Refresher Units in Mathematics

a prelude to

Modern Engineering Mathematics

Fifth edition

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Before you start

This set of Refresher Units was designed for students whose school mathematics is a bit rusty, who may be have had a year out before going to university or have not studied mathematics for a couple of years.

Progress in mathematics often depends on a sound familiarity with school-work so if you have problems with your mathematics courses these refresher exercises, with their explanations, will help you. You may also find them helpful when you revise for your end of course examinations.

A comprehensive contents section is included so that individual students can select what they need and plan their way through the set of units.

From a teaching point of view, the units could constitute a scheduled bridging course or a directed learning programme with support from a Mathematics Support Centre

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Refresher Unit 1: Arithmetic Skills

1.1 Precedence rule

- When we work out any complicated arithmetic expression, we have to observe strict rules of priority. In particular multiplication and division must be carried out before addition and subtraction. For example

  $2 \times 3 - 1$ is worked out first multiplying 2×3 and then subtracting 1 from the result to give the answer 5.

- We can over-ride the rule by using brackets to show we want the calculation performed in a different way. For example

  $2 \times (3 - 1)$ is worked out first evaluating the term in the brackets and then multiplying by 2 to give the answer 4.

Some reminders

- When we multiply the same number by itself many times we use the power notation. The term a^n , where a is a number and n is a positive whole number, is read

  '*a* to the power of *n*'

  and denotes a multiplied by itself n times. For example,

  $6^3 = 6 \times 6 \times 6 = 216$

- If two numbers are multiplied or divided then:

  (a) when only one of the numbers is negative, the answer is negative;

  (b) when both numbers are negative, the answer is positive.

  As examples:

  $(-2) \times (-3) = 6$

  $(4) \times (-2) = -8$

  $(4) \div (-2) = -2$

  $(-10) \div (-5) = 2$

  $(-2)^3 = (-2) \times (-2) \times (-2) = -8$

Summary

When evaluating an arithmetic expression:

- the powers are calculated first
- then multiplications and/or divisions
- then additions and /or subtractions

When operations of equal precedence are adjacent, for example, $5 - 3 + 2$, the left-hand operation is performed first so that

$$5 - 3 + 2 = 4$$

These rules are over-ridden by using brackets so that

$$5 - (3 + 2) = 0$$

Consequently quantities inside brackets must be calculated first, subject to the preceding rules.

Practice Questions 1.1

Calculate the following arithmetic expressions (do not use a calculator)

(a) $7 + 5 - 3$ (b) $5 + 6 \times 2$ (c) $10 - 12 \div 3$

(d) $(-5) \times (-7 + 8)$ (e) $(-8) \div 4 + 2$ (f) $1 - (-2) \div (-3 \times 2 + 5)$

(g) $(-6) \div (-3) - (-5 + 5) \times (7)$ (h) $1 - (-4) + 3$ (i) $(-5) \times (-7) + 8$

(j) $(-3)^5 - (-2)^4$ (k) $2 + 6 - 3^2$ (l) $2^2 + 4^2$

(m) $(2 + 3)^2$ (n) $(3 - 4 \times 2) + (9 \div 3 - 5)$

1.2. Fractions: some definitions

- A fraction is a number of the form $\frac{p}{q}$, where p and q are whole numbers. The number p on top of the dividing line is called the numerator and the number q on the bottom is called the denominator. An example of a fraction is $\frac{3}{5}$ in which 3 is the numerator and 5 is the denominator.

  Note: By whole numbers we mean the counting numbers 1, 2, 3, 4,.... , which are also referred to as the natural numbers.

- The value of a fraction remains unchanged if the numerator and denominator are multiplied or divided by the same number (which cannot be zero). For example

$$\frac{4}{7} = \frac{16}{28} \quad \text{and} \quad \frac{10}{12} = \frac{5}{6}$$

- If the numerator and denominator have common factors and we divide both by these factors until no further reduction is possible then the resulting fraction is said to be in its simplest (or lowest) form. For example

$$\frac{4}{16}=\frac{2}{8}=\frac{1}{4}$$

so $\frac{1}{4}$ is $\frac{4}{16}$ expressed in its lowest form.

- If p is less than q then the fraction $\frac{p}{q}$ is less than 1 and is said to be a proper fraction.

- A fraction may have a value greater than 1. In such cases p is greater than q and the fraction is said to be an improper fraction. Such a fraction may be expressed as the sum of a whole number and a proper fraction. For example,

  $\frac{13}{3}$ is an improper fraction. Dividing 13 by 3 gives 4 and a remainder of 1*,* so we have that $\frac{13}{3}=4+\frac{1}{3}$, which we write as $4\frac{1}{3}$ (read as 'four and one third' – note the use of the word 'and' to mean +)

- A fraction expressed in the form $4\frac{1}{3}$ is called a mixed number. Clearly, by reversing the above process, a mixed number can be expressed as an improper fraction.

1.3 Fractions: addition and subtraction

- To add or subtract fractions we first find a number, which the denominators of all the fractions divide into. This is called a common denominator. Each fraction is then expressed in terms of this common denominator so they become 'like fractions', which can be readily added or subtracted.

