Solutions Manual to Accompany

Fourth Edition
Introduction to Random Signals and Applied Kalman Filtering with MATLAB Exercises

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Note from the authors:

There are numerous MATLAB m-files contained throughout the following problem solutions. They were prepared at various times in the course of this book preparation and with some variations in the many versions of MATLAB. Thus, the enclosed m-files should be used with some caution and not taken too literally as "bug-free".

CONTENTS

CHAPTER 1

1.1 Total no. of possible hands $= \binom{52}{5}$

$$= \frac{52!}{5!\,47!} = 2,598,960$$

Number of Heart flushes $= \binom{13}{5}$

$$= \frac{13!}{5!\,8!} = 1287$$

Number of Spade flushes $= 1287$

Same for Clubs and Diamonds

$\therefore$ Total possible flushes $= 4 \cdot 1287$

$$\therefore P(\text{Flush}) = \frac{4 \cdot 1287}{2,598,960} \approx \frac{1}{500}$$

1.2 Total no. of Black Jack deals

$$= \binom{52}{2} = \frac{52!}{2!\,50!} = \frac{52 \cdot 51}{2} = 26 \cdot 51$$

Total Black Jack combinations of ace and face card:

A of Spades
- K, K, K, K (any suit)
- Q, Q, Q, Q (any suit)
- J, J, J, J (any suit)
- 10, 10, 10, 10 (any suit)

16 total

A of Hearts ← etc.

$$\therefore P\left[\begin{matrix}\text{Any Ace and} \\ \text{K, Q, J, or 10}\end{matrix}\right] = \frac{4 \cdot 16}{26 \cdot 51} = \frac{32}{663} \approx \frac{1}{20}$$

1.3 (a) Note that the entries of each of the columns in the table total approx. 100%. Thus, these numbers (in decimal form) can be interpreted as conditional probability. For example, the middle number 51 would be the probability of "strongly opposed" given "Republican".

(b) No. The joint probabilities cannot be reconstructed from the numbers in the table. Note in Bayes rule you need the unconditional probabilities $p(x)$ and $p(y)$ in order to compute $p(x,y)$, and $p(x)$ and $p(y)$ are not known from the table.

