \\ -XA+C & -XB+D \end{bmatrix}$$

5. Compute the left side of the equation: $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} AI+BX & A0+BY \\ CI+0X & C0+0Y \end{bmatrix}$

Set this equal to the right side of the equation:

$$\begin{bmatrix} A+BX & BY \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix} \text{ so that } \begin{matrix} A+BX=0 & BY=I \\ C=Z & 0=0 \end{matrix}$$

Since the (2, 1) blocks are equal, $Z = C$. Since the (1, 2) blocks are equal, $BY = I$. To proceed further, assume that B and Y are square. Then the equation $BY = I$ implies that B is invertible, by the IMT, and $Y = B^{-1}$. (See the boxed remark that follows the IMT.) Finally, from the equality of the (1, 1) blocks, $BX = -A$, $B^{-1}BX = B^{-1}(-A)$, and $X = -B^{-1}A$.

The order of the factors for X is crucial.

Note: For simplicity, statements (j) and (k) in the Invertible Matrix Theorem involve square matrices C and D. Actually, if A is $n \times n$ and if C is any matrix such that AC is the $n \times n$ identity matrix, then C must be $n \times n$, too. (For AC to be defined, C must have n rows, and the equation $AC = I$ implies that C has n columns.) Similarly, $DA = I$ implies that D is $n \times n$. Rather than discuss this in class, I expect that in Exercises 5–8, when students see an equation such as $BY = I$, they will decide that *both* B and Y should be square in order to use the IMT.

6. Compute the left side of the equation: $\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} XA+0B & X0+0C \\ YA+ZB & Y0+ZC \end{bmatrix} = \begin{bmatrix} XA & 0 \\ YA+ZB & ZC \end{bmatrix}$

Set this equal to the right side of the equation: $\begin{bmatrix} XA & 0 \\ YA+ZB & ZC \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ so that $\begin{matrix} XA=I & 0=0 \\ YA+ZB=0 & ZC=I \end{matrix}$

To use the equality of the (1, 1) blocks, assume that A and X are square. By the IMT, the equation $XA = I$ implies that A is invertible and $X = A^{-1}$. (See the boxed remark that follows the IMT.) Similarly, if C and Z are assumed to be square, then the equation $ZC = I$ implies that C is invertible, by the IMT, and $Z = C^{-1}$. Finally, use the (2, 1) blocks and right-multiplication by A^{-1} to get $YA = -ZB = -C^{-1}B$, then $YAA^{-1} = (-C^{-1}B)A^{-1}$, and $Y = -C^{-1}BA^{-1}$. The order of the factors for Y is crucial.

7. Compute the left side of the equation: $\begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} XA+0+0B & XZ+0+0I \\ YA+0+IB & YZ+0+II \end{bmatrix}$

Set this equal to the right side of the equation:

$$\begin{bmatrix} XA & XZ \\ YA+B & YZ+I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ so that } \begin{matrix} XA=I & XZ=0 \\ YA+B=0 & YZ+I=I \end{matrix}$$

To use the equality of the (1, 1) blocks, assume that A and X are square. By the IMT, the equation $XA = I$ implies that A is invertible and $X = A^{-1}$. (See the boxed remark that follows the IMT) Also, X is invertible. Since $XZ = 0$, $X^{-1}XZ = X^{-1}0 = 0$, so Z must be 0. Finally, from the equality of the (2, 1) blocks, $YA = -B$. Right-multiplication by A^{-1} shows that $YAA^{-1} = -BA^{-1}$ and $Y = -BA^{-1}$. The order of the factors for Y is crucial.

8. Compute the left side of the equation: $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} AX+B0 & AY+B0 & AZ+BI \\ 0X+I0 & 0Y+I0 & 0Z+II \end{bmatrix}$

Set this equal to the right side of the equation: $\begin{bmatrix} AX & AY & AZ+B \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$

To use the equality of the (1, 1) blocks, assume that A and X are square. By the IMT, the equation $XA = I$ implies that A is invertible and $X = A^{-1}$. (See the boxed remark that follows the IMT. Since $AY = 0$, from the equality of the (1, 2) blocks, left-multiplication by A^{-1} gives $A^{-1}AY = A^{-1}0 = 0$, so $Y = 0$. Finally, from the (1, 3) blocks, $AZ = -B$. Left-multiplication by A^{-1} gives $A^{-1}AZ = A^{-1}(-B)$, and $Z = -A^{-1}B$. The order of the factors for Z is crucial.

Note: The *Study Guide* tells students, "Problems such as 5–10 make good exam questions. Remember to mention the IMT when appropriate, and remember that matrix multiplication is generally not commutative." When a problem statement includes a condition that a matrix is square, I expect my students to mention this fact when they apply the IMT.

9. Compute the left side of the equation:

$$\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} IA_{11}+0A_{21}+0A_{31} & IA_{12}+0A_{22}+0A_{32} \\ XA_{11}+IA_{21}+0A_{31} & XA_{12}+IA_{22}+0A_{32} \\ YA_{11}+0A_{21}+IA_{31} & YA_{12}+0A_{22}+IA_{32} \end{bmatrix}$$

Set this equal to the right side of the equation: $\begin{bmatrix} A_{11} & A_{12} \\ XA_{11}+A_{21} & XA_{12}+A_{22} \\ YA_{11}+A_{31} & YA_{12}+A_{32} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}$

$$\text{so that } \begin{matrix} A_{11}=B_{11} & A_{12}=B_{12} \\ XA_{11}+A_{21}=0 & XA_{12}+A_{22}=B_{22} \\ YA_{11}+A_{31}=0 & YA_{12}+A_{32}=B_{32} \end{matrix}.$$

Since the (2,1) blocks are equal, $XA_{11} + A_{21} = 0$ and $XA_{11} = -A_{21}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives $X = -A_{21}A_{11}^{-1}$. Likewise since the (3,1) blocks are equal, $YA_{11} + A_{31} = 0$ and $YA_{11} = -A_{31}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives $Y = -A_{31}A_{11}^{-1}$. Finally, from the (2,2) entries, $XA_{12} + A_{22} = B_{22}$. Since $X = -A_{21}A_{11}^{-1}$, $B_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

10. Since the two matrices are inverses, $\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$

Compute the left side of the equation:

$$\begin{bmatrix} I & 0 & 0 \\ C & I & 0 \\ A & B & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ Z & I & 0 \\ X & Y & I \end{bmatrix} = \begin{bmatrix} II + 0Z + 0X & I0 + 0I + 0Y & I0 + 00 + 0I \\ CI + IZ + 0X & C0 + II + 0Y & C0 + I0 + 0I \\ AI + BZ + IX & A0 + BI + IY & A0 + B0 + II \end{bmatrix}$$

Set this equal to the right side of the equation: $\begin{bmatrix} I & 0 & 0 \\ C + Z & I & 0 \\ A + BZ + X & B + Y & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$

$$\begin{array}{lll} I = I & 0 = 0 & 0 = 0 \\ \text{so that} \quad C + Z = 0 & I = I & 0 = 0 \\ A + BZ + X = 0 & B + Y = 0 & I = I \end{array}$$

Since the (2,1) blocks are equal, $C + Z = 0$ and $Z = -C$. Likewise since the (3, 2) blocks are equal, $B + Y = 0$ and $Y = -B$. Finally, from the (3,1) entries, $A + BZ + X = 0$ and $X = -A - BZ$. Since $Z = -C$, $X = -A - B(-C) = -A + BC$.

11. **a**. True. See the subsection *Addition and Scalar Multiplication*.

b. False. See the paragraph before Example 3.

12. **a**. True. See the paragraph before Example 4.

b. False. See the paragraph before Example 3.

13. You are asked to establish an *if and only if* statement. First, supose that A is invertible, and let $A^{-1} = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$. Then $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} BD & BE \\ CF & CG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

Since B is square, the equation $BD = I$ implies that B is invertible, by the IMT. Similarly, $CG = I$ implies that C is invertible. Also, the equation $BE = 0$ imples that $E = B^{-1}0 = 0$. Similarly $F = 0$. Thus

$$A^{-1} = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} D & E \\ E & G \end{bmatrix} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \qquad (*)$$

This proves that A is invertible *only if* B and C are invertible. For the "*if*" part of the statement, suppose that B and C are invertible. Then (*) provides a likely candidate for A^{-1} which can be used to show that A is invertible. Compute: $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} BB^{-1} & 0 \\ 0 & CC^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$.

Since A is square, this calculation and the IMT imply that A is invertible. (Don't forget this final sentence. Without it, the argument is incomplete.) Instead of that sentence, you could add the equation:

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} B^{-1}B & 0 \\ 0 & C^{-1}C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

14. You are asked to establish an *if and only if* statement. First suppose that A is invertible. Example 5 shows that A_{11} and A_{22} are invertible. This proves that A is invertible *only if* A_{11} and A_{22} are invertible. For the *if* part of this statement, suppose that A_{11} and A_{22} are invertible. Then the formula in Example 5 provides a likely candidate for A^{-1} which can be used to show that A is invertible . Compute:

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^{-1} + A_{12}0 & A_{11}(-A_{11}^{-1})A_{12}\,A_{22}^{-1} + A_{12}\,A_{22}^{-1} \\ 0A_{11}^{-1} + A_{22}0 & 0(-A_{11}^{-1})\,A_{12}\,A_{22}^{-1} + A_{22}A_{22}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & -(A_{11}A_{11}^{-1})\,A_{12}\,A_{22}^{-1} + A_{12}\,A_{22}^{-1} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & -A_{12}\,A_{22}^{-1} + A_{12}\,A_{22}^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since A is square, this calculation and the IMT imply that A is invertible.

15. Compute the right side of the equation:

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ XA_{11} & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{bmatrix}$$

Set this equal to the left side of the equation:

$$\begin{bmatrix} A_{11} & A_{11}Y \\ XA_{11} & XA_{11}Y + S \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ so that } \begin{matrix} A_{11} = A_{11} & A_{11}Y = A_{12} \\ XA_{11} = A_{21} & XA_{11}Y + S = A_{22} \end{matrix}$$

Since the (1, 2) blocks are equal, $A_{11}Y = A_{12}$. Since A_{11} is invertible, left multiplication by A_{11}^{-1} gives $Y = A_{11}^{-1}A_{12}$. Likewise since the (2,1) blocks are equal, $XA_{11} = A_{21}$. Since A_{11} is invertible, right multiplication by A_{11}^{-1} gives that $X = A_{21}A_{11}^{-1}$. One can check that the matrix S as given in the exercise satisfies the equation $XA_{11}Y + S = A_{22}$ with the calculated values of X and Y given above.

16. Suppose that A and A_{11} are invertible. First note that $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Since the matrices $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ and $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ are square, they are both invertible by the IMT. Equation (7) may be left multipled by $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1}$ and right multipled by $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1}$ to find $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}^{-1} A \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}^{-1}$.

Thus by Theorem 6, the matrix $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$ is invertible as the product of invertible matrices. Finally, Exercise 13 above may be used to show that S is invertible.

17. The column-row expansions of G_k and G_{k+1} are:

$$\begin{aligned} G_k &= X_k X_k^T \\ &= \text{col}_1(X_k)\,\text{row}_1(X_k^T) + \cdots + \text{col}_k(X_k)\,\text{row}_k(X_k^T) \end{aligned}$$

and

$$\begin{aligned} G_{k+1} &= X_{k+1} X_{k+1}^T \\ &= \text{col}_1(X_{k+1})\,\text{row}_1(X_{k+1}^T) + \cdots + \text{col}_k(X_{k+1})\,\text{row}_k(X_{k+1}^T) + \text{col}_{k+1}(X_{k+1})\,\text{row}_{k+1}(X_{k+1}^T) \\ &= \text{col}_1(X_k)\,\text{row}_1(X_k^T) + \cdots + \text{col}_k(X_k)\,\text{row}_k(X_k^T) + \text{col}_{k+1}(X_{k+1})\,\text{row}_{k+1}(X_k^T) \\ &= G_k + \text{col}_{k+1}(X_{k+1})\,\text{row}_{k+1}(X_k^T) \end{aligned}$$

since the first k columns of X_{k+1} are identical to the first k columns of X_k. Thus to update G_k to produce G_{k+1}, the number $\text{col}_{k+1}(X_{k+1})\,\text{row}_{k+1}(X_k^T)$ should be added to G_k.

18. Since $W = \begin{bmatrix} X & \mathbf{x}_0 \end{bmatrix}$, the product $W^T W = \begin{bmatrix} X^T \\ \mathbf{x}_0^T \end{bmatrix} [\, X \quad \mathbf{x}_0 \,] = \begin{bmatrix} X^T X & X^T \mathbf{x}_0 \\ \mathbf{x}_0^T X & \mathbf{x}_0^T \mathbf{x}_0 \end{bmatrix}$. By applying the formula for S from Exercise 15, S may be computed:

$$\begin{aligned} S &= \mathbf{x}_0^T \mathbf{x}_0 - \mathbf{x}_0^T X (X^T X)^{-1} X^T \mathbf{x}_0 \\ &= \mathbf{x}_0^T (I_m - X(X^T X)^{-1} X^T) \mathbf{x}_0 \\ &= \mathbf{x}_0^T M \mathbf{x}_0 \end{aligned}$$

19. The matrix equation (8) in the text is equivalent to $(A - sI_n)\mathbf{x} + B\mathbf{u} = 0$ and $C\mathbf{x} + \mathbf{u} = \mathbf{y}$

Rewrite the first equation as $(A - sI_n)\mathbf{x} = -B\mathbf{u}$. When $A - sI_n$ is invertible,

$\mathbf{x} = (A - sI_n)^{-1}(-B\mathbf{u}) = -(A - sI_n)^{-1} B\mathbf{u}$

Substitute this formula for $\mathbf{x}$ into the second equation above:

$C(-(A - sI_n)^{-1} B\mathbf{u}) + \mathbf{u} = \mathbf{y}$, so that $I_m\,\mathbf{u} - C(A - sI_n)^{-1} B\mathbf{u} = \mathbf{y}$

Thus $\mathbf{y} = (I_m - C(A - sI_n)^{-1} B)\mathbf{u}$. If $W(s) = I_m - C(A - sI_n)^{-1} B$, then $\mathbf{y} = W(s)\mathbf{u}$. The matrix W(s) is the Schur complement of the matrix $A - sI_n$ in the system matrix in equation (8)

20. The matrix in question is $\begin{bmatrix} A - BC - sI_n & B \\ -C & I_m \end{bmatrix}$. By applying the formula for S from Exercise 15, S may be computed:

$$\begin{aligned} S &= I_m - (-C)(A - BC - sI_m)^{-1} B \\ &= I_m + C(A - BC - sI_m)^{-1} B \end{aligned}$$

21. **a**. $A^2 = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 3-3 & 0+(-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $M^2 = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} = \begin{bmatrix} A^2 + 0 & 0+0 \\ A - A & 0+(-A)^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

22. Let C be any nonzero 2×3 matrix. Following the pattern in 21(a) with block matrices instead of numbers, set $M = \begin{bmatrix} I_3 & 0 \\ C & -I_2 \end{bmatrix}$ and verify $M^2 = \begin{bmatrix} I_3 & 0 \\ C & -I_2 \end{bmatrix}\begin{bmatrix} I_3 & 0 \\ C & -I_2 \end{bmatrix} = \begin{bmatrix} I_3 + 0 & 0 + 0 \\ CI_3 - I_2C & 0 + (-I_2)^2 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & I_2 \end{bmatrix}$.

23. The product of two 1×1 "lower triangular" matrices is "lower triangular." Suppose that for $n = k$, the product of two $k \times k$ lower triangular matrices is lower triangular, and consider any $(k+1) \times (k+1)$ matrices A_1 and B_1. Partition these matrices as $A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}$, $B_1 = \begin{bmatrix} b & \mathbf{0}^T \\ \mathbf{w} & B \end{bmatrix}$

where A and B are $k \times k$ matrices, $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^k$, and a and b are scalars. Since A_1 and B_1 are lower triangular, so are A and B. Then

$$A_1B_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}\begin{bmatrix} b & \mathbf{0}^T \\ \mathbf{w} & B \end{bmatrix} = \begin{bmatrix} ab + \mathbf{0}^T\mathbf{w} & a\mathbf{0}^T + \mathbf{0}^TB \\ \mathbf{v}b + A\mathbf{w} & \mathbf{v}\mathbf{0}^T + AB \end{bmatrix} = \begin{bmatrix} ab & \mathbf{0}^T \\ b\mathbf{v} + A\mathbf{w} & AB \end{bmatrix}$$

Since A and B are $k \times k$, AB is lower triangular. The form of A_1B_1 shows that it, too, is lower triangular. Thus the statement about lower triangular matrices is true for $n = k+1$ if it is true for $n = k$. By the principle of induction, the statement is true for all $n \geq 1$.

