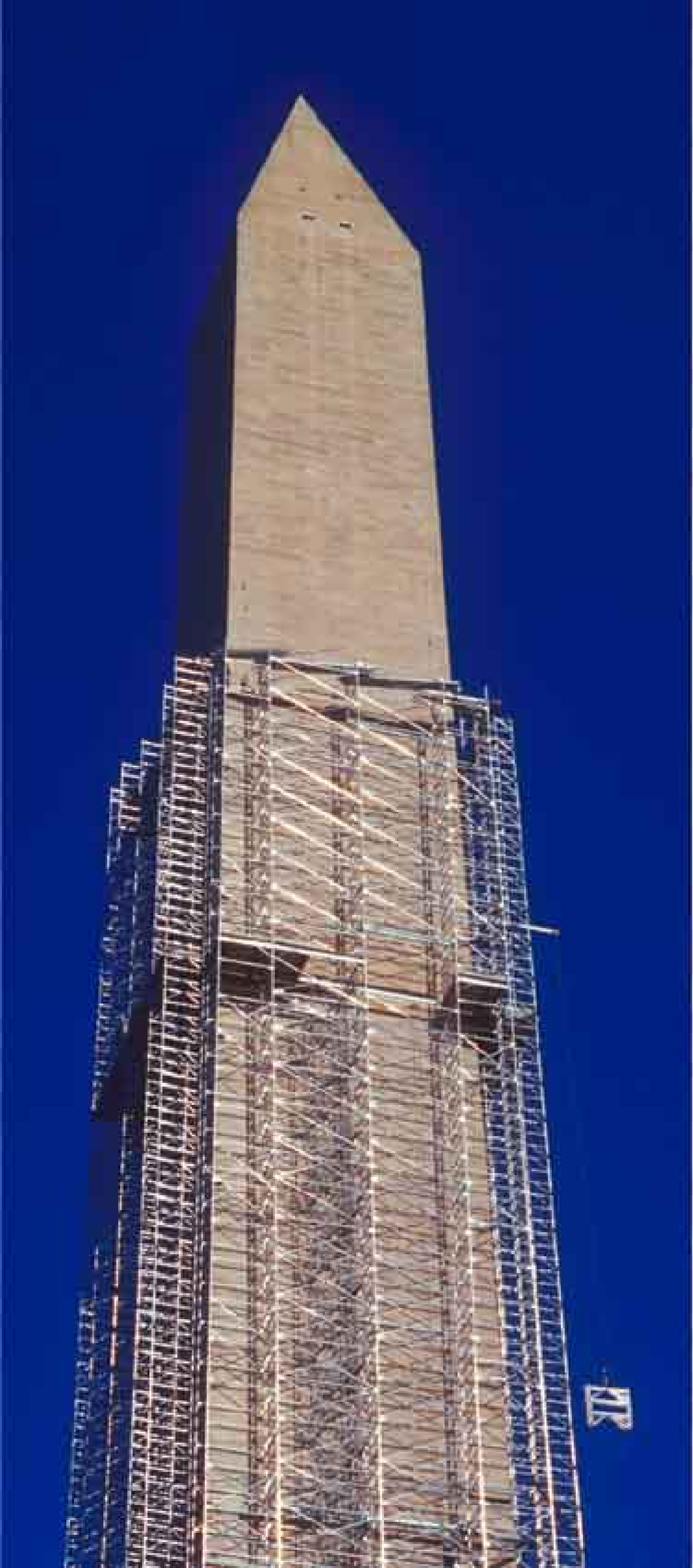

Matrix Analysis and Applied Linear Algebra

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siam

Solutions for Chapter 1

Solutions for exercises in section 1. 2

1.2.1. $(1, 0, 0)$

1.2.2. $(1, 2, 3)$

1.2.3. $(1, 0, -1)$

1.2.4. $(-1/2, 1/2, 0, 1)$

1.2.5. $\left(\begin{array}{c|c|c} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{array}\right)$

1.2.6. Every row operation is reversible. In particular the "inverse" of any row operation is again a row operation of the same type.

1.2.7. $\frac{\pi}{2}, \pi, 0$

1.2.8. The third equation in the triangularized form is $0x_3 = 1$, which is impossible to solve.

1.2.9. The third equation in the triangularized form is $0x_3 = 0$, and all numbers are solutions. This means that you can start the back substitution with any value whatsoever and consequently produce infinitely many solutions for the system.

1.2.10. $\alpha = -3$, $\beta = \frac{11}{2}$, and $\gamma = -\frac{3}{2}$

1.2.11. (a) If $x_i =$ the number initially in chamber $\#i$, then

$$\begin{aligned} .4x_1 + 0x_2 + 0x_3 + .2x_4 &= 12 \\ 0x_1 + .4x_2 + .3x_3 + .2x_4 &= 25 \\ 0x_1 + .3x_2 + .4x_3 + .2x_4 &= 26 \\ .6x_1 + .3x_2 + .3x_3 + .4x_4 &= 37 \end{aligned}$$

and the solution is $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, and $x_4 = 40$.

(b) 16, 22, 22, 40

1.2.12. To interchange rows i and j, perform the following sequence of Type II and Type III operations.

$$\begin{array}{ll} R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow R_i - R_j & \text{(replace row } i \text{ by the difference of row } i \text{ and } j) \\ R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow -R_i & \text{(replace row } i \text{ by its negative)} \end{array}$$

1.2.13. (a) This has the effect of interchanging the order of the unknowns— x_j and x_k are permuted. (b) The solution to the new system is the same as the solution to the old system except that the solution for the j^{th} unknown of the new system is $\hat{x}_j = \frac{1}{\alpha}x_j$. This has the effect of "changing the units" of the j^{th} unknown. (c) The solution to the new system is the same as the solution for the old system except that the solution for the k^{th} unknown in the new system is $\hat{x}_k = x_k - \alpha x_j$.

1.2.14. $h_{ij} = \frac{1}{i+j-1}$

1.2.16. If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ are two different solutions, then

$$\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2} = \begin{pmatrix} \frac{x_1+y_1}{2} \\ \frac{x_2+y_2}{2} \\ \vdots \\ \frac{x_m+y_m}{2} \end{pmatrix}$$

is a third solution different from both $\mathbf{x}$ and $\mathbf{y}$.

