MP CHAPTER 2 SOLUTIONS

Section 2.1

1a. -A= $\begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix}$

1b. 3A= $\begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{bmatrix}$

1c. A+2B is undefined.

1d. A^T= $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

1e. B^T= $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \end{bmatrix}$

1f. AB= $\begin{bmatrix} 4 & 6 \\ 10 & 15 \\ 16 & 24 \end{bmatrix}$

1g. BA is undefined.

2. $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} .50 & 0 & .10 \\ .30 & .70 & .30 \\ .20 & .30 & .60 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3. Let A = (a_{ij}), B = (b_{ij}), and C = (c_{ij}). We must show that A(BC) = (AB)C. The i-j'th element of A(BC) is given by

$$\sum_x a_{ix}\Big(\sum_y b_{xy}c_{yj}\Big) = \sum_x \sum_y a_{ix}b_{xy}c_{yj}$$

The i-j'th element of (AB)C is given by

$$\sum_y \Big(\sum_x a_{ix}b_{xy}\Big)c_{yj} = \sum_y \sum_x a_{ix}b_{xy}c_{yj} = \sum_x \sum_y a_{ix}b_{xy}c_{yj}$$

Thus A(BC) = (AB)C

4. The i-j'th element of $(AB)^T$ = element j-i of AB = Scalar product of row j of A with column i of B.
Now element i-j of B^TA^T = Scalar product of row i of B^T with column j of A^T = Scalar product of row j of A with column i of B. Thus
$(AB)^T = B^TA^T$.

5a. From problem 4 we know that $(AA^T)^T = AA^T$, which proves the desired result.

5b. $(A + A^T)_{ij} = a_{ij} + a_{ji}$. Also $(A + A^T)_{ji} = a_{ji} + a_{ij}$ which proves the desired result.

6. To compute each of the n^2 elements of AB requires n multiplications. Therefore a total of n^2 multiplications are needed to compute AB. To compute each of the n^2 elements of AB requires n – 1 additions. Therefore a total of $n^2(n - 1) = n^3 - n^2$ additions are needed to compute AB.

7a. Trace A+ B $= \sum_i (a_{ii} + b_{ii}) = \sum_i a_{ii} + \sum_i b_{ii} = Trace\ A + Trace\ B.$

7b. Trace(AB) = $\sum_i \sum_k a_{ik} b_{ki}$ and Trace(BA) = $\sum_k \sum_i b_{ik} a_{ki}$.
Note that the terms used to compute the i’th entry of BA duplicate the i’th term used to compute each entry of AB. Therefore exactly the same terms are used to compute the sum of all entries in AB and BA.

Section 2.2 Solutions

1. $\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 6 \\ 1 & 3 & 8 \end{bmatrix}$

Section 2.3 Solutions

1. $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 4 \\ 1 & 2 & 1 & 1 & 8 \end{array}\right]$ $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 0 & 5 \end{array}\right]$

$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

The last row of the last matrix indicates that the original system has no solution.

2. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 0 & 6 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & 2 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right]$

This system has an infinite number of solutions of the form
$x_3 = k$, $x_1 = 2 - 2k$, $x_2 = 2 + k$.

3. $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 3 & 2 & 4 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

This system has the unique solution $x_1 = 2$ $x_2 = -1$.

4. $\left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 6 \\ 1 & 1 & 1 & 0 & 4 \end{array}\right]$ $\left[\begin{array}{cccc|c} 1 & -1/2 & 1/2 & 1/2 & 3 \\ 1 & 1 & 1 & 0 & 4 \end{array}\right]$

$\left[\begin{array}{cccc|c} 1 & -1/2 & 1/2 & 1/2 & 3 \\ 0 & 3/2 & 1/2 & -1/2 & 1 \end{array}\right]$

$$\left[\begin{array}{cccc|c} 1 & -1/2 & 1/2 & 1/2 & 3 \\ 0 & 1 & 1/3 & -1/3 & 2/3 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 2/3 & 1/3 & 10/3 \\ 0 & 1 & 1/3 & -1/3 & 2/3 \end{array}\right]$$

This system has an infinite number of solutions of the form
$x_3 = c$, $x_4 = k$, $x_1 = 10/3 - 2c/3 - k/3$, $x_2 = 2/3 - c/3 + k/3$.