Note: Exercise 23 is good for mathematics and computer science students. The solution of Exercise 23 in the *Study Guide* shows students how to use the principle of induction. The *Study Guide* also has an appendix on "The Principle of Induction," at the end of Section 2.4. The text presents more applications of induction in Section 3.2 and in the Supplementary Exercises for Chapter 3.

24. Let $A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$, $B_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & \cdots & -1 & 1 \end{bmatrix}$.

By direct computation $A_2B_2 = I_2$. Assume that for $n = k$, the matrix A_kB_k is I_k, and write

$$A_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A_k \end{bmatrix} \quad \text{and} \quad B_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{w} & B_k \end{bmatrix}$$

where $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^k$, $\mathbf{v}^T = [1 \ \ 1 \ \cdots \ 1]$, and $\mathbf{w}^T = [-1 \ \ 0 \ \cdots \ 0]$. Then

$$A_{k+1}B_{k+1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & A_k \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{w} & B_k \end{bmatrix} = \begin{bmatrix} 1 + \mathbf{0}^T\mathbf{w} & \mathbf{0}^T + \mathbf{0}^TB_k \\ \mathbf{v} + A_k\mathbf{w} & \mathbf{v}\mathbf{0}^T + A_kB_k \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_k \end{bmatrix} = I_{k+1}$$

The (2,1)-entry is $\mathbf{0}$ because $\mathbf{v}$ equals the first column of A_k., and $A_k\mathbf{w}$ is –1 times the first column of A_k. By the principle of induction, $A_nB_n = I_n$ for all $n \geq 2$. Since A_n and B_n are square, the IMT shows that these matrices are invertible, and $B_n = A_n^{-1}$.

Note: An induction proof can also be given using partitions with the form shown below. The details are slightly more complicated.

$$A_{k+1} = \begin{bmatrix} A_k & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix} \quad \text{and} \quad B_{k+1} = \begin{bmatrix} B_k & \mathbf{0} \\ \mathbf{w}^T & 1 \end{bmatrix}$$

$$A_{k+1}B_{k+1} = \begin{bmatrix} A_k & \mathbf{0} \\ \mathbf{v}^T & 1 \end{bmatrix}\begin{bmatrix} B_k & \mathbf{0} \\ \mathbf{w}^T & 1 \end{bmatrix} = \begin{bmatrix} A_kB_k + \mathbf{0}\mathbf{w}^T & A^k\mathbf{0} + \mathbf{0} \\ \mathbf{v}^TB_k + \mathbf{w}^T & \mathbf{v}^T0 + 1 \end{bmatrix} = \begin{bmatrix} I_k & \mathbf{0} \\ \mathbf{0^T} & 1 \end{bmatrix} = I_{k+1}$$

The (2,1)-entry is $\mathbf{0}^T$ because $\mathbf{v}^T$ times a column of B_k equals the sum of the entries in the column, and all of such sums are zero except the last, which is 1. So $\mathbf{v}^TB_k$ is the negative of $\mathbf{w}^T$. By the principle of induction, $A_nB_n = I_n$ for all $n \geq 2$. Since A_n and B_n are square, the IMT shows that these matrices are invertible, and $B_n = A_n^{-1}$.

25. First, visualize a partition of A as a 2×2 block–diagonal matrix, as below, and then visualize the (2,2)-block itself as a block-diagonal matrix. That is,

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right], \text{ where } A_{22} = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 7 & 8 \\ 0 & 5 & 6 \end{array}\right] = \begin{bmatrix} 2 & 0 \\ 0 & B \end{bmatrix}$$

Observe that B is invertible and $B^{-1} = \begin{bmatrix} 3 & -4 \\ -2.5 & 3.5 \end{bmatrix}$. By Exercise 13, the block diagonal matrix A_{22} is invertible, and $A_{22}^{-1} = \left[\begin{array}{c|cc} .5 & 0 & 0 \\ \hline 0 & 3 & -4 \\ 0 & -2.5 & 3.5 \end{array}\right] = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & -2.5 & 3.5 \end{bmatrix}$

Next, observe that A_{11} is also invertible, with inverse $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. By Exercise 13, A itself is invertible, and its inverse is block diagonal:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} = \left[\begin{array}{cc|ccc} -5 & 2 & & 0 & \\ 3 & -1 & & & \\ \hline & & .5 & 0 & 0 \\ & 0 & 0 & 3 & -4 \\ & & 0 & -2.5 & 3.5 \end{array}\right] = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -2.5 & 3.5 \end{bmatrix}$$

26. **[M]** This exercise and the next, which involve large matrices, are more appropriate for MATLAB, Maple, and Mathematica, than for the graphic calculators.

a. Display the submatrix of A obtained from rows 15 to 20 and columns 5 to 10.

MATLAB: `A(15:20, 5:10)`

Maple: `submatrix(A, 15..20, 5..10)`

Mathematica: `Take[ A, {15,20}, {5,10} ]`

b. Insert a 5×10 matrix B into rows 10 to 14 and columns 20 to 29 of matrix A:

MATLAB: `A(10:14, 20:29) = B ;` The semicolon suppresses output display.

Maple: `copyinto(B, A, 10, 20):` The colon suppresses output display.

Mathematica:
```
For [ i=10, i<=14, i++,
  For [ j=20, j<=29, j++,
    A [[ i,j ]] = B [[ i-9, j-19 ]] ] ];
```
Colon suppresses output.

c. To create $B = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$ with MATLAB, build B out of four blocks:

```
B = [A zeros(30,20); zeros(20,30) A'];
```

Another method: first enter `B = A ;` and then enlarge B with the command

```
B(21:50, 31:50) = A';
```

This places A^T in the (2, 2) block of the larger B and fills in the (1, 2) and (2, 1) blocks with zeros.

For Maple:

```
B := matrix(50,50,0):
copyinto(A, B, 1, 1):
copyinto( transpose(A), B, 21, 31):
```

For Mathematica:

```
B = BlockMatrix[ {{A, ZeroMatrix[30,20]},{ZeroMatrix[20,30],
Transpose[A]}} ]
```

27. **a**. **[M]** Construct A from four blocks, say C_{11}, C_{12}, C_{21}, and C_{22}, for example with C_{11} a 30×30 matrix and C_{22} a 20×20 matrix.

MATLAB:
```
C11 = A(1:30, 1:30) + B(1:30, 1:30)
C12 = A(1:30, 31:50) + B(1:30, 31:50)
C21 = A(31:50, 1:30)+ B(31:50, 1:30)
C22 = A(31:50, 31:50) + B(31:50, 31:50)
C = [C11 C12; C21 C22]
```

The commands in Maple and Mathematica are analogous, but with different syntax. The first commands are:

Maple: `C11 := submatrix(A, 1..30, 1..30} + submatrix(B, 1..30, 1..30)`

Mathematica: `c11 := Take[ A, {1,30}, {1,30} ] + Take[B, {1,30), {1,30} ]`

b. The algebra needed comes from block matrix multiplication:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Partition both A and B, for example with 30×30 (1, 1) blocks and 20×20 (2, 2) blocks. The four necessary submatrix computations use syntax analogous to that shown for (a).

c. The algebra needed comes from the block matrix equation $\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{x}_1$ and $\mathbf{b}_1$ are in $\mathbb{R}^{30}$ and $\mathbf{x}_2$ and $\mathbf{b}_2$ are in $\mathbb{R}^{20}$. Then $A_{11}\mathbf{x}_1 = \mathbf{b}_1$, which can be solved to produce $\mathbf{x}_1$. Once $\mathbf{x}_1$ is found, rewrite the equation $A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 = \mathbf{b}_2$ as $A_{22}\mathbf{x}_2 = \mathbf{c}$, where $\mathbf{c} = \mathbf{b}_2 - A_{21}\mathbf{x}_1$, and solve $A_{22}\mathbf{x}_2 = \mathbf{c}$ for $\mathbf{x}_2$.

Notes: The following may be used in place of Example 5:

Example 5: Use equation (*) to find formulas for X, Y, and Z in terms of A, B, and C. Mention any assumptions you make in order to produce the formulas.

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \qquad (*)$$

Solution:

This matrix equation provides four equations that can be used to find X, Y, and Z:

$X + 0 = I$, $\qquad 0 = 0$

$YI + ZA = C$, $\qquad Y0 + ZB = I$ (Note the order of the factors.)

The first equation says that $X = I$. To solve the fourth equation, $ZB = I$, assume that B and Z are square. In this case, the equation $ZB = I$ implies that B and Z are invertible, by the IMT. (Actually, it suffices to assume either that B is square or that Z is square.) Then, right-multiply each side of $ZB = I$ to get $ZBB^{-1} = IB^{-1}$ and $Z = B^{-1}$. Finally, the third equation is $Y + ZA = C$. So, $Y + B^{-1}A = C$, and $Y = C - B^{-1}A$.

The following counterexample shows that Z need not be square for the equation (*) above to be true.

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 3 & 1 \\ 3 & 4 & 1 & 0 & -1 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 2 & 5 \\ 1 & 1 & -1 & -3 \\ 1 & -1 & 2 & 4 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 6 & 5 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right]$$

Note that Z is not determined by A, B, and C, when B is not square. For instance, another Z that works in this counterexample is $Z = \begin{bmatrix} 3 & 5 & 0 \\ -1 & -2 & 0 \end{bmatrix}$.

2.5 SOLUTIONS

Notes: Modern algorithms in numerical linear algebra are often described using matrix factorizations. For numerical work, this section is more important than Sections 4.7 and 5.4, even though matrix factorizations are explained nicely in terms of change of bases. Computational exercises in this section emphasize the use of the LU factorization to solve linear systems. The LU factorization is performed using the algorithm explained in the paragraphs before Example 2, and performed in Example 2. The text discusses how to build L when no interchanges are needed to reduce the given matrix to U. An appendix in the *Study Guide* discusses how to build L in permuted unit lower triangular form when row interchanges are needed. Other factorizations are introduced in Exercises 22–26.

1. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$.

$$[L \ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & 16 \end{bmatrix}$$ The only arithmetic is in column 4

$$\sim\begin{bmatrix}1&0&0&-7\\0&1&0&-2\\0&0&1&6\end{bmatrix},\ \text{so}\ \mathbf{y}=\begin{bmatrix}-7\\-2\\6\end{bmatrix}.$$

Next, solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation).

$$[U\ \ \mathbf{y}]=\begin{bmatrix}3&-7&-2&-7\\0&-2&-1&-2\\0&0&-1&6\end{bmatrix}\sim\begin{bmatrix}3&-7&-2&-7\\0&-2&-1&-2\\0&0&1&-6\end{bmatrix}\sim\begin{bmatrix}3&-7&0&-19\\0&-2&0&-8\\0&0&1&-6\end{bmatrix}$$

$$\sim\begin{bmatrix}3&-7&0&-19\\0&1&0&4\\0&0&1&-6\end{bmatrix}\sim\begin{bmatrix}3&0&0&9\\0&1&0&4\\0&0&1&-6\end{bmatrix}\sim\begin{bmatrix}1&0&0&3\\0&1&0&4\\0&0&1&-6\end{bmatrix},\ \text{so}\ \mathbf{x}=\begin{bmatrix}3\\4\\-6\end{bmatrix}.$$

To confirm this result, row reduce the matrix $[A\ \mathbf{b}]$:

$$[A\ \ \mathbf{b}]=\begin{bmatrix}3&-7&-2&-7\\-3&5&1&5\\6&-4&0&2\end{bmatrix}\sim\begin{bmatrix}3&-7&-2&-7\\0&-2&-1&-2\\0&10&4&16\end{bmatrix}\sim\begin{bmatrix}3&-7&-2&-7\\0&-2&-1&-2\\0&0&-1&6\end{bmatrix}$$

From this point the row reduction follows that of $[U\ \ \mathbf{y}]$ above, yielding the same result.

2. $L=\begin{bmatrix}1&0&0\\-1&1&0\\2&0&1\end{bmatrix}, U=\begin{bmatrix}4&3&-5\\0&-2&2\\0&0&2\end{bmatrix}, \mathbf{b}=\begin{bmatrix}2\\-4\\6\end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$:

$$[L\ \ \mathbf{b}]=\begin{bmatrix}1&0&0&2\\-1&1&0&-4\\2&0&1&6\end{bmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-2\\0&0&1&2\end{bmatrix},\ \text{so}\ \mathbf{y}=\begin{bmatrix}2\\-2\\2\end{bmatrix}.$$

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U\ \ \mathbf{y}]=\begin{bmatrix}4&3&-5&2\\0&-2&2&-2\\0&0&2&2\end{bmatrix}\sim\begin{bmatrix}4&3&-5&2\\0&-2&2&-2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}4&3&0&7\\0&-2&0&-4\\0&0&1&1\end{bmatrix}$$

$$\sim\begin{bmatrix}4&3&0&7\\0&1&0&2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}4&0&0&1\\0&1&0&2\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1/4\\0&1&0&2\\0&0&1&1\end{bmatrix},\text{so}\ \mathbf{x}=\begin{bmatrix}1/4\\2\\1\end{bmatrix}.$$

To confirm this result, row reduce the matrix $[A\ \ \mathbf{b}]$:

$$[A\ \ \mathbf{b}]=\begin{bmatrix}4&3&-5&2\\-4&-5&7&-4\\8&6&-8&6\end{bmatrix}\sim\begin{bmatrix}4&3&-5&2\\0&-2&2&-2\\0&0&2&2\end{bmatrix}$$

From this point the row reduction follows that of $[U\ \ \mathbf{y}]$ above, yielding the same result.

3. $L=\begin{bmatrix}1&0&0\\-3&1&0\\4&-1&1\end{bmatrix}, U=\begin{bmatrix}2&-1&2\\0&-3&4\\0&0&1\end{bmatrix}, \mathbf{b}=\begin{bmatrix}1\\0\\4\end{bmatrix}$. First, solve $L\mathbf{y}=\mathbf{b}$:

$$[L\ \ \mathbf{b}]=\begin{bmatrix}1&0&0&1\\-3&1&0&0\\4&-1&1&4\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&3\\0&-1&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&3\\0&0&1&3\end{bmatrix},\ \text{so}\ \mathbf{y}=\begin{bmatrix}1\\3\\3\end{bmatrix}.$$

Next solve $U\mathbf{x}=\mathbf{y}$, using back-substitution (with matrix notation):

$$[U\ \ \mathbf{y}]=\begin{bmatrix}2&-1&2&1\\0&-3&4&3\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&-1&0&-5\\0&-3&0&-9\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&-1&0&-5\\0&1&0&3\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&0&0&-2\\0&1&0&3\\0&0&1&3\end{bmatrix},\ \text{so}\ \mathbf{x}=\begin{bmatrix}-1\\3\\3\end{bmatrix}.$$

4. $L=\begin{bmatrix}1&0&0\\1/2&1&0\\3/2&-5&1\end{bmatrix}, U=\begin{bmatrix}2&-2&4\\0&-2&-1\\0&0&-6\end{bmatrix}, \mathbf{b}=\begin{bmatrix}0\\-5\\7\end{bmatrix}$. First, solve $L\mathbf{y}=\mathbf{b}$:

$$[L\ \ \mathbf{b}]=\begin{bmatrix}1&0&0&0\\1/2&1&0&-5\\3/2&-5&1&7\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&-5\\0&-5&1&7\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&-5\\0&0&1&-18\end{bmatrix},\ \text{so}\ \mathbf{y}=\begin{bmatrix}0\\-5\\-18\end{bmatrix}.$$

Next solve $U\mathbf{x}=\mathbf{y}$, using back-substitution (with matrix notation):

$$[U\ \ \mathbf{y}]=\begin{bmatrix}2&-2&4&0\\0&-2&-1&-5\\0&0&-6&-18\end{bmatrix}\sim\begin{bmatrix}2&-2&4&0\\0&-2&-1&-5\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&-2&0&-12\\0&-2&0&-2\\0&0&1&3\end{bmatrix}$$

$$\sim\begin{bmatrix}2&-2&0&-12\\0&1&0&1\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}2&0&0&-10\\0&1&0&1\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}1&0&0&-5\\0&1&0&1\\0&0&1&3\end{bmatrix},\ \text{so}\ \mathbf{x}=\begin{bmatrix}-5\\1\\3\end{bmatrix}.$$