Solutions for exercises in section 1. 3

1.3.1. $(1, 0, -1)$

1.3.2. $(2, -1, 0, 0)$

1.3.3. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Solutions for exercises in section 1. 4

1.4.2. Use $y'(t_k) = y'_k \approx \dfrac{y_{k+1} - y_{k-1}}{2h}$ and $y''(t_k) = y''_k \approx \dfrac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$ to write

$$f(t_k) = f_k = y''_k - y'_k \approx \frac{2y_{k-1} - 4y_k + 2y_{k+1}}{2h^2} - \frac{hy_{k+1} - hy_{k-1}}{2h^2}, \quad k = 1, 2, \ldots, n,$$

with $y_0 = y_{n+1} = 0$. These discrete approximations form the tridiagonal system

$$\begin{pmatrix} -4 & 2-h & & & \\ 2+h & -4 & 2-h & & \\ & \ddots & \ddots & \ddots & \\ & & 2+h & -4 & 2-h \\ & & & 2+h & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = 2h^2 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}.$$

Solutions for exercises in section 1. 5

1.5.1. (a) $(0,-1)$ (c) $(1,-1)$ (e) $\left(\frac{1}{1.001}, \frac{-1}{1.001}\right)$

1.5.2. (a) $(0,1)$ (b) $(2,1)$ (c) $(2,1)$ (d) $\left(\frac{2}{1.0001}, \frac{1.0003}{1.0001}\right)$

1.5.3. Without PP: (1.01, 1.03) With PP: (1, 1) Exact: (1, 1)

1.5.4. (a) $$\left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ .500 & .333 & .250 & .333 \\ .333 & .250 & .200 & .200 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .083 & .083 & .166 \\ 0 & .083 & .089 & .089 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .083 & .083 & .166 \\ 0 & 0 & .006 & -.077 \end{array}\right) \qquad z = -.077/.006 = -12.8,$$

$$y = (.166 - .083z)/.083 = 14.8, \quad x = .333 - (.5y + .333z) = -2.81$$

(b) $$\left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ .500 & .333 & .250 & .333 \\ .333 & .250 & .200 & .200 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 1 & .666 & .500 & .666 \\ 1 & .751 & .601 & .601 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .166 & .167 & .333 \\ 0 & .251 & .268 & .268 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .251 & .268 & .268 \\ 0 & .166 & .167 & .333 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .251 & .268 & .268 \\ 0 & 0 & -.01 & .156 \end{array}\right) \qquad z = -.156/.01 = -15.6,$$

$$y = (.268 - .268z)/.251 = 17.7, \quad x = .333 - (.5y + .333z) = -3.33$$

(c) $$\left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ .500 & .333 & .250 & .333 \\ .333 & .250 & .200 & .200 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 1 & .666 & .500 & .666 \\ 1 & .751 & .601 & .601 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .166 & .167 & .333 \\ 0 & .251 & .268 & .268 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .994 & 1 & 1.99 \\ 0 & .937 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & .500 & .333 & .333 \\ 0 & .994 & 1 & 1.99 \\ 0 & 0 & .057 & -.880 \end{array}\right) \qquad z = -.88/.057 = -15.4,$$

$$y = (1.99 - z)/.994 = 17.5, \quad x = .333 - (.5y + .333z) = -3.29$$

(d) $x = -3, \quad y = 16, \quad z = -14$

1.5.5. (a)

$$\begin{aligned} .0055x + .095y + 960z &= 5000 \\ .0011x + .\ 01y + 112z &= 600 \\ .0093x + .025y + 560z &= 3000 \end{aligned}$$

(b) 3-digit solution $= (55,900$ lbs. silica, $8,600$ lbs. iron, 4.04 lbs. gold). Exact solution (to 10 digits) $= (56,753.68899,\ 8,626.560726,\ 4.029511918)$. The relative error (rounded to 3 digits) is $e_r = 1.49 \times 10^{-2}$.

(c) Let $u = x/2000$, $v = y/1000$, and $w = 12z$ to obtain the system

$$\begin{aligned} 11u + 95v + \quad 80w &= 5000 \\ 2.2u + 10v + 9.33w &= 600 \\ 18.6u + 25v + 46.7w &= 3000. \end{aligned}$$

(d) 3-digit solution $= (28.5$ tons silica, 8.85 half-tons iron, 48.1 troy oz. gold). Exact solution (to 10 digits) $= (28.82648317,\ 8.859282804,\ 48.01596023)$. The relative error (rounded to 3 digits) is $e_r = 5.95 \times 10^{-3}$. So, partial pivoting applied to the column-scaled system yields higher relative accuracy than partial pivoting applied to the unscaled system.

1.5.6. (a) $(-8.1, -6.09) =$ 3-digit solution with partial pivoting but no scaling.
(b) No! Scaled partial pivoting produces the exact solution—the same as with complete pivoting.

1.5.7. (a) 2^{n-1} (b) 2

(c) This is a famous example that shows that there are indeed cases where partial pivoting will fail due to the large growth of some elements during elimination, but complete pivoting will be successful because all elements remain relatively small and of the same order of magnitude.

1.5.8. Use the fact that with partial pivoting no multiplier can exceed 1 together with the triangle inequality $|\alpha + \beta| \leq |\alpha| + |\beta|$ and proceed inductively.

Solutions for exercises in section 1. 6

1.6.1. (a) There are no 5-digit solutions. (b) This doesn't help—there are now infinitely many 5-digit solutions. (c) 6-digit solution $= (1.23964, -1.3)$ and exact solution $= (1, -1)$ (d) $r_1 = r_2 = 0$ (e) $r_1 = -10^{-6}$ and $r_2 = 10^{-7}$ (f) Even if computed residuals are 0, you can't be sure you have the exact solution.

1.6.2. (a) $(1, -1.0015)$ (b) Ill-conditioning guarantees that the solution will be very sensitive to *some* small perturbation but not necessarily to *every* small perturbation. It is usually difficult to determine beforehand those perturbations for which an ill-conditioned system will not be sensitive, so one is forced to be pessimistic whenever ill-conditioning is suspected.

1.6.3. (a) $m_1(5) = m_2(5) = -1.2519$, $m_1(6) = -1.25187$, and $m_2(6) = -1.25188$
(c) An optimally well-conditioned system represents orthogonal (i.e., perpendicular) lines, planes, etc.

1.6.4. They rank as (b) = Almost optimally well-conditioned. (a) = Moderately well-conditioned. (c) = Badly ill-conditioned.

