INSTRUCTOR'S SOLUTIONS MANUAL
MULTIVARIABLE

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CALCULUS
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AND
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SECOND EDITION

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Contents

Chapter 8

Sequences and Infinite Series

8.1 An Overview

8.1.1 A *sequence* is an ordered list of numbers $a_1, a_2, a_3, \ldots$, often written $\{a_1, a_2, \ldots\}$ or $\{a_n\}$. For example, the natural numbers $\{1, 2, 3, ...\}$ are a sequence where $a_n = n$ for every n.

8.1.2 $a_1 = \frac{1}{1} = 1$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{3}$; $a_4 = \frac{1}{4}$; $a_5 = \frac{1}{5}$.

8.1.3 $a_1 = 1$ (given); $a_2 = 1 \cdot a_1 = 1$; $a_3 = 2 \cdot a_2 = 2$; $a_4 = 3 \cdot a_3 = 6$; $a_5 = 4 \cdot a_4 = 24$.

8.1.4 A *finite sum* is the sum of a finite number of items, for example the sum of a finite number of terms of a sequence.

8.1.5 An *infinite series* is an infinite sum of numbers. Thus if $\{a_n\}$ is a sequence, then $a_1+a_2+\cdots = \sum_{k=1}^{\infty} a_k$ is an infinite series. For example, if $a_k = \frac{1}{k}$, then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is an infinite series.

8.1.6 $S_1 = \sum_{k=1}^{1} k = 1$; $S_2 = \sum_{k=1}^{2} k = 1 + 2 = 3$; $S_3 = \sum_{k=1}^{3} k = 1 + 2 + 3 = 6$; $S_4 = \sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = 10$.

8.1.7 $S_1 = \sum_{k=1}^{1} k^2 = 1$; $S_2 = \sum_{k=1}^{2} k^2 = 1 + 4 = 5$; $S_3 = \sum_{k=1}^{3} k^2 = 1 + 4 + 9 = 14$; $S_4 = \sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30$.

8.1.8 $S_1 = \sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} = 1$; $S_2 = \sum_{k=1}^{2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$; $S_3 = \sum_{k=1}^{3} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$; $S_4 = \sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

8.1.9 $a_1 = \dfrac{1}{10}; a_2 = \dfrac{1}{100}; a_3 = \dfrac{1}{1000}; a_4 = \dfrac{1}{10000}$.

8.1.10 $a_1 = 3(1) + 1 = 4$. $a_2 = 3(2) + 1 = 7$, $a_3 = 3(3) + 1 = 10$, $a_4 = 3(4) + 1 = 13$.

8.1.11 $a_1 = \frac{-1}{2}$, $a_2 = \frac{1}{2^2} = \frac{1}{4}$. $a_3 = \frac{-2}{2^3} = \frac{-1}{8}$, $a_4 = \frac{1}{2^4} = \frac{1}{16}$.

8.1.12 $a_1 = 2 - 1 = 1$. $a_2 = 2 + 1 = 3$, $a_3 = 2 - 1 = 1$, $a_4 = 2 + 1 = 3$.

8.1.13 $a_1 = \frac{2^2}{2+1} = \frac{4}{3}$. $a_2 = \frac{2^3}{2^2+1} = \frac{8}{5}$. $a_3 = \frac{2^4}{2^3+1} = \frac{16}{9}$. $a_4 = \frac{2^5}{2^4+1} = \frac{32}{17}$.

8.1.14 $a_1 = 1 + \frac{1}{1} = 2$; $a_2 = 2 + \frac{1}{2} = \frac{5}{2}$; $a_3 = 3 + \frac{1}{3} = \frac{10}{3}$; $a_4 = 4 + \frac{1}{4} = \frac{17}{4}$.

8.1.15 $a_1 = 1+\sin(\pi/2) = 2$; $a_2 = 1+\sin(2\pi/2) = 1+\sin\pi = 1$; $a_3 = 1+\sin(3\pi/2) = 0$; $a_4 = 1+\sin(4\pi/2) = 1 + \sin 2\pi = 1$.