Example 1.1

Determine $\frac{3}{4}+\frac{1}{5}$

Answer

A common denominator for 4 and 5 is 20 so

$$\frac{3}{4}+\frac{1}{5}=\frac{15}{20}+\frac{4}{20}=\frac{15+4}{20}=\frac{19}{20}$$

Example 1.2

Determine $\frac{2}{3}+\frac{1}{4}-\frac{2}{5}$

Answer

A common denominator for 3, 4 and 5 is 60 giving

$$\frac{2}{3}+\frac{1}{4}-\frac{2}{5}=\frac{(2\times 20)+(1\times 15)-(2\times 12)}{60}=\frac{40+15-24}{60}=\frac{31}{60}$$

1.4 Fractions: multiplication

- To multiply fractions simply multiply the numerator terms and multiply the denominator terms. For example:

$$\frac{2}{3}\times\frac{4}{5}=\frac{2\times4}{3\times5}=\frac{8}{15}$$

- In the case of multiplication factors common to both numerator terms and denominator terms can be cancelled before doing the multiplication. This should be done as it makes the calculation easier

Example 1.3

Calculate $\frac{2}{3}\times\frac{5}{6}\times\frac{7}{15}$

Answer

Here 2 is a common factor of both 2 and 6 and 5 is a common factor of both 5 and 15. Cancelling we have:

$$\frac{1}{3}\times\frac{1}{3}\times\frac{7}{3}=\frac{7}{27}$$

1.5 Fractions: division

- The problem of dividing one fraction by another can be turned into a multiplication.

- To divide one fraction by another invert (turn 'upside-down') the divisor (the fraction after the ÷ sign) and multiply. For example,

$$\frac{2}{3}\div\frac{5}{7}=\frac{2}{3}\times\frac{7}{5}=\frac{14}{15}$$

Practice Questions 1.2

Express as fractions in their simplest form (Do not use a calculator)

1. (a) $\frac{5}{12}+\frac{3}{8}$ (b) $\frac{3}{4}-\frac{11}{12}$ (c) $1-\frac{1}{2}+\frac{1}{3}$

   (d) $\frac{2}{7}+\frac{1}{6}-\frac{1}{3}$ (e) $1\frac{3}{8}+2\frac{1}{6}$ (f) $1\frac{7}{8}+1\frac{1}{4}-2\frac{1}{2}$

   (g) $\frac{2}{3}\times\frac{5}{6}\times\frac{3}{4}$ (h) $\frac{7}{8}\div\frac{3}{4}$ (i) $\frac{2}{3}+(\frac{2}{5}\times\frac{1}{3})\div\frac{1}{3}$

(j) $3\frac{3}{4}\times 1\frac{1}{5}$ (k) $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{5}}$ (l) $-\frac{1}{2}\times(-\frac{5}{2}\div\frac{1}{4})$

(m) $1\frac{3}{4}\times 1\frac{3}{4}\div 1\frac{1}{4}$ (n) $1\frac{2}{3}\times 4$ (o) $2\frac{1}{2}\div 3\frac{7}{8}\times 1\frac{3}{4}$

2. The line segment *AC* as a fraction of the line segment *AB* in Figure 1.1

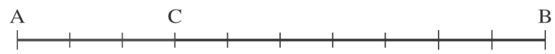


Figure 1.1 Fraction of line

3. The shaded area as a fraction of the rectangle ABCD in Figure 1.2

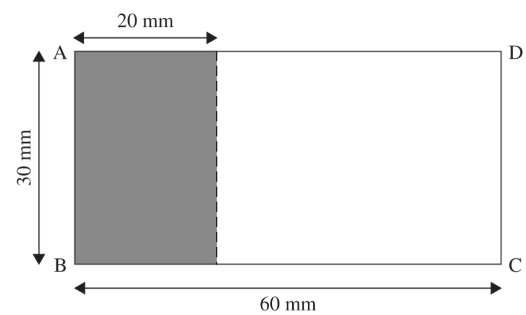


Figure 1.2 Fraction of rectangle

4. The sector OAB as *a* fraction of the area of the circle in Figure 1.3

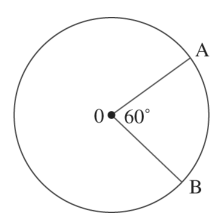


Figure 1.3 Fraction of circle

1.6 Powers and laws of indices

- We have already introduced the power notation a^n (read as a to the power *n*) to denote *a* multiplied by itself *n* times. Here *n* is called the index or exponent. Operations with powering obey simple rules, which we remind you of in this section