5. $L=\begin{bmatrix}1&0&0&0\\2&1&0&0\\-1&0&1&0\\-4&3&-5&1\end{bmatrix}, U=\begin{bmatrix}1&-2&-4&-3\\0&-3&1&0\\0&0&2&1\\0&0&0&1\end{bmatrix}, \mathbf{b}=\begin{bmatrix}1\\7\\0\\3\end{bmatrix}$. First solve $L\mathbf{y}=\mathbf{b}$:

$$[L\ \ \mathbf{b}]=\begin{bmatrix}1&0&0&0&1\\2&1&0&0&7\\-1&0&1&0&0\\-4&3&-5&1&3\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&1\\0&1&0&0&5\\0&0&1&0&1\\0&3&-5&1&7\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&1\\0&1&0&0&5\\0&0&1&0&1\\0&0&-5&1&-8\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&1\\0&1&0&0&5\\0&0&1&0&1\\0&0&0&1&-3\end{bmatrix},$$

so $\mathbf{y}=\begin{bmatrix}1\\5\\1\\-3\end{bmatrix}$. Next solve $U\mathbf{x}=\mathbf{y}$, using back-substitution (with matrix notation):

$$[U \quad \mathbf{y}] = \begin{bmatrix} 1 & -2 & -4 & -3 & 1 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -4 & 0 & -8 \\ 0 & -3 & 1 & 0 & 5 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}, \quad \text{so } \mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 2 \\ -3 \end{bmatrix}.$$

6. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$:

$$[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 & -2 \\ 3 & -2 & 1 & 0 & -1 \\ -5 & 4 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & -4 \\ 0 & 4 & -1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \text{so } \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U \quad \mathbf{y}] = \begin{bmatrix} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 2 & 1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 3 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \text{so } \mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

7. Place the first pivot column of $\begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix}$ into L, after dividing the column by 2 (the pivot), then add 3/2 times row 1 to row 2, yielding U.

$$A = \begin{bmatrix} \textcircled{2} & 5 \\ -3 & \textcircled{-4} \end{bmatrix} \sim \begin{bmatrix} 2 & 5 \\ 0 & \textcircled{7/2} \end{bmatrix} = U$$

$$\begin{bmatrix} \textcircled{2} \\ -3 \end{bmatrix} \quad [\textcircled{7/2}]$$

$$\div 2 \qquad \div 7/2$$

$$\begin{bmatrix} 1 & \\ -3/2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}$$

8.Row reduce A to echelon form using only row replacement operations. Then follow the algorithm in Example 2 to find L.

$$A = \begin{bmatrix} \textcircled{6} & 9 \\ 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 6 & 9 \\ 0 & \textcircled{-1} \end{bmatrix} = U$$

$$\begin{bmatrix} \textcircled{6} \\ 4 \end{bmatrix} \quad [\textcircled{-1}]$$

$$\div 6 \qquad \div -1$$

$$\begin{bmatrix} 1 & \\ 2/3 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}$$

9. $$A = \begin{bmatrix} \textcircled{3} & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & \textcircled{-3} & 12 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & \textcircled{-8} \end{bmatrix} = U$$

$$\begin{bmatrix} \textcircled{3} \\ -3 \\ 9 \end{bmatrix} \quad \begin{bmatrix} \textcircled{-3} \\ -2 \end{bmatrix} \quad [\textcircled{-8}]$$

$$\div 3 \qquad \div -3 \qquad \div -8$$

$$\begin{bmatrix} 1 & & \\ -1 & 1 & \\ 3 & 2/3 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}$$

10. $A = \begin{bmatrix} \textcircled{-5} & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & \textcircled{-2} & -1 \\ 0 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & \textcircled{9} \end{bmatrix} = U$

$\begin{bmatrix} \textcircled{-5} \\ 10 \\ 15 \end{bmatrix} \begin{bmatrix} \textcircled{-2} \\ 10 \end{bmatrix} [\textcircled{9}]$

$\div -5 \quad \div -2 \quad \div 9$

$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -3 & -5 & 1 \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} \textcircled{3} & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & \textcircled{5} & -4 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & \textcircled{5} \end{bmatrix} = U$

$\begin{bmatrix} \textcircled{3} \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} \textcircled{5} \\ 5 \end{bmatrix} [\textcircled{5}]$

$\div 3 \quad \div 5 \quad \div 5$

$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1/3 & 1 & 1 \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix}$

12. Row reduce A to echelon form using only row replacement operations. Then follow the algorithm in Example 2 to find L. Use the last column of I_3 to make L unit lower triangular.

$$A = \begin{bmatrix} \textcircled{2} & -4 & 2 \\ 1 & 5 & -4 \\ -6 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2 \\ 0 & \textcircled{7} & -5 \\ 0 & -14 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{bmatrix} \textcircled{2} \\ 1 \\ -6 \end{bmatrix} \quad \begin{bmatrix} \textcircled{7} \\ -14 \end{bmatrix}$$

$$\div 2 \qquad \div 7$$

$$\begin{bmatrix} 1 & \\ 1/2 & 1 \\ -3 & -2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}$$

13. $\begin{bmatrix} \textcircled{1} & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & \textcircled{-2} & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$ No more pivots!

$$\begin{bmatrix} \textcircled{1} \\ -1 \\ 4 \\ -2 \end{bmatrix} \quad \begin{bmatrix} \textcircled{-2} \\ -10 \\ 2 \end{bmatrix}$$

Use the last two columns of I_4 to make L unit lower triangular.

$$\div 1 \qquad \div -2$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 4 & 5 & 1 & \\ -2 & -1 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}$$

14. $A = \begin{bmatrix} \textcircled{1} & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \\ -1 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & \textcircled{-5} & 1 & -6 \\ 0 & 5 & -1 & 6 \\ 0 & 10 & -2 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

$\begin{bmatrix} \textcircled{1} \\ 3 \\ -2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \textcircled{-5} \\ 5 \\ 10 \end{bmatrix}$ Use the last two columns of I_4 to make L unit lower triangular.

$\div 1 \quad \div -5$

$\begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ -2 & -1 & 1 & \\ -1 & -2 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$

15. $A = \begin{bmatrix} \textcircled{2} & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & \textcircled{3} & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & \textcircled{5} \end{bmatrix} = U$

$\begin{bmatrix} \textcircled{2} \\ 6 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \textcircled{3} \\ -6 \end{bmatrix} \quad [\textcircled{5}]$

$\div 2 \quad \div 3 \quad \div 5$

$\begin{bmatrix} 1 & & \\ 3 & 1 & \\ -1/2 & -2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$

16. $A = \begin{bmatrix} \textcircled{2} & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & \textcircled{-7} & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

$$\begin{bmatrix} \textcircled{2} \\ -4 \\ 3 \\ -6 \\ 8 \end{bmatrix} \quad \begin{bmatrix} \textcircled{-7} \\ 14 \\ -14 \\ 21 \end{bmatrix}$$ Use the last three columns of I_5 to make L unit lower triangular.

$$\div 2 \quad \div -7$$

$$\begin{bmatrix} 1 & & & & \\ -2 & 1 & & & \\ 3/2 & -2 & 1 & & \\ -3 & 2 & 0 & 1 & \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix}$$

17. $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ To find L^{-1}, use the method of Section 2.2; that is, row reduce $[L \quad I]$:

$$[L \quad I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} = [I \quad L^{-1}],$$

so $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Likewise to find U^{-1}, row reduce $[U \; I]$:

$$[U \quad I] = \begin{bmatrix} 4 & 3 & -5 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & 0 & 1 & 0 & 5/2 \\ 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 0 & 0 & 1 & 3/2 & 1 \\ 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/4 & 3/8 & 1/4 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{bmatrix} = [I \quad U^{-1}],$$

so $U^{-1} = \begin{bmatrix} 1/4 & 3/8 & 1/4 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$. Thus

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/4 & 3/8 & 1/4 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/8 & 3/8 & 1/4 \\ -3/2 & -1/2 & 1/2 \\ -1 & 0 & 1/2 \end{bmatrix}$$

18. $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ To find L^{-1}, row reduce $[L \quad I]$:

$$[L \quad I] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 & 1 & 0 \\ 4 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & -1 & 1 & -4 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & L^{-1} \end{bmatrix},$$

so $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. Likewise to find U^{-1}, row reduce $\begin{bmatrix} U & I \end{bmatrix}$:

$$\begin{bmatrix} U & I \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 & 1 & 0 & 0 \\ 0 & -3 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & -2 \\ 0 & -3 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1/3 & 4/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & 1 & -1/3 & -2/3 \\ 0 & 1 & 0 & 0 & -1/3 & 4/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 & -1/6 & -1/3 \\ 0 & 1 & 0 & 0 & -1/3 & 4/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = [I \quad U^{-1}],$$

so $U^{-1} = \begin{bmatrix} 1/2 & -1/6 & -1/3 \\ 0 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}$. Thus

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1/2 & -1/6 & -1/3 \\ 0 & -1/3 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 & -1/3 \\ -7/3 & 1 & 4/3 \\ -1 & 1 & 1 \end{bmatrix}$$

19. Let A be a lower-triangular $n \times n$ matrix with nonzero entries on the diagonal, and consider the augmented matrix $[A \quad I]$.

The (1, 1)-entry can be scaled to 1 and the entries below it can be changed to 0 by adding multiples of row 1 to the rows below. This affects only the first column of A and the first column of I. So the (2, 2)-entry in the new matrix is still nonzero and now is the only nonzero entry of row 2 in the first n columns (because A was lower triangular).

The (2, 2)-entry can be scaled to 1, the entries below it can be changed to 0 by adding multiples of row 2 to the rows below. This affects only columns 2 and $n + 2$ of the augmented matrix. Now the (3, 3) entry in A is the only nonzero entry of the third row in the first n columns, so it can be scaled to 1 and then used as a pivot to zero out entries below it. Continuing in this way, A is eventually reduced to I, by scaling each row with a pivot and then using only row operations that add multiples of the pivot row to rows below.

The row operations just described only add rows to rows below, so the I on the right in $[A \; I]$ changes into a lower triangular matrix. By Theorem 7 in Section 2.2, that matrix is A^{-1}.

20. Let $A = LU$ be an LU factorization for A. Since L is unit lower triangular, it is invertible by Exercise 19. Thus by the Invertible Matrix Theroem, L may be row reduced to I. But L is unit lower triangular, so it can be row reduced to I by adding suitable multiples of a row to the rows below it, beginning with the top row. Note that all of the described row operations done to L are row-replacement operations. If elementary matrices $E_1, E_2, \ldots E_p$ implement these row-replacement operations, then

$$E_p \cdots E_2 E_1 A = (E_p \cdots E_2 E_1)LU = IU = U$$

This shows that A may be row reduced to U using only row-replacement operations.

21. (Solution in *Study Guide*.) Suppose $A = BC$, with B invertible. Then there exist elementary matrices $E_1, \ldots, E_p$ corresponding to row operations that reduce B to I, in the sense that $E_p \cdots E_1 B = I$. Applying the same sequence of row operations to A amounts to left-multiplying A by the product $E_p \cdots E_1$. By associativity of matrix multiplication.

$$E_p \cdots E_1 A = E_p \cdots E_1 BC = IC = C$$

so the same sequence of row operations reduces A to C.

22. First find an LU factorization for A. Row reduce A to echelon form using only row replacement operations:

$$A = \begin{bmatrix} \textcircled{2} & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & \textcircled{3} & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

then follow the algorithm in Example 2 to find L. Use the last two columns of I_5 to make L unit lower triangular.

$$\begin{bmatrix} \textcircled{2} \\ 6 \\ 2 \\ 4 \\ -6 \end{bmatrix} \begin{bmatrix} \textcircled{3} \\ -3 \\ 6 \\ -9 \end{bmatrix} \begin{bmatrix} \textcircled{5} \\ -5 \\ 10 \end{bmatrix}$$

$\div 2 \quad \div 3 \quad \div 5$

$$\begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 1 & -1 & 1 & & \\ 2 & 2 & -1 & 1 & \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & 3 & 2 & 0 & 1 \end{bmatrix}$$

Now notice that the bottom two rows of U contain only zeros. If one uses the row-column method to find LU, the entries in the final two columns of L will not be used, since these entries will be multiplied zeros from the bottom two rows of U. So let B be the first three columns of L and let C be the top three rows of U. That is,

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & 3 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Then B and C have the desired sizes and $BC = LU = A$. We can generalize this process to the case where A in $m \times n$, $A = LU$, and U has only three non-zero rows: let B be the first three columns of L and let C be the top three rows of U.

23. **a**. Express each row of D as the transpose of a column vector. Then use the multiplication rule for partitioned matrices to write

$$A = CD = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{bmatrix} \begin{bmatrix} \mathbf{d}_1^T \\ \mathbf{d}_2^T \\ \mathbf{d}_3^T \\ \mathbf{d}_4^T \end{bmatrix} = \mathbf{c}_1\mathbf{d}_1^T + \mathbf{c}_2\mathbf{d}_2^T + \mathbf{c}_3\mathbf{d}_3^T + \mathbf{c}_4\mathbf{d}_4^T$$

which is the sum of four outer products.

b. Since A has $400 \times 100 = 40000$ entries, C has $400 \times 4 = 1600$ entries and D has $4 \times 100 = 400$ entries, to store C and D together requires only 2000 entries, which is 5% of the amount of entries needed to store A directly.

24. Since Q is square and $Q^TQ = I$, Q is invertible by the Invertible Matrix Theorem and $Q^{-1} = Q^T$. Thus A is the product of invertible matrices and hence is invertible. Thus by Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$. From $A\mathbf{x} = \mathbf{b}$, we have $QR\mathbf{x} = \mathbf{b}$, $Q^TQR\mathbf{x} = Q^T\mathbf{b}$, $R\mathbf{x} = Q^T\mathbf{b}$, and finally $\mathbf{x} = R^{-1}Q^T\mathbf{b}$. A good algorithm for finding $\mathbf{x}$ is to compute $Q^T\mathbf{b}$ and then row reduce the matrix $[\, R \;\; Q^T\mathbf{b} \,]$. See Exercise 11 in Section 2.2 for details on why this process works. The reduction is fast in this case because R is a triangular matrix.

25. $A = UDV^T$. Since U and V^T are square, the equations $U^TU = I$ and $V^TV = I$ imply that U and V^T are invertible, by the IMT, and hence $U^{-1} = U^T$ and $(V^T)^{-1} = V$. Since the diagonal entries $\sigma_1, \ldots, \sigma_n$ in D are nonzero, D is invertible, with the inverse of D being the diagonal matrix with $\sigma_1^{-1}, \ldots, \sigma_n^{-1}$ on the diagonal. Thus A is a product of invertible matrices. By Theorem 6, A is invertible and $A^{-1} = (UDV^T)^{-1} = (V^T)^{-1}D^{-1}U^{-1} = VD^{-1}U^T$.

26. If $A = PDP^{-1}$, where P is an invertible 3×3 matrix and D is the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

Then $A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$ and since

$$D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^2 & 0 \\ 0 & 0 & 1/3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}$$

$$A^2 = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} P^{-1}$$

Likewise, $A^3 = PD^3P^{-1}$, so $A^3 = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^3 & 0 \\ 0 & 0 & 1/3^3 \end{bmatrix}P^{-1} = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & 0 & 1/27 \end{bmatrix}P^{-1}$

In general, $A^k = PD^kP^{-1}$, so $A^k = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^k & 0 \\ 0 & 0 & 1/3^k \end{bmatrix}P^{-1}$

27. First consider using a series circuit with resistance R_1 followed by a shunt circuit with resistance R_2 for the network. The transfer matrix for this network is

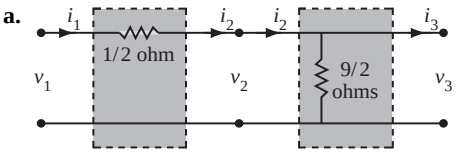
$$\begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -R_1 \\ -1/R_2 & (R_1 + R_2)/R_2 \end{bmatrix}$$

For an input of 12 volts and 6 amps to produce an output of 9 volts and 4 amps, the transfer matrix must satisfy

$$\begin{bmatrix} 1 & -R_1 \\ -1/R_2 & (R_1 + R_2)/R_2 \end{bmatrix}\begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 - 6R_1 \\ (-12 + 6R_1 + 6R_2)/R_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Equate the top entries and obtain $R_1 = \frac{1}{2}\text{ohm}$. Substitute this value in the bottom entry and solve to obtain $R_2 = \frac{9}{2}\text{ohms}$. The ladder network is

a.