8.1.16 $a_1 = 2\cdot 1^2 - 3\cdot 1+1 = 0$; $a_2 = 2\cdot 2^2 - 3\cdot 2+1 = 3$; $a_3 = 2\cdot 3^2 - 3\cdot 3+1 = 10$; $a_4 = 2\cdot 4^2 - 3\cdot 4+1 = 21$.

8.1.17 $a_1 = 2$, $a_2 = 2 \cdot 2 = 4$, $a_3 = 2(4) = 8$, $a_4 = 2 \cdot 8 = 16$.

8.1.18 $a_1 = 32$, $a_2 = 32/2 = 16$, $a_3 = 16/2 = 8$, $a_4 = 8/2 = 4$.

8.1.19 $a_1 = 10$ (given); $a_2 = 3 \cdot a_1 - 12 = 30 - 12 = 18$; $a_3 = 3 \cdot a_2 - 12 = 54 - 12 = 42$; $a_4 = 3 \cdot a_3 - 12 = 126 - 12 = 114$.

8.1.20 $a_1 = 1$ (given); $a_2 = a_1^2 - 1 = 0$; $a_3 = a_2^2 - 1 = -1$; $a_4 = a_3^2 - 1 = 0$.

8.1.21 $a_1 = 0$ (given); $a_2 = 3 \cdot a_1^2 + 1 + 1 = 2$; $a_3 = 3 \cdot a_2^2 + 2 + 1 = 15$; $a_4 = 3 \cdot a_3^2 + 3 + 1 = 679$.

8.1.22 $a_0 = 1$ (given); $a_1 = 1$ (given); $a_2 = a_1 + a_0 = 2$; $a_3 = a_2 + a_1 = 3$; $a_4 = a_3 + a_2 = 5$.

8.1.23

a. $\frac{1}{32}, \frac{1}{64}$.

b. $a_1 = 1$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{1}{2^{n-1}}$.

8.1.24

a. -6, 7.

b. $a_1 = 1$; $a_{n+1} = (-1)^n(|a_n| + 1)$.

c. $a_n = (-1)^{n+1}n$.

8.1.25

a. -5, 5.

b. $a_1 = -5$, $a_{n+1} = -a_n$.

c. $a_n = (-1)^n \cdot 5$.

8.1.26

a. 14, 17.

b. $a_1 = 2$; $a_{n+1} = a_n + 3$.

c. $a_n = -1 + 3n$.

8.1.27

a. 32, 64.

b. $a_1 = 1$; $a_{n+1} = 2a_n$.

c. $a_n = 2^{n-1}$.

8.1.28

a. 36, 49.

b. $a_1 = 1$; $a_{n+1} = (\sqrt{a_n} + 1)^2$.

c. $a_n = n^2$.

8.1.29

a. 243, 729.

b. $a_1 = 1$; $a_{n+1} = 3a_n$.

c. $a_n = 3^{n-1}$.

8.1.30

a. 2, 1.

b. $a_1 = 64$; $a_{n+1} = \frac{a_n}{2}$.

c. $a_n = \frac{64}{2^{n-1}} = 2^{7-n}$.

8.1.31 $a_1 = 9$, $a_2 = 99$, $a_3 = 999$, $a_4 = 9999$. This sequence diverges, because the terms get larger without bound.

8.1.32 $a_1 = 2$, $a_2 = 17$, $a_3 = 82$, $a_4 = 257$. This sequence diverges, because the terms get larger without bound.

8.1.33 $a_1 = \frac{1}{10}$, $a_2 = \frac{1}{100}$, $a_3 = \frac{1}{1000}$, $a_4 = \frac{1}{10{,}000}$. This sequence converges to zero.

8.1.34 $a_1 = \frac{1}{10}$, $a_2 = \frac{1}{100}$, $a_3 = \frac{1}{1000}$, $a_4 = \frac{1}{10{,}000}$. This sequence converges to zero.