1.6.5. Original solution $= (1, 1, 1)$. Perturbed solution $= (-238, 490, -266)$. System is ill-conditioned.

Solutions for Chapter 2

Solutions for exercises in section 2. 1

2.1.1. (a) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is one possible answer. Rank = 3 and the basic columns are $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*4}\}$. (b) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is one possible answer. Rank = 3 and every column in $\mathbf{A}$ is basic.

(c) $\begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 0 & 0 & 2 & -2 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is one possible answer. The rank is 3, and the basic columns are $\{\mathbf{A}_{*1}, \mathbf{A}_{*3}, \mathbf{A}_{*5}\}$.

2.1.2. (c) and (d) are in row echelon form.

2.1.3. (a) Since any row or column can contain at most one pivot, the number of pivots cannot exceed the number of rows nor the number of columns. (b) A zero row cannot contain a pivot. (c) If one row is a multiple of another, then one of them can be annihilated by the other to produce a zero row. Now the result of the previous part applies. (d) One row can be annihilated by the associated combination of row operations. (e) If a column is zero, then there are fewer than n basic columns because each basic column must contain a pivot.

2.1.4. (a) $rank\,(\mathbf{A}) = 3$ (b) 3-digit $rank\,(\mathbf{A}) = 2$ (c) With PP, 3-digit $rank\,(\mathbf{A}) = 3$

2.1.5. 15

2.1.6. (a) No, consider the form $\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{array}\right)$ (b) Yes—in fact, $\mathbf{E}$ is a row echelon form obtainable from $\mathbf{A}$.

Solutions for exercises in section 2. 2

2.2.1. (a) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{A}_{*3} = 2\mathbf{A}_{*1} + \frac{1}{2}\mathbf{A}_{*2}$

(b) $\begin{pmatrix} 1 & \frac{1}{2} & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and

$\mathbf{A}_{*2} = \frac{1}{2}\mathbf{A}_{*1}, \quad \mathbf{A}_{*4} = 2\mathbf{A}_{*1} - \mathbf{A}_{*3}, \quad \mathbf{A}_{*6} = 2\mathbf{A}_{*1} - 3\mathbf{A}_{*5}, \quad \mathbf{A}_{*7} = \mathbf{A}_{*3} + \mathbf{A}_{*5}$

2.2.2. No.

2.2.3. The same would have to hold in $\mathbf{E_A}$, and there you can see that this means not all columns can be basic. Remember, $rank(\mathbf{A}) =$ number of basic columns.

2.2.4. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\mathbf{A}_{*3}$ is almost a combination of $\mathbf{A}_{*1}$ and $\mathbf{A}_{*2}$. In particular, $\mathbf{A}_{*3} \approx -\mathbf{A}_{*1} + 2\mathbf{A}_{*2}$.

2.2.5. $\mathbf{E}_{*1} = 2\mathbf{E}_{*2} - \mathbf{E}_{*3}$ and $\mathbf{E}_{*2} = \frac{1}{2}\mathbf{E}_{*1} + \frac{1}{2}\mathbf{E}_{*3}$

Solutions for exercises in section 2. 3

2.3.1. (a), (b)—There is no need to do any arithmetic for this one because the right-hand side is entirely zero so that you know (0,0,0) is automatically one solution. (d), (f)

2.3.3. It is always true that $rank(\mathbf{A}) \le rank[\mathbf{A}|\mathbf{b}] \le m$. Since $rank(\mathbf{A}) = m$, it follows that $rank[\mathbf{A}|\mathbf{b}] = rank(\mathbf{A})$.

2.3.4. Yes—Consistency implies that $\mathbf{b}$ and $\mathbf{c}$ are each combinations of the basic columns in $\mathbf{A}$. If $\mathbf{b} = \sum \beta_i \mathbf{A}_{*b_i}$ and $\mathbf{c} = \sum \gamma_i \mathbf{A}_{*b_i}$ where the $\mathbf{A}_{*b_i}$'s are the basic columns, then $\mathbf{b} + \mathbf{c} = \sum(\beta_i + \gamma_i)\mathbf{A}_{*b_i} = \sum \xi_i \mathbf{A}_{*b_i}$, where $\xi_i = \beta_i + \gamma_i$ so that $\mathbf{b} + \mathbf{c}$ is also a combination of the basic columns in $\mathbf{A}$.

2.3.5. Yes—because the 4×3 system $\alpha + \beta x_i + \gamma x_i^2 = y_i$ obtained by using the four given points (x_i, y_i) is consistent.

2.3.6. The system is inconsistent using 5-digits but consistent when 6-digits are used.

2.3.7. If x, y, and z denote the number of pounds of the respective brands applied, then the following constraints must be met.

$$\begin{aligned} \text{total \# units of phosphorous} &= 2x + y + z = 10 \\ \text{total \# units of potassium} &= 3x + 3y \quad\;\; = 9 \\ \text{total \# units of nitrogen} &= 5x + 4y + z = 19 \end{aligned}$$

Since this is a consistent system, the recommendation can be satisfied exactly. Of course, the solution tells how much of each brand to apply.

2.3.8. No—if one or more such rows were ever present, how could you possibly eliminate all of them with row operations? You could eliminate all but one, but then there is no way to eliminate the last remaining one, and hence it would have to appear in the final form.

Solutions for exercises in section 2. 4

2.4.1. (a) $x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$ (b) $y\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$ (c) $x_3\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$

(d) The trivial solution is the only solution.

2.4.2. $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-\frac{1}{2}\\0\end{pmatrix}$

2.4.3. $x_2\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix}$

2.4.4. $rank\,(\mathbf{A})=3$

2.4.5. (a) 2—because the maximum rank is 4. (b) 5—because the minimum rank is 1.

2.4.6. Because $r=rank\,(\mathbf{A})\le m<n \implies n-r>0$.

2.4.7. There are many different correct answers. One approach is to answer the question "What must $\mathbf{E_A}$ look like?" The form of the general solution tells you that $rank\,(\mathbf{A})=2$ and that the first and third columns are basic. Consequently, $\mathbf{E_A}=\begin{pmatrix}1&\alpha&0&\beta\\0&0&1&\gamma\\0&0&0&0\end{pmatrix}$ so that $x_1=-\alpha x_2-\beta x_4$ and $x_3=-\gamma x_4$ gives rise to the general solution $x_2\begin{pmatrix}-\alpha\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-\beta\\0\\-\gamma\\1\end{pmatrix}$. Therefore, $\alpha=2$, $\beta=3$, and $\gamma=-2$. Any matrix $\mathbf{A}$ obtained by performing row operations to $\mathbf{E_A}$ will be the coefficient matrix for a homogeneous system with the desired general solution.

