Linear Algebra Done Right
Solutions Manual

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This solutions manual has not been subjected to the same amount of scrutiny as the book, so errors are more likely. I would be grateful for information about any errors that you notice. If you know nicer solutions to any of the exercises than the solutions given here, please let me know so that I can improve future versions of this solutions manual.

Please check my web site for errata and other information about *Linear Algebra Done Right*. I welcome comments about either the book or the solutions manual.

Have fun!

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CHAPTER 1

Vector Spaces

1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c+di.$$

SOLUTION: Multiplying both the numerator and the denominator of the left side of the equation above by $a-bi$ gives

$$\frac{a-bi}{a^2+b^2} = c+di.$$

Thus we must have

$$c = \frac{a}{a^2+b^2} \quad \text{and} \quad d = \frac{-b}{a^2+b^2};$$

because a and b are not both 0, we are not dividing by 0.

COMMENT: Note that these formulas for c and d are derived under the assumption that $a+bi$ has a multiplicative inverse. However, we can forget about the derivation and verify (using the definition of complex multiplication) that

$$(a+bi)\Big(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\Big) = 1,$$

which shows that every nonzero complex number does indeed have a multiplicative inverse.

2. Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

SOLUTION: Using the definition of complex multiplication, we have

$$\Big(\frac{-1+\sqrt{3}i}{2}\Big)^2 = \frac{-1-\sqrt{3}i}{2}.$$

Thus

$$\begin{aligned}\Big(\frac{-1+\sqrt{3}i}{2}\Big)^3 &= \Big(\frac{-1-\sqrt{3}i}{2}\Big)\Big(\frac{-1+\sqrt{3}i}{2}\Big)\\ &= 1.\end{aligned}$$

3. Prove that $-(-v) = v$ for every $v \in V$.

SOLUTION: Let $v \in V$. By the definition of additive inverse, we have

$$v + (-v) = 0.$$

The additive inverse of $-v$, which by definition is $-(-v)$, is the unique vector that when added to $-v$ gives 0. The equation above shows that v has this property. Thus $-(-v) = v$.

COMMENT: Using 1.6 twice leads to another proof that $-(-v) = v$. However, the proof given above uses only the additive structure of V, whereas a proof using 1.6 also uses the multiplicative structure.

4. Prove that if $a \in \mathbf{F}$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.

SOLUTION: Suppose that $a \in \mathbf{F}$, $v \in V$, and

$$av = 0.$$

We want to prove that $a = 0$ or $v = 0$. If $a = 0$, then we are done. So suppose that $a \neq 0$. Multiplying both sides of the equation above by $1/a$ gives

$$\frac{1}{a}(av) = \frac{1}{a}0.$$

The associative property shows that the left side of the equation above equals $1v$, which equals v. The right side of the equation above equals 0 (by 1.5). Thus $v = 0$, completing the proof.

5. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$.

SOLUTION: (a) Let

$$U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}.$$

To show that U is a subspace of $\mathbf{F}^3$, first note that $(0, 0, 0) \in U$, so U is nonempty.

Next, suppose that $(x_1, x_2, x_3) \in U$ and $(y_1, y_2, y_3) \in U$. Then

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ y_1 + 2y_2 + 3y_3 &= 0. \end{aligned}$$

Adding these equations, we have

$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = 0,$$

which means that $(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U$. Thus U is closed under addition.

Next, suppose that $(x_1, x_2, x_3) \in U$ and $a \in \mathbf{F}$. Then

$$x_1 + 2x_2 + 3x_3 = 0.$$

Multiplying this equation by a, we have

$$(ax_1) + 2(ax_2) + 3(ax_3) = 0,$$

which means that $(ax_1, ax_2, ax_3) \in U$. Thus U is closed under scalar multiplication.

Because U is a nonempty subset of $\mathbf{F}^3$ that is closed under addition and scalar multiplication, U is a subspace of $\mathbf{F}^3$.

(b) Let

$$U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}.$$

Then $(4, 0, 0) \in U$ but $0(4, 0, 0)$, which equals $(0, 0, 0)$, is not in U. Thus U is not closed under scalar multiplication. Thus U is not a subspace of $\mathbf{F}^3$.