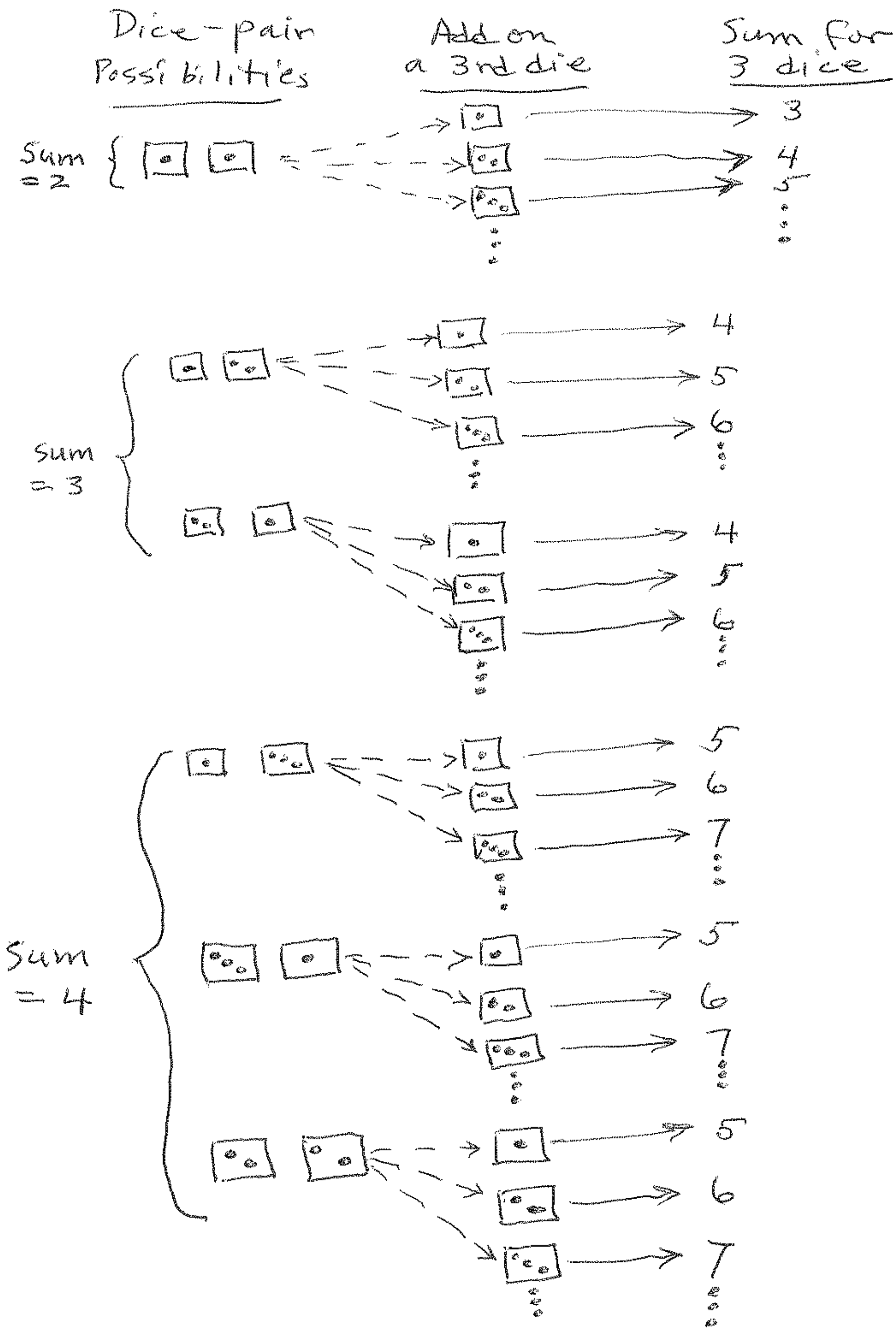
1.4 There are $6^3 = 216$ discrete possible results that can occur with the roll of 3 dice. However, we are only interested in the sums in this experiment. Therefore, we will group together elemental events that achieve the same sum, and this can be done with the chart on the next page for a few of the smaller sums. It is clear from the chart that we can only achieve a sum of 3 in one of 216 ways, a sum of 4 in 3 ways, a sum of 5 in 6 ways, ... and so forth. Therefore our sample space will have 16 elements which can be labeled $3, 4, 5 \cdots 18$, and their associated probabilities will be

$$P_3 = 1/216, \; P_4 = 3/216, \; P_5 = 6/216, \ldots,$$

and $\text{Prob}(3 \text{ or } 4) = 4/216$.

1.4 (cont.)

Chart for computing probabilities for Prob. 1.4



Etc.

1.5 Assume a "typical" statistical situation:

The player makes a 1-chip single-number bet on each of 38 spins of the wheel. He loses on 37 spins and wins on 1 spin. On the play where he wins he gets back his 1 chip plus 35 more from the casino. So after 38 spins he has spent 38 chips and gets back 36 chips. His return is then

(a) Return = $36/38 \approx .9474$ or 94.74%

The casino's percentage is then:

(b) Casino "take" = $1 - .9474 = 5.26\%$

(c) With a "corner" bet, the player wins 4 times in 38 spins, and thus his total return is:

Return = $8 * 4$ + (4 of his own coins)
= 36 chips

Thus, the "corner" bet is the same identical <u>percentage</u> bet as the single-number bet.

There are many other bets that can be made at the roulette table. The <u>American Casino Guide</u> referenced at the end of Chap. 1 has an especially good section on roulette.