5. $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0.5 & 1 \\ 0 & 0 & 2 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 0.5 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row of the final matrix indicates that the original system has no solution.

6. $$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus original system has a unique solution $x_1 = x_2 = x_3 = 1$.

7. $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 1 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus the original system has the unique solution $x_1 = 1$,
$x_2 = 1$, $x_3 = 2$.

8. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$ $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

We now must pass over the third column, because it has no nonzero element below row 2. We now work on column 4 and obtain

$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$ $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

We find that the original system has an infinite number of solutions of the form $x_1 = 2 + k$, $x_2 = -1 - 2k$, $x_3 = k$, $x_4 = 3$.

9. In this situation there must be at least one non-basic variable. Thus N is nonempty and Case 2 cannot apply. Thus the original system cannot have a unique solution.

Section 2.4 Solutions

1. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Row of 0's indicates that V is linearly dependent.

2. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 3/2 & 0 \\ 3 & 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 3/2 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rank of the last matrix is 3, so V is linearly independent.

3. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1/2 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1/2 \\ 0 & 3/2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since rank of last matrix =2, V is linearly independent.

4. $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Since rank of last matrix = 1, V is linearly dependent. This also follows from

$$-3/2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5. Since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ V is linearly dependent.

This also follows from the fact that the matrix $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$ has rank 2.

6. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since the last matrix has rank 3, V is linearly independent.

7. If A**x**=**b** has a solution $\mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ then

$$x_1 \begin{bmatrix} a_{11} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} + \ldots\ x_n \begin{bmatrix} a_{1n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

Thus if A**x**=**b** has a solution, then **b** is a linear combination of the columns of A with x_i being the weight of column i in the linear combination.

Similarly, if **b** is a linear combination of the columns of A with weights $x_1, x_2, \ldots x_n$, respectively then the vector

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$ is a solution to the linear system A**x**=**b**

8. Let V={$\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_m}$} be any collection of m (m_>3) two-dimensional row vectors. Now let

$$A=\begin{bmatrix} Row\ 1\ is\ v_1 \\ Row\ 2\ is\ v_2 \\ . \\ . \\ . \\ Row\ m\ is\ v_m \end{bmatrix}$$

Since the Gauss-Jordan Method is complete after the second column is transformed, rank A≤2. Thus rank A cannot equal m (remember m≥3). This means that V cannot be a linearly independent set of vectors.

9. If V is linearly dependent, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$, where at least one c_i (say c_1) is nonzero. Dividing through by c_1, we now find that $\mathbf{v}_1$ may be written as a linear combination of the other vectors in V.

Suppose $\mathbf{v}_1$ may be written as a linear combination of the other vectors in V. Then we know that $\mathbf{v}_1 = c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$.
Rearranging this equation we obtain $\mathbf{v}_1 - c_2\mathbf{v}_2 - \ldots - c_n\mathbf{v}_n = \mathbf{0}$.
This shows that V is a linearly dependent set of vectors.

Section 2.5 Solutions

1. $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$
$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$

Thus $A^{-1}=\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

2. $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$

Thus $A^{-1}=$ $\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1 & -2 & 3 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$

3. $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$

Since we can never transform what is to the left of | into I_3,
A^{-1} does not exist.

4. $$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

Again, A^{-1} does not exist.