8.1.35 $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = -\frac{1}{8}$, $a_4 = \frac{1}{16}$. This sequence converges to 0 because each term is smaller in absolute value than the preceding term and they get arbitrarily close to zero.

8.1.36 $a_1 = 0.9$, $a_2 = 0.99$, $a_3 = 0.999$, $a_4 = .9999$. This sequence converges to 1.

8.1.37 $a_1 = 1 + 1 = 2$, $a_2 = 1 + 1 = 2$, $a_3 = 2$, $a_4 = 2$. This constant sequence converges to 2.

8.1.38 $a_1 = 9 + \frac{9}{10} = 9.9$, $a_2 = 9 + \frac{9.9}{10} = 9.99$, $a_3 = 9 + \frac{9.99}{10} = 9.999$, $a_4 = 9 + \frac{9.999}{10} = 9.9999$. This sequence converges to 10.

8.1.39 $a_1 = \frac{50}{11} + 50 \approx 54.545$, $a_2 = \frac{54.545}{11} + 50 \approx 54.959$, $a_3 = \frac{54.959}{11} + 50 \approx 54.996$, $a_4 = \frac{54.996}{11} + 50 \approx 55.000$. This sequence converges to 55.

8.1.40 $a_1 = 0 - 1 = -1$. $a_2 = -10 - 1 = -11$, $a_3 = -110 - 1 = -111$, $a_4 = -1110 - 1 = -1111$. This sequence diverges.

8.1.41

n	1	2	3	4	4	6	7	8	9	10
a_n	0.4636	0.2450	0.1244	0.0624	0.0312	0.0156	0.0078	0.0039	0.0020	0.0010

This sequence appears to converge to 0.

8.1.42

n	1	2	3	4	5	6	7	8	9	10
a_n	3.1396	3.1406	3.1409	3.1411	3.1412	3.1413	3.1413	3.1413	3.1414	3.1414

This sequence appears to converge to π.

8.1.43

n	1	2	3	4	5	6	7	8	9	10
a_n	0	2	6	12	20	30	42	56	72	90

This sequence appears to diverge.

8.1.44

n	1	2	3	4	5	6	7	8	9	10
a_n	9.9	9.95	9.9667	9.975	9.98	9.9833	9.9857	9.9875	9.9889	9.99

This sequence appears to converge to 10.

8.1.45

n	1	2	3	4	5	6	7	8	9	10
a_n	0.83333	0.96154	0.99206	0.99840	0.99968	0.99994	0.99999	1.0000	1.0000	1.0000

This sequence appears to converge to 1.

8.1.46

n	1	2	3	4	5	6	7	8	9	10	11
a_n	0.9589	0.9896	0.9974	0.9993	0.9998	1.000	1.000	1.0000	1.000	1.000	1.000

This sequence converges to 1.

8.1.47

a. 2.5, 2.25, 2.125, 2.0625.

b. The limit is 2.

8.1.48

a. 1.33333, 1.125, 1.06667, 1.04167.

b. The limit is 1.

8.1.49

n	0	1	2	3	4	5	6	7	8	9	10
a_n	3	3.500	3.750	3.875	3.938	3.969	3.984	3.992	3.996	3.998	3.999

This sequence converges to 4.

8.1.50

n	0	1	2	3	4	5	6	7	8	9
a_n	1	−2.75	−3.688	−3.922	−3.981	−3.995	−3.999	−4.000	−4.000	−4.000

This sequence converges to −4.

8.1.51

n	0	1	2	3	4	5	6	7	8	9	10
a_n	0	1	3	7	15	31	63	127	255	511	1023

This sequence diverges.

8.1.52

n	0	1	2	3	4	5	6	7	8	9	10
a_n	10	4	3.4	3.34	3.334	3.333	3.333	3.333	3.333	3.333	3.333

This sequence converges to $\frac{10}{3}$.

8.1.53

n	0	1	2	3	4	5	6	7	8	9
a_n	1000	18.811	5.1686	4.1367	4.0169	4.0021	4.0003	4.0000	4.0000	4.0000

This sequence converges to 4.