Next consider using a shunt circuit with resistance R_1 followed by a series circuit with resistance R_2 for the network. The transfer matrix for this network is

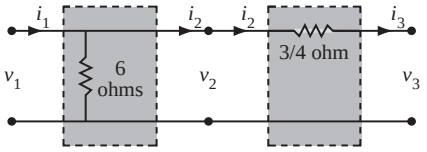
$$\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} (R_1 + R_2)/R_1 & -R_2 \\ -1/R_1 & 1 \end{bmatrix}$$

For an input of 12 volts and 6 amps to produce an output of 9 volts and 4 amps, the transfer matrix must satisfy

$$\begin{bmatrix} (R_1 + R_2)/R_1 & -R_2 \\ -1/R_1 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} (12R_1 + 12R_2)/R_1 - 6R_2 \\ -12/R_1 + 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Equate the bottom entries and obtain $R_1 = 6$ ohms. Substitute this value in the top entry and solve to obtain $R_2 = \frac{3}{4}\text{ohms}$. The ladder network is

b.



28. The three shunt circuits have transfer matrices

$$\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}$$

respectively. To find the transfer matrix for the series of circuits, multiply these matrices

$$\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(1/R_1 + 1/R_2 + 1/R_3) & 1 \end{bmatrix}$$

Thus the resulting network is itself a shunt circuit with resistance $1/R_1 + 1/R_2 + 1/R_3$.

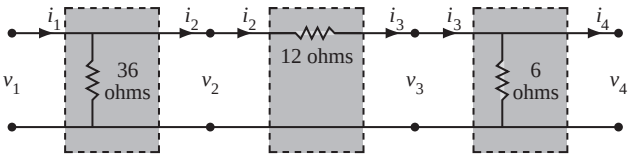
29. **a**. The first circuit is a shunt circuit with resistance R_1 ohms, so its transfer matrix is $\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix}$. The second circuit is a series circuit with resistance R_2 ohms, so its transfer matrix is $\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}$. The third circuit is a shunt circuit with resistance R_3 ohms so its transfer matrix is $\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}$. The transfer matrix of the network is the product of these matrices, in *right-to-left* order:

$$\begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} (R_1 + R_2)/R_1 & -R_2 \\ -(R_1 + R_2 + R_3)/(R_1R_3) & (R_2 + R_3)/R_3 \end{bmatrix}$$

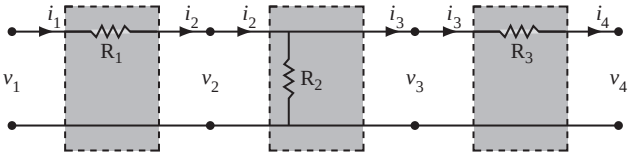
b. To find a ladder network with a structure like that in part (a) and with the given transfer matrix A, we must find resistances R_1, R_2, and R_3 such tha $A = \begin{bmatrix} 4/3 & -12 \\ -1/4 & 3 \end{bmatrix} = \begin{bmatrix} (R_1 + R_2)/R_1 & -R_2 \\ -(R_1 + R_2 + R_3)/R_3 & (R_2 + R_3)/R_3 \end{bmatrix}$

From the (1, 2) entries, $R_2 = 12$ ohms. The (1, 1) entries now give $(R_1 + 12)/R_1 = 4/3,$ which may be solved to obtain $R_1 = 36$ ohms. Likewise the (2, 2) entries give $(R_3 + 12)/R_3 = 3,$ which also may be solved to obtain $R_3 = 6$ ohms. Thus the matrix A may be factored as

$A = \begin{bmatrix} 1 & 0 \\ -1/R_3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/6 & 1 \end{bmatrix}\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/36 & 1 \end{bmatrix}$. The ladder network is



30. Answers may vary. The network below interchanges the series and shunt circuits.



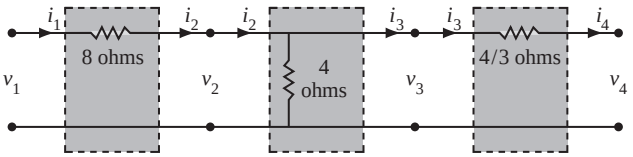
The transfer matrix of this network is the product of the individual transfer matrices, in *right-to-left* order. $\begin{bmatrix} 1 & -R_3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (R_2+R_3)/R_2 & -R_3 - R_1(R_2+R_3)/R_2 \\ -1/R_2 & (R_1+R_2)/R_2 \end{bmatrix}$

By setting the matrix A from the previous exercise equal to this matrix, one may find that

$$\begin{bmatrix} (R_2+R_3)/R_2 & -R_3 - R_1(R_2+R_3)/R_2 \\ -1/R_2 & (R_1+R_2)/R_2 \end{bmatrix} = \begin{bmatrix} 4/3 & -12 \\ -1/4 & 3 \end{bmatrix}$$

Set the (2, 1) entries equal and obtain $R_2 = 4$ ohms. Substitute this value for R_2, equating the (2, 2) entries and solving gives $R_1 = 8$ ohms. Likewise equating the (1, 1) entries gives $R_3 = 4/3$ ohms.

The ladder network is



Note: The *Study Guide's* MATLAB box for Section 2.5 suggests that for most LU factorizations in this section, students can use the **`gauss`** command repeatedly to produce U, and use paper and mental arithmetic to write down the columns of L as the row reduction to U proceeds. This is because for Exercises 7–16 the pivots are integers and other entries are simple fractions. However, for Exercises 31 and 32 this is not reasonable, and students are expected to solve an elementary programming problem. (The *Study Guide* provides no hints.)

31. **[M]** Store the matrix A in a temporary matrix B and create L initially as the 8×8 identity matrix. The following sequence of MATLAB commands fills in the entries of L below the diagonal, one column at a time, until the first seven columns are filled. (The eighth column is the final column of the identity matrix.)

```
L(2:8, 1) = B(2:8, 1)/B(1, 1)
B = gauss(B, 1)
L(3:8, 2) = B(3:8, 2)/B(2, 2)
B = gauss(B, 2)
⋮
L(8:8, 7) = B(8:8, 7)/B(7, 7)
U = gauss(B,7)
```

Of course, some students may realize that a loop will speed up the process. The **`for..end`** syntax is illustrated in the MATLAB box for Section 5.6. Here is a MATLAB program that includes the initial setup of B and L:

```
B = A
L = eye(8)
for j=1:7
  L(j+1:8, j) = B(j+1:8, j)/B(j, j)
  B = gauss(B, j)
end
U = B
```

a. To four decimal places, the results of the LU decomposition are

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & -.0667 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.2667 & -.2857 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.2679 & -.0833 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.2917 & -.2921 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.2697 & -.0861 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.2948 & -.2931 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.75 & -.25 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.7333 & -1.0667 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4286 & -.2857 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.7083 & -1.0833 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.3919 & -.2921 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7052 & -1.0861 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3868 \end{bmatrix}$$

b. The result of solving $L\mathbf{y} = \mathbf{b}$ and then $U\mathbf{x} = \mathbf{y}$ is

$\mathbf{x} = (3.9569, 6.5885, 4.2392, 7.3971, 5.6029, 8.7608, 9.4115, 12.0431)$

c. $A^{-1} = \begin{bmatrix} .2953 & .0866 & .0945 & .0509 & .0318 & .0227 & .0010 & .0082 \\ .0866 & .2953 & .0509 & .0945 & .0227 & .0318 & .0082 & .0100 \\ .0945 & .0509 & .3271 & .1093 & .1045 & .0591 & .0318 & .0227 \\ .0509 & .0945 & .1093 & .3271 & .0591 & .1045 & .0227 & .0318 \\ .0318 & .0227 & .1045 & .0591 & .3271 & .1093 & .0945 & .0509 \\ .0227 & .0318 & .0591 & .1045 & .1093 & .3271 & .0509 & .0945 \\ .0010 & .0082 & .0318 & .0227 & .0945 & .0509 & .2953 & .0866 \\ .0082 & .0100 & .0227 & .0318 & .0509 & .0945 & .0866 & .2953 \end{bmatrix}$

32. **a.** **[M]** $A = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$. The commands shown for Exercise 31, but modified for 5×5 matrices, produce $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & -\frac{3}{8} & 1 & 0 & 0 \\ 0 & 0 & -\frac{8}{21} & 1 & 0 \\ 0 & 0 & 0 & -\frac{21}{55} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & -1 & 0 & 0 \\ 0 & 0 & \frac{21}{8} & -1 & 0 \\ 0 & 0 & 0 & \frac{55}{21} & -1 \\ 0 & 0 & 0 & 0 & \frac{144}{55} \end{bmatrix}$

b. Let $\mathbf{s}_{k+1}$ be the solution of $L\mathbf{s}_{k+1} = \mathbf{t}_k$ for $k = 0, 1, 2, \ldots$. Then $\mathbf{t}_{k+1}$ is the solution of $U\mathbf{t}_{k+1} = \mathbf{s}_{k+1}$ for $k = 0, 1, 2, \ldots$. The results are

$$\mathbf{s}_1 = \begin{bmatrix} 10.0000 \\ 15.3333 \\ 17.7500 \\ 18.7619 \\ 17.1636 \end{bmatrix}, \mathbf{t}_1 = \begin{bmatrix} 6.5556 \\ 9.6667 \\ 10.4444 \\ 9.6667 \\ 6.5556 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 6.5556 \\ 11.8519 \\ 14.8889 \\ 15.3386 \\ 12.4121 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 4.7407 \\ 7.6667 \\ 8.5926 \\ 7.6667 \\ 4.7407 \end{bmatrix},$$

$$\mathbf{s}_3 = \begin{bmatrix} 4.7407 \\ 9.2469 \\ 12.0602 \\ 12.2610 \\ 9.4222 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 3.5988 \\ 6.0556 \\ 6.9012 \\ 6.0556 \\ 3.5988 \end{bmatrix}, \mathbf{s}_4 = \begin{bmatrix} 3.5988 \\ 7.2551 \\ 9.6219 \\ 9.7210 \\ 7.3104 \end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix} 2.7922 \\ 4.7778 \\ 5.4856 \\ 4.7778 \\ 2.7922 \end{bmatrix}.$$

2.6 SOLUTIONS

Notes: This section is independent of Section 1.10. The material here makes a good backdrop for the series expansion of $(I–C)^{-1}$ because this formula is actually used in some practical economic work. Exercise 8 gives an interpretation to entries of an inverse matrix that could be stated without the economic context.

1. The answer to this exercise will depend upon the order in which the student chooses to list the sectors. The important fact to remember is that each column is the unit consumption vector for the appropriate sector. If we order the sectors manufacturing, agriculture, and services, then the consumption matrix is

$$C = \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & 0 \\ .30 & .10 & .10 \end{bmatrix}$$

The intermediate demands created by the production vector $\mathbf{x}$ are given by $C\mathbf{x}$. Thus in this case the intermediate demand is

$$C\mathbf{x} = \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 20 \\ 10 \end{bmatrix}$$

2. Solve the equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$ for $\mathbf{d}$:

$$\mathbf{d} = \mathbf{x} - C\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .9x_1 - .6x_2 - .6x_3 \\ -.3x_1 + .8x_2 \\ -.3x_1 - .1x_2 + .9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ 0 \end{bmatrix}$$

This system of equations has the augmented matrix $\begin{bmatrix} .90 & -.60 & -.60 & 0 \\ -.30 & .80 & .00 & 18 \\ -.30 & -.10 & .90 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 33.33 \\ 0 & 1 & 0 & 35.00 \\ 0 & 0 & 1 & 15.00 \end{bmatrix}$

So $\mathbf{x} = \begin{bmatrix} 33.33 \\ 35.00 \\ 15.00 \end{bmatrix}$.

3. Solving as in Exercise 2:

$$\mathbf{d} = \mathbf{x} - C\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .9x_1 - .6x_2 - .6x_3 \\ -.3x_1 + .8x_2 \\ -.3x_1 - .1x_2 + .9x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix} .90 & -.60 & -.60 & 18 \\ -.30 & .80 & .00 & 0 \\ -.30 & -.10 & .90 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 40.00 \\ 0 & 1 & 0 & 15.00 \\ 0 & 0 & 1 & 15.00 \end{bmatrix}, \text{ so } \mathbf{x} = \begin{bmatrix} 40.00 \\ 15.00 \\ 15.00 \end{bmatrix}.$$

4. Solving as in Exercise 2:

$$\mathbf{d} = \mathbf{x} - C\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .9x_1 - .6x_2 - .6x_3 \\ -.3x_1 + .8x_2 \\ -.3x_1 - .1x_2 + .9x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \\ 0 \end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix} .90 & -.60 & -.60 & 18 \\ -.30 & .80 & .00 & 18 \\ -.30 & -.10 & .90 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 73.33 \\ 0 & 1 & 0 & 50.00 \\ 0 & 0 & 1 & 30.00 \end{bmatrix}, \quad \text{so } \mathbf{x} = \begin{bmatrix} 73.33 \\ 50.00 \\ 30.00 \end{bmatrix}.$$

Note: Exercises 2–4 may be used by students to discover the linearity of the Leontief model.

5. $\mathbf{x} = (I - C)^{-1}\mathbf{d} = \begin{bmatrix} 1 & -.5 \\ -.6 & .8 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 30 \end{bmatrix} = \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix} \begin{bmatrix} 50 \\ 30 \end{bmatrix} = \begin{bmatrix} 110 \\ 120 \end{bmatrix}$

6. $\mathbf{x} = (I - C)^{-1}\mathbf{d} = \begin{bmatrix} .9 & -.6 \\ -.5 & .8 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 11 \end{bmatrix} = \begin{bmatrix} 40/21 & 30/21 \\ 25/21 & 45/21 \end{bmatrix} \begin{bmatrix} 18 \\ 11 \end{bmatrix} = \begin{bmatrix} 50 \\ 45 \end{bmatrix}$

7. **a**. From Exercise 5, $(I - C)^{-1} = \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix}$ so $\mathbf{x}_1 = (I - C)^{-1}\mathbf{d}_1 = \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 1.2 \end{bmatrix}$

which is the first column of $(I - C)^{-1}$.

b. $\mathbf{x}_2 = (I - C)^{-1}\mathbf{d}_2 = \begin{bmatrix} 1.6 & 1 \\ 1.2 & 2 \end{bmatrix} \begin{bmatrix} 51 \\ 30 \end{bmatrix} = \begin{bmatrix} 111.6 \\ 121.2 \end{bmatrix}$

c. From Exercise 5, the production **x** corressponding to $\mathbf{d} = \begin{bmatrix} 50 \\ 20 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 110 \\ 120 \end{bmatrix}$.

Note that $\mathbf{d}_2 = \mathbf{d} + \mathbf{d}_1$. Thus

$$\begin{aligned} \mathbf{x}_2 &= (I - C)^{-1}\mathbf{d}_2 \\ &= (I - C)^{-1}(\mathbf{d} + \mathbf{d}_1) \\ &= (I - C)^{-1}\mathbf{d} + (I - C)^{-1}\mathbf{d}_1 \\ &= \mathbf{x} + \mathbf{x}_1 \end{aligned}$$

8. **a**. Given $(I - C)\mathbf{x} = \mathbf{d}$ and $(I - C)\Delta\mathbf{x} = \Delta\mathbf{d}$,

$$(I - C)(\mathbf{x} + \Delta\mathbf{x}) = (I - C)\mathbf{x} + (I - C)\Delta\mathbf{x} = \mathbf{d} + \Delta\mathbf{d}$$

Thus $\mathbf{x} + \Delta\mathbf{x}$ is the production level corresponding to a demand of $\mathbf{d} + \Delta\mathbf{d}$.

b. Since $\Delta\mathbf{x} = (I - C)^{-1}\Delta\mathbf{d}$ and $\Delta\mathbf{d}$ is the first column of I, $\Delta\mathbf{x}$ will be the first column of $(I - C)^{-1}$.

9. In this case $I - C = \begin{bmatrix} .8 & -.2 & .0 \\ -.3 & .9 & -.3 \\ -.1 & .0 & .8 \end{bmatrix}$. Row reduce $[\,I - C \quad \mathbf{d}\,]$ to find

$$\begin{bmatrix} .8 & -.2 & .0 & 40.0 \\ -.3 & .9 & -.3 & 60.0 \\ -.1 & .0 & .8 & 80.0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 82.8 \\ 0 & 1 & 0 & 131.0 \\ 0 & 0 & 1 & 110.3 \end{bmatrix}.$$ So $\mathbf{x}$ = (82.8, 131.0, 110.3).

10. From Exercise 8, the (i, j) entry in $(I - C)^{-1}$ corresponds to the effect on production of sector i when the final demand for the output of sector j increases by one unit. Since these entries are all positive, an increase in the final demand for any sector will cause the production of all sectors to increase. Thus an increase in the demand for any sector will lead to an increase in the demand for all sectors.