- Multiplication: When multiplying powers of the same quantity we add the indices giving

$$a^n a^m = a^{n+m}$$

so, for example

$$a^6 a^4 = a^{6+4} = a^{10}$$

- Division: When dividing powers of the same quantity we subtract the indices, giving

$$a^n \div a^m = \frac{a^n}{a^m} = a^{n-m}$$

so, for example

$$\frac{a^8}{a^3} = a^{8-3} = a^5$$

- Raising to a power: When raising the power of a quantity to a power we multiply the indices, giving

$$(a^n)^m = a^{nm}$$

so, for example

$$(a^2)^3 = a^6$$

- Negative powers: We define a^{-n} to be the reciprocal of a^n, giving

$$a^{-n} = \frac{1}{a^n} \text{ provided } a \text{ is not zero}$$

so, for example

$$a^{-3} = \frac{1}{a^3}$$

- Zero power: Any quantity to the power of zero is equal to 1, giving

$$a^0 = 1$$

so, for example

$$5^0 = 1$$

- Fractional powers:

(1) $a^{\frac{1}{n}}$ is defined as the nth root of *a,* giving

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

so, for example

$a^{\frac{1}{2}} = \sqrt{a}$ 'square root with a positive'

$a^{\frac{1}{3}} = \sqrt[3]{a}$ 'cube root with a unrestricted'

(**Note:** $a^{\frac{1}{2}} > 0$, whilst $a^{\frac{1}{3}} > 0$ if $a > 0$ and $a^{\frac{1}{3}} < 0$ if $a < 0$)

(2) $a^{\frac{m}{n}}$ is defined as the nth root of a^m, giving

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

so, for example

$$a^{\frac{3}{2}} = \sqrt{a^3}$$

Practice Questions 1.3

1. Simplify the following:

(a) $2^3 \times 2^{-4}$ (b) $\dfrac{5^6}{5^4}$ (c) $3^{\frac{1}{3}} \times 3^{\frac{5}{3}}$

(d) $(5^3)^2$ (e) $32^{-\frac{1}{5}}$ (f) $\dfrac{3^2 \times 3^5 \times 3^6}{3^6 \times 3^4}$

2. Evaluate the following:

(a) 2^4 (b) $25^{\frac{1}{2}}$ (c) $16^{-\frac{1}{2}}$

(d) $16^{\frac{3}{4}}$ (e) $25^{-\frac{3}{2}}$ (f) $81^{\frac{1}{4}}$

3. Find the values of:

(a) $27^{\frac{1}{3}}$ (b) $-(8)^{\frac{2}{3}}$ (c) $(-8)^{\frac{2}{3}}$ (d) $16^{-\frac{3}{2}}$

(e) $-(2)^{-2}$ (f) $(-\frac{1}{8})^{-\frac{2}{3}}$ (g) $9^{-\frac{1}{2}}$

1.7 Decimals

- In practice, especially when using a calculator or computer, decimals are used as an alternative to fractions.

- To convert a fraction into a decimal we simply carry out the division. For example,

$$\frac{2}{5} = 2 \div 5 = 0.4$$

the 'dot' in this case is referred to as a decimal point. Digits to the right of the decimal point denote the number of tenths, hundreds, thousandths, etc., respectively. Thus decimals are readily converted into fractions as illustrated by the following examples;

$$0.4 = \frac{4}{10} = \frac{2}{5}$$

$$0.45 = \frac{4}{10} + \frac{5}{100} = \frac{40+5}{100} = \frac{45}{100} = \frac{9}{20}$$

$$0.456 = \frac{4}{10} + \frac{5}{100} + \frac{6}{1000} = \frac{400+50+6}{1000} = \frac{456}{1000} = \frac{57}{125}$$

- When doing calculations by hand care must be taken with placement of the decimal point. It is easy to make copying errors so always take care to write the leading zero before the decimal point (0.1 rather than .1) as this practice reduces the risk!

- When adding or subtracting decimals we line up the decimal point and add or subtract as usual.

Example 1.4

Evaluate 3.125 + 0.32 + 0.056

Answer

Line up the numbers beneath each other with the decimal points vertically underneath each other.

$$\begin{array}{r} 3.125 \\ 0.32 \\ +\,0.056 \\ \hline 3.501 \end{array}$$

Adding as usual gives the answer 3.501

- When multiplying two decimals we first multiply ignoring the decimal points. We count the total number of digits after the decimal point in both. The decimal point in the answer is then placed so that there is the same number of digits after the decimal point.

Example 1.5

Evaluate 6.32×0.6

Answer

First calculate 632×6

$$\begin{array}{r} 632 \\ 6\times \\ \hline 3792 \end{array}$$

The total number of digits after the decimal point in the two numbers is 2+1=3, so there will be three digits after the decimal point in the answer. This gives the answer as 3.792

- When dividing two decimals by hand the first step is to ensure that the divisor (the number we are dividing by) is a whole number. To do this we must multiply both numbers by an appropriate power of 10. We then divide in the usual way.