1.6 Imagine 27 specific cards exposed to the declarer including 11 specific trumps. This leaves 2 trumps outstanding that will be denoted T1 and T2. The possible opposing hands may be categorized as follows:

	Left Opponent	Right Opponent
(a)	T1, T2, x x x ⋯ x (10 x's)	x x, x, ⋯ x (13 x's)
(b)	T1, x, x, x ⋯ x (11 x's)	T2, x, x, x ⋯ x (12 x's)
(c)	T2, x, x, x ⋯ x (11 x's)	T1, x, x, x, ⋯ x (12 x's)
(d)	x, x, x, ⋯ x (12 x's)	T1, T2, x, x, ⋯ x (11 x's)

Calculation of number of possibilities for each of the (a) (b) (c) and (d) categories:

category (a): $\dfrac{23!}{10!\,13!}$ = No. of hands with T1, T2 left

Category (b): $\dfrac{23!}{11!\,12!}$ = No. of hands with T1 left, T2 Rt.

category (c) $\dfrac{23!}{11!\,12!}$ = No. of hands with T2 left, T1 Rt.

category (d) $\dfrac{23!}{12!\,11!}$ = No. of hands with T1, T2 Right

Part (a): P[T1, T2 are to the left] =

$$\frac{\dfrac{23!}{10!\,13!}}{\dfrac{23!}{10!\,13!}+\dfrac{23!}{11!\,12!}+\dfrac{23!}{11!\,12!}+\dfrac{23!}{12!\,11!}} = \frac{11}{50}$$

1.6 (cont.)

Part (b): $P[T1, T2 \text{ are to the right}] =$

$$\frac{23!/11!\,12!}{\text{Sum as in (a)}} = \frac{13}{50}$$

Part (c): $P[T1 \text{ and } T2 \text{ are split}] =$

$$\left(23!/_{11!\,12!} + 23!/_{12!\,11!}\right)/\text{sum as in (a)} = 26/50$$

1.7

Prob(Rolling 12) = 1/36

Consider "typical" 36-roll experiment.

Player makes unit bet 36 times.

Wins 1 roll; loses 35 rolls

Money spent = 36

Money returned = 30 + 1 (bet returned on win)

= 31

(a) Ave. percentage return = $31/36 \approx 86.11\%$

(b) Casino "take" = $(1 - 31/36) \approx 13.9\%$

This "1-roll" bet is a relatively poor bet when compared with throwing the dice. (See Example 1.4, Section 1.4.)

1.8 (a) Prob(Starter card is a jack) = $4/52 \approx .0769$

(b) Dealer has seen 6 cards and no jack is in hand. Therefore, there are 46 remaining unknown cards from dealer's viewpoint, and they contain 4 jacks. Therefore, when deck is cut

$$\text{Prob(starter card is a jack)} = 4/46 \approx .0870$$

(C) Again, there are 46 unknown cards, but this time they contain only 3 jacks, (not 4). Therefore,

$$\text{Prob(starter card is a jack)} = 3/4 \approx .0652$$

1.9

Use total itemization approach. Let "0" denote boy, "1" denote girl. The 16 possibilities are:

0000	0100	1000	→ 1100
0001	→ 0101	→ 1001	1101
0010	→ 0110	→ 1010	1110
→ 0011	0111	1011	1111

The 6 "favorable" events are denoted "→".

$\therefore P[2 \text{ boys}, 2 \text{ girls}] = 6/16 = 3/8$

1.10 Use heuristic approach.

(a) Example of "typical" 3 zeros and 3 ones:

001101

$P[\text{above combination}] = \left(\frac{1}{2}\right)^6$

But there are $\binom{6}{3} = \frac{6!}{3!3!} = 20$ possible arrangements of 3 zeros and 3 ones.

$\therefore P[\text{Exactly 3 zeros, 3 ones}] = 20 \cdot \left(\frac{1}{2}\right)^6 = 5/16$

(b) Using arguments similar to those in (a):

$P[\text{Exactly 4 zeros, 2 ones}] = \binom{6}{4} \cdot \left(\frac{1}{2}\right)^6 = 15/64$

(c) Similarily,

$P[\text{Exactly 5 zeros, 1 one}] = \binom{6}{5} \cdot \left(\frac{1}{2}\right)^6 = 3/32$

(d) $P[\text{6 zeros}] = (1/2)^6 = 1/64$

1.11 Use heuristic approach just as in Prob 1.9.

Typical sequence with say 2 errors:

Error locations

$$\underbrace{[1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1]}_{n \text{ bits}}$$

$P[\text{above situation}] = p^2 \cdot (1-p)^{n-2} \cdot \binom{\text{No. of arrangements}}{\text{of 2 errors}}$

Generalization for k errors:

$$P[k \text{ errors}] = \binom{n}{k} p^k (1-p)^{n-k}$$

1.12

(a) Deal: 4 Hearts, 1 Spade

Discard: Throw the spade; keep the 4 hearts.