11. (Solution in *study Guide*) Following the hint in the text, compute $\mathbf{p}^T\mathbf{x}$ in two ways. First, take the transpose of both sides of the price equation, $\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$, to obtain

$\mathbf{p}^T = (C^T\mathbf{p} + \mathrm{v})^T = (C^T\mathbf{p})^T + \mathbf{v}^T = \mathbf{p}^T C + \mathbf{v}^T$ and right-multiply by $\mathbf{x}$ to get

$\mathbf{p}^T\mathbf{x} = (\mathbf{p}^T C + \mathbf{v}^T)\mathbf{x} = \mathbf{p}^T C\mathbf{x} + \mathbf{v}^T\mathbf{x}$

Another way to compute $\mathbf{p}^T\mathbf{x}$ starts with the production equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$. Left multiply by $\mathbf{p}^T$ to get $\mathbf{p}^T\mathbf{x} = \mathbf{p}^T(C\mathbf{x} + \mathbf{d}) = \mathbf{p}^T C\mathbf{x} + \mathbf{p}^T\mathbf{d}$. The two expression for $\mathbf{p}^T\mathbf{x}$ show that $\mathbf{p}^T C\mathbf{x} + \mathbf{v}^T\mathbf{x} = \mathbf{p}^T C\mathbf{x} + \mathbf{p}^T\mathbf{d}$ so $\mathbf{v}^T\mathbf{x} = \mathbf{p}^T\mathbf{d}$. The *Study Guide* also provides a slightly different solution.

12. Since $D_{m+1} = I + C + C^2 + \ldots + C^{m+1} = I + C(I + C + \ldots + C^m) = I + CD_m$, D_{m+1} may be found iteratively by $D_{m+1} = I + CD_m$.

13. **[M]** The matrix $I - C$ is $\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 \end{bmatrix}$ so the augmented matrix $[\,I - C \quad \mathbf{d}\,]$ may be row reduced to find

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 & 74000 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 & 56000 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 & 10500 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 & 25000 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 & 17500 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 & 196000 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 & 5000 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 99576 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 97703 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 51231 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 131570 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 49488 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 329554 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13835 \end{bmatrix}$$

so **x** = (99576, 97703, 51321, 131570, 49488, 329554, 13835). Since the entries in **d** seem to be accurate to the nearest thousand, a more realistic answer would be **x** = (100000, 98000, 51000, 132000, 49000, 330000, 14000).

14. **[M]** The augmented matrix $[\,I - C \quad \mathbf{d}\,]$ in this case may be row reduced to find

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 & 99640 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 & 75548 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 & 14444 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 & 33501 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 & 23527 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 & 263985 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 & 6526 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 134034 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 131687 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 69472 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 176912 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 66596 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 443773 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 18431 \end{bmatrix}$$

so **x** = (134034, 131687, 69472, 176912, 66596, 443773, 18431). To the nearest thousand, **x** = (134000, 132000, 69000, 177000, 67000, 444000, 18000).

15. **[M]** Here are the iterations rounded to the nearest tenth:

$$\mathbf{x}^{(0)} = (74000.0, 56000.0, 10500.0, 25000.0, 17500.0, 196000.0, 5000.0)$$
$$\mathbf{x}^{(1)} = (89344.2, 77730.5, 26708.1, 72334.7, 30325.6, 265158.2, 9327.8)$$
$$\mathbf{x}^{(2)} = (94681.2, 87714.5, 37577.3, 100520.5, 38598.0, 296563.8, 11480.0)$$
$$\mathbf{x}^{(3)} = (97091.9, 92573.1, 43867.8, 115457.0, 43491.0, 312319.0, 12598.8)$$
$$\mathbf{x}^{(4)} = (98291.6, 95033.2, 47314.5, 123202.5, 46247.0, 320502.4, 13185.5)$$
$$\mathbf{x}^{(5)} = (98907.2, 96305.3, 49160.6, 127213.7, 47756.4, 324796.1, 13493.8)$$
$$\mathbf{x}^{(6)} = (99226.6, 96969.6, 50139.6, 129296.7, 48569.3, 327053.8, 13655.9)$$
$$\mathbf{x}^{(7)} = (99393.1, 97317.8, 50656.4, 130381.6, 49002.8, 328240.9, 13741.1)$$
$$\mathbf{x}^{(8)} = (99480.0, 97500.7, 50928.7, 130948.0, 49232.5, 328864.7, 13785.9)$$
$$\mathbf{x}^{(9)} = (99525.5, 97596.8, 51071.9, 131244.1, 49353.8, 329192.3, 13809.4)$$
$$\mathbf{x}^{(10)} = (99549.4, 97647.2, 51147.2, 131399.2, 49417.7, 329364.4, 13821.7)$$
$$\mathbf{x}^{(11)} = (99561.9, 97673.7, 51186.8, 131480.4, 49451.3, 329454.7, 13828.2)$$
$$\mathbf{x}^{(12)} = (99568.4, 97687.6, 51207.5, 131523.0, 49469.0, 329502.1, 13831.6)$$

so $\mathbf{x}^{(12)}$ is the first vector whose entries are accurate to the nearest thousand. The calculation of $\mathbf{x}^{(12)}$ takes about 1260 flops, while the row reduction above takes about 550 flops. If C is larger than 20×20, then fewer flops are required to compute $\mathbf{x}^{(12)}$ by iteration than by row reduction. The advantage of the iterative method increases with the size of C. The matrix C also becomes more sparse for larger models, so fewer iterations are needed for good accuracy.

2.7 SOLUTIONS

Notes: The content of this section seems to have universal appeal with students. It also provides practice with composition of linear transformations. The case study for Chapter 2 concerns computer graphics – see this case study (available as a project on the website) for more examples of computer graphics in action. The *Study Guide* encourages the student to examine the book by Foley referenced in the text. This section could form the beginning of an independent study on computer graphics with an interested student.

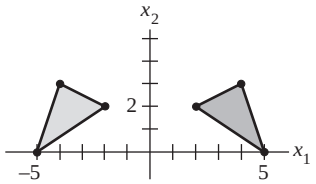
1. Refer to Example 5. The representation in homogenous coordinates can be written as a partitioned matrix of the form $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, where A is the matrix of the linear transformation. Since in this case $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$, the representation of the transformation with respect to homogenous coordinates is $\begin{bmatrix} 1 & .25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: The *Study Guide* shows the student why the action of $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$ on the vector $\begin{bmatrix} \mathrm{x} \\ 1 \end{bmatrix}$ corresponds to the action of A on $\mathbf{x}$.

2. The matrix of the transformation is $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, so the transformed data matrix is

$$AD = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix}$$

Both the original triangle and the transformed triangle are shown in the following sketch.



3. Following Examples 4–6, $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & \sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 2\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} .8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .8 & 0 & -1.6 \\ 0 & 1.2 & 3.6 \\ 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ -1/2 & -\sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. A 60° rotation about the origin is given in homogeneous coordinates by the matrix $\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To rotate about the point (6, 8), first translate by (–6, –8), then rotate about the origin, then translate back by (6, 8) (see the Practice Problem in this section). A 60° rotation about (6, 8)

is thus given in homogeneous coordinates by the matrix

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$$

8. A 45° rotation about the origin is given in homogeneous coordinates by the matrix $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To rotate about the point (3, 7), first translate by (–3, –7), then rotate about the origin, then translate back by (3, 7) (see the Practice Problem in this section). A 45° rotation about (3, 7) is thus is given in homogeneous coordinates by the matrix

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 3+2\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2}/2 & 7-5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

9. To produce each entry in BD two multiplications are necessary. Since BD is a 2×200 matrix, it will take $2 \times 2 \times 200 = 800$ multiplications to compute BD. By the same reasoning it will take $2 \times 2 \times 200 = 800$ multiplications to compute $A(BD)$. Thus to compute $A(BD)$ from the beginning will take 800 + 800 = 1600 multiplications.

To compute the 2×2 matrix AB it will take $2 \times 2 \times 2 = 8$ multiplications, and to compute $(AB)D$ it will take $2 \times 2 \times 200 = 800$ multiplications. Thus to compute $(AB)D$ from the beginning will take 8 + 800 = 808 multiplications.

For computer graphics calculations that require applying multiple transformations to data matrices, it is thus more efficient to compute the product of the transformation matrices before applying the result to the data matrix.

10. Let the transformation matrices in homogeneous coordinates for the dilation, rotation, and translation be called respectively D, and R, and T. Then for some value of s, φ, h, and k,

$$D = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; R = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the products of these matrices:

$$DR = \begin{bmatrix} s\cos\varphi & -s\sin\varphi & 0 \\ s\sin\varphi & s\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; RD = \begin{bmatrix} s\cos\varphi & -s\sin\varphi & 0 \\ s\sin\varphi & s\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$DT = \begin{bmatrix} s & 0 & sh \\ 0 & s & sk \\ 0 & 0 & 1 \end{bmatrix}, TD = \begin{bmatrix} s & 0 & h \\ 0 & s & k \\ 0 & 0 & 1 \end{bmatrix},$$

$$RT = \begin{bmatrix} \cos\varphi & -\sin\varphi & h\cos\varphi - k\sin\varphi \\ \sin\varphi & \cos\varphi & h\sin\varphi + k\cos\varphi \\ 0 & 0 & 1 \end{bmatrix}, TR = \begin{bmatrix} \cos\varphi & -\sin\varphi & h \\ \sin\varphi & \cos\varphi & k \\ 0 & 0 & 1 \end{bmatrix}$$

Since $DR = RD$, $DT \neq TD$ and $RT \neq TR$, D and R commute, D and T do not commute and R and T do not commute.

11. To simplify A_2A_1 completely, the following trigonometric identities will be needed:

1. $-\tan\varphi\cos\varphi = -\frac{\sin\varphi}{\cos\varphi}\cos\varphi = -\sin\varphi$

2. $\sec\varphi - \tan\varphi\sin\varphi = \frac{1}{\cos\varphi} - \frac{\sin\varphi}{\cos\varphi}\sin\varphi = \frac{1-\sin^2\varphi}{\cos\varphi} = \frac{\cos^2\varphi}{\cos\varphi} = \cos\varphi$

Using these identities,

$$A_2A_1 = \begin{bmatrix} \sec\varphi & -\tan\varphi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sec\varphi - \tan\varphi\sin\varphi & -\tan\varphi\cos\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is the transformation matrix in homogeneous coordinates for a rotation in $\mathbb{R}^2$.

12. To simplify this product completely, the following trigonometric identity will be needed:

$$\tan\varphi/2 = \frac{1-\cos\varphi}{\sin\varphi} = \frac{\sin\varphi}{1+\cos\varphi}$$

This identity has two important consequences:

$$1 - (\tan\varphi/2)(\sin\varphi) = 1 - \frac{1-\cos\varphi}{\sin\varphi}\sin\varphi = \cos\varphi$$

$$(\cos\varphi)(-\tan\varphi/2) - \tan\varphi/2 = -(\cos\varphi+1)\tan\varphi/2 = -(\cos\varphi+1)\frac{\sin\varphi}{1+\cos\varphi} = -\sin\varphi$$

The product may be computed and simplified using these results:

$$\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-(\tan\varphi/2)(\sin\varphi) & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\tan\varphi/2 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan\varphi/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & (\cos\varphi)(-\tan\varphi/2) - \tan\varphi/2 & 0 \\ \sin\varphi & -(\sin\varphi)(\tan\varphi/2) + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, which is the transformation matrix in homogeneous coordinates for a rotation in $\mathbb{R}^2$.

13. Consider first applying the linear transformation on $\mathbb{R}^2$ whose matrix is A, then applying a translation by the vector $\mathbf{p}$ to the result. The matrix representation in homogeneous coordinates of the linear transformation is $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, while the matrix representation in homogeneous coordinates of the translation is $\begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix}$. Applying these transformations in order leads to a transformation whose matrix representation in homogeneous coordinates is $\begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix}$ which is the desired matrix.

14. The matrix for the transformation in Exercise 7 was found to be $\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$ This matrix is of the form $\begin{bmatrix} A & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix}$, where $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 3+4\sqrt{3} \\ 4-3\sqrt{3} \end{bmatrix}$. By Exercise 13, this matrix may be written as $\begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, that is, the composition of a linear transformation on $\mathbb{R}^2$ and a translation. The matrix A is the matrix of a rotation about the origin in $\mathbb{R}^2$. Thus the transformation in Exercise 7 is the composition of a rotation about the origin and a translation by $\mathbf{p} = \begin{bmatrix} 3+4\sqrt{3} \\ 4-3\sqrt{3} \end{bmatrix}$.

15. Since $(X, Y, Z, H) = (\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \frac{1}{24})$, the corresponding point in $\mathbb{R}^3$ has coordinates

$$(x, y, z) = \left(\frac{X}{H}, \frac{Y}{H}, \frac{Z}{H}\right) = \left(\frac{\frac{1}{2}}{\frac{1}{24}}, \frac{-\frac{1}{4}}{\frac{1}{24}}, \frac{\frac{1}{8}}{\frac{1}{24}}\right) = (12, -6, 3)$$

16. The homogeneous coordinates (1, –2, 3, 4) represent the point $(1/4, -2/4, 3/4) = (1/4, -1/2, 3/4)$ while the homogeneous coordinates (10, –20, 30, 40) represent the point $(10/40, -20/40, 30/40) = (1/4, -1/2, 3/4)$

so the two sets of homogeneous coordinates represent the same point in $\mathbb{R}^3$.

17. Follow Example 7a by first constructing that 3×3 matrix for this rotation. The vector $\mathbf{e}_1$ is not changed by this rotation. The vector $\mathbf{e}_2$ is rotated 60° toward the positive z-axis, ending up at the point (0, cos 60°, sin 60°) = $(0, 1/2, \sqrt{3}/2)$. The vector $\mathbf{e}_3$ is rotated 60° toward the negative y-axis, stopping at the point (0, cos 150°, sin 150°) = $(0, -\sqrt{3}/2, 1/2)$. The matrix A for this rotation is thus $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$. so in homogeneous coordinates the transformation is represented by the matrix $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

18. First construct the 3×3 matrix for the rotation. The vector $\mathbf{e}_1$ is rotated 30° toward the negative y-axis, ending up at the point (cos(–30)°, sin (–30)°, 0) = $(\sqrt{3}/2, -1/2, 0)$. The vector $\mathbf{e}_2$ is rotated 60° toward the positive x-axis, ending up at the point (cos 60°, sin 60°, 0) = $(1/2, \sqrt{3}/2, 0)$. The vector $\mathbf{e}_3$ is not changed by the rotation. The matrix A for the rotation is thus $A = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so in homogeneous coordinates the rotation is represented by the matrix $\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Following Example 7b, in homogeneous coordinates the translation by the vector (5, –2, 1) is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Thus the complete transformation is represented in homogeneous coordinates by the matrix $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

19. Referring to the material preceding Example 8 in the text, we find that the matrix P that performs a perspective projection with center of projection (0, 0, 10) is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}$.

The homogeneous coordinates of the vertices of the triangle may be written as (4.2, 1.2, 4, 1), (6, 4, 2, 1), and (2, 2, 6, 1), so the data matrix for S is $\begin{bmatrix} 4.2 & 6 & 2 \\ 1.2 & 4 & 2 \\ 4 & 2 & 6 \\ 1 & 1 & 1 \end{bmatrix}$ and the data matrix for the transformed triangle is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}\begin{bmatrix} 4.2 & 6 & 2 \\ 1.2 & 4 & 2 \\ 4 & 2 & 6 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4.2 & 6 & 2 \\ 1.2 & 4 & 2 \\ 0 & 0 & 0 \\ .6 & .8 & .4 \end{bmatrix}$. Finally, the columns of this matrix may be converted from homogeneous coordinates by dividing by the final coordinate:

$(4.2, 1.2, 0, .6) \to (4.2/.6, 1.2/.6, 0/.6) = (7, 2, 0)$

$(6, 4, 0, .8) \to (6/.8, 4/.8, 0/.8) = (7.5, 5, 0)$

$(2, 2, 0, .4) \to (2/.4, 2/.4, 0/.4) = (5, 5, 0)$

So the coordinates of the vertices of the transformed triangle are (7, 2, 0), (7.5, 5, 0), and (5, 5, 0).