Example 1.6

Evaluate $19.11 \div 1.5$

Answer

To convert the divisor 1.5 to a whole number we must first multiply by 10. We must do the same to the number being divided, so

$$19.11 \div 1.5 \text{ is the same as } 191.1 \div 15$$

Dividing in the usual way:

$$15\overline{\smash{)}191.10}\quad = 12.74$$

gives the answer as 12.74

Practice Questions 1.4

(Do not use a calculator except to check your answers)

1. Express $\frac{5}{8}$ as a decimal.
2. Express 0.15 as a proper fraction in its simplest form.
3. Evaluate the following:

(a) $0.375 + 0.625$ (b) $10.24 + 2.341 + 0.027$ (c) $0.156 - 0.045$

(d) $18.231 - 7.25$ (e) $6.71 - 0.0325$ (f) $4.201 + 1.82 - 3.516$

(g) $(0.2)^2$ (h) $(0.02)^2$ (i) 3.65×3.502

(j) 2.015×1.45 (k) $19.24 \div 2.6$ (l) $21.03 \div 0.03$

(m) $\frac{90.03}{0.03} + \frac{14.04}{0.002}$ (n) $\sqrt{0.0025}$ (o) $\sqrt{0.16}$ (p) $\sqrt{1.21}$

1.8 Decimals: rounding off

Some numbers, such as $\frac{10}{3}$, have decimal representations which do not end. In practical problems these numbers are 'rounded' to a sensible number of decimal places, using the closest decimal number. For example

$$\frac{150}{101} = 1.485\ 148\ 514\ 851\ 485\ \ldots$$

= 1 to 0 decimal places (0dp)
= 1.5 to 1 decimal place (1dp)
= 1.49 to 2 decimal places (2dp)
= 1.485 to 3 decimal places (3dp)
= 1.4851 to 4 decimal places (4dp)

Practice Questions 1.5

(Do not use a calculator)

1. Round off the number 0.05651 to 3 decimal places.
2. Round off the number 0.05649 to 3 decimal places.
3. Round off the number –0.0035 to 2 decimal places.
4. Express $\frac{2}{23}$ as a decimal number correct to 3 decimal places.
5. Evaluate 2.8 ÷5.2 – 5.21 correct to 1 decimal place.
6. Express $\pi = 3.1415927 \ldots$ correct to 3 decimal places.

1.9 Decimals: significant figures

- Rounded values and experimental data are approximations to the true value. The figures (digits) given are significant (or meaningful) within the context in which they are used. The number of such significant figures is a measure of its relative accuracy. Thus 3.142 which has 4 significant figures (sf) is ten times more accurate as an approximation to π than 3.14 which has 3 significant figures (3sf). Notice that the leading zeros after the decimal point of decimal fractions do not give information about relative accuracy, so that the correctly rounded numbers

  2.031 and 0.00005291

  are both correct to 4 significant figures.

Practice Questions 1.6

How many decimal places (dp) and how many significant figures (sf) do the following correctly rounded numbers have?

(a) 13.0567 (b) 0.345 (c) –0.0034 (d) 251

1.10 Decimals: scientific notation

- Scientific notation represent numbers in the form $a \times 10^n$, where a is a number between –1 and 1 and n is an integer. For example:

$$931.5671 = 9.315671 \times 10^2$$

and

$$-0.00213 = -2.13 \times 10^{-3}$$

Practice Questions 1.7

(Do not use a calculator)

Express in scientific notation:

(a) 0.345 (b) –0.0035 (c) 251 (d) 532.21

(e) $1.5 \times 10^5 + 2 \times 10^2$ (f) $(1.5 \times 10^5) \times (2 \times 10^2)$ (g) $(1.5 \times 10^5) \div (2 \times 10^2)$

1.11 Percentages

- The term 'per cent' (which is denoted by the symbol %) means 'per one hundred'. For example, 20% means '20 per hundred' and we write

$$20\% = \frac{20}{100}$$

so that 20% is simply another representation of the fraction

$$\frac{20}{100} = \frac{1}{5}$$

- In general any fraction $\frac{p}{q}$ may be represented as a percentage r by expressing it in the equivalent form $\frac{r}{100}$. Thus, for example

$$\frac{1}{4} = \frac{25}{100}$$

giving 25% as the percentage representation of $\frac{1}{4}$.

- Since $\frac{p}{q} = \frac{r}{100}$ we have that

$$r = \frac{p}{q} \times 100$$

so to find the percentage representation of a fraction we multiply by 100. For example the percentage representation of the fraction $\frac{7}{20}$ is

$$\frac{7}{20} \times 100 = 35\%$$

- Conversely to express a percentage as a fraction we divide by 100, so for example

$$35\% = \frac{35}{100} = \frac{7}{100}$$

- In the same way we can express percentages as decimals and decimals as percentages. To express a percentage as a decimal we simply divide by 100 and to express a decimal as a percentage we multiply by 100. The following examples illustrate

0.6 expressed as a percentage is $0.6 \times 100 = 60\%$

6% expressed as a decimal is $\frac{6}{100} = 0.06$

- To obtain a percentage r of a given quantity Q we first express r as a fraction (or decimal) and then multiply by the quantity Q. Thus

$$\boxed{r\% \text{ of } Q = \frac{r}{100} \times Q}$$

Example 1.7

What is 25% of £50 ?

Answer

$$25\% \text{ of } £50 = £\ \frac{25}{100} \times 50 = £12.50$$

Example 1.8

23% of a consignment of bananas is bad. There is 34.5kg of bad bananas. How many kilograms of bananas are there in the consignment?