$P[\text{Drawing a heart}] = \frac{13-4}{52-5} = \frac{9}{47}$

$P[\text{Not drawing a heart}] = 38/47$

$\text{Average return} = 0 \cdot \frac{38}{47} + 5 \cdot \frac{9}{47} = \frac{45}{47}$

(Conditioned on an initial deal of 4 of one suit, of course.)

(b) Average return to players for this hypothetical situation would be

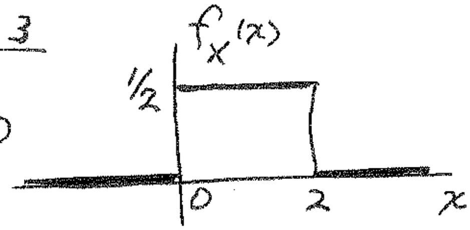
$$\text{Return} = .1 * (100.2) + .9 * (98.0)$$

$$\approx 98.2\%$$

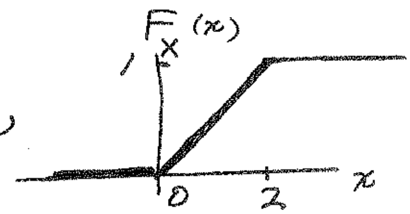
Note that the casino would still have 1.8% "take", for this hypothetical situation.

1.13

(a)



(b)



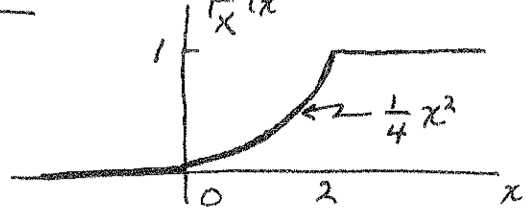
(c) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^2 x \cdot \frac{1}{2}\,dx = 1$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^2 x^2 \cdot \frac{1}{2}\,dx = 4/3$

$\text{Var } X = E(X^2) - [E(X)]^2 = 4/3 - 1 = 1/3$

1.14

(a)



(b) $\text{Var}\, X = E(X^2) - [E(X)]^2$

$$= \int_0^2 x^2 \cdot \tfrac{1}{2}x\,dx - \left[\int_0^2 x \cdot \tfrac{1}{2}x\,dx\right]^2$$

$$= 2 - (4/3)^2 = 2/9$$

1.15

Let X = Time to failure. (exponentially distributed according to $\alpha e^{-\alpha x}$)

Then, P[Failure occurs between 0 and time T]

$$= \int_0^T \alpha e^{-\alpha x}\,dx = 1 - e^{-\alpha T}$$

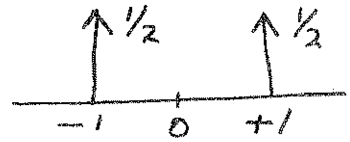
Average lifetime $= E(X)$

$$= \int_0^\infty x \cdot \alpha e^{-\alpha x}\,dx = \alpha \cdot \frac{1}{\alpha^2} = \frac{1}{\alpha}$$

For 10,000 hr lifetime, $\alpha = \dfrac{1}{10{,}000}\ \text{hr}^{-1}$

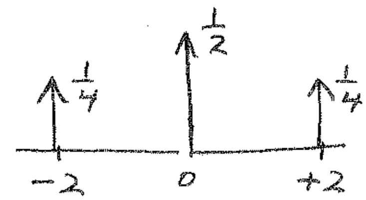
1.16

(a) One coin



(b) 2 Coins

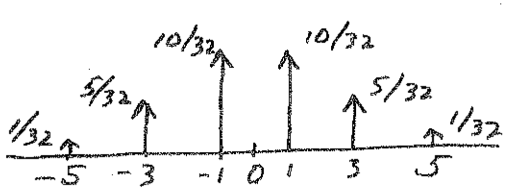
Events	Sum	Prob.
+1, +1	2	1/4
+1, −1	0	1/2
−1, +1	0	
−1, −1	−2	1/4