(c) Let

$$U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}.$$

Then $(1, 1, 0) \in U$ and $(0, 0, 1) \in U$, but the sum of these two vectors, which equals $(1, 1, 1)$, is not in U. Thus U is not closed under addition. Thus U is not a subspace of $\mathbf{F}^3$.

(d) Let

$$U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}.$$

To show that U is a subspace of $\mathbf{F}^3$, first note that $(0, 0, 0) \in U$, so U is nonempty.

Next, suppose that $(x_1, x_2, x_3) \in U$ and $(y_1, y_2, y_3) \in U$. Then

$$\begin{aligned} x_1 &= 5x_3 \\ y_1 &= 5y_3. \end{aligned}$$

Adding these equations, we have

$$x_1 + y_1 = 5(x_3 + y_3),$$

which means that $(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U$. Thus U is closed under addition.

Next, suppose that $(x_1, x_2, x_3) \in U$ and $a \in \mathbf{F}$. Then

$$x_1 = 5x_3.$$

Multiplying this equation by a, we have

$$ax_1 = 5(ax_3),$$

which means that $(ax_1, ax_2, ax_3) \in U$. Thus U is closed under scalar multiplication.

Because U is a nonempty subset of $\mathbf{F}^3$ that is closed under addition and scalar multiplication, U is a subspace of $\mathbf{F}^3$.

6. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

SOLUTION: Let $U = \{(m, n) : m \text{ and } n \text{ are integers}\}$. Then clearly U is closed under addition and under taking additive inverses. However, $(1, 1) \in U$ but $\frac{1}{2}(1, 1)$, which equals $(\frac{1}{2}, \frac{1}{2})$, is not in U, so U is not closed under scalar multiplication. Thus U is not a subspace of $\mathbf{R}^2$.

Of course there are also many other examples.

7. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

SOLUTION: Let U be the union of the two coordinate axes in $\mathbf{R}^2$. More precisely, let

$$U = \{(x, 0) : x \in \mathbf{R}\} \cup \{(0, y) : y \in \mathbf{R}\}.$$

Then clearly U is closed under scalar multiplication. However, $(1, 0)$ and $(0, 1)$ are in U but their sum, which equals $(1, 1)$ is not in U, so U is not closed under addition. Thus U is not a subspace of $\mathbf{R}^2$.

Of course there are also many other examples.

8. Prove that the intersection of any collection of subspaces of V is a subspace of V.

SOLUTION: Suppose $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection of subspaces of V; here Γ is an arbitrary index set. We need to prove that $\bigcap_{\alpha \in \Gamma} U_\alpha$, which equals the set of vectors that are in U_α for every $\alpha \in \Gamma$, is a subspace of V.

The additive identity 0 is in U_α for every $\alpha \in \Gamma$ (because each U_α is a subspace of V). Thus $0 \in \bigcap_{\alpha \in \Gamma} U_\alpha$. In particular, $\bigcap_{\alpha \in \Gamma} U_\alpha$ is a nonempty subset of V.

Suppose $u, v \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Then $u, v \in U_\alpha$ for every $\alpha \in \Gamma$. Thus $u + v \in U_\alpha$ for every $\alpha \in \Gamma$ (because each U_α is a subspace of V). Thus $u + v \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Thus $\bigcap_{\alpha \in \Gamma} U_\alpha$ is closed under addition.

Suppose $u \in \bigcap_{\alpha \in \Gamma} U_\alpha$ and $a \in \mathbf{F}$. Then $u \in U_\alpha$ for every $\alpha \in \Gamma$. Thus $au \in U_\alpha$ for every $\alpha \in \Gamma$ (because each U_α is a subspace of V). Thus $au \in \bigcap_{\alpha \in \Gamma} U_\alpha$. Thus $\bigcap_{\alpha \in \Gamma} U_\alpha$ is closed under scalar multiplication.

Because $\bigcap_{\alpha \in \Gamma} U_\alpha$ is a nonempty subset of V that is closed under addition and scalar multiplication, $\bigcap_{\alpha \in \Gamma} U_\alpha$ is a subspace of V.