5. $$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 3 \\ 2 & -1 \end{bmatrix}$$

Thus $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

6. $$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1 & -2 & 3 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

Thus $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1 & 2 & 3 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

7. Suppose A is an mxm matrix .If rows of A are

linearly independent, then rank A = m and we can perform ero's on A that yield the identity matrix. This means that the Gauss-Jordan method for finding the inverse will not yield a row of 0's. Thus if rows of A are linearly independent, then A will have an inverse.

If the rows of A are not linearly independent, then rank A < m, and ero's on A cannot yield the identity matrix. Hence if rows of A are not linearly independent, then A will have no inverse.

8a. Since $100B(B^{-1}/100) = I$, we have that

$(100B)^{-1} = B^{-1}/100$

8b. Multiply each entry in column 1 of B^{-1} by 1/2.

8c. Multiply each entry in row 1 of B^{-1} by 1/2.

9. $(B^{-1}A^{-1})AB = I$ and $(AB)(B^{-1}A^{-1}) = I$, so $(AB)^{-1} = B^{-1}A^{-1}$

10. We know that

(1) $AA^{-1} = I$.

By Problem 4 of Section 2.1 taking the transpose of both sides of (1) yields

$$(A^{-1})^T A^T = I^T = I.$$

Thus the inverse of A^T is $(A^{-1})^T$

11. For $i \neq j$, the scalar product of row i of A with row j of A must equal 0. This implies, by the way, that the vector associated with row i is orthogonal, or perpendicular, to the vector associated with row j.

Also, for each row i, the scalar product of row i with itself must equal 1. This means that the "length" of each row of A (when viewed as a vector) must equal 1

Section 2.6 Solutions

1. Expansion by row 2 yields

$(-1)^{2+1}(4)(-6) + (-1)^{2+2}(5)(-12) + (-1)^{2+3}6(-6) = 0$

Expansion by row 3 yields

$(-1)^{3+1}(7)(-3) + (-1)^{3+2}(8)(-6) + (-1)^{3+3}(9)(-3) = 0$

2. By row 2 cofactors we obtain

$$2\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

But $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \det \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = 15$

Thus the determinant of the original matrix is 30.

3. Any upper triangular matrix A may be written as

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Expanding by row 3 cofactors we find

$\det A = f \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = adf$ = product of diagonal entries of A

4a. and 4b. For any 1x1 matrix $\det -A = -a_{11} = -\det A$. For any 2x2 matrix $\det(-A) = (-a_{11})(-a_{22}) - (-a_{21})(-a_{12}) = \det A$. For any 3x3 matrix

$$\det(-A) = -a_{11} \det \begin{bmatrix} -a_{22} & -a_{23} \\ -a_{32} & -a_{33} \end{bmatrix} + a_{12} \det$$

$$\begin{bmatrix} -a_{21} & -a_{23} \\ -a_{31} & -a_{33} \end{bmatrix}$$

$$-a_{13} \det \begin{bmatrix} -a_{21} & -a_{22} \\ -a_{31} & -a_{32} \end{bmatrix}$$

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Since det A = det (-A) holds for 2x2 matrices we find that for any 3x3 matrix det (-A) = -det A. A similar argument works for 4x4 matrices to show that det (-A) = det A.

4c. For any nxn matrix where n is even, det (-A) = det A, while if n is odd det (-A) = -det A.

Solutions to Review Problems

1. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 1 & 5 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ We now find that $x_3 = k$, $x_1 = -1 + k$, $x_2 = 3 - k$

2. $\left[\begin{array}{cc|cc} 0 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{cc|cc} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & 1/2 & 0 & 1/2 \\ 0 & 3 & 1 & 0 \end{array}\right]$

Thus $\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/6 & 1/2 \\ 1/3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} U_{t+1} \\ T_{t+1} \end{bmatrix} = \begin{bmatrix} .75 & 0 \\ .20 & .90 \end{bmatrix} \begin{bmatrix} U_t \\ T_t \end{bmatrix}$