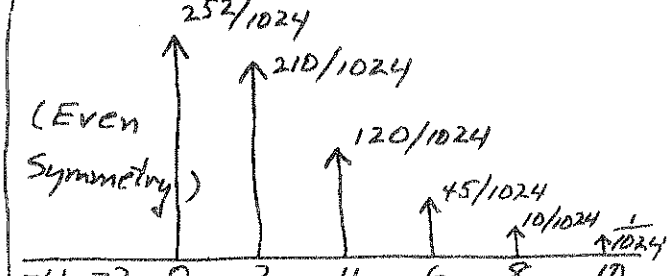
(c) 5 Coins

	Events (Let "0" denote −1)	Sum	Prob.
	0 0 0 0 0	−5	1/32
5 combinations with 1 zero	0 0 0 0 1 0 0 0 1 0 ⋮ 1 0 0 0 0	−3	5/32
$\binom{5}{2} = 10$ comb. with 2 zeros	0 0 0 1 1 0 0 1 0 1 ⋮	−1	10/32
	ETC.	1	10/32
		3	5/32
		5	1/32

Final Result for 3 coins:



Part (d). Result for 10 coins:



1.17 Refer to Table 1.1 for probabilities of sums.

$$E(X) = 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36}$$
$$+ 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36}$$
$$= \frac{252}{36} = 7$$

$$E(X^2) = (2)^2\cdot\frac{1}{36} + (3)^2\cdot\frac{2}{36} + \cdots + (12)^2\cdot\frac{1}{36}$$
$$= \frac{1974}{36} = 54.833\ldots$$

$$\operatorname{Var} X = E(X^2) - [E(X)]^2 = 54.833 - 7^2$$
$$= 5.8333\ldots$$

1.18

X \ Y	1	3	5	Marginal Prob. P(X)
1	1/18	1/18	1/18	1/6
3	1/18	1/18	1/6	5/18
5	1/18	1/6	1/3	5/9
Marginal Prob. P(Y)	1/6	5/18	5/9	

(a) Test for independence: Does $P(X,Y) = P(X)\cdot P(Y)$?
For example, try $X=1$ and $Y=1$.

$P(X=1, Y=1) = 1/18$ (from table)

$$P(X=1)\cdot P(Y=1) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$$

We need not check any further. Independence criterion is <u>not</u> satisfied

1.18 (cont.)

(b) From the marginal probabilities, we have

$$P(Y=5) = 5/9$$

(c) $P(Y=5 \mid X=3) = \dfrac{P(Y=5, X=3)}{P(X=3)} = \dfrac{1/6}{5/18} = 3/5$

1.19 Refer to Fig. Problem 1.19 and first compute the joint probabilities:

$P(X=0, Y=0) = (.9)(.75) = .675$

$P(X=0, Y=1) = (.1)(.75) = .075$

$P(X=1, Y=0) = (.2)(.25) = .05$

$P(X=1, Y=1) = (.8)(.25) = .20$

Part (c): Joint and marginal probabilities

X \ Y	0	1	
0	.675	.075	.75
1	.05	.20	.25
	.725	.275	

Part (b): Unconditional probabilities

$P(Y=0) = .725$

$P(Y=1) = .275$

Part (a): Conditional probabilities

$P(X=0 \mid Y=1) = \dfrac{.075}{.275} = \dfrac{3}{11}$

$P(X=0 \mid Y=0) = \dfrac{.675}{.725} = \dfrac{27}{29}$

1.20

Rayleigh density fcn $= \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$, $r > 0$

(a) Calculation of mean:

$$E(R) = \int_0^\infty r \cdot \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} dr = \frac{1}{2}\frac{\sqrt{2\pi}\sigma}{\sigma^2}\int_{-\infty}^{\infty} \frac{r^2}{\sqrt{2\pi}\sigma} e^{\frac{-r^2}{2\sigma^2}} dr$$

The integral is just the variance of $N(0,\sigma^2)$.
Therefore,

$$E(X) = \frac{\sqrt{2\pi}\,\sigma}{2\sigma^2} \cdot \sigma^2 = \sqrt{\frac{\pi}{2}}\,\sigma$$