COMMENT: For many students, the hardest part of this exercise is understanding the meaning of an arbitrary intersection of sets. Instructors who do not want to deal with this issue should change the exercise to "Prove that the intersection of any finite collection of subspaces of V is a subspace of V." Many students will then prove that the intersection of two subspaces of V is a subspace of V and use induction to get the result for finite collections of subspaces.

9. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

SOLUTION: Suppose U and W are subspaces of V such that $U \cup W$ is a subspace of V. We will use proof by contradiction to show that $U \subset W$ or $W \subset U$. Suppose that our desired result is false. Then $U \not\subset W$ and $W \not\subset U$. This means that there exists $u \in U$ such that $u \notin W$ and there exists $w \in W$ such that $w \notin U$. Because u and w are both in $U \cup W$, which is a subspace of V, we can conclude that $u + w \in U \cup W$. Thus $u + w \in U$ or $u + w \in W$.

First consider the possibility that $u+w \in U$. In this case w, which equals $(u + w) + (-u)$, would be in the sum of two elements of U and hence we would have $w \in U$, contradicting our assumption that $w \notin U$.

Now consider the possibility that $u+w \in W$. In this case u, which equals $(u + w) + (-w)$, would be in the sum of two elements of W and hence we would have $u \in W$, contradicting our assumption that $u \notin W$.

The two paragraphs above show that $u + w \notin U$ and $u + w \notin W$, contradicting the final sentence of the first paragraph of this solution. This contradiction completes our proof that $U \subset W$ or $W \subset U$.

The other direction of this exercise is trivial: if we have two subspaces of V, one of which is contained in the other, then the union of these two subspaces equals the larger of them, which is a subspace of V.

10. Suppose that U is a subspace of V. What is $U + U$?

SOLUTION: By definition, $U+U = \{u+v : u, v \in U\}$. Clearly $U \subset U+U$ because if $u \in U$, then u equals $u + 0$, which expresses u as a sum of two elements of U. Conversely, $U+U \subset U$ because the sum of two elements of U is an element of U (because U is a subspace of V). Conclusion: $U+U = U$.

11. Is the operation of addition on the subspaces of V commutative? Associative? (In other words, if U_1, U_2, U_3 are subspaces of V, is $U_1+U_2 = U_2+U_1$? Is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?)

SOLUTION: Suppose U_1, U_2, U_3 are subspaces of V.

A typical element of U_1+U_2 is a vector of the form u_1+u_2, where $u_1 \in U_1$ and $u_2 \in U_2$. Because addition of vectors is commutative, $u_1 + u_2$ equals

$u_2 + u_1$, which is a typical element of $U_2 + U_1$. Thus $U_1 + U_2 = U_2 + U_1$. In other words, the operation of addition on the subspaces of V is commutative.

A typical element of $(U_1 + U_2) + U_3$ is a vector of the form $(u_1 + u_2) + u_3$, where $u_1 \in U_1$, $u_2 \in U_2$, and $u_3 \in U_3$. Because addition of vectors is associative, $(u_1 + u_2) + u_3$ equals $u_1 + (u_2 + u_3)$, which is a typical element of $U_1 + (U_2 + U_3)$. Thus $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$. In other words, the operation of addition on the subspaces of V is associative.

12. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

SOLUTION: The subspace $\{0\}$ is an additive identity for the operation of addition on the subspaces of V. More precisely, if U is a subspace of V, then $U + \{0\} = \{0\} + U = U$.

For a subspace U of V to have an additive inverse, there would have to be another subspace W of V such that $U + W = \{0\}$. Because both U and W are contained in $U + W$, this is possible only if $U = W = \{0\}$. Thus $\{0\}$ is the only subspace of V that has an additive inverse.

13. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

SOLUTION: To construct a counterexample for the assertion above, choose V to be any nonzero vector space. Let $U_1 = \{0\}$, $U_2 = V$, and $W = V$. Then $U_1 + W$ and $U_2 + W$ are both equal to V, but $U_1 \neq U_2$.

