1.1 The beam of a rectangular thin-walled section (i.e., *t* is very small) is designed to carry both bending moment *M* and torque *T*. If the total wall contour length $L = 2(a+b)$ (see Fig. 1.16) is fixed, find the optimum *b/a* ratio to achieve the most efficient section if $M = T$ and $\sigma_{allowable} = 2\tau_{allowable}$. Note that for closed thin-walled sections such as the one in Fig.1.16, the shear stress due to torsion is

$$\tau = \frac{T}{2abt}$$

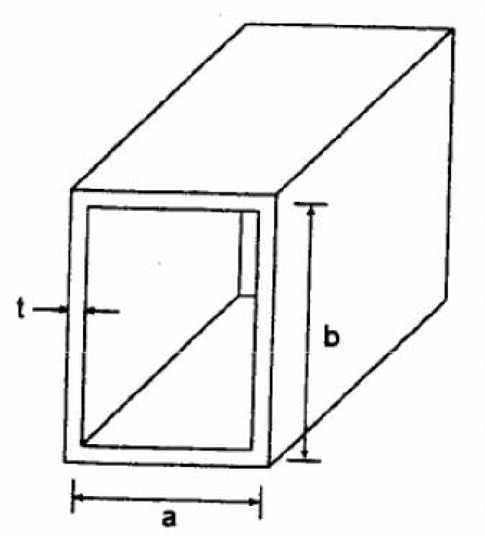

Figure 1.16 Closed thin-walled section

<u>Solution:</u>

(1) The bending stress of beams is $\sigma = \frac{My}{I}$, where y is the distance from the neutral axis. The moment of inertia *I* of the cross-section can be calculated by considering the four segments of thin walls and using the formula for a rectangular section with height h and width w. $I = \sum(\frac{1}{12}wh^3 + Ad^2)$ in which A is the cross-sectional area of the segment and d is the distance of the centroid of the segment to the neutral axis. Note that the Parallel Axis Theorem is applied. The result is $I = 2\cdot\frac{1}{12}tb^3 + 2\cdot[\frac{1}{12}\cdot at^3 + (at)\cdot(\frac{b}{2})^2] \approx \frac{tb^2}{6}(3a+b)$, assuming that *t* is very small.

(2) The shear stress due to torsion for a closed thin-walled section shown above is $\tau = \frac{T}{2abt}$.

(3) Two approaches are employed to find the solution.

(i) Assume that the bending stress reaches the allowable $\sigma_{allowable}$ first and find the corresponding bending maximum bending moment. Then apply the stated loading condition of $T = M$ to check whether the corresponding $\tau_{\max}$ has exceeded the allowable shear stress $\tau_{allowable}$. If this condition is violated, then the optimized b/a ratio is not valid.

(a) $$\sigma|_{y=\frac{b}{2}} = \frac{My}{I} = \frac{M \cdot \frac{b}{2}}{\frac{tb^2}{6}(3a+b)} = \frac{3M}{tb(3a+b)}$$

When given $L = 2(a+b)$ as a constant, a can be expressed in terms of b and L as $a = \frac{L}{2} - b$. Then we can minimize

$$S = \frac{tb(3a+b)}{3} = \frac{tb(3L-4b)}{6}$$ in order to maximize σ, i.e.,

$$\frac{\partial S}{\partial b} = 0 \Rightarrow \frac{t}{6}(3L-8b) = 0 \Rightarrow b = \frac{3L}{8}, \text{ so } a = \frac{L}{2} - b = \frac{L}{8}$$

where the optimum ratio is $\frac{b}{a} = 3$

Thus, $$\sigma_{\max} = \frac{3M}{tb(3a+b)} = \frac{3M}{t \cdot (3L/8) \cdot (3 \cdot L/8 + 3L/8)} = \frac{32M}{3tL^2}$$

(b) Check $\tau_{\max}$ with $T = M$ and b/a = 3 and check whether $\tau_{\max}$ is within the allowable shear stress $\tau_{allowable}$.

$$\tau_{\max} = \frac{T}{2abt} = \frac{M}{2 \cdot (L/8) \cdot (3L/8) \cdot t} = \frac{32M}{3tL^2} = \sigma_{\max} = \sigma_{allowable}$$
$$> \tau_{allowable} = \frac{\sigma_{allowable}}{2}$$

The result above means that under this assumption, shear stress τ would reach the allowable stress $\tau_{allowable}$ before σ reaches $\sigma_{allowable}$. Consequently, the optimal ratio obtained is not valid and different assumption needs to be made.

(ii) Assume now that failure is controlled by shear stress. We assume that $\tau_{\max} = \tau_{allowable}$ is reached first and then find the corresponding bending stress according to the loading condition $M = T$.

(a) $$\tau = \frac{T}{2abt}$$

Again we minimize $S = 2abt = (L-2b)bt$ in order to maximize τ, i.e.,

$$\frac{\partial S}{\partial b} = 0 \Rightarrow (L - 4b) = 0 \Rightarrow b = \frac{L}{4}, \text{ so } \; a = \frac{L}{2} - b = \frac{L}{4}$$

and the optimum ratio is $\frac{b}{a} = 1$

and $\tau_{\max} = \frac{T}{2abt} = \frac{T}{2 \cdot (L/4) \cdot (L/4) \cdot t} = \frac{8T}{tL^2}$

(b) Then corresponding $\sigma_{\max}$ under the optimum condition stated above can be obtained using $M = T$. We have

$$\sigma_{\max} = \frac{3M}{tb(3a+b)} = \frac{3T}{t \cdot (L/4) \cdot (3 \cdot L/4 + L/4)} = \frac{12T}{tL^2} = \frac{3}{2}\tau_{\max} = \frac{3}{2}\tau_{allowable}$$
$$< \sigma_{allowable} = 2\tau_{allowable}$$

This means that when the structure fails in shear, the bending stress is still within the allowable stress level. Thus the optimum ratio $\frac{b}{a} = 1$ is valid.

(4) In conclusion, $\frac{b}{a} = 1$ achieves the most efficient section for the stated conditions.

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