Calculation of variance:

$$E(X^2) = \int_0^\infty \frac{r^3}{\sigma^2} e^{-r^2/2\sigma^2} dr$$

The above integral must be integrated by parts.
The result is $2\sigma^2$.

$$\therefore \operatorname{Var} X = E(X^2) - [E(X)]^2 = 2\sigma^2 - \left(\sqrt{\frac{\pi}{2}}\sigma\right)^2$$

$$= \sigma^2\left(2 - \frac{\pi}{2}\right)$$

(b) The mode is the peak value of $f_R(r)$.
Therefore, differentiate and set equal to zero.

$$\frac{df_R}{dr} = \frac{r}{\sigma^2} \cdot e^{-r^2/2\sigma^2} \cdot \left(-\frac{1}{\sigma^2} r\right) + \frac{1}{\sigma^2} \cdot e^{-r^2/2\sigma^2} = 0$$

Or $\quad -r^2 \frac{1}{\sigma^2} = -1$

Or $\quad r = \sigma$ (Mode is σ)

1.21

(a) $$P[R < R_o] = \int_0^{R_o} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = 1 - e^{-R_o^2/2\sigma^2}$$

(b) To get CEP, let the result of part (a) equal ½.

$$1 - e^{-R_o^2/2\sigma^2} = 1/2$$

Now solve for R_o

$$e^{-R_o^2/2\sigma^2} = 1/2$$

or $$R_o = \sqrt{2\sigma^2 \ln 2} = \sigma\sqrt{2 \ln 2}$$

$$\approx 1.177\,\sigma$$

(c) To get R_{95} repeat part (b) with R set at R_{95} rather than R_o

$$1 - e^{-R_{95}^2/2\sigma^2} = .95$$

or $$e^{-R_{95}^2/2\sigma^2} = .05$$

or $$e^{R_{95}^2/2\sigma^2} = 20$$

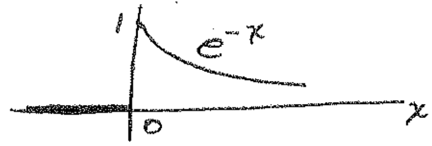
$$R_{95}^2/2\sigma^2 = \ln(20)$$

$$R_{95} = 2\sigma^2 \ln(20)$$

or $$R_{95} \approx 2.45\,\sigma$$

1.22

$$f_X = \begin{cases} e^{-x}, & x>0 \\ 0, & x<0 \end{cases}$$



(a) $P(X \geq 2) = \int_2^\infty e^{-x}\,dx = e^{-2} \approx .135$

(b) $P(1 \leq X \leq 2) = \int_1^2 e^{-x} = e^{-1} - e^{-2} \approx .232$

(c) $E(X) = \int_0^\infty x e^{-x}\,dx = 1$

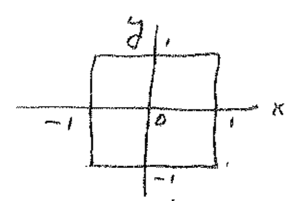
$E(X^2) = \int_0^\infty x^2 e^{-x}\,dx = 2$

Hint: These are just Laplace transform integrals, so tables can be used letting $s=1$ in the tables.

$\text{Var}\,X = E(X^2) - [E(X)]^2 = 2 - 1^2 = 1$

1.23

$$f_{XY} = \begin{cases} .25, & \text{in square} \\ 0, & \text{otherwise} \end{cases}$$



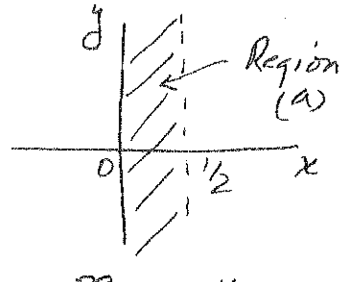
$$f_X(x) = \begin{cases} \int_{-1}^{1} .25\,dy, & \text{for } -1<x<+1, \\ 0, & \text{otherwise} \end{cases} = .25y\Big|_{-1}^{1} = 1/2$$

Similarly, $f_Y(y) = \begin{cases} 1/2, & -1<y<1 \\ 0, & \text{otherwise} \end{cases}$

$\therefore$ Test of $f_{XY} = f_X \cdot f_Y$ is satisfied and X and Y are statistically independent.