Of course there are also many other examples.

14. Suppose U is the subspace of $\mathcal{P}(\mathbf{F})$ consisting of all polynomials p of the form

$$p(z) = az^2 + bz^5,$$

where $a, b \in \mathbf{F}$. Find a subspace W of $\mathcal{P}(\mathbf{F})$ such that $\mathcal{P}(\mathbf{F}) = U \oplus W$.

SOLUTION: Let W be the set of all polynomials (with coefficients in $\mathbf{F}$) whose z^2-coefficient and z^5-coefficient both equal 0. Then every polynomial in $\mathcal{P}(\mathbf{F})$ can be written uniquely in the form $p + q$, where $p \in U$ and $q \in W$. Thus $\mathcal{P}(\mathbf{F}) = U \oplus W$.

COMMENT: There are other possible choices for W that give a correct solution to this exercise, but the choice for W made above is certainly the most natural one.

15. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

SOLUTION: To construct a counterexample for the assertion above, let $V = \mathbf{F}^2$, let $U_1 = \{(x, 0) : x \in \mathbf{F}\}$, let $U_2 = \{(0, y) : y \in \mathbf{F}\}$, and let $W = \{(z, z) : z \in \mathbf{F}\}$. Then

$$\mathbf{F}^2 = U_1 \oplus W \quad \text{and} \quad \mathbf{F}^2 = U_2 \oplus W,$$

as is easy to verify, but $U_1 \neq U_2$.

Of course there are also many other examples.

CHAPTER 2

Finite-Dimensional Vector Spaces

1. Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

SOLUTION: Suppose $(v_1, \ldots, v_n)$ spans V. Let $v \in V$. To show that $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$, we need to find $a_1, \ldots, a_n \in \mathbf{F}$ such that

$$v = a_1(v_1 - v_2) + a_2(v_2 - v_3) + \ldots a_{n-1}(v_{n-1} - v_n) + a_n v_n.$$

Rearranging terms of the equation above, we see that we need to find $a_1, \ldots, a_n \in \mathbf{F}$ such that

(a) $$v = a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + \cdots + (a_n - a_{n-1})v_n.$$

Because $(v_1, \ldots, v_n)$ spans V, there exist $b_1, \ldots, b_n \in \mathbf{F}$ such that

(b) $$v = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots + b_n v_n.$$

Comparing equations (a) and (b), we see that (a) will be satisfied if we choose a_1 to equal b_1 and then choose a_2 to equal $b_2 + a_1$ and then choose a_3 to equal $b_3 + a_2$, and so on.

2. Prove that if $(v_1, \ldots, v_n)$ is linearly independent in V, then so is the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

SOLUTION: Suppose $(v_1, \ldots, v_n)$ is linearly independent in V. To prove that the list displayed above is linearly independent, suppose $a_1, \ldots, a_n \in \mathbf{F}$ are such that

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_n v_n = 0.$$

Rearranging terms, the equation above can be rewritten as

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + \cdots + (a_n - a_{n-1})v_n = 0.$$

Because $(v_1, \ldots, v_n)$ is linearly independent, the equation above implies that

$$\begin{aligned} a_1 &= 0 \\ a_2 - a_1 &= 0 \\ a_3 - a_2 &= 0 \\ &\vdots \\ a_n - a_{n-1} &= 0. \end{aligned}$$

The first equation above tells us that $a_1 = 0$. That information, combined with the second equation, tells us that $a_2 = 0$. That information, combined with the third equation, tells us that $a_3 = 0$. Continue in this fashion, getting $a_1 = \cdots = a_n = 0$. Thus $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$ is linearly independent.

3. Suppose $(v_1, \ldots, v_n)$ is linearly independent in V and $w \in V$. Prove that if $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

SOLUTION: Suppose $(v_1 + w, \ldots, v_n + w)$ is linearly dependent. Then there exist scalars $a_1, \ldots, a_n$, not all 0, such that

$$a_1(v_1 + w) + \cdots + a_n(v_n + w) = 0.$$

Rearranging this equation, we have

$$a_1 v_1 + \cdots + a_n v_n = -(a_1 + \cdots + a_n)w.$$

If $a_1 + \cdots + a_n$ were 0, then the equation above would contradict the linear independence of $(v_1, \ldots, v_n)$. Thus $a_1 + \cdots + a_n \neq 0$. Hence we can divide both sides of the equation above by $-(a_1 + \cdots + a_n)$, showing that $w \in \operatorname{span}(v_1, \ldots, v_n)$.