4. $\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right] \left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right] \left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 0 & -1/2 & -1/2 \\ 1 & 2 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 0 & -1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{array}\right] \left[\begin{array}{cc|c} 1 & 3/2 & 3/2 \\ 0 & 1 & 1 \\ 0 & 1/2 & 1/2 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1/2 & 1/2 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

Thus $x_1 = 0$ and $x_2 = 1$ is the unique solution

5. $\left[\begin{array}{cc|cc} 0 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & 1/2 & 0 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & -3/2 & 1 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$

Thus $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

6. $\begin{bmatrix} GPA_1 \\ GPA_2 \end{bmatrix} = \begin{bmatrix} 3.6 & 3. & 2.6 & 3.4 \\ 2.7 & 3.1 & 2.9 & 3.6 \end{bmatrix} \begin{bmatrix} 4/14 \\ 4/14 \\ 3/14 \\ 3/14 \end{bmatrix}$

7. $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & -1 & 0 \end{array}\right] \left[\begin{array}{cc|c} 1 & 1/2 & 3/2 \\ 3 & 1 & 4 \\ 1 & -1 & 0 \end{array}\right] \left[\begin{array}{cc|c} 1 & 1/2 & 3/2 \\ 0 & -1/2 & -1/2 \\ 1 & -1 & 0 \end{array}\right]$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 3/2 \\ 0 & -1/2 & -1/2 \\ 0 & -3/2 & -3/2 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 1/2 & 3/2 \\ 0 & 1 & 1 \\ 0 & -3/2 & -3/2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -3/2 & 3/2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$ Thus the unique solution is $x_1 = 1$, $x_2 = 1$.

8. $$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

Thus $$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

9. $C_{t+1} = .94C_t$ and $A_{t+1} = .05C_t + .97A_t$. Using matrix notation we obtain

$$\begin{bmatrix} C_{t+1} \\ A_{t+1} \end{bmatrix} = \begin{bmatrix} .94 & 0 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} C_t \\ A_t \end{bmatrix}$$

10. $$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 5 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The row [0 0 0 |-1] indicates that the original equation system has no solution.

11. $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

Thus $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

12. $R_{t+1} = .9R_t + .2C_t$ and $C_{t+1} = .1R_t + .8C_t$. Using matrix
notation we obtain

$$\begin{bmatrix} R_{t+1} \\ C_{t+1} \end{bmatrix} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} R_t \\ C_t \end{bmatrix}$$

13. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{bmatrix}$

The last matrix has rank 2, thus the original set of vectors is linearly independent.

14. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The last matrix has rank 2<3, so the original set of vectors is linearly dependent.

15. Only if a, b, c and d are all non-zero will rank A=4. Thus
A^{-1} exists if and only if all of a, b, c, and d are non-zero.

15b. Applying the Gauss-Jordan Method we find that if a, b, c, and d are all non-zero

$$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix}$$

16. $$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_4 = c$, $x_1 = 1 + c$, $x_2 = 1 - c$, $x_3 = 3 - c$ yields an infinite number of solutions to the original system of equations.

17. Let s=state tax paid. f=federal tax paid, and b=bonus paid to employees. Then s, f, and b must satisfy the following system of equations:

$$b = .05(60{,}000 - f - s)$$
$$s = .05(60{,}000 - b)$$
$$f = .40(60{,}000 - b - s)$$

18. Expanding by row 2 cofactors we obtain

$$(-1)^{2+1}(1) \det \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix} = -4$$

19. Let

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

be a 2x2 matrix that does not have an inverse.

If a = b = c = d = 0, then det A = 0. Now assume at least one element of A is nonzero (for the sake of definiteness assume that a/=0). If det A /=0 (that is, if ad - bc/=0) then in applying ero's to the matrix A|I_2, A is transformed into the identity and A will have an inverse. However, if det A = 0, then the second row of A will be transformed into [0 0], and A will not have an inverse.

20a. Since rank A = m and the system has m variables, N will be empty and the unique solution to A**x** = **0** will be $x_1 = x_2 = \ldots = x_m = 0$.