2.4.8. If $\sum_i x_{f_i}\mathbf{h}_i$ is the general solution, then there must exist scalars α_i and β_i such that $\mathbf{c}_1=\sum_i\alpha_i\mathbf{h}_i$ and $\mathbf{c}_2=\sum_i\beta_i\mathbf{h}_i$. Therefore, $\mathbf{c}_1+\mathbf{c}_2=\sum_i(\alpha_i+\beta_i)\mathbf{h}_i$, and this shows that $\mathbf{c}_1+\mathbf{c}_2$ is the solution obtained when the free variables x_{f_i} assume the values $x_{f_i}=\alpha_i+\beta_i$.

Solutions for exercises in section 2. 5

2.5.1. (a) $\begin{pmatrix}1\\0\\2\\0\end{pmatrix}+x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$ (b) $\begin{pmatrix}1\\0\\2\end{pmatrix}+y\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}$

2.5.2. From Example 2.5.1, the solutions of the linear equations are:

$$\begin{aligned} x_1 &= 1 - x_3 - 2x_4 \\ x_2 &= 1 - x_3 \\ x_3 &\text{ is free} \\ x_4 &\text{ is free} \\ x_5 &= -1 \end{aligned}$$

Substitute these into the two constraints to get $x_3 = \pm 1$ and $x_4 = \pm 1$. Thus there are exactly four solutions:

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

2.5.3. (a) $\{(3,0,4),\ (2,1,5),\ (1,2,6),\ (0,3,7)\}$ See the solution to Exercise 2.3.7 for the underlying system. (b) $(3,0,4)$ costs \$15 and is least expensive.

2.5.4. (a) Consistent for all α. (b) $\alpha \neq 3$, in which case the solution is $(1,-1,0)$.

(c) $\alpha = 3$, in which case the general solution is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$.

2.5.5. No

2.5.6.

$$\mathbf{E_A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times n}$$

2.5.7. See the solution to Exercise 2.4.7.

2.5.8. (a) $\begin{pmatrix} -.3976 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -.7988 \\ 1 \\ 0 \end{pmatrix}$ (b) There are no solutions in this case.

(c) $\begin{pmatrix} 1.43964 \\ -2.3 \\ 1 \end{pmatrix}$

Solutions for exercises in section 2. 6

2.6.1. (a) $(1/575)(383, 533, 261, 644, -150, -111)$
2.6.2. $(1/211)(179, 452, 36)$
2.6.3. (18, 10)
2.6.4. (a) 4 (b) 6 (c) 7 loops but only 3 simple loops. (d) Show that $rank\,([\mathbf{A}|\mathbf{b}]) = 3$ (g) 5/6

I fear explanations explanatory of things explained.
— *Abraham Lincoln* (1809–1865)

Solutions for Chapter 3

Solutions for exercises in section 3. 2

3.2.1. (a) $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ (b) $x = -\frac{1}{2}$, $y = -6$, and $z = 0$

3.2.2. (a) Neither (b) Skew symmetric (c) Symmetric (d) Neither

3.2.3. The 3×3 zero matrix trivially satisfies all conditions, and it is the only possible answer for part (a). The only possible answers for (b) are real symmetric matrices. There are many nontrivial possibilities for (c).

3.2.4. $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T \implies (\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A} + \mathbf{B}$. Yes—the skew-symmetric matrices are also closed under matrix addition.

3.2.5. (a) $\mathbf{A} = -\mathbf{A}^T \implies a_{ij} = -a_{ji}$. If $i = j$, then $a_{jj} = -a_{jj} \implies a_{jj} = 0$.

(b) $\mathbf{A} = -\mathbf{A}^* \implies a_{ij} = -\overline{a_{ji}}$. If $i = j$, then $a_{jj} = -\overline{a_{jj}}$. Write $a_{jj} = x + \mathrm{i}y$ to see that $a_{jj} = -\overline{a_{jj}} \implies x + \mathrm{i}y = -x + \mathrm{i}y \implies x = 0 \implies a_{jj}$ is pure imaginary.

(c) $\mathbf{B}^* = (\mathrm{i}\mathbf{A})^* = -\mathrm{i}\mathbf{A}^* = -\mathrm{i}\overline{\mathbf{A}}^T = -\mathrm{i}\mathbf{A}^T = -\mathrm{i}\mathbf{A} = -\mathbf{B}$.

3.2.6. (a) Let $\mathbf{S} = \mathbf{A} + \mathbf{A}^T$ and $\mathbf{K} = \mathbf{A} - \mathbf{A}^T$. Then $\mathbf{S}^T = \mathbf{A}^T + \mathbf{A}^{T^T} = \mathbf{A}^T + \mathbf{A} = \mathbf{S}$. Likewise, $\mathbf{K}^T = \mathbf{A}^T - \mathbf{A}^{T^T} = \mathbf{A}^T - \mathbf{A} = -\mathbf{K}$.

(b) $\mathbf{A} = \frac{\mathbf{S}}{2} + \frac{\mathbf{K}}{2}$ is one such decomposition. To see it is unique, suppose $\mathbf{A} = \mathbf{X} + \mathbf{Y}$, where $\mathbf{X} = \mathbf{X}^T$ and $\mathbf{Y} = -\mathbf{Y}^T$. Thus, $\mathbf{A}^T = \mathbf{X}^T + \mathbf{Y}^T = \mathbf{X} - \mathbf{Y} \implies \mathbf{A} + \mathbf{A}^T = 2\mathbf{X}$, so that $\mathbf{X} = \frac{\mathbf{A}+\mathbf{A}^T}{2} = \frac{\mathbf{S}}{2}$. A similar argument shows that $\mathbf{Y} = \frac{\mathbf{A}-\mathbf{A}^T}{2} = \frac{\mathbf{K}}{2}$.