20. As in the previous exercise, the matrix P that performs the perspective projection is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}$

The homogeneous coordinates of the vertices of the triangle may be written as (9, 3, –5, 1), (12, 8, 2, 1), and (1.8, 2.7, 1, 1), so the data matrix for S is $\begin{bmatrix} 9 & 12 & 1.8 \\ 3 & 8 & 2.7 \\ -5 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the data matrix for the transformed triangle is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 12 & 1.8 \\ 3 & 8 & 2.7 \\ -5 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 1.8 \\ 3 & 8 & 2.7 \\ 0 & 0 & 0 \\ 1.5 & .8 & .9 \end{bmatrix}$. Finally, the columns of this matrix may be converted from homogeneous coordinates by dividing by the final coordinate:

$(9, 3, 0, 1.5) \to (9/1.5, 3/1.5, 0/1.5) = (6, 2, 0)$

$(12, 8, 0, .8) \to (12/.8, 8/.8, 0/.8) = (15, 10, 0)$

$(1.8, 2.7, 0, .9) \to (1.8/.9, 2.7/.9, 0/.9) = (2, 3, 0)$

So the coordinates of the vertices of the transformed triangle are (6, 2, 0), (15, 10, 0), and (2, 3, 0).

21. **[M]** Solve the given equation for the vector (R, G, B), giving

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} .61 & .29 & .15 \\ .35 & .59 & .063 \\ .04 & .12 & .787 \end{bmatrix}^{-1}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 2.2586 & -1.0395 & -.3473 \\ -1.3495 & 2.3441 & .0696 \\ .0910 & -.3046 & 1.2777 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

22. **[M]** Solve the given equation for the vector (R, G, B), giving

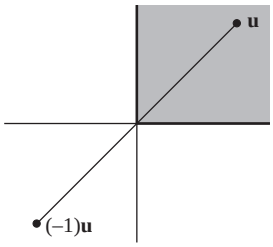
$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} .299 & .587 & .114 \\ .596 & -.275 & -.321 \\ .212 & -.528 & .311 \end{bmatrix}^{-1} \begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} 1.0031 & .9548 & .6179 \\ .9968 & -.2707 & -.6448 \\ 1.0085 & -1.1105 & 1.6996 \end{bmatrix} \begin{bmatrix} Y \\ I \\ Q \end{bmatrix}$$

2.8 SOLUTIONS

Notes: Cover this section only if you plan to skip most or all of Chapter 4. This section and the next cover everything you need from Sections 4.1–4.6 to discuss the topics in Section 4.9 and Chapters 5–7 (except for the general inner product spaces in Sections 6.7 and 6.8). Students may use Section 4.2 for review, particularly the Table near the end of the section. (The final subsection on linear transformations should be omitted.) Example 6 and the associated exercises are critical for work with eigenspaces in Chapters 5 and 7. Exercises 31–36 review the Invertible Matrix Theorem. New statements will be added to this theorem in Section 2.9.

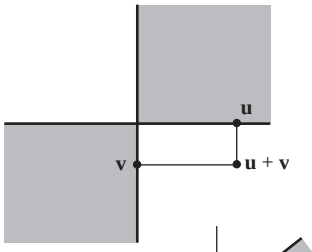
Key Exercises: 5–20 and 23–26.

1. The set is closed under sums but not under multiplication by a negative scalar. A counterexample to the subspace condition is shown at the right.

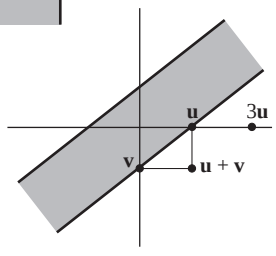


Note: Most students prefer to give a geometric counterexample, but some may choose an algebraic calculation. The four exercises here should help students develop an understanding of subspaces, but they may be insufficient if you want students to be able to analyze an unfamiliar set on an exam. Developing that skill seems more appropriate for classes covering Sections 4.1–4.6.

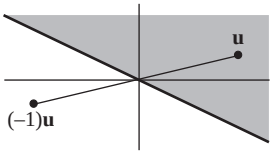
2. The set is closed under scalar multiples but not sums. For example, the sum of the vectors **u** and **v** shown here is not in H.



3. No. The set is not closed under sums or scalar multiples. The subset consisting of the points on the line $x_2 = x_1$ is a subspace, so any "counterexample" must use at least one point not on this line. Here are two counterexamples to the subspace conditions:



4. No. The set is closed under sums, but not under multiplication by a negative scalar.



5. The vector **w** is in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{w}$ is consistent. The row operations below show that **w** is *not* in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{w}] \sim \begin{bmatrix} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & -2 & 11 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -4 & 8 \\ 0 & \textcircled{1} & -10 \\ 0 & 0 & \textcircled{-9} \end{bmatrix}$$

6. The vector **u** is in the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{u}$ is consistent. The row operations below show that **u** is *not* in the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{u}] \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & -7 & -14 & 9 \\ 0 & -5 & -10 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 4 & 5 & -4 \\ 0 & \textcircled{1} & 2 & 2 \\ 0 & 0 & 0 & \textcircled{23} \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Note: For a quiz, you could use $\mathbf{w} = (1, -3, 11, 8)$, which is *in* Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. **a**. There are three vectors: $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. There are infinitely many vectors in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ = Col A.

c. Deciding whether **p** is in Col A requires calculation:

$$[A \ \ \mathbf{p}] \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -3 & -4 & 6 \\ 0 & \textcircled{-4} & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{p}$ has a solution, so **p** is in Col A.

8. $$[A \ \ \mathbf{p}] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & -2 & 0 & 1 \\ 0 & \textcircled{2} & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes, the augmented matrix $[A \ \ \mathbf{p}]$ corresponds to a consistent system, so **p** is in Col A.

9. To determine whether **p** is in Nul A, simply compute $A\mathbf{p}$. Using A and **p** as in Exercise 7,

$$A\mathbf{p} = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}.$$ Since $A\mathbf{p} \neq \mathbf{0}$, **p** is *not* in Nul A.

10. To determine whether **u** is in Nul A, simply compute $A\mathbf{u}$. Using A as in Exercise 7 and $\mathbf{u} = (-2, 3, 1)$,

$$A\mathbf{u} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Yes, **u** is in Nul A.

11. $p = 4$ and $q = 3$. Nul A is a subspace of $\mathbb{R}^4$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 4 entries, to match the columns of A. Col A is a subspace of $\mathbb{R}^3$ because each column vector has 3 entries.

12. $p = 3$ and $q = 4$. Nul A is a subspace of $\mathbb{R}^3$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 3 entries, to match the columns of A. Col A is a subspace of $\mathbb{R}^4$ because each column vector has 4 entries.

13. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$. (Include an augmented column of zeros, to avoid errors.)

$$\begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ -9 & -4 & 1 & 7 & 0 \\ 9 & 2 & -5 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & -4 & -8 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 1 & 0 \\ 0 & \textcircled{1} & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} \textcircled{x_1} \quad & - x_3 + x_4 = 0 \\ \textcircled{x_2} \; & + 2x_3 - 4x_4 = 0 \\ & 0 = 0 \end{aligned}$$

The general solution is $x_1 = x_3 - x_4$, and $x_2 = -2x_3 + 4x_4$, with x_3 and x_4 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any values for x_3 and x_4 (not both zero). For instance, set $x_3 = 1$ and $x_4 = 0$ to obtain the vector $(1, -2, 1, 0)$ in Nul A.

Note: Section 2.8 of *Study Guide* introduces the **`ref`** command (or **`rref`**, depending on the technology), which produces the reduced echelon form of a matrix. This will greatly speed up homework for students who have a matrix program available.

14. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1/3 & 0 \\ 0 & \textcircled{1} & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = (1/3)x_3$ and $x_2 = (-5/3)\,x_3$, with x_3 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any nonzero value of x_3. For instance, set $x_3 = 3$ to obtain the vector $(1, -5, 3)$ in Nul A.

15. Yes. Let A be the matrix whose columns are the vectors given. Then A is invertible because its determinant is nonzero, and so its columns form a basis for $\mathbb{R}^2$, by the Invertible Matrix Theorem (or by Example 5). (Other reasons for the invertibility of A could be given.)

16. No. One vector is a multiple of the other, so they are linearly dependent and hence cannot be a basis for any subspace.

17. Yes. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A = \begin{bmatrix} 0 & 5 & 6 \\ 1 & -7 & 3 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ -2 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & -10 & 11 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -7 & 3 \\ 0 & \textcircled{5} & 6 \\ 0 & 0 & \textcircled{23} \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbb{R}^3$ (as pointed out in Example 5).

18. Yes. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A = \begin{bmatrix} 1 & -5 & 7 \\ 1 & -1 & 0 \\ -2 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 7 \\ 0 & 4 & -7 \\ 0 & -8 & 9 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -5 & 7 \\ 0 & \textcircled{4} & -7 \\ 0 & 0 & \textcircled{-5} \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbb{R}^3$ (as pointed out in Example 5).

19. No. The vectors cannot be a basis for $\mathbb{R}^3$ because they only span a plane in $\mathbb{R}^3$. Or, point out that the columns of the matrix $\begin{bmatrix} 3 & 6 \\ -8 & 2 \\ 1 & -5 \end{bmatrix}$ cannot possibly span $\mathbb{R}^3$ because the matrix cannot have a pivot in every row. So the columns are not a basis for $\mathbb{R}^3$.

Note: The *Study Guide* warns students NOT to say that the two vectors here are a basis for $\mathbb{R}^2$.

20. No. The vectors are linearly dependent because there are more vectors in the set than entries in each vector. (Theorem 8 in Section 1.7.) So the vectors cannot be a basis for any subspace.

21. **a**. False. See the definition at the beginning of the section. The critical phrases "for each" are missing.

b. True. See the paragraph before Example 4.

c. True. See Theorem 12.

d. True. See Example 5.

e. True. See the first part of the solution of Example 8.

22. **a**. False. See the definition at the beginning of the section. The condition about the zero vector is only one of the conditions for a subspace.

b. True. See Example 3.

c. True. See Theorem 12.

d. False. See the paragraph after Example 4.

e. False. See the Warning that follows Theorem 13.

23. (Solution in *Study Guide*) $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & 6 & -5 \\ 0 & \textcircled{1} & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The echelon form identifies columns 1 and 2 as the pivot columns. A basis for Col A uses columns 1 and 2 of A: $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$. This is not the only choice, but it is the "standard" choice. A *wrong* choice is to select columns 1 and 2 of the echelon form. These columns have zero in the third entry and could not possibly generate the columns displayed in A.

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$\begin{bmatrix} \textcircled{1} & 0 & -4 & 7 & 0 \\ 0 & \textcircled{1} & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This corresponds to: $\begin{aligned} \textcircled{x_1} \quad - 4x_3 + 7x_4 &= 0 \\ \textcircled{x_2} + 5x_3 - 6x_4 &= 0 \\ 0 &= 0 \end{aligned}$.

Solve for the basic variables and write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 - 7x_4 \\ -5x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

. Basis for Nul A: $\begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$.

Notes: (1) A basis is a *set* of vectors. For simplicity, the answers here and in the text list the vectors without enclosing the list inside set brackets. This style is also easier for students. I am careful, however, to distinguish between a matrix and the set or list whose elements are the columns of the matrix.

(2) Recall from Chapter 1 that students are encouraged to use the augmented matrix when solving $A\mathbf{x} = \mathbf{0}$, to avoid the common error of misinterpreting the reduced echelon form of A as itself the augmented matrix for a nonhomogeneous system.

(3) Because the concept of a basis is just being introduced, I insist that my students write the parametric vector form of the solution of $A\mathbf{x} = \mathbf{0}$. They see how the basis vectors span the solution space and are obviously linearly independent. A shortcut, which some instructors might introduce later in the course, is only to solve for the basic variables and to produce each basis vector one at a time. Namely, set all free variables equal to zero except for one free variable, and set that variable equal to a suitable nonzero number.

24. $A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 6 & 9 \\ 0 & 0 & \textcircled{4} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Basis for Col A: $\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$.

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$\begin{bmatrix} \textcircled{1} & -3 & 0 & 1.50 & 0 \\ 0 & 0 & \textcircled{1} & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This corresponds to: $\begin{aligned} \textcircled{x_1} - 3x_2 \quad + 1.50x_4 &= 0 \\ \textcircled{x_3} + 1.25x_4 &= 0 \\ 0 &= 0 \end{aligned}$.

Solve for the basic variables and write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 1.5x_4 \\ x_2 \\ -1.25x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}.$$

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 4 & 8 & 0 & 5 \\ 0 & \textcircled{2} & 5 & 0 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Basis for Col A: $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$.

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} \textcircled{1} & 0 & -2 & 0 & 7 & 0 \\ 0 & \textcircled{1} & 2.5 & 0 & -.5 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{aligned} \textcircled{x_1} \quad - \quad 2x_3 \quad + \quad 7x_5 &= 0 \\ \textcircled{x_2} + \; 2.5x_3 \quad - \quad .5x_5 &= 0 \\ \textcircled{x_4} + \quad 4x_5 &= 0 \\ 0 &= 0 \end{aligned}.$$

$$\text{The solution of } A\mathbf{x} = 0 \text{ in parametric vector form: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 7x_5 \\ -2.5x_3 + .5x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ .5 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

$\mathbf{u}$ $\mathbf{v}$

Basis for Nul A: $\{\mathbf{u}, \mathbf{v}\}$.

Note: The solution above illustrates how students could write a solution on an exam, when time is precious, namely, describe the basis by giving names to appropriate vectors found in the calculations.

26. $A = \begin{bmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{3} & -1 & 7 & 0 & 6 \\ 0 & \textcircled{2} & 4 & 0 & 3 \\ 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Basis for Col A: $\begin{bmatrix} 3 \\ -2 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 3 \end{bmatrix}$.

For Nul A,

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 & 2.5 & 0 \\ 0 & \textcircled{1} & 2 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{aligned} \textcircled{x_1} \quad + \; 3x_3 \quad + \; 2.5x_5 &= 0 \\ \textcircled{x_2} + \; 2x_3 \quad + \; 1.5x_5 &= 0 \\ \textcircled{x_4} + \quad x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - 2.5x_5 \\ -2x_3 - 1.5x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2.5 \\ -1.5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

. Basis for Nul A: $\{\mathbf{u}, \mathbf{v}\}$.

$\mathbf{u}$ $\mathbf{v}$

27. Construct a nonzero 3×3 matrix A and construct $\mathbf{b}$ to be almost any convenient linear combination of the columns of A.

28. The easiest construction is to write a 3×3 matrix in echelon form that has only 2 pivots, and let $\mathbf{b}$ be any vector in $\mathbb{R}^3$ whose third entry is nonzero.

29. (Solution in *Study Guide*) A simple construction is to write any nonzero 3×3 matrix whose columns are obviously linearly dependent, and then make $\mathbf{b}$ a vector of weights from a linear dependence relation among the columns. For instance, if the first two columns of A are equal, then $\mathbf{b}$ could be $(1, -1, 0)$.

30. Since Col A is the set of all linear combinations of $\mathbf{a}_1, \ldots, \mathbf{a}_p$, the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ spans Col A. Because $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is also linearly independent, it is a basis for Col A. (There is no need to discuss pivot columns and Theorem 13, though a proof could be given using this information.)

31. If Col $F \neq \mathbb{R}^5$, then the columns of F do not span $\mathbb{R}^5$. Since F is square, the IMT shows that F is not invertible and the equation $F\mathbf{x} = \mathbf{0}$ has a nontrivial solution. That is, Nul F contains a nonzero vector. Another way to describe this is to write Nul $F \neq \{\mathbf{0}\}$.

32. If Nul R contains nonzero vectors, then the equation $R\mathbf{x} = \mathbf{0}$ has nontrivial solutions. Since R is square, the IMT shows that R is not invertible and the columns of R do not span $\mathbb{R}^6$. So Col R is a subspace of $\mathbb{R}^6$, but Col $R \neq \mathbb{R}^6$.

33. If Col $Q = \mathbb{R}^4$, then the columns of Q span $\mathbb{R}^4$. Since Q is square, the IMT shows that Q is invertible and the equation $Q\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^4$. Also, each solution is unique, by Theorem 5 in Section 2.2.

34. If Nul $P = \{\mathbf{0}\}$, then the equation $P\mathbf{x} = \mathbf{0}$ has only the trivial solution. Since P is square, the IMT shows that P is invertible and the equation $P\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^5$. Also, each solution is unique, by Theorem 5 in Section 2.2.

35. If the columns of B are linearly independent, then the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution. That is, Nul $B = \{\mathbf{0}\}$.

36. If the columns of A form a basis, they are linearly independent. This means that A cannot have more columns than rows. Since the columns also span $\mathbb{R}^m$, A must have a pivot in each row, which means that A cannot have more rows than columns. As a result, A must be a square matrix.