8.1.54

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1.4212	1.5538	1.5981	1.6119	1.6161	1.6174	1.6179	1.6180	1.6180	1.6180

This sequence converges to $\frac{1+\sqrt{5}}{2} \approx 1.618$.

8.1.55

a. 20, 10, 5, 2.5.

b. $h_n = 20(0.5)^n$.

8.1.56

a. 10, 9, 8.1, 7.29.

b. $h_n = 10(0.9)^n$.

8.1.57

a. 30, 7.5, 1.875, 0.46875.

b. $h_n = 30(0.25)^n$.

8.1.58

a. 20, 15, 11.25, 8.438

b. $h_n = 20(0.75)^n$.

8.1.59 $S_1 = 0.3$, $S_2 = 0.33$, $S_3 = 0.333$, $S_4 = 0.3333$. It appears that the infinite series has a value of $0.3333\ldots = \frac{1}{3}$.

8.1.60 $S_1 = 0.6$, $S_2 = 0.66$, $S_3 = 0.666$, $S_4 = 0.6666$. It appears that the infinite series has a value of $0.6666\ldots = \frac{2}{3}$.

8.1.61 $S_1 = 4$, $S_2 = 4.9$, $S_3 = 4.99$, $S_4 = 4.999$. The infinite series has a value of $4.999\cdots = 5$.

8.1.62 $S_1 = 1$, $S_2 = \frac{3}{2} = 1.5$, $S_3 = \frac{7}{4} = 1.75$, $S_4 = \frac{15}{8} = 1.875$. The infinite series has a value of 2.

8.1.63

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{4}{5}$, $S_3 = \frac{6}{7}$, $S_4 = \frac{8}{9}$.

b. It appears that $S_n = \frac{2n}{2n+1}$.

c. The series has a value of 1 (the partial sums converge to 1).

8.1.64

a. $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$.

b. $S_n = 1 - \frac{1}{2^n}$.

c. The partial sums converge to 1, so that is the value of the series.

8.1.65

a. $S_1 = \frac{1}{3}$, $S_2 = \frac{2}{5}$, $S_3 = \frac{3}{7}$, $S_4 = \frac{4}{9}$.

b. $S_n = \frac{n}{2n+1}$.

c. The partial sums converge to $\frac{1}{2}$, which is the value of the series.

8.1.66

a. $S_1 = \frac{2}{3}$, $S_2 = \frac{8}{9}$, $S_3 = \frac{26}{27}$, $S_4 = \frac{80}{81}$.

b. $S_n = 1 - \frac{1}{3^n}$.

c. The partial sums converge to 1, which is the value of the series.

8.1.67

a. True. For example, $S_2 = 1 + 2 = 3$, and $S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$.

b. False. For example, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$ where $a_n = 1 - \frac{1}{2^n}$ converges to 1, but each term is greater than the previous one.

c. True. In order for the partial sums to converge, they must get closer and closer together. In order for this to happen, the difference between successive partial sums, which is just the value of a_n, must approach zero.

8.1.68 The height at the n^{th} bounce is given by the recurrence $h_n = r \cdot h_{n-1}$; an explicit form for this sequence is $h_n = h_0 \cdot r^n$. The distance traveled by the ball between the n^{th} and the $(n+1)^{\text{st}}$ bounce is thus $2h_n = 2h_0 \cdot r^n$, so that $S_{n+1} = \sum_{i=0}^{n} 2h_0 \cdot r^i$.

a. Here $h_0 = 20$, $r = 0.5$, so $S_1 = 40$, $S_2 = 40 + 40 \cdot 0.5 = 60$, $S_3 = S_2 + 40 \cdot (0.5)^2 = 70$, $S_4 = S_3 + 40 \cdot (0.5)^3 = 75$, $S_5 = S_4 + 40 \cdot (0.5)^4 = 77.5$

b.

n	1	2	3	4	5	6
a_n	40	60	70	75	77.5	78.75
n	7	8	9	10	11	12
a_n	79.375	79.688	79.844	79.922	79.961	79.980
n	13	14	15	16	17	18
a_n	79.990	79.995	79.998	79.999	79.999	80.000
n	19	20	21	22	23	24
a_n	80.000	80.000	80.000	80.000	80.000	80.000

The sequence converges to 80.