3.2.7. (a) $[(\mathbf{A}+\mathbf{B})^*]_{ij} = [\overline{\mathbf{A}+\mathbf{B}}]_{ji} = [\overline{\mathbf{A}} + \overline{\mathbf{B}}]_{ji} = [\overline{\mathbf{A}}]_{ji} + [\overline{\mathbf{B}}]_{ji} = [\mathbf{A}^*]_{ij} + [\mathbf{B}^*]_{ij} = [\mathbf{A}^* + \mathbf{B}^*]_{ij}$

(b) $[(\alpha\mathbf{A})^*]_{ij} = [\overline{\alpha\mathbf{A}}]_{ji} = [\bar{\alpha}\overline{\mathbf{A}}]_{ji} = \bar{\alpha}[\overline{\mathbf{A}}]_{ji} = \bar{\alpha}[\mathbf{A}^*]_{ij}$

3.2.8. $k \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$

Solutions for exercises in section 3. 3

3.3.1. Functions (b) and (f) are linear. For example, to check if (b) is linear, let $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and check if $f(\mathbf{A}+\mathbf{B}) = f(\mathbf{A}) + f(\mathbf{B})$ and

$f(\alpha\mathbf{A}) = \alpha f(\mathbf{A})$. Do so by writing

$$f(\mathbf{A}+\mathbf{B}) = f\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix} = \begin{pmatrix} a_2+b_2 \\ a_1+b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = f(\mathbf{A}) + f(\mathbf{B}),$$

$$f(\alpha\mathbf{A}) = f\begin{pmatrix} \alpha a_1 \\ \alpha a_2 \end{pmatrix} = \begin{pmatrix} \alpha a_2 \\ \alpha a_1 \end{pmatrix} = \alpha \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = \alpha f(\mathbf{A}).$$

3.3.2. Write $f(\mathbf{x}) = \sum_{i=1}^n \xi_i x_i$. For all points $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, and for all scalars α, it is true that

$$\begin{aligned} f(\alpha\mathbf{x}+\mathbf{y}) &= \sum_{i=1}^n \xi_i(\alpha x_i + y_i) = \sum_{i=1}^n \xi_i \alpha x_i + \sum_{i=1}^n \xi_i y_i \\ &= \alpha \sum_{i=1}^n \xi_i x_i + \sum_{i=1}^n \xi_i y_i = \alpha f(\mathbf{x}) + f(\mathbf{y}). \end{aligned}$$

3.3.3. There are many possibilities. Two of the simplest and most common are Hooke's law for springs that says that $F = kx$ (see Example 3.2.1) and Newton's second law that says that $F = ma$ (i.e., force = mass × acceleration).

3.3.4. They are all linear. To see that rotation is linear, use trigonometry to deduce that if $\mathbf{p} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $f(\mathbf{p}) = \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where

$$\begin{aligned} u_1 &= (\cos\theta)x_1 - (\sin\theta)x_2 \\ u_2 &= (\sin\theta)x_1 + (\cos\theta)x_2. \end{aligned}$$

f is linear because this is a special case of Example 3.3.2. To see that reflection is linear, write $\mathbf{p} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $f(\mathbf{p}) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$. Verification of linearity is straightforward. For the projection function, use the Pythagorean theorem to conclude that if $\mathbf{p} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $f(\mathbf{p}) = \frac{x_1+x_2}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Linearity is now easily verified.

Solutions for exercises in section 3. 4

3.4.1. Refer to the solution for Exercise 3.3.4. If $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{P}$ denote the matrices associated with the rotation, reflection, and projection, respectively, then

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

3.4.2. Refer to the solution for Exercise 3.4.1 and write

$$\mathbf{RQ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}.$$

If $Q(\mathbf{x})$ is the rotation function and $R(\mathbf{x})$ is the reflection function, then the composition is

$$R\big(Q(\mathbf{x})\big) = \begin{pmatrix} (\cos\theta)x_1 - (\sin\theta)x_2 \\ -(\sin\theta)x_1 - (\cos\theta)x_2 \end{pmatrix}.$$

3.4.3. Refer to the solution for Exercise 3.4.1 and write

$$\begin{aligned}\mathbf{PQR} &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta + \sin\theta & \sin\theta - \cos\theta \\ \cos\theta + \sin\theta & \sin\theta - \cos\theta \end{pmatrix}.\end{aligned}$$

Therefore, the composition of the three functions in the order asked for is

$$P\Big(Q\big(R(\mathbf{x})\big)\Big) = \frac{1}{2}\begin{pmatrix} (\cos\theta + \sin\theta)x_1 + (\sin\theta - \cos\theta)x_2 \\ (\cos\theta + \sin\theta)x_1 + (\sin\theta - \cos\theta)x_2 \end{pmatrix}.$$

Solutions for exercises in section 3. 5

3.5.1. (a) $\mathbf{AB} = \begin{pmatrix} 10 & 15 \\ 12 & 8 \\ 28 & 52 \end{pmatrix}$ (b) $\mathbf{BA}$ does not exist (c) $\mathbf{CB}$ does not exist

(d) $\mathbf{C}^T\mathbf{B} = (\,10 \quad 31\,)$ (e) $\mathbf{A}^2 = \begin{pmatrix} 13 & -1 & 19 \\ 16 & 13 & 12 \\ 36 & -17 & 64 \end{pmatrix}$ (f) $\mathbf{B}^2$ does not exist

(g) $\mathbf{C}^T\mathbf{C} = 14$ (h) $\mathbf{CC}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ (i) $\mathbf{BB}^T = \begin{pmatrix} 5 & 8 & 17 \\ 8 & 16 & 28 \\ 17 & 28 & 58 \end{pmatrix}$

(j) $\mathbf{B}^T\mathbf{B} = \begin{pmatrix} 10 & 23 \\ 23 & 69 \end{pmatrix}$ (k) $\mathbf{C}^T\mathbf{AC} = 76$

3.5.2. (a) $\mathbf{A}=\begin{pmatrix}2&1&1\\4&0&2\\2&2&0\end{pmatrix},\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\mathbf{b}=\begin{pmatrix}3\\10\\-2\end{pmatrix}$ (b) $\mathbf{s}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$

(c) $\mathbf{b}=\mathbf{A}_{*1}-2\mathbf{A}_{*2}+3\mathbf{A}_{*3}$

3.5.3. (a) $\mathbf{EA}=\begin{pmatrix}\mathbf{A}_{1*}\\\mathbf{A}_{2*}\\3\mathbf{A}_{1*}+\mathbf{A}_{3*}\end{pmatrix}$ (b) $\mathbf{AE}=\left(\mathbf{A}_{*1}+3\mathbf{A}_{*3}\quad\mathbf{A}_{*2}\quad\mathbf{A}_{*3}\right)$

3.5.4. (a) $\mathbf{A}_{*j}$ (b) $\mathbf{A}_{i*}$ (c) a_{ij}

3.5.5. $\mathbf{Ax}=\mathbf{Bx}\ \forall\ \mathbf{x}\implies\mathbf{Ae}_j=\mathbf{Be}_j\ \forall\ \mathbf{e}_j\implies\mathbf{A}_{*j}=\mathbf{B}_{*j}\ \forall\ j\implies\mathbf{A}=\mathbf{B}$.
(The symbol $\forall$ is mathematical shorthand for the phrase "for all.")