37. **[M]** Use the command that produces the reduced echelon form in one step (**`ref`** or **`rref`** depending on the program). See the Section 2.8 in the *Study Guide* for details. By Theorem 13, the pivot columns of A form a basis for Col A.

$$A=\begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 2.5 & -4.5 & 3.5 \\ 0 & \textcircled{1} & 1.5 & -2.5 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Basis for Col A: $\begin{bmatrix} 3 \\ -7 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ 7 \\ -7 \end{bmatrix}$

For Nul A, obtain the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{aligned} \textcircled{x_1} \quad &+ 2.5x_3 - 4.5x_4 + 3.5x_5 = 0 \\ \textcircled{x_2} \; &+ 1.5x_3 - 2.5x_4 + 1.5x_5 = 0 \end{aligned}$$

Solution: $\begin{cases} x_1 = -2.5x_3 + 4.5x_4 - 3.5x_5 \\ x_2 = -1.5x_3 + 2.5x_4 - 1.5x_5 \\ x_3, x_4, \text{and } x_5 \text{ are free} \end{cases}$

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} -2.5x_3+4.5x_4-3.5x_5 \\ -1.5x_3+2.5x_4-1.5x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=x_3\begin{bmatrix} -2.5 \\ -1.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}+x_4\begin{bmatrix} 4.5 \\ 2.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}+x_5\begin{bmatrix} -3.5 \\ -1.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}=x_3\mathbf{u}+x_4\mathbf{v}+x_5\mathbf{w}$$

By the argument in Example 6, a basis for Nul A is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

38. **[M]** $A=\begin{bmatrix} 5 & 2 & 0 & -8 & -8 \\ 4 & 1 & 2 & -8 & -9 \\ 5 & 1 & 3 & 5 & 19 \\ -8 & -5 & 6 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 60 & 122 \\ 0 & \textcircled{1} & 0 & -154 & -309 \\ 0 & 0 & \textcircled{1} & -47 & -94 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The pivot columns of A form a basis for Col A: $\begin{bmatrix} 5 \\ 4 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 6 \end{bmatrix}$.

For Nul A, solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} \textcircled{x_1} \quad &+ 60x_4 + 122x_5 = 0 \\ \textcircled{x_2} \; &- 154x_4 - 309x_5 = 0 \\ \textcircled{x_3} \; &- 47x_4 - 94x_5 = 0 \end{aligned}$$

Solution: $\begin{cases} x_1 = -60x_4 - 122x_5 \\ x_2 = 154x_4 + 309x_5 \\ x_3 = 47x_4 + 94x_5 \\ x_4 \text{ and } x_5 \text{ are free} \end{cases}$

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} -60x_4-122x_5 \\ 154x_4+309x_5 \\ 47x_4+94x_5 \\ x_4 \\ x_5 \end{bmatrix}=x_4\begin{bmatrix} -60 \\ 154 \\ 47 \\ 1 \\ 0 \end{bmatrix}+x_5\begin{bmatrix} -122 \\ 309 \\ 94 \\ 0 \\ 1 \end{bmatrix}=x_4\mathbf{u}+x_5\mathbf{v}$$. By the method of Example 6, a basis for Nul A is $\{\mathbf{u}, \mathbf{v}\}$.

Note: The *Study Guide* for Section 2.8 gives directions for students to construct a review sheet for the concept of a subspace and the two main types of subspaces, Col A and Nul A, and a review sheet for the concept of a basis. I encourage you to consider making this an assignment for your class.

2.9 SOLUTIONS

Notes: This section contains the ideas from Sections 4.4–4.6 that are needed for later work in Chapters 5–7. If you have time, you can enrich the geometric content of "coordinate systems" by discussing crystal lattices (Example 3 and Exercises 35 and 36 in Section 4.4.) Some students might profit from reading Examples 1–3 from Section 4.4 and Examples 2, 4, and 5 from Section 4.6. Section 4.5 is probably *not* a good reference for students who have not considered general vector spaces.

Coordinate vectors are important mainly to give an intuitive and geometric feeling for the isomorphism between a k-dimensional subspace and $\mathbb{R}^k$. If you plan to omit Sections 5.4, 5.6, 5.7 and 7.2, you can safely omit Exercises 1–8 here. Exercises 1–16 may be assigned after students have read as far as Example 2. Exercises 19 and 20 use the Rank Theorem, but they can also be assigned before the Rank Theorem is discussed.

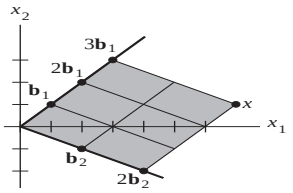
The Rank Theorem in this section omits the nontrivial fact about Row A which is included in the Rank Theorem of Section 4.6, but that is used only in Section 7.4. The row space itself can be introduced in Section 6.2, for use in Chapter 6 and Section 7.4.

Exercises 9–16 include important review of techniques taught in Section 2.8 (and in Sections 1.2 and 2.5). They make good test questions because they require little arithmetic. My students need the practice here.

Nearly every time I teach the course and start Chapter 5, I find that at least one or two students cannot find a basis for a two-dimensional eigenspace!

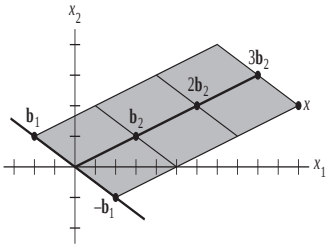
1. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $\mathbf{x}$ is formed from $\mathbf{b}_1$ and $\mathbf{b}_2$ using weights 3 and 2:

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$



2. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $\mathbf{x}$ is formed from $\mathbf{b}_1$ and $\mathbf{b}_2$ using weights –1 and 3:

$$\mathbf{x} = (-1)\mathbf{b}_1 + 3\mathbf{b}_2 = (-1)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$$



3. To find c_1 and c_2 that satisfy $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, row reduce the augmented matrix:

$$[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}] = \begin{bmatrix} 1 & -2 & -3 \\ -4 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \end{bmatrix}$$

. Or, one can write a matrix equation as

suggested by Exercise 7 and solve using the matrix inverse. In either case,
$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$.

4. As in Exercise 3, $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}] = \begin{bmatrix} 1 & -3 & -7 \\ -3 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -7 \\ 0 & -4 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}$, and

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

5. $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}] = \begin{bmatrix} 1 & -3 & 4 \\ 5 & -7 & 10 \\ -3 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & 8 & -10 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -5/4 \end{bmatrix}$.

6. $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}] = \begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 22 & 11 \\ 0 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}$.

7. Fig. 1 suggests that $\mathbf{w} = 2\mathbf{b}_1 - \mathbf{b}_2$ and $\mathbf{x} = 1.5\mathbf{b}_1 + .5\mathbf{b}_2$, in which case, $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1.5 \\ .5 \end{bmatrix}$. To confirm $[\mathbf{x}]_{\mathcal{B}}$, compute $1.5\mathbf{b}_1 + .5\mathbf{b}_2 = 1.5\begin{bmatrix} 3 \\ 0 \end{bmatrix} + .5\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{x}$

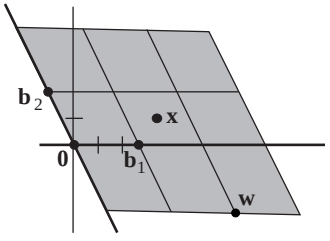


Figure 1

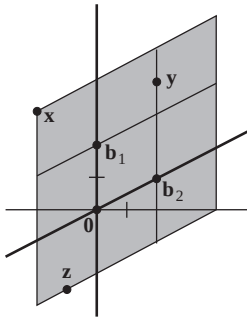


Figure 2

Note: Figures 1 and 2 display what Section 4.4 calls $\mathcal{B}$-graph paper.

8. Fig. 2 suggests that $\mathbf{x} = 2\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{y} = 1.5\mathbf{b}_1 + \mathbf{b}_2$, and $\mathbf{z} = -\mathbf{b}_1 - .5\mathbf{b}_2$. If so, then $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$, and $[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -.5 \end{bmatrix}$. To confirm $[\mathbf{y}]_{\mathcal{B}}$ and $[\mathbf{z}]_{\mathcal{B}}$, compute

$1.5\mathbf{b}_1 + \mathbf{b}_2 = 1.5\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \mathbf{y}$ and $-\mathbf{b}_1 - .5\mathbf{b}_2 = -1\begin{bmatrix} 0 \\ 2 \end{bmatrix} - .5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix} = \mathbf{z}$.

9. The information $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & -4 \\ 0 & 0 & \textcircled{5} & -7 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is enough to see that columns 1, 3, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$.

For Nul A, use the reduced echelon form, augmented with a zero column to insure that the equation $A\mathbf{x} = \mathbf{0}$ is kept in mind:

$$\begin{bmatrix} \textcircled{1} & -3 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} \textcircled{x_1} - 3x_2 &= 0 \\ \textcircled{x_3} &= 0 \\ \textcircled{x_4} &= 0 \end{aligned}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \text{ So } \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ is}$$

x_2 is the free variable

a basis for Nul A. From this information, dim Col A = 3 (because A has three pivot columns) and dim Nul A = 1 (because the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable).

10. The information $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 9 & 5 & 4 \\ 0 & \textcircled{1} & -3 & 0 & -7 \\ 0 & 0 & 0 & \textcircled{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$. For Nul A,

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 & 0 & 0 \\ 0 & \textcircled{1} & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} \textcircled{x_1} + 3x_3 &= 0 \\ \textcircled{x_2} - 3x_3 - 7x_5 &= 0 \\ \textcircled{x_4} - 2x_5 &= 0 \end{aligned}$$

x_3 and x_5 are free variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

11. The information $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & -5 & 0 & -1 \\ 0 & \textcircled{1} & 2 & 4 & 5 \\ 0 & 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix}$. For Nul A,

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} \textcircled{1} & 0 & -9 & 0 & 5 & 0 \\ 0 & \textcircled{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \textcircled{x_1} - 9x_3 + 5x_5 = 0 \\ \textcircled{x_2} + 2x_3 - 3x_5 = 0 \\ \textcircled{x_4} + 2x_5 = 0 \\ x_3 \text{ and } x_5 \text{ are free variables} \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9x_3 - 5x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

12. The information $A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & -4 & 3 & 3 \\ 0 & 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{-5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 3, and 5 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix}$. For Nul A

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} \textcircled{1} & 2 & 0 & -5 & 0 & 0 \\ 0 & 0 & \textcircled{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \textcircled{x_1} + 2x_2 - 5x_4 = 0 \\ \textcircled{x_3} - 2x_4 = 0 \\ \textcircled{x_5} = 0 \\ x_2 \text{ and } x_4 \text{ are free variables} \end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

13. The four vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 10 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & -4 \\ 0 & 0 & \textcircled{5} & -7 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 3, and 4 of the original matrix form a basis for H, so dim $H = 3$.

Note: Either Exercise 13 or 14 should be assigned because there are always one or two students who confuse Col A with Nul A. Or, they wrongly connect "set of linear combinations" with "parametric vector form" (of the general solution of $A\mathbf{x} = \mathbf{0}$).

14. The five vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -20 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & 0 & -1 & 3 \\ 0 & \textcircled{-1} & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 of the original matrix form a basis for H, so dim $H = 2$.

15. Col $A = \mathbb{R}^3$, because A has a pivot in each row and so the columns of A span $\mathbb{R}^3$. Nul A *cannot* equal $\mathbb{R}^2$, because Nul A is a subspace of $\mathbb{R}^{5\,5}$. It is true, however, that Nul A is two-dimensional. Reason: the equation $A\mathbf{x} = \mathbf{0}$ has two free variables, because A has five columns and only three of them are pivot columns.

16. Col A *cannot* be $\mathbb{R}^3$ because the columns of A have four entries. (In fact, Col A is a 3-dimensional subspace of $\mathbb{R}^{4\,4}$, because the 3 pivot columns of A form a basis for Col A.) Since A has 7 columns and 3 pivot columns, the equation $A\mathbf{x} = \mathbf{0}$ has 4 free variables. So, dim Nul $A = 4$.

17. **a**. True. This is the definition of a B-coordinate vector.

b. False. Dimension is defined only for a subspace. A line must be through the origin in $\mathbb{R}^n$ to be a subspace of $\mathbb{R}^n$.

c. True. The sentence before Example 1 concludes that the number of pivot columns of A is the rank of A, which is the dimension of Col A by definition.

d. True. This is equivalent to the Rank Theorem because rank A *is* the dimension of Col A.

e. True, by the Basis Theorem. In this case, the spanning set is automatically a linearly independent set.

18. **a**. True. This fact is justified in the second paragraph of this section.

b. True. See the second paragraph after Fig. 1.

c. False. The dimension of Nul A is the number of *free* variables in the equation $A\mathbf{x} = \mathbf{0}$. See Example 2.

d. True, by the definition of *rank*.

e. True, by the Basis Theorem. In this case, the linearly independent set is automatically a spanning set.

19. The fact that the solution space of $A\mathbf{x} = \mathbf{0}$ has a basis of three vectors means that dim Nul $A = 3$. Since a 5×7 matrix A has 7 columns, the Rank Theorem shows that rank $A = 7 - \dim \text{Nul } A = 4$.

Note: One can solve Exercises 19–22 without explicit reference to the Rank Theorem. For instance, in Exercise 19, if the null space of a matrix A is three-dimensional, then the equation $A\mathbf{x} = \mathbf{0}$ has three free variables, and three of the columns of A are nonpivot columns. Since a 5×7 matrix has seven columns, A must have four pivot columns (which form a basis of Col A). So rank A = dim Col A = 4.

20. A 4×5 matrix A has 5 columns. By the Rank Theorem, rank A = 5 – dim Nul A. Since the null space is three-dimensional, rank A = 2.

21. A 7×6 matrix has 6 columns. By the Rank Theorem, dim Nul A = 6 – rank A. Since the rank is four, dim Nul A = 2. That is, the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is two.

22. Suppose that the subspace $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is four-dimensional. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ were linearly independent, it would be a basis for H. This is impossible, by the statement just before the definition of *dimension* in Section 2.9, which essentially says that *every* basis of a p-dimensional subspace consists of p vectors. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ must be linearly dependent.

23. A 3×4 matrix A with a two-dimensional column space has two pivot columns. The remaining two columns will correspond to free variables in the equation $A\mathbf{x} = \mathbf{0}$. So the desired construction is possible. There are six possible locations for the two pivot columns, one of which is $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A simple construction is to take two vectors in $\mathbb{R}^3$ that are obviously not linearly dependent, and put two copies of these two vectors in any order. The resulting matrix will obviously have a two-dimensional column space. There is no need to worry about whether Nul A has the correct dimension, since this is guaranteed by the Rank Theorem: dim Nul A = 4 – rank A.

24. A rank 1 matrix has a one-dimensional column space. Every column is a multiple of some fixed vector. To construct a 4×3 matrix, choose any nonzero vector in $\mathbb{R}^4$, and use it for one column. Choose any multiples of the vector for the other two columns.

25. The p columns of A span Col A by definition. If dim Col $A = p$, then the spanning set of p columns is automatically a basis for Col A, by the Basis Theorem. In particular, the columns are linearly independent.

26. If columns $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, and $\mathbf{a}_6$ of A are linearly independent and if dim Col A = 4, then $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6\}$ is a linearly independent set in a 4-dimensional column space. By the Basis Theorem, this set of four vectors is a basis for the column space.

27. **a**. Start with $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_q]$, where $q > p$. For $j = 1, \ldots, q$, the vector $\mathbf{a}_j$ is in W. Since the columns of B span W, the vector $\mathbf{a}_j$ is in the column space of B. That is, $\mathbf{a}_j = B\mathbf{c}_j$ for some vector $\mathbf{c}_j$ of weights. Note that $\mathbf{c}_j$ is in $\mathbb{R}^p$ because B has p columns.

b. Let $C = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_q]$. Then C is a $p \times q$ matrix because each of the q columns is in $\mathbb{R}^p$. By hypothesis, q is larger than p, so C has more columns than rows. By a theorem, the columns of C are linearly dependent and there exists a nonzero vector $\mathbf{u}$ in $\mathbb{R}^q$ such that $C\mathbf{u} = \mathbf{0}$.

c. From part (a) and the definition of matrix multiplication $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_q] = [B\mathbf{c}_1 \ \cdots \ B\mathbf{c}_q] = BC$. From part (b), $A\mathbf{u} = (BC)\mathbf{u} = B(C\mathbf{u}) = B\mathbf{0} = \mathbf{0}$. Since $\mathbf{u}$ is nonzero, the columns of A are linearly dependent.

28. If $\mathcal{A}$ contained more vectors than $\mathcal{B}$, then $\mathcal{A}$ would be linearly dependent, by Exercise 27, because $\mathcal{B}$ spans W. Repeat the argument with $\mathcal{B}$ and $\mathcal{A}$ interchanged to conclude that $\mathcal{B}$ cannot contain more vectors than $\mathcal{A}$.