1.24

$$f_{XY} = \begin{cases} e^{-(x+y)}, & 1^{st} \text{ quadrant} \\ 0, & \text{otherwise} \end{cases}$$

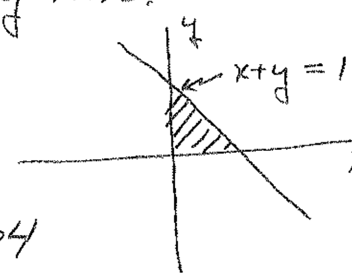


(a) Integrate over region (a)

$$P[X \le 1/2] = \int_0^{\infty}\int_0^{1/2} e^{-x}\cdot e^{-y}\,dx\,dy = \int_0^{\infty} e^{-y}\int_0^{1/2} e^{-x}\,dx\,dy$$

$$= 1 - e^{-1/2} \approx .393$$

(b) Integrate over region below x+y line.

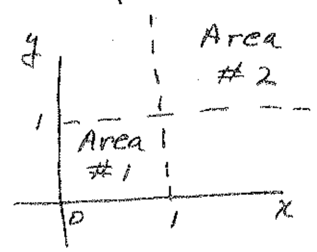


$$P[(X+Y) \le 1] = \int_0^{1}\int_0^{1-y} e^{-x}\cdot e^{-y}\,dx\,dy$$

$$= 1 - 2e^{-1} \approx .264$$

(c) $P[(X \text{ or } Y) \ge 1] = 1 - \iint_{\text{Area 1}} f_{XY}\,dx\,dy$

$$= 1 - (1-e^{-1})^2 \approx .60$$



(d) $P[(X \text{ and } Y) \ge 1] = \iint_{\text{Area 2}} f_{XY}\,dx\,dy$

$$= e^{-2} \approx .135$$

1.25

In order to be stat. ind. $f_{XY} = f_X \cdot f_Y$

Check: First compute f_X:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \begin{cases} \int_0^{\infty} e^{-(x+y)}\,dy = e^{-x}, & \text{for } x \ge 0 \\ 0, & \text{for } x < 0 \end{cases}$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \begin{cases} e^{-y}, & \text{for } y \ge 0 \\ 0, & \text{for } y < 0 \end{cases}$$

Clearly, the product $f_X \cdot f_Y = f_{XY}$ for all x and y, so X and Y are statistically independent.

1.26
$$f_X(x) = \frac{1}{2}e^{-|x|} \quad , \quad f_Y = e^{-2|y|}$$

Define Z to be: $Z = X + Y$

Then,
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(u)\cdot f_Y(z-u)\,du$$

Rather than integrate above int. directly, use Fourier transform theory.

$$\mathcal{F}[f_Z] = \mathcal{F}[f_X]\cdot\mathcal{F}[f_Y]$$

$$= \frac{1}{\omega^2+1}\cdot\frac{4}{\omega^2+4}$$

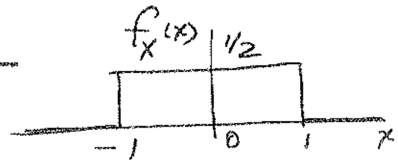
Now use partial fraction expansion.

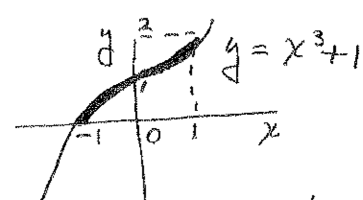
$$\mathcal{F}[f_Z] = \frac{4/3}{\omega^2+1} + \frac{-4/3}{\omega^2+4}$$

We now recognize the inverse of each term.

$$\therefore f_Z(z) = \frac{2}{3}e^{-|z|} - \frac{1}{3}e^{-2|z|}$$

1.27





The function $y = x^3 + 1$ is one-to-one, so solve for x in terms of y and use Eq. (1.14.6)

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X[h(y)]$$