8.1.69 Using the work from the previous problem:

a. Here $h_0 = 20$, $r = 0.75$, so $S_1 = 40$, $S_2 = 40 + 40 \cdot 0.75 = 70$, $S_3 = S_2 + 40 \cdot (0.75)^2 = 92.5$, $S_4 = S_3 + 40 \cdot (0.75)^3 = 109.375$, $S_5 = S_4 + 40 \cdot (0.75)^4 = 122.03125$

b.

n	1	2	3	4	5	6
a_n	40	70	92.5	109.375	122.031	131.523
n	7	8	9	10	11	12
a_n	138.643	143.982	147.986	150.990	153.242	154.932
n	13	14	15	16	17	18
a_n	156.199	157.149	157.862	158.396	158.797	159.098
n	19	20	21	22	23	24
a_n	159.323	159.493	159.619	159.715	159.786	159.839

The sequence converges to 160.

8.1.70

a. $s_1 = -1$, $s_2 = 0$, $s_3 = -1$, $s_4 = 0$.

b. The limit does not exist.

8.1.71

a. 0.9, 0.99, 0.999, .9999.

b. The limit is 1.

8.1.72

a. 1.5, 3.75, 7.125, 12.1875.

b. The limit does not exist.

8.1.73

a. $\frac{1}{3}$, $\frac{4}{9}$, $\frac{13}{27}$, $\frac{40}{81}$.

b. The limit is 1/2.

8.1.74

a. 1, 3, 6, 10.

b. The limit does not exist.

8.1.75

a. -1, 0, -1, 0.

b. The limit does not exist.

8.1.76

a. -1, 1, -2, 2.

b. The limit does not exist.

8.1.77

a. $\frac{3}{10} = 0.3$, $\frac{33}{100} = 0.33$, $\frac{333}{1000} = 0.333$, $\frac{3333}{10000} = 0.3333$.

b. The limit is 1/3.

8.1.78

a. $p_0 = 250$, $p_1 = 250 \cdot 1.03 = 258$, $p_2 = 250 \cdot 1.03^2 = 265$, $p_3 = 250 \cdot 1.03^3 = 273$, $p_4 = 250 \cdot 1.03^4 = 281$.

b. The initial population is 250, so that $p_0 = 250$. Then $p_n = 250 \cdot (1.03)^n$, because the population increases by 3 percent each month.

c. $p_{n+1} = p_n \cdot 1.03$.

d. The population increases without bound.

8.1.79

a. $M_0 = 20$, $M_1 = 20 \cdot 0.5 = 10$, $M_2 = 20 \cdot 0.5^2 = 5$, $M_3 = 20 \cdot 0.5^3 = 2.5$, $M_4 = 20 \cdot 0.5^4 = 1.25$

b. $M_n = 20 \cdot 0.5^n$.

c. The initial mass is $M_0 = 20$. We are given that 50% of the mass is gone after each decade, so that $M_{n+1} = 0.5 \cdot M_n$, $n \geq 0$.

d. The amount of material goes to 0.

8.1.80

a. $c_0 = 100$, $c_1 = 103$, $c_2 = 106.09$, $c_3 = 109.27$, $c_4 = 112.55$.

b. $c_n = 100(1.03)^n$ for $n \geq 0$.

c. We are given that $c_0 = 100$ (where year 0 is 1984); because it increases by 3% per year, $c_{n+1} = 1.03 \cdot c_n$.

d. The sequence diverges.

8.1.81

a. $d_0 = 200$, $d_1 = 200 \cdot .95 = 190$, $d_2 = 200 \cdot .95^2 = 180.5$, $d_3 = 200 \cdot .95^3 = 171.475$, $d_4 = 200 \cdot .95^4 = 162.90125$.

b. $d_n = 200(0.95)^n$, $n \geq 0$.

c. We are given $d_0 = 200$; because 5% of the drug is washed out every hour, that means that 95% of the preceding amount is left every hour, so that $d_{n+1} = 0.95 \cdot d_n$.

d. The sequence converges to 0.