29. **[M]** Apply the matrix command **ref** or **rref** to the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}]$:

$$\begin{bmatrix} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1.667 \\ 0 & \textcircled{1} & 2.667 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}$ is consistent, so $\mathbf{x}$ is in the subspace H. The decimal approximations suggest $c_1 = -5/3$ and $c_2 = 8/3$, and it can be checked that these values are precise. Thus, the $\mathcal{B}$-coordinate of $\mathbf{x}$ is $(-5/3, 8/3)$.

30. **[M]** Apply the matrix command **ref** or **rref** to the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}]$:

$$\begin{bmatrix} -6 & 8 & -9 & 4 \\ 4 & -3 & 5 & 7 \\ -9 & 7 & -8 & -8 \\ 4 & -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 3 \\ 0 & \textcircled{1} & 0 & 5 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}]$ are pivot columns, so $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Thus $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis $\mathcal{B}$ for the subspace H which they span. View $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}]$ as an augmented matrix for $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{x}$. The reduced echelon form shows that $\mathbf{x}$ is in H and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

Notes: The *Study Guide* for Section 2.9 contains a complete list of the statements in the Invertible Matrix Theorem that have been given so far. The format is the same as that used in Section 2.3, with three columns: statements that are logically equivalent for any $m \times n$ matrix and are related to existence concepts, those that are equivalent only for any $n \times n$ matrix, and those that are equivalent for any $n \times p$ matrix and are related to uniqueness concepts. Four statements are included that are not in the text's official list of statements, to give more symmetry to the three columns.

The *Study Guide* section also contains directions for making a review sheet for "dimension" and "rank."

Chapter 2 SUPPLEMENTARY EXERCISES

1. **a**. True. If A and B are $m \times n$ matrices, then B^T has as many rows as A has columns, so AB^T is defined. Also, A^TB is defined because A^T has m columns and B has m rows.

b. False. B must have 2 columns. A has as many columns as B has rows.

c. True. The ith row of A has the form $(0, \ldots, d_i, \ldots, 0)$. So the ith row of AB is $(0, \ldots, d_i, \ldots, 0)B$, which is d_i times the ith row of B.

d. False. Take the zero matrix for B. Or, construct a matrix B such that the equation $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions, and construct C and D so that $C \neq D$ and the columns of $C - D$ satisfy the equation $B\mathbf{x} = \mathbf{0}$. Then $B(C - D) = 0$ and $BC = BD$.

e. False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

f. False. $(A + B)(A - B) = A^2 - AB + BA - B^2$. This equals $A^2 - B^2$ if and only if A commutes with B.

g. True. An $n \times n$ replacement matrix has $n + 1$ nonzero entries. The $n \times n$ scale and interchange matrices have n nonzero entries.

h. True. The transpose of an elementary matrix is an elementary matrix of the same type.

i. True. An $n \times n$ elementary matrix is obtained by a row operation on I_n.

j. False. Elementary matrices are invertible, so a product of such matrices is invertible. But not every square matrix is invertible.

k. True. If A is 3×3 with three pivot positions, then A is row equivalent to I_3.

l. False. A must be square in order to conclude from the equation $AB = I$ that A is invertible.

m. False. AB is invertible, but $(AB)^{-1} = B^{-1}A^{-1}$, and this product is not always equal to $A^{-1}B^{-1}$.

n. True. Given $AB = BA$, left-multiply by A^{-1} to get $B = A^{-1}BA$, and then right-multiply by A^{-1} to obtain $BA^{-1} = A^{-1}B$.

o. False. The correct equation is $(rA)^{-1} = r^{-1}A^{-1}$, because $(rA)(r^{-1}A^{-1}) = (rr^{-1})(AA^{-1}) = 1 \cdot I = I$.

p. True. If the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then there are no free variables in this equation, which means that A must have three pivot positions (since A is 3×3). By the Invertible Matrix Theorem, A is invertible.

2. $C = (C^{-1})^{-1} = \dfrac{1}{-2}\begin{bmatrix} 7 & -5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -7/2 & 5/2 \\ 3 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Next, $(I - A)(I + A + A^2) = I + A + A^2 - A(I + A + A^2) = I + A + A^2 - A - A^2 - A^3 = I - A^3$.

Since $A^3 = 0$, $(I - A)(I + A + A^2) = I$.

4. From Exercise 3, the inverse of $I - A$ is probably $I + A + A^2 + \cdots + A^{n-1}$. To verify this, compute

$$(I - A)(I + A + \cdots + A^{n-1}) = I + A + \cdots + A^{n-1} - A(I + A + \cdots + A^{n-1}) = I - AA^{n-1} = I - A^n$$

If $A^n = 0$, then the matrix $B = I + A + A^2 + \cdots + A^{n-1}$ satisfies $(I - A)B = I$. Since $I - A$ and B are square, they are invertible by the Invertible Matrix Theorem, and B is the inverse of $I - A$.

5. $A^2 = 2A - I$. Multiply by A: $A^3 = 2A^2 - A$. Substitute $A^2 = 2A - I$: $A^3 = 2(2A - I) - A = 3A - 2I$.
Multiply by A again: $A^4 = A(3A - 2I) = 3A^2 - 2A$. Substitute the identity $A^2 = 2A - I$ again.
Finally, $A^4 = 3(2A - I) - 2A = 4A - 3I$.

6. Let $A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. By direct computation, $A^2 = I$, $B^2 = I$, and $AB=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = -BA$.

7. (Partial answer in *Study Guide*) Since $A^{-1}B$ is the solution of $AX = B$, row reduction of $[A \quad B]$ to $[I \quad X]$ will produce $X = A^{-1}B$. See Exercise 12 in Section 2.2.

$$[A \quad B]=\left[\begin{array}{ccc|cc}1 & 3 & 8 & -3 & 5\\ 2 & 4 & 11 & 1 & 5\\ 1 & 2 & 5 & 3 & 4\end{array}\right]\sim\begin{bmatrix}1 & 3 & 8 & -3 & 5\\ 0 & -2 & -5 & 7 & -5\\ 0 & -1 & -3 & 6 & -1\end{bmatrix}\sim\begin{bmatrix}1 & 3 & 8 & -3 & 5\\ 0 & 1 & 3 & -6 & 1\\ 0 & -2 & -5 & 7 & -5\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 3 & 8 & -3 & 5\\ 0 & 1 & 3 & -6 & 1\\ 0 & 0 & 1 & -5 & -3\end{bmatrix}\sim\begin{bmatrix}1 & 3 & 0 & 37 & 29\\ 0 & 1 & 0 & 9 & 10\\ 0 & 0 & 1 & -5 & -3\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0 & 10 & -1\\ 0 & 1 & 0 & 9 & 10\\ 0 & 0 & 1 & -5 & -3\end{bmatrix}.$$ Thus, $A^{-1}B=\begin{bmatrix}10 & -1\\ 9 & 10\\ -5 & -3\end{bmatrix}$.

8. By definition of matrix multiplication, the matrix A satisfies $A\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}=\begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}$.

Right-multiply both sides by the inverse of $\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$. The left side becomes A. Thus,

$$A=\begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 4 & -1\end{bmatrix}.$$

9. Given $AB=\begin{bmatrix}5 & 4\\ -2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}7 & 3\\ 2 & 1\end{bmatrix}$, notice that $ABB^{-1} = A$. Since $\det B = 7 - 6 = 1$,

$$B^{-1}=\begin{bmatrix}1 & -3\\ -2 & 7\end{bmatrix} \text{ and } A=(AB)B^{-1}=\begin{bmatrix}5 & 4\\ -2 & 3\end{bmatrix}\begin{bmatrix}1 & -3\\ -2 & 7\end{bmatrix}=\begin{bmatrix}-3 & 13\\ -8 & 27\end{bmatrix}$$

Note: Variants of this question make simple exam questions.

10. Since A is invertible, so is A^T, by the Invertible Matrix Theorem. Then A^TA is the product of invertible matrices and so is invertible. Thus, the formula $(A^TA)^{-1}A^T$ makes sense. By Theorem 6 in Section 2.2,

$(A^TA)^{-1}\cdot A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}I = A^{-1}$

An alternative calculation: $(A^TA)^{-1}A^T\cdot A = (A^TA)^{-1}(A^TA) = I$. Since A is invertible, this equation shows that its inverse is $(A^TA)^{-1}A^T$.

11. **a**. For $i = 1,\ldots, n$, $p(x_i) = c_0 + c_1x_i + \cdots + c_{n-1}x_i^{n-1} = \text{row}_i(V)\cdot\begin{bmatrix}c_0\\ \vdots\\ c_{n-1}\end{bmatrix} = \text{row}_i(V)\mathbf{c}$.

By a property of matrix multiplication, shown after Example 6 in Section 2.1, and the fact that $\mathbf{c}$ was chosen to satisfy $V\mathbf{c} = \mathbf{y}$, $\text{row}_i(V)\mathbf{c} = \text{row}_i(V\mathbf{c}) = \text{row}_i(\mathbf{y}) = y_i$

Thus, $p(x_i) = y_i$. To summarize, the entries in $V\mathbf{c}$ are the values of the polynomial $p(x)$ at $x_1, \ldots, x_n$.

b. Suppose $x_1, \ldots, x_n$ are distinct, and suppose $V\mathbf{c} = \mathbf{0}$ for some vector $\mathbf{c}$. Then the entries in $\mathbf{c}$ are the coefficients of a polynomial whose value is zero at the distinct points $x_1, \ldots, x_n$. However, a nonzero polynomial of degree $n - 1$ cannot have n zeros, so the polynomial must be identically zero. That is, the entries in $\mathbf{c}$ must all be zero. This shows that the columns of V are linearly independent.

c. (Solution in *Study Guide*) When $x_1, \ldots, x_n$ are distinct, the columns of V are linearly independent, by (b). By the Invertible Matrix Theorem, V is invertible and its columns span $\mathbb{R}^n$. So, for every $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, there is a vector $\mathbf{c}$ such that $V\mathbf{c} = \mathbf{y}$. Let p be the polynomial whose coefficients are listed in $\mathbf{c}$. Then, by (a), p is an interpolating polynomial for $(x_1, y_1), \ldots, (x_n, y_n)$.

12. If $A = LU$, then $\text{col}_1(A) = L \cdot \text{col}_1(U)$. Since $\text{col}_1(U)$ has a zero in every entry except possibly the first, $L \cdot \text{col}_1(U)$ is a linear combination of the columns of L in which all weights except possibly the first are zero. So $\text{col}_1(A)$ is a multiple of $\text{col}_1(L)$.

Similarly, $\text{col}_2(A) = L \cdot \text{col}_2(U)$, which is a linear combination of the columns of L using the first two entries in $\text{col}_2(U)$ as weights, because the other entries in $\text{col}_2(U)$ are zero. Thus $\text{col}_2(A)$ is a linear combination of the first two columns of L.

13. **a**. $P^2 = (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{u}(1)\mathbf{u}^T = P$, because $\mathbf{u}$ satisfies $\mathbf{u}^T\mathbf{u} = 1$.

b. $P^T = (\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^{TT}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T = P$

c. $Q^2 = (I - 2P)(I - 2P) = I - I(2P) - 2PI + 2P(2P)$
$= I - 4P + 4P^2 = I$, because of part (a).

14. Given $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, define P and Q as in Exercise 13 by

$$P = \mathbf{u}\mathbf{u}^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = I - 2P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

If $\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$, then $P\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ and $Q\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

15. Left-multiplication by an elementary matrix produces an elementary row operation: $B \sim E_1B \sim E_2E_1B \sim E_3E_2E_1B = C$, so B is row equivalent to C. Since row operations are reversible, C is row equivalent to B. (Alternatively, show C being changed into B by row operations using the inverse of the E_i.)

16. Since A is not invertible, there is a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{v} = \mathbf{0}$. Place n copies of $\mathbf{v}$ into an $n \times n$ matrix B. Then $AB = A[\mathbf{v} \ \cdots \ \mathbf{v}] = [A\mathbf{v} \ \cdots \ A\mathbf{v}] = 0$.

17. Let A be a 6×4 matrix and B a 4×6 matrix. Since B has more columns than rows, its six columns are linearly dependent and there is a nonzero $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. Thus $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. This shows that the matrix AB is not invertible, by the IMT. (Basically the same argument was used to solve Exercise 22 in Section 2.1.)

Note: (In the *Study Guide*) It is possible that BA is invertible. For example, let C be an invertible 4×4 matrix and construct $A = \begin{bmatrix} C \\ 0 \end{bmatrix}$ and $B = [C^{-1} \ \ 0]$. Then $BA = I_4$, which is invertible.

18. By hypothesis, A is 5×3, C is 3×5, and $AC = I_3$. Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$. Then $CA\mathbf{x} = C\mathbf{b}$. Since $CA = I$, $\mathbf{x}$ must be $C\mathbf{b}$. This shows that $C\mathbf{b}$ is the only solution of $A\mathbf{x} = \mathbf{b}$.

19. **[M]** Let $A = \begin{bmatrix} .4 & .2 & .3 \\ .3 & .6 & .3 \\ .3 & .2 & .4 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} .31 & .26 & .30 \\ .39 & .48 & .39 \\ .30 & .26 & .31 \end{bmatrix}$. Instead of computing A^3 next, speed up the calculations by computing

$$A^4 = A^2A^2 = \begin{bmatrix} .2875 & .2834 & .2874 \\ .4251 & .4332 & .4251 \\ .2874 & .2834 & .2875 \end{bmatrix}, \quad A^8 = A^4A^4 = \begin{bmatrix} .2857 & .2857 & .2857 \\ .4285 & .4286 & .4285 \\ .2857 & .2857 & .2857 \end{bmatrix}$$

To four decimal places, as k increases, $A^k \to \begin{bmatrix} .2857 & .2857 & .2857 \\ .4286 & .4286 & .4286 \\ .2857 & .2857 & .2857 \end{bmatrix}$, or, in rational format,

$$A^k \to \begin{bmatrix} 2/7 & 2/7 & 2/7 \\ 3/7 & 3/7 & 3/7 \\ 2/7 & 2/7 & 2/7 \end{bmatrix}.$$

If $B = \begin{bmatrix} 0 & .2 & .3 \\ .1 & .6 & .3 \\ .9 & .2 & .4 \end{bmatrix}$, then $B^2 = \begin{bmatrix} .29 & .18 & .18 \\ .33 & .44 & .33 \\ .38 & .38 & .49 \end{bmatrix}$, $B^4 = \begin{bmatrix} .2119 & .1998 & .1998 \\ .3663 & .3764 & .3663 \\ .4218 & .4218 & .4339 \end{bmatrix}$,

$B^8 = \begin{bmatrix} .2024 & .2022 & .2022 \\ .3707 & .3709 & .3707 \\ .4269 & .4269 & .4271 \end{bmatrix}$. To four decimal places, as k increases, $B^k \to \begin{bmatrix} .2022 & .2022 & .2022 \\ .3708 & .3708 & .3708 \\ .4270 & .4270 & .4270 \end{bmatrix}$,

or, in rational format, $B^k \to \begin{bmatrix} 18/89 & 18/89 & 18/89 \\ 33/89 & 33/89 & 33/89 \\ 38/89 & 38/89 & 38/89 \end{bmatrix}$.

20. **[M]** The 4×4 matrix A_4 is the 4×4 matrix of ones, minus the 4×4 identity matrix. The MATLAB command is `A4 = ones(4) - eye(4)`. For the inverse, use `inv(A4)`.

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_4^{-1} = \begin{bmatrix} -2/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & -2/3 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_5^{-1} = \begin{bmatrix} -3/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & -3/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -3/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & -3/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & -3/4 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_6^{-1} = \begin{bmatrix} -4/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & -4/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & -4/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & -4/5 \end{bmatrix}$$

The construction of A_6 and the appearance of its inverse suggest that the inverse is related to I_6. In fact, $A_6^{-1} + I_6$ is 1/5 times the 6×6 matrix of ones. Let J denotes the $n \times n$ matrix of ones. The conjecture is:

$$A_n = J - I_n \quad \text{and} \quad A_n^{-1} = \frac{1}{n-1} \cdot J - I_n$$

Proof: (Not required) Observe that $J^2 = nJ$ and $A_n J = (J - I)J = J^2 - J = (n-1)J$. Now compute $A_n((n-1)^{-1}J - I) = (n-1)^{-1} A_n J - A_n = J - (J - I) = I$. Since A_n is square, A_n is invertible and its inverse is $(n-1)^{-1}J - I$.