Answer

23% of bananas weighed 34.5kg, so

100% (total consignment) weighed $\frac{34.5}{23} \times 100 = 150kg$.

Practice Questions 1.8

1. Express the following fractions as percentages
   (a) $\frac{3}{20}$ (b) $\frac{4}{5}$ (c) $\frac{9}{25}$

2. Express the following decimals as percentages
   (a) 0.52 (b) 0.03 (c) 0.455

3. Express the following percentages as decimals and as proper fractions
   (a) 17.5% (b) 40% (c) 3.25%

4. Express 16 as a percentage of 320.

5. There are 22 defective bulbs in a batch of 550. What percentage of the bulbs is defective?

6. What percentage of the rectangle ABCD, of Figure 1.4, is shaded

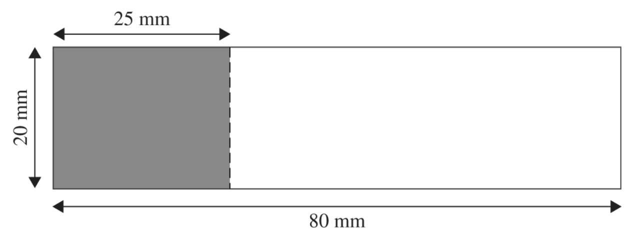


Figure 1.4 Percentage of rectangle

7. Calculate the profit as a percentage of the cost price of a refrigerator bought for £275 and sold for £330.

8. Calculate the selling price of a refrigerator, which is sold with a 20% discount, if its list price is £330.

9. After winning a cash prize Sally gave 25% of the prize to her brother Dick. If the amount of money she gave to Dick was £120 what was the value of her cash prize?

10. Paul scored 27 marks out of a possible 45 marks in a science test. What percentage was this?

1.12 Ratios and proportional parts

- A ratio is a representation of the relationship of one quantity p to a second quantity q by means of the fraction $\frac{p}{q}$. For example, in Figure 1.5

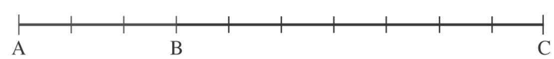


Figure 1.5 Ratio of line segments

the ratio of the line segment AB to the line segment BC is $\frac{3}{7}$. This is often written using the : symbol as

$$\text{AB:BC} = 3:7$$

The line segment *AB* expressed as a fraction of the full line AC is

$$\frac{\text{AB}}{\text{AC}} = \frac{\text{AB}}{\text{AB}+\text{BC}} = \frac{3}{3+7} = \frac{3}{10}$$

Example 1.9

Regulations for taking a party of school children on a trip abroad requires that the ratio of accompanied teachers to children travelling must be 1:6. If 54 children sign up for the trip how any teachers must travel with the party?

Answer

$$\frac{\text{number of teachers}}{\text{number of children}} = \frac{1}{6}$$

so

$$\text{number of teachers} = \frac{1}{6} \times 54 = 9$$

Example 1.10

The ratio of males to females in a class is 7:5. If there are 60 children in the class how many are females?

Answer

Fraction of children females = $\dfrac{5}{7+5} = \dfrac{5}{12}$, so

$$\text{number of females} = \frac{5}{12} \times 60 = 25$$

- If a fixed amount Q is divided between two parties A and B in the ratio $a{:}b$ then

  A will receive $\dfrac{a}{a+b} \times Q$, and

  B will receive $\dfrac{b}{a+b} \times Q$

This can be extended to sharing a fixed amount in proportional parts between more than two parties. For example, if Q is divided between three parties A,B and C in the ratio $a{:}b{:}c$ then

A will receive $\dfrac{a}{a+b+c} \times Q$

B will receive $\dfrac{b}{a+b+c} \times Q$, and

C will receive $\dfrac{c}{a+b+c} \times Q$

Example 1.11

A prize of £7500 is to be divided between John, Jane and Mary in the ratios 3:4:8. How much will each receive?

Answer

John will receive $\dfrac{3}{3+4+8} \times 7500 = \dfrac{3}{15} \times 7500 = £1500$

Jane will receive $\dfrac{4}{3+4+8} \times 7500 = \dfrac{4}{15} \times 7500 = £2000$

Mary will receive $\dfrac{8}{3+4+8} \times 7500 = \dfrac{8}{15} \times 7500 = £4000$

Practice Questions 1.9

1. In Figure 1.6 what is the ratio AB:AC ?

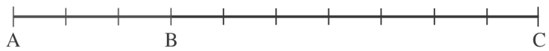


Figure 1.6 Ratio of line to segment

2. Mark on Figure 1.6 the point D which is such that

   AD:DC = 2:5

3. A mortar mixture contains cement and sand in the ratio 2:3. The total weight is 10kg. What weights of cement and sand does it contain?

4. The ratio of the sides of two squares is 2:3. What is the ratio of their areas?

5. In a small town the ratio of the number of males to the number of females is 80:81. If there are 9680 males in the town how many females are there in the town?