3.5.6. The limit is the zero matrix.

3.5.7. If $\mathbf{A}$ is $m\times p$ and $\mathbf{B}$ is $p\times n$, write the product as

$$\mathbf{AB}=\left(\mathbf{A}_{*1}\quad\mathbf{A}_{*2}\quad\cdots\quad\mathbf{A}_{*p}\right)\begin{pmatrix}\mathbf{B}_{1*}\\\mathbf{B}_{2*}\\\vdots\\\mathbf{B}_{p*}\end{pmatrix}=\mathbf{A}_{*1}\mathbf{B}_{1*}+\mathbf{A}_{*2}\mathbf{B}_{2*}+\cdots+\mathbf{A}_{*p}\mathbf{B}_{p*}$$
$$=\sum_{k=1}^{p}\mathbf{A}_{*k}\mathbf{B}_{k*}.$$

3.5.8. (a) $[\mathbf{AB}]_{ij}=\mathbf{A}_{i*}\mathbf{B}_{*j}=\left(0\ \cdots\ 0\ \ a_{ii}\ \cdots\ a_{in}\right)\begin{pmatrix}b_{1j}\\\vdots\\b_{jj}\\0\\\vdots\\0\end{pmatrix}$ is 0 when $i>j$.

(b) When $i=j$, the only nonzero term in the product $\mathbf{A}_{i*}\mathbf{B}_{*i}$ is $a_{ii}b_{ii}$.
(c) Yes.

3.5.9. Use $[\mathbf{AB}]_{ij}=\sum_k a_{ik}b_{kj}$ along with the rules of differentiation to write

$$\frac{d[\mathbf{AB}]_{ij}}{dt}=\frac{d\left(\sum_k a_{ik}b_{kj}\right)}{dt}=\sum_k\frac{d(a_{ik}b_{kj})}{dt}$$
$$=\sum_k\left(\frac{da_{ik}}{dt}b_{kj}+a_{ik}\frac{db_{kj}}{dt}\right)=\sum_k\frac{da_{ik}}{dt}b_{kj}+\sum_k a_{ik}\frac{db_{kj}}{dt}$$
$$=\left[\frac{d\mathbf{A}}{dt}\mathbf{B}\right]_{ij}+\left[\mathbf{A}\frac{d\mathbf{B}}{dt}\right]_{ij}=\left[\frac{d\mathbf{A}}{dt}\mathbf{B}+\mathbf{A}\frac{d\mathbf{B}}{dt}\right]_{ij}.$$

3.5.10. (a) $[\mathbf{Ce}]_i=$ the total number of paths *leaving* node i.
(b) $[\mathbf{e}^T\mathbf{C}]_i=$ the total number of paths *entering* node i.

3.5.11. At time t, the concentration of salt in tank i is $\frac{x_i(t)}{V}$ lbs/gal. For tank 1,

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = 0\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_1(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= -\frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}}.\end{aligned}$$

For tank 2,

$$\begin{aligned}\frac{dx_2}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = \frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_2(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= \frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}} - \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} = \frac{r}{V}\Big(x_1(t) - x_2(t)\Big),\end{aligned}$$

and for tank 3,

$$\begin{aligned}\frac{dx_3}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_3(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} - \frac{r}{V}x_3(t)\frac{\text{lbs}}{\text{sec}} = \frac{r}{V}\Big(x_2(t) - x_3(t)\Big).\end{aligned}$$

This is a system of three linear first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= \tfrac{r}{V}\Big(-x_1(t)\Big) \\ \frac{dx_2}{dt} &= \tfrac{r}{V}\Big(x_1(t) - x_2(t)\Big) \\ \frac{dx_3}{dt} &= \tfrac{r}{V}\Big(x_2(t) - x_3(t)\Big)\end{aligned}$$

that can be written as a single matrix differential equation

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \frac{r}{V}\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Solutions for exercises in section 3. 6

3.6.1.

$$\mathbf{AB} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \end{pmatrix} \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B}_1 + \mathbf{A}_{12}\mathbf{B}_2 + \mathbf{A}_{13}\mathbf{B}_3 \\ \mathbf{A}_{21}\mathbf{B}_1 + \mathbf{A}_{22}\mathbf{B}_2 + \mathbf{A}_{23}\mathbf{B}_3 \end{pmatrix}$$

$$= \left(\begin{array}{cc} -10 & -19 \\ -10 & -19 \\ \hline -1 & -1 \end{array}\right)$$

3.6.2. Use block multiplication to verify $\mathbf{L}^2 = \mathbf{I}$—be careful not to commute any of the terms when forming the various products.

3.6.3. Partition the matrix as $\mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$, where $\mathbf{C} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and observe that $\mathbf{C}^2 = \mathbf{C}$. Use this together with block multiplication to conclude that

$$\mathbf{A}^k = \begin{pmatrix} \mathbf{I} & \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots + \mathbf{C}^k \\ \mathbf{0} & \mathbf{C}^k \end{pmatrix} = \begin{pmatrix} \mathbf{I} & k\mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}.$$

Therefore, $\mathbf{A}^{300} = \begin{pmatrix} 1 & 0 & 0 & 100 & 100 & 100 \\ 0 & 1 & 0 & 100 & 100 & 100 \\ 0 & 0 & 1 & 100 & 100 & 100 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}$.

3.6.4. $(\mathbf{A}^*\mathbf{A})^* = \mathbf{A}^*\mathbf{A}^{**} = \mathbf{A}^*\mathbf{A}$ and $(\mathbf{A}\mathbf{A}^*)^* = \mathbf{A}^{**}\mathbf{A}^* = \mathbf{A}\mathbf{A}^*$.

3.6.5. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{BA} = \mathbf{AB}$. It is easy to construct a 2×2 example to show that this need not be true when $\mathbf{AB} \neq \mathbf{BA}$.