4. Suppose m is a positive integer. Is the set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and with degree equal to m a subspace of $\mathcal{P}(\mathbf{F})$?

SOLUTION: The set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and with degree equal to m is not a subspace of $\mathcal{P}(\mathbf{F})$ because it is not closed under addition. Specifically, the sum of two polynomials of degree m may be a polynomial with degree less than m. For example, suppose $m = 2$. Then $7 + 4z + 5z^2$ and $1 + 2z - 5z^2$ are both polynomials of degree 2 but their sum, which equals $8 + 6z$, is a polynomial of degree 1.

5. Prove that $\mathbf{F}^\infty$ is infinite dimensional.

SOLUTION: For each positive integer m, let e_m be the element of $\mathbf{F}^\infty$ whose m^{th} coordinate equals 1 and whose other coordinates equal 0:

$$e_m = (0, \ldots, 0, \underset{\substack{\uparrow \\ m^{\text{th}} \text{ coordinate}}}{1}, 0, \ldots).$$

Then $(e_1, \ldots, e_m)$ is a linearly independent list of vectors in $\mathbf{F}^\infty$, as is easy to verify. This implies, by the marginal comment attached to 2.6, that $\mathbf{F}^\infty$ is infinite dimensional.

6. Prove that the real vector space consisting of all continuous real-valued functions on the interval $[0, 1]$ is infinite dimensional.

SOLUTION: Let V denote the real vector space of all continuous real-valued functions on the interval $[0, 1]$. For each positive integer m, the list $(1, x, \ldots, x^m)$ is linearly independent in V (because if $a_0, \ldots, a_m \in \mathbf{R}$ are such that

$$a_0 + a_1 x + \cdots + a_m x^m = 0$$

for every $x \in [0, 1]$, then the polynomial above has infinitely many roots and hence all its coefficients must equal 0). This implies, by the marginal comment attached to 2.6, that V is infinite dimensional.

7. Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer n.

SOLUTION: First suppose that V is infinite dimensional. Choose v_1 to be any nonzero vector in V. Choose $v_2, v_3, \ldots$ by the following inductive process: suppose that $v_1, \ldots, v_{n-1}$ have been chosen; choose any vector $v_n \in V$ such that $v_n \notin \operatorname{span}(v_1, \ldots, v_{n-1})$—because V is not finite dimensional, $\operatorname{span}(v_1, \ldots, v_{n-1})$ cannot equal V so choosing v_n in this fashion is possible. The linear dependence lemma (2.4) implies that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer n, as desired.

Conversely, suppose there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer n. This implies, by the marginal comment attached to 2.6, that V is infinite dimensional.

8. Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

SOLUTION: Obviously

$$U = \{(3x_2, x_2, 7x_4, x_4, x_5) : x_2, x_4, x_5 \in \mathbf{R}\}.$$

From this representation of U, we see easily that

$$\big((3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)\big)$$

is a basis of U.

Of course there are also other possible choices of bases of U.

9. Prove or disprove: there exists a basis (p_0, p_1, p_2, p_3) of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

SOLUTION: Define $p_0, p_1, p_2, p_3 \in \mathcal{P}_3(\mathbf{F})$ by

$$\begin{aligned} p_0(z) &= 1, \\ p_1(z) &= z, \\ p_2(z) &= z^2 + z^3, \\ p_3(z) &= z^3. \end{aligned}$$

None of the polynomials p_0, p_1, p_2, p_3 has degree 2, but (p_0, p_1, p_2, p_3) is a basis of $\mathcal{P}_3(\mathbf{F})$, as is easy to verify.

Of course there are also other possible choices of bases of $\mathcal{P}_3(\mathbf{F})$ without using polynomials of degree 2.