6. An Ordinance Survey road atlas of Great Britain adopts the scale 1cm to 2.5 km. Express this as a ratio.

7. Two neighbouring villages *A* and *B* have populations of 336 and 240 respectively. The two villages are to share a grant of £10,728 in proportion to their populations. How much will each village receive?

8. An aircraft carries 2880 litres of fuel contained in three tanks *A, B* and *C* in the ratios 3:5:4. What is the quantity of fuel in each tank?

Answers to Practice Questions Unit 1

Practice Questions 1.1

(a) 9 (b) 17 (c) 6 (d) –5 (e) 0

(f) –1 (g) 2 (h) 8 (i) 43 (j) –259

(k) –1 (l) 20 (m) 25 (n) –7

Practice Questions 1.2

1. (a) $\frac{19}{24}$ (b) $-\frac{1}{6}$ (c) $\frac{5}{6}$ (d) $\frac{5}{42}$

   (e) $3\frac{13}{24}$ (f) $\frac{5}{8}$ (g) $\frac{5}{12}$ (h) $\frac{7}{6}=1\frac{1}{6}$

   (i) $\frac{16}{15}=1\frac{1}{15}$ (j) $\frac{9}{2}=4\frac{1}{2}$ (k) $\frac{25}{6}=4\frac{1}{6}$ (l) 5

   (m) $\frac{49}{20}=2\frac{9}{20}$ (n) $\frac{20}{3}=6\frac{2}{3}$ (o) $\frac{35}{31}=1\frac{4}{31}$

2. $\frac{3}{10}$ 3. $\frac{1}{3}$ 4. $\frac{1}{6}$

Practice Questions 1.3

1. (a) $\frac{1}{2}$ (b) 25 (c) 9 (d) 5^6 (e) $\frac{1}{2}$ (f) 27

2. (a) 16 (b) 5 (c) $\frac{1}{4}$ (d) 8 (e) $\frac{1}{125}$ (f) 3

3. (a) 3 (b) –4 (c) 4 (d) $\frac{1}{64}$ (e) $-\frac{1}{4}$ (f) $\frac{1}{3}$

Practice Questions 1.4

1. 0.625 2. $\frac{3}{20}$

3. (a) 1 (b) 12.608 (c) 0.111 (d) 10.981

   (e) 6.6775 (f) 2.505 (g) 0.04 (h) 0.0004

   (i) 12.7823 (j) 2.92175 (k) 7.4 (l) 701

   (m) 10021 (n) 0.05 (o) 0.4 (p) 1.1

Practice Questions 1.5

1. 0.057 2. 0.056 3. –0.00

4. 0.087 5. –4.7 6. 3.142

Practice Questions 1.6

(a) 4dp, 6sf (b) 3dp, 3sf (c) 4dp, 2sf (d) 0dp, 3sf

Practice Questions 1.7

(a) 3.45×10^{-1} (b) -3.5×10^{-3} (c) 2.51×10^{2} (d) 5.3221×10^{2}

(e) 1.502×10^{5} (f) 3.0×10^{7} (g) 7.5×10^{2}

Practice Questions 1.8

1. (a) 15% (b) 80% (c) 36%

2. (a) 52% (b) 3% (c) 45.5%

3. (a) $\frac{9}{40}$, 0.175 (b) $\frac{2}{5}$, 0.4 (c) $\frac{13}{400}$, 0.0325

4. 5%

5. 4%

6. 31.25%

7. 20%

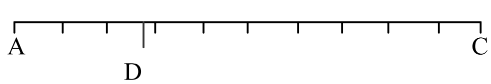
8. £264

9. £480

10. 60%

Practice Questions 1.9

1. 3 : 10

2.


(AD = 2.86 AC)

3. 4kg, 6kg

4. 4 : 9

5. 9801

6. 1 : 250,000

7. £6258, £4470

8. *A* 720 litres, *B* 1200 litres, *C* 960 litres

End of Refresher Unit 1

Refresher Unit 2: Algebraic Skills

2.1 Basic operations

- In algebra letters of the alphabet are used to represent variables. All the operations, such as $+$, $-$, $\times$ and $\div$, used in arithmetic can be applied in algebra.

- The $\times$ sign for multiplication is not normally used in algebra to avoid confusion with the letter x, which is frequently used as a variable. Consequently

$$a \times b \text{ is written as } ab$$

2.2 Addition and subtraction

- As in arithmetic order of addition is not important so that

$$a + b = b + a \qquad \text{‘commutative law of addition’}$$

but order does matter in subtraction and

$$a - b \neq b - a$$

- Addition and subtraction of algebraic quantities can only be performed on like terms. The processes of addition and subtraction are normally used to simplify algebraic expressions.