3.6.6.

$$\begin{aligned} [(\mathbf{D}+\mathbf{E})\mathbf{F}]_{ij} &= (\mathbf{D}+\mathbf{E})_{i*}\mathbf{F}_{*j} = \sum_k [\mathbf{D}+\mathbf{E}]_{ik}[\mathbf{F}]_{kj} = \sum_k \left([\mathbf{D}]_{ik} + [\mathbf{E}]_{ik}\right)[\mathbf{F}]_{kj} \\ &= \sum_k \left([\mathbf{D}]_{ik}[\mathbf{F}]_{kj} + [\mathbf{E}]_{ik}[\mathbf{F}]_{kj}\right) = \sum_k [\mathbf{D}]_{ik}[\mathbf{F}]_{kj} + \sum_k [\mathbf{E}]_{ik}[\mathbf{F}]_{kj} \\ &= \mathbf{D}_{i*}\mathbf{F}_{*j} + \mathbf{E}_{i*}\mathbf{F}_{*j} = [\mathbf{DF}]_{ij} + [\mathbf{EF}]_{ij} \\ &= [\mathbf{DF} + \mathbf{EF}]_{ij}. \end{aligned}$$

3.6.7. If a matrix $\mathbf{X}$ did indeed exist, then

$$\begin{aligned} \mathbf{I} = \mathbf{AX} - \mathbf{XA} &\implies \mathit{trace}\,(\mathbf{I}) = \mathit{trace}\,(\mathbf{AX} - \mathbf{XA}) \\ &\implies n = \mathit{trace}\,(\mathbf{AX}) - \mathit{trace}\,(\mathbf{XA}) = 0, \end{aligned}$$

which is impossible.

3.6.8. (a) $\mathbf{y}^T\mathbf{A} = \mathbf{b}^T \implies (\mathbf{y}^T\mathbf{A})^T = \mathbf{b}^{T^T} \implies \mathbf{A}^T\mathbf{y} = \mathbf{b}$. This is an $n \times m$ system of equations whose coefficient matrix is $\mathbf{A}^T$. (b) They are the same.

3.6.9. Draw a transition diagram similar to that in Example 3.6.3 with North and South replaced by ON and OFF, respectively. Let x_k be the proportion of switches in the ON state, and let y_k be the proportion of switches in the OFF state after k clock cycles have elapsed. According to the given information,

$$\begin{aligned} x_k &= x_{k-1}(.1) + y_{k-1}(.3) \\ y_k &= x_{k-1}(.9) + y_{k-1}(.7) \end{aligned}$$

so that $\mathbf{p}_k = \mathbf{p}_{k-1}\mathbf{P}$, where

$$\mathbf{p}_k = \begin{pmatrix} x_k & y_k \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} .1 & .9 \\ .3 & .7 \end{pmatrix}.$$

Just as in Example 3.6.3, $\mathbf{p}_k = \mathbf{p}_0\mathbf{P}^k$. Compute a few powers of $\mathbf{P}$ to find

$$\mathbf{P}^2 = \begin{pmatrix} .280 & .720 \\ .240 & .760 \end{pmatrix}, \quad \mathbf{P}^3 = \begin{pmatrix} .244 & .756 \\ .252 & .748 \end{pmatrix}$$

$$\mathbf{P}^4 = \begin{pmatrix} .251 & .749 \\ .250 & .750 \end{pmatrix}, \quad \mathbf{P}^5 = \begin{pmatrix} .250 & .750 \\ .250 & .750 \end{pmatrix}$$

and deduce that $\mathbf{P}^\infty = \lim_{k\to\infty} \mathbf{P}^k = \begin{pmatrix} 1/4 & 3/4 \\ 1/4 & 3/4 \end{pmatrix}$. Thus

$$\mathbf{p}_k \to \mathbf{p}_0\mathbf{P}^\infty = \left(\tfrac{1}{4}(x_0 + y_0) \quad \tfrac{3}{4}(x_0 + y_0) \right) = \left(\tfrac{1}{4} \quad \tfrac{3}{4} \right).$$

For practical purposes, the device can be considered to be in equilibrium after about 5 clock cycles—regardless of the initial proportions.

3.6.10. $\begin{pmatrix} -4 & 1 & -6 & 5 \end{pmatrix}$

3.6.11. (a) $\mathit{trace}\,(\mathbf{ABC}) = \mathit{trace}\,(\mathbf{A}\{\mathbf{BC}\}) = \mathit{trace}\,(\{\mathbf{BC}\}\mathbf{A}) = \mathit{trace}\,(\mathbf{BCA})$. The other equality is similar. (b) Use almost any set of 2×2 matrices to construct an example that shows equality need not hold. (c) Use the fact that $\mathit{trace}\,(\mathbf{C}^T) = \mathit{trace}\,(\mathbf{C})$ for all square matrices to conclude that

$$\begin{aligned} \mathit{trace}\,(\mathbf{A}^T\mathbf{B}) &= \mathit{trace}\left((\mathbf{A}^T\mathbf{B})^T\right) = \mathit{trace}\left(\mathbf{B}^T\mathbf{A}^{T^T}\right) \\ &= \mathit{trace}\,(\mathbf{B}^T\mathbf{A}) = \mathit{trace}\,(\mathbf{A}\mathbf{B}^T). \end{aligned}$$

3.6.12. (a) $\mathbf{x}^T\mathbf{x} = 0 \iff \sum_{k=1}^n x_i^2 = 0 \iff x_i = 0$ for each $i \iff \mathbf{x} = \mathbf{0}$.