10. Suppose that V is finite dimensional, with $\dim V = n$. Prove that there exist one-dimensional subspaces $U_1, \dots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

SOLUTION: Let $(v_1, \dots, v_n)$ be a basis of V. For each j, let U_j equal $\operatorname{span}(v_j)$; in other words, $U_j = \{av_j : a \in \mathbf{F}\}$. Because $(v_1, \dots, v_n)$ is a basis of V, each vector in V can be written uniquely in the form

$$a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \dots, a_n \in \mathbf{F}$ (see 2.8). By definition of direct sum, this means that $V = U_1 \oplus \cdots \oplus U_n$.

11. Suppose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

SOLUTION: Let $(u_1, \dots, u_n)$ be a basis of U. Thus $n = \dim U$, and by hypothesis we also have $n = \dim V$. Thus $(u_1, \dots, u_n)$ is a linearly independent (because it is a basis of U) list of vectors in V with length $\dim V$. From 2.17, we see that $(u_1, \dots, u_n)$ is a basis of V. In particular every vector in V is a linear combination of $(u_1, \dots, u_n)$. Because each $u_j \in U$, this implies that $U = V$.

12. Suppose that $p_0, p_1, \dots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $(p_0, p_1, \dots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

SOLUTION: Because $p_j(2) = 0$ for each j, the constant polynomial 1 is not in $\operatorname{span}(p_0, \dots, p_m)$. Thus $(p_0, \dots, p_m)$ is not a basis of $\mathcal{P}_m(\mathbf{F})$. Because $(p_0, \dots, p_m)$ is a list of length $m+1$ and $\mathcal{P}_m(\mathbf{F})$ has dimension $m+1$, this implies (by 2.17) that $(p_0, \dots, p_m)$ is not linearly independent.

13. Suppose U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $U \cap W = \{0\}$.

SOLUTION: We know (from 2.18) that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Because $\dim(U+W) = 8$, $\dim U = 3$, and $\dim W = 5$, this implies that $\dim(U \cap W) = 0$. Thus $U \cap W = \{0\}$.

14. Suppose that U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

SOLUTION: Using 2.18 we have

$$\begin{aligned}9 &\geq \dim(U+W)\\ &= \dim U + \dim W - \dim(U \cap W)\\ &= 10 - \dim(U \cap W).\end{aligned}$$

Thus $\dim(U \cap W) \geq 1$. In particular, $U \cap W \neq \{0\}$.

15. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}&\dim(U_1 + U_2 + U_3)\\ &\qquad = \dim U_1 + \dim U_2 + \dim U_3\\ &\qquad\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3)\\ &\qquad\quad + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

SOLUTION: To give a counterexample, let $V = \mathbf{R}^2$, and let

$$\begin{aligned}U_1 &= \{(x,0) : x \in \mathbf{R}\},\\ U_2 &= \{(0,y) : y \in \mathbf{R}\},\\ U_3 &= \{(x,x) : x \in \mathbf{R}\}.\end{aligned}$$

Then $U_1 + U_2 + U_3 = \mathbf{R}^2$, so $\dim(U_1 + U_2 + U_3) = 2$. However,

$$\dim U_1 = \dim U_2 = \dim U_3 = 1$$

and

$$\dim(U_1 \cap U_2) = \dim(U_1 \cap U_3) = \dim(U_2 \cap U_3) = \dim(U_1 \cap U_2 \cap U_3) = 0.$$

Thus in this case our guess would reduce to the formula $2 = 3$, which obviously is false.

Of course there are also many other examples.

16. Prove that if V is finite dimensional and $U_1, \dots, U_m$ are subspaces of V, then

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

SOLUTION: For each $j = 1, \dots m$, choose a basis for U_j. Put these bases together to form a single list of vectors in V. Clearly this list spans $U_1 + \cdots + U_m$. Hence the dimension of $U_1 + \cdots + U_m$ is less than or equal to the number of vectors in this list (by 2.10), which equals $\dim U_1 + \cdots + \dim U_m$. In other words,

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

17. Suppose V is finite dimensional. Prove that if $U_1, \dots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m.$$

COMMENT: This exercise deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a finite set is written as a disjoint union of subsets, then the number of elements in the set equals the sum of the number of elements in the disjoint subsets.