Example 2.1

Simplify the following algebraic expressions

(a) $5x + 2x - x$
(b) $(2a + 3b) - (a - b)$
(c) $2m - [3m + 2n - (4m + n)]$

Answer

(a) As in arithmetic when operations of equal precedence are adjacent the left hand operation is performed first, so

$$5x + 2x - x = 7x - x \qquad \text{‘adding } 5x \text{ and } 2x \text{ first’}$$
$$= 6x \qquad \text{‘subtracting } x \text{ from } 7x\text{’}$$

(b) Removing the brackets first

$$(2a + 3b) - (a - b) = 2a + 3b - a + b$$ ‘note that when – precedes a bracket then all signs are changed when bracket removed’
$$= (2a - a) + (3b + b) \qquad \text{‘ collecting together like terms’}$$
$$= a + 4b$$

(c) When we have brackets inside brackets then inner bracket is removed first

$$2m - [3m + 2n - (4m + n)] = 2m - [3m + 2n - 4m - n] \quad \text{‘removing inner bracket’}$$

$= 2m - 3m -2n +4m +n$ ‘removing remaining bracket’
$= (2m -3m +4m) + (-2n +n)$ ‘collecting like terms’
$= (3m) + (-n)$
$= 3m - n$

Practice Questions 2.1

Simplify the following algebraic expressions.

(a) $6a - 7a + 2a$ (b) $10x -3x +7x -13x$ (c) $(7a +3b) - (3a - 5b)$

(d) $(m - n) + (2m - 2n) - (5m - 7n)$ (e) $5a + b - (2a + 3b)$

(f) $3x +2y - [15x + 3y -(9x - 2y)]$ (g) $7p - 5q + [2p -3q - (14p -6q)]$

(h) $5x - 6y - (-2x -3y)$ (i) $(a +2b +3c) - (a -2b -3c)$ (j) $(x +2y +3z) - (x -2y +7z)$

2.3 Multiplication

- Remember that it is normal to omit the $\times$ sign when multiplying and write

  $a \times b$ as ab

- Order of multiplication is not important so that

  $ab = ba$ ‘commutative law of multiplication’

- As in arithmetic

  $a \times a$ is written as a^2 ‘read as a squared’
  $a \times a \times a$ is written as a^3 ‘read as a cubed’
  $a \times a \times a \times a$ is written as a^4 ‘read as a to power of four’
  and so on

- $a(b + c)$ means multiply everything inside the bracket by what is outside so that

  $a(b + c) = ab +ac$ ‘distributive law of multiplication over addition’

  Likewise $a(b - c) = ab - ac$

Example 2.2

Simplify the following expressions

(a) $2(2a + 3) + 3(3a +4)$

(b) $x(3 +2x) + 2(3 +5x^2)$

Answer

(a) Multiplying out the brackets

$$2(2a + 3) + 3(3a + 4) = 4a + 6 + 9a + 12$$
$$= (4a + 9a) + (6 + 12) \qquad \text{‘collecting like terms’}$$
$$= 13a + 18$$

(b) Multiplying out the brackets

$$x(3 + 2x) + 2(3 + 5x^2) = 3x + 2x^2 + 6 + 10x^2$$

$$= (2x^2 + 10x^2) + 3x + 6 \qquad \text{‘collecting like terms’}$$
$$= 12x^2 + 3x + 6$$

Practice Questions 2.2

Simplify the following expressions

(a) $4(c - d) - 3(c - 2d)$ (b) $3x(y + 2) + 2y(x - 1)$

(c) $m(m + 2n) - n(3n + 2m - 4)$ (d) $3x(x - y) - 2y(2x - 3y)$

(e) $7(2a - 4b - 5) - 3(2a - 8b + 5)$ (f) $p^2 + 7p - [5p^2 - (3p - 4p^2)]$

2.4 Multiplying brackets $(a + b)(c + d)$

- This can be written as

$$(a + b)(c + d) = a(c + d) + b(c + d) \qquad \text{‘extension of distributive law’}$$

$$= ac + ad + bc + bd$$

Example 2.3

Multiply out the following

(a) $(2x + 1)(y - 3)$

(b) $(m + n)(n - m) - (n - 2m)(m - 3n)$

Answer

(a) $(2x + 1)(y - 3) = 2x(y - 3) + 1(y - 3)$

$$= 2xy - 6x + y - 3$$

(b) $(m + n)(n - m) - (n - 2m)(m - 3n)$

$$= m(n - m) + n(n - m) - n(m - 3n) + 2m(m - 3n)$$

$= mn - m^2 + n^2 - nm - nm + 3n^2 + 2m^2 - 6mn$

$= (-m^2 + 2m^2) + (n^2 + 3n^2) + (mn - nm - nm - 6mn)$ ‘collecting like terms’

$= m^2 + 4n^2 - 7mn$ ‘since $mn = nm$’

Practice Questions 2.3

Multiply out the following:

(a) $(2x + 1)(y - 3)$ (b) $(-3a + 4)(2a - 7)$

(c) $(m - 3)(2m + 1) - (2 - m)(m + 5)$ (d) $(a + 2b - c)(a - 2b)$

(e) $(x + y)(y + z) - (x + y)(x + z) + (z + y)(y + x)$

(f) $(p - 2)(3p^2 + 2p - 1)$

2.5 Some important results

Write out these proofs by hand and make sure you understand each step

Result 1: $(a + b)^2 = a^2 + 2ab + b^2$ ‘perfect square’