(b)
$$\begin{aligned} \mathit{trace}\,(\mathbf{A}^T\mathbf{A}) = 0 &\iff \sum_i [\mathbf{A}^T\mathbf{A}]_{ii} = 0 \iff \sum_i (\mathbf{A}^T)_{i*}\mathbf{A}_{*i} = 0 \\ &\iff \sum_i \sum_k [\mathbf{A}^T]_{ik}[\mathbf{A}]_{ki} = 0 \iff \sum_i \sum_k [\mathbf{A}]_{ki}[\mathbf{A}]_{ki} = 0 \\ &\iff \sum_i \sum_k [\mathbf{A}]_{ki}^2 = 0 \\ &\iff [\mathbf{A}]_{ki} = 0 \text{ for each } k \text{ and } i \iff \mathbf{A} = \mathbf{0} \end{aligned}$$

Solutions for exercises in section 3. 7

3.7.1. (a) $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ (b) Singular (c) $\begin{pmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{pmatrix}$ (d) Singular

(e) $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

3.7.2. Write the equation as $(\mathbf{I}-\mathbf{A})\mathbf{X}=\mathbf{B}$ and compute

$$\mathbf{X}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}=\begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 3 \end{pmatrix}.$$

3.7.3. In each case, the given information implies that $rank(\mathbf{A})<n$—see the solution for Exercise 2.1.3.

3.7.4. (a) If $\mathbf{D}$ is diagonal, then $\mathbf{D}^{-1}$ exists if and only if each $d_{ii}\neq 0$, in which case

$$\begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}^{-1}=\begin{pmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{pmatrix}.$$

(b) If $\mathbf{T}$ is triangular, then $\mathbf{T}^{-1}$ exists if and only if each $t_{ii}\neq 0$. If $\mathbf{T}$ is upper (lower) triangular, then $\mathbf{T}^{-1}$ is also upper (lower) triangular with $[\mathbf{T}^{-1}]_{ii}=1/t_{ii}$.

3.7.5. $\left(\mathbf{A}^{-1}\right)^T=\left(\mathbf{A}^T\right)^{-1}=\mathbf{A}^{-1}$.

3.7.6. Start with $\mathbf{A}(\mathbf{I}-\mathbf{A})=(\mathbf{I}-\mathbf{A})\mathbf{A}$ and apply $(\mathbf{I}-\mathbf{A})^{-1}$ to both sides, first on one side and then on the other.

3.7.7. Use the result of Example 3.6.5 that says that $trace(\mathbf{AB})=trace(\mathbf{BA})$ to write

$$m=trace(\mathbf{I}_m)=trace(\mathbf{AB})=trace(\mathbf{BA})=trace(\mathbf{I}_n)=n.$$

3.7.8. Use the reverse order law for inversion to write

$$\left[\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}\right]^{-1}=\mathbf{B}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{A}^{-1}=\mathbf{B}^{-1}+\mathbf{A}^{-1}$$

and

$$\left[\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}\right]^{-1}=\mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{B}^{-1}=\mathbf{B}^{-1}+\mathbf{A}^{-1}.$$

3.7.9. (a) $(\mathbf{I}-\mathbf{S})\mathbf{x}=\mathbf{0} \implies \mathbf{x}^T(\mathbf{I}-\mathbf{S})\mathbf{x}=0 \implies \mathbf{x}^T\mathbf{x}=\mathbf{x}^T\mathbf{S}\mathbf{x}$. Taking transposes on both sides yields $\mathbf{x}^T\mathbf{x}=-\mathbf{x}^T\mathbf{S}\mathbf{x}$, so that $\mathbf{x}^T\mathbf{x}=0$, and thus $\mathbf{x}=\mathbf{0}$

(recall Exercise 3.6.12). The conclusion follows from property (3.7.8).

(b) First notice that Exercise 3.7.6 implies that $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1} = (\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})$. By using the reverse order laws, transposing both sides yields exactly the same thing as inverting both sides.

3.7.10. Use block multiplication to verify that the product of the matrix with its inverse is the identity matrix.

3.7.11. Use block multiplication to verify that the product of the matrix with its inverse is the identity matrix.

3.7.12. Let $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{pmatrix}$. The hypothesis implies that $\mathbf{MX} = \mathbf{I}$, and hence (from the discussion in Example 3.7.2) it must also be true that $\mathbf{XM} = \mathbf{I}$, from which the conclusion follows. **Note:** This problem appeared on a past Putnam Exam—a national mathematics competition for undergraduate students that is considered to be quite challenging. This means that you can be proud of yourself if you solved it before looking at this solution.

Solutions for exercises in section 3. 8

3.8.1. (a) $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$

(b) Let $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{d}^T = (0 \quad 2 \quad 1)$ to obtain $\mathbf{C}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & -4 & 2 \end{pmatrix}$

3.8.2. $\mathbf{A}_{*j}$ needs to be removed, and $\mathbf{b}$ needs to be inserted in its place. This is accomplished by writing $\mathbf{B} = \mathbf{A} + (\mathbf{b} - \mathbf{A}_{*j})\mathbf{e}_j^T$. Applying the Sherman–Morrison formula with $\mathbf{c} = \mathbf{b} - \mathbf{A}_{*j}$ and $\mathbf{d}^T = \mathbf{e}_j^T$ yields

$$\mathbf{B}^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}(\mathbf{b} - \mathbf{A}_{*j})\mathbf{e}_j^T\mathbf{A}^{-1}}{1 + \mathbf{e}_j^T\mathbf{A}^{-1}(\mathbf{b} - \mathbf{A}_{*j})} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{b}\mathbf{e}_j^T\mathbf{A}^{-1} - \mathbf{e}_j\mathbf{e}_j^T\mathbf{A}^{-1}}{1 + \mathbf{e}_j^T\mathbf{A}^{-1}\mathbf{b} - \mathbf{e}_j^T\mathbf{e}_j}$$

$$= \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{b}[\mathbf{A}^{-1}]_{j*} - \mathbf{e}_j[\mathbf{A}^{-1}]_{j*}}{[\mathbf{A}^{-1}]_{j*}\mathbf{b}} = \mathbf{A}^{-1} - \frac{\left(\mathbf{A}^{-1}\mathbf{b} - \mathbf{e}_j\right)[\mathbf{A}^{-1}]_{j*}}{[\mathbf{A}^{-1}]_{j*}\mathbf{b}}.$$

3.8.3. Use the Sherman–Morrison formula to write

$$\mathbf{z} = (\mathbf{A} + \mathbf{c}\mathbf{d}^T)^{-1}\mathbf{b} = \left(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}}{1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}\right)\mathbf{b} = \mathbf{A}^{-1}\mathbf{b} - \frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}\mathbf{b}}{1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}$$

$$= \mathbf{x} - \frac{\mathbf{y}\mathbf{d}^T\mathbf{x}}{1 + \mathbf{d}^T\mathbf{y}}.$$

3.8.4. (a) For a nonsingular matrix $\mathbf{A}$, the Sherman–Morrison formula guarantees that $\mathbf{A} + \alpha\mathbf{e}_i\mathbf{e}_j^T$ is also nonsingular when $1 + \alpha\left[\mathbf{A}^{-1}\right]_{ji} \neq 0$, and this certainly will be true if α is sufficiently small.