20b. If rank A < m, the Gauss-Jordan Method will yield at least one row of 0's on the bottom (with the right hand side for each of these equations still being 0). Thus N will be non-empty and we will be in Case 3 (an infinite number of solutions).

21. The given system may be written in the form A**x** = **0** where

$$A = \begin{bmatrix} 1-P_{11} & -P_{21} & \cdots & -P_{n1} \\ -P_{12} & 1-P_{22} & \cdots & -P_{n2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -P_{1n} & -P_{2n} & \cdots & -P_{nn} \end{bmatrix}$$

Now use ero's as follows: Replace row 1 of A by (row 1 of A) + (row 2 of A). Then replace the new row 1 by (new row 1) + (row 3). Continue in this fashion until you have replaced the current row 1 by (current row 1) + (row n). The resulting matrix will have the following as its first row:

$[1-p_{11}-p_{12}-...-p_{1n} \quad 1-p_{21}-p_{22}-...-p_{2n} \quad ... \quad 1-p_{n1}-p_{n2}-...-p_{nn}]$

= [0 0 ... 0], where the last equality follows from the fact that the sum of all the entries in each row of A is equal to 1.

This shows that A has at most n - 1 independent rows. Thus rank A can be at most n - 1. Problem 20 now shows that $A\mathbf{x}=\mathbf{0}$ (and the original system) has an infinite number of solutions.

22a. $A = \begin{array}{c|ccc} & \text{steel} & \text{cars} & \text{machines} \\ \hline \text{steel} & .30 & .45 & .40 \\ \text{cars} & .15 & .20 & .10 \\ \text{machines} & .40 & .10 & .45 \end{array}$

Since (Total Steel Produced) = (Steel Consumed) + (Steel used to produce steel, cars and machines),

$$s = d_s + .3s + .45c + .4m.$$

Similarly we find that

$$c = d_c + .15s + .20c + .10m$$

and

$$m = d_m + .40s + .10c + .45m.$$

In matrix form we obtain the following linear system

$$\begin{bmatrix} s \\ c \\ m \end{bmatrix} = \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix} + \begin{bmatrix} .30 & .45 & .40 \\ .15 & .20 & .10 \\ .40 & .10 & .45 \end{bmatrix} \begin{bmatrix} s \\ c \\ m \end{bmatrix}$$

22c. Just write $\begin{bmatrix} s \\ c \\ m \end{bmatrix}$ as I $\begin{bmatrix} s \\ c \\ m \end{bmatrix}$ and subtract A $\begin{bmatrix} s \\ c \\ m \end{bmatrix}$ from both sides of the answer to 22b.

22d. From our answer to 22c, we find that

$$\begin{bmatrix} s \\ c \\ m \end{bmatrix} = (I-A)^{-1} \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$

If $s \geq 0, c \geq 0$, and $m \geq 0$, then Seriland can meet the required demands. If any of s, c, and m are negative, then Seriland cannot meet the required demands.

22e. Before we increase the amount of required steel by $1,

$$\begin{bmatrix} s \\ c \\ m \end{bmatrix} = (I-A)^{-1} \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$

After increasing amount of required steel by 1,

$$\begin{bmatrix} s \\ c \\ m \end{bmatrix} = (I-A)^{-1} \begin{bmatrix} d_s + 1 \\ d_c \\ d_m \end{bmatrix} = \text{Original} \begin{bmatrix} s \\ c \\ m \end{bmatrix} + (I-A)^{-1}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \text{Original} \begin{bmatrix} s \\ c \\ m \end{bmatrix} + (\text{First column of } (I-A)^{-1}) \ .$$

Thus increasing steel requirements by $1 increases demand for steel by element 1-1 of $(I-A)^{-1}$,increases demand for cars by element 2-1 of $(I-A)^{-1}$, and increases demand for machines by element 3-1 of $(I-A)^{-1}$.