Proof $(a + b)^2 = (a + b)(a + b)$

$= a(a + b) + b(a + b)$

$= a^2 + ab + ba + b^2$

$= a^2 + 2ab + b^2$

Result 2: $(a - b)^2 = a^2 - 2ab + b^2$ ‘perfect square’

Proof $(a - b)^2 = (a - b)(a - b)$

$= a(a - b) - b(a - b)$

$= a^2 - 2ab + b^2$

Result 3: $(a - b)(a + b) = a^2 - b^2$ ‘difference of squares’

Proof $(a - b)(a + b) = a(a + b) - b(a + b)$

$= a^2 + ab - ba - b^2$

$= a^2 - b^2$ ‘since $ab = ba$’

These results are used in many different ways, sometimes to expand a bracket, sometimes to express as a single term.

Example 2.4

Simplify the expression $(x + 3)^2 - 2x - 5$

Answer

Using Result 1

$$(x + 3)^2 - 2x - 5 = x^2 + 6x + 9 - 2x - 5$$

$$= x^2 + 4x + 4 \qquad \text{‘collecting like terms’}$$

$$= (x + 2)^2$$

Example 2.5

Express as a single term $9x^2 - 30x + 25$

Answer

$$9x^2 - 30x + 25 = (3x)^2 - 2(5)(3x) + 5^2$$

$$= (3x - 5)^2 \qquad \text{‘using Result 1’}$$

- Sometimes the results are useful in arithmetic

Example 2.6

Without using a calculator calculate 119^2

Answer

$$119^2 = (120 - 1)^2 = 120^2 - 240 + 1 \qquad \text{‘using Result 2’}$$

$$= 14400 - 240 + 1$$
$$= 14161$$

Example 2.7

Without using a calculator calculate $13^2 - 11^2$

Answer

Using Result 3 for difference of two squares

$$13^2 - 11^2 = (13 - 11)(13 + 11)$$
$$= 2 \times 24$$
$$= 48$$

Practice Questions 2.4

1. Expand $(x + 2y)^2$

2. Expand $(2a - 1)^2$

3. Expand and simplify $(2m + 3n)^2 - (5m - 7n)^2$

4. Evaluate $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$ without using a calculator.

5. Simplify $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$

6. Express as a single term $4x^2 + 20x + 25$

7. Express as a single term $x^2 - 6x + 9$

8. Use Result 1 to calculate 121^2

9. Use the difference of two squares to calculate $13^2 - 7^2$

10. Show that $ab = \frac{1}{4}\{(a+b)^2 - (a-b)^2\}$

11. Use the formula of Question 10 to evaluate 75×25 (without using a calculator)

12. Expand $(x + 2y + 1)^2$

2.6 Factorisation

- When factoring, the objective is to express an algebraic expression as the product of two expressions (called factors). This is similar to factoring in arithmetic where, for example, two factors of 15 are 5 and 3 since $15 = 5 \times 3$.

- If there is a factor common to each term of an expression then take this outside a bracket to factorise

Example 2.8

Factorise the following expressions

(a) $5x + 10y$ (b) $6a^2 - 12ab + 3a$ (c) $6y^2 - 2y$

Answer

(a) 5 is common to both terms so taking it outside a bracket gives

$$5x + 10y = 5(x + 2y)$$

so 5 and $(x + 2y)$ are factors of $5x + 10y$

(b) $3a$ is common to all terms so taking it outside a bracket gives

$$6a^2 - 12ab + 3a = 3a\,(2a - 4b + 1)$$

so $3a$ and $(2a - 4b + 1)$ are factors of $6a^2 - 12ab + 3a$

(c) $2y$ is common to both terms so taking it outside a bracket gives

$$6y^2 - 2y = 2y\,(3y - 1)$$

so $2y$ and $(3y - 1)$ are factors of $6y^2 - 2y$

- Sometimes factors may be found by grouping together terms having a common factor

Example 2.9

Factorise the following expressions

(a) $ax - ay + bx - by$ (b) $a^2 + bc + ab + ac$

Answer

(a) $ax - ay + bx - by = (ax - ay) + (bx - by)$

$$= a(x - y) + b(x - y)$$

$$= (x - y)(a + b)$$

so $(x - y)$ and $(a+b)$ are factors of $ax - ay + bx - by$

(b) $a^2 + bc + ab + ac = (a^2 + ab) + (bc + ac)$

$$= a\,(a + b) + c\,(b + a)$$

$$= (a + b)(a + c)$$

so that $(a + b)$ and $(a + c)$ are factors of $a^2 + bc + ab + ac$

- A quadratic expression in x *is* an expression of the form

$$ax^2 + bx + c$$

where a b and c are normally numbers. Sometimes it is easy to spot how a quadratic expression can be factorised. In general this uses the formula

$$(px + q)(rx + t) = pr\,x^2 + (qr + pt)\,x + qt$$

$$= ax^2 + bx + c$$

so to factorise we inspect the factors of a and c and investigate whether a specific combination of those factors will yield b.