SOLUTION: Suppose that $U_1, \dots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$. For each $j = 1, \dots m$, choose a basis for U_j. Put these bases together to form a single list B of vectors in V. Clearly B spans $U_1 + \cdots + U_m$, which equals V. If we show that B is also linearly independent, then it will be a basis of V. Thus the dimension of V will equal the number of vectors B. In other words, we will have

$$\dim V = \dim U_1 + \cdots + \dim U_m,$$

as desired.

We still need to show that B is linearly independent. To do this, suppose that some linear combination of B equals 0. Write this linear combination as $u_1 + \cdots + u_m$, where we have grouped together the terms that come from the basis vectors of U_1 and called their sum u_1, and similarly up to u_m. Thus we have

$$u_1 + \cdots + u_m = 0,$$

where each $u_j \in U_j$. Because $V = U_1 \oplus \cdots \oplus U_m$, this implies that each u_j equals 0. Because each u_j is a linear combination of our basis of U_j, all the coefficients in the linear combination defining u_j must equal 0. Thus all the coefficients in our original linear combination of B must equal 0. In other words, B is linearly independent, completing our proof.

CHAPTER 3

Linear Maps

1. Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

SOLUTION: Suppose $\dim V = 1$ and $T \in \mathcal{L}(V, V)$. Let u be any nonzero vector in V. Then every vector in V is a scalar multiple of u. In particular, $Tu = au$ for some $a \in \mathbf{F}$.

Now consider a typical vector $v \in V$. There exists $b \in \mathbf{F}$ such that $v = bu$. Thus

$$\begin{aligned} Tv &= T(bu) \\ &= bT(u) \\ &= b(au) \\ &= a(bu) \\ &= av. \end{aligned}$$

2. Give an example of a function $f \colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$f(av) = af(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but f is not linear.

SOLUTION: Define $f \colon \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x, y) = (x^3 + y^3)^{1/3}.$$

Then $f(av) = af(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$. However, f is not linear because $f(1,0) = 1$ and $f(0,1) = 1$ but

$$\begin{aligned} f\big((1,0)+(0,1)\big) &= f(1,1) \\ &= 2^{1/3} \\ &\neq f(1,0)+f(0,1). \end{aligned}$$

Of course there are also many other examples.

COMMENT: This exercise shows that homogeneity alone is not enough to imply that a function is a linear map. Additivity alone is also not enough to imply that a function is a linear map, although the proof of this involves advanced tools that are beyond the scope of this book.

3. Suppose that V is finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

SOLUTION: Suppose U is a subspace of V and $S \in \mathcal{L}(U, W)$. Let $(u_1, \dots, u_m)$ be a basis of U. Then $(u_1, \dots, u_m)$ is a linearly independent list of vectors in V, and so can be extended to a basis $(u_1, \dots, u_m, v_1, \dots, v_n)$ of V (by 2.12). Define $T \in \mathcal{L}(V, W)$ by

$$T(a_1u_1 + \dots a_mu_m + b_1v_1 + \dots b_nv_n) = a_1Su_1 + \cdots + a_mSu_m.$$

Then $Tu = Su$ for all $u \in U$.

COMMENT: Defining $T\colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U; \\ 0 & \text{if } v \notin U. \end{cases}$$

does not work because this map is not linear.

4. Suppose that T is a linear map from V to $\mathbf{F}$. Prove that if $u \in V$ is not in $\operatorname{null} T$, then

$$V = \operatorname{null} T \oplus \{au : a \in \mathbf{F}\}.$$

SOLUTION: Suppose $u \in V$ is not in $\operatorname{null} T$. If $a \in \mathbf{F}$ and $au \in \operatorname{null} T$, then $0 = T(au) = aTu$, which implies that $a = 0$ (because $Tu \neq 0$). Thus

$$\operatorname{null} T \cap \{au : a \in \mathbf{F}\} = \{0\}.$$

If $v \in V$, then