8.1.82

a. Using the recurrence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{10}{a_n}\right)$, we build a table:

n	0	1	2	3	4	5
a_n	10	5.5	3.659090909	3.196005081	3.162455622	3.162277665

The true value is $\sqrt{10} \approx 3.162277660$, so the sequence converges with an error of less than 0.01 after only 4 iterations, and is within 0.0001 after only 5 iterations.

b. The recurrence is now $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

c	$\sqrt{c}$	0	1	2	3	4	5	6
2	1.414	2	1.5	1.417	1.414	1.414	1.414	1.414
3	1.732	3	2	1.750	1.732	1.732	1.732	1.732
4	2.000	4	2.5	2.050	2.001	2.000	2.000	2.000
5	2.236	5	3	2.333	2.238	2.236	2.236	2.236
6	2.449	6	3.6	2.607	2.454	2.449	2.449	2.449
7	2.646	7	4	2.875	2.655	2.646	2.646	2.646
8	2.828	8	4.5	3.139	2.844	2.828	2.828	2.828
9	3.000	9	5.0	3.400	3.024	3.000	3.000	3.000
10	3.162	10	5.5	3.659	3.196	3.162	3.162	3.162

For $c = 2$ the sequence converges to within 0.01 after two iterations.
For $c = 3$, 4, 5, 6, and 7 the sequence converges to within 0.01 after three iterations.
For $c = 8$, 9, and 10 it requires four iterations.

8.2 Sequences

8.2.1 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing) and has a limit of 0.

8.2.2 Again there are many examples; one is $a_n = \ln(n)$. It is increasing, and has no limit.

8.2.3 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing), is bounded above by 1 and below by 0, and has a limit of 0.

8.2.4 For example, $a_n = (-1)^n$. For all values of n we have $|a_n| = 1$, so it is bounded. All the odd terms are -1 and all the even terms are 1, so the sequence does not have a limit.

8.2.5 $\{r^n\}$ converges for $-1 < r \le 1$. It diverges for all other values of r (see Theorem 8.3).

8.2.6 By Theorem 8.1, if we can find a function $f(x)$ such that $f(n) = a_n$ for all positive integers n, then if $\lim\limits_{x\to\infty} f(x)$ exists and is equal to L, we then have $\lim\limits_{n\to\infty} a_n$ exists and is also equal to L. This means that we can apply function-oriented limit methods such as L'Hôpital's rule to determine limits of sequences.

8.2.7 $\{e^{n/100}\}$ grows faster than $\{n^{100}\}$ as $n \to \infty$.

8.2.8 The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

8.2.9 Divide numerator and denominator by n^4 to get $\lim\limits_{n\to\infty} \frac{1/n}{1+\frac{1}{n^4}} = 0$.

8.2.10 Divide numerator and denominator by n^{12} to get $\lim\limits_{n\to\infty} \frac{1}{3+\frac{4}{n^{12}}} = \frac{1}{3}$.

8.2.11 Divide numerator and denominator by n^3 to get $\lim\limits_{n\to\infty} \frac{3-n^{-3}}{2+n^{-3}} = \frac{3}{2}$.

8.2.12 Divide numerator and denominator by e^n to get $\lim\limits_{n\to\infty} \frac{2+(1/e^n)}{1} = 2$.

8.2.13 Divide numerator and denominator by 3^n to get $\lim\limits_{n\to\infty} \frac{3+(1/3^{n-1})}{1} = 3$.

8.2.14 Divide numerator by k and denominator by $k = \sqrt{k^2}$ to get $\lim\limits_{k\to\infty} \frac{1}{\sqrt{9+(1/k^2)}} = \frac{1}{3}$.

8.2.15 $\lim\limits_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$.

8.2.16 Multiply by $\dfrac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$ to obtain

$$\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+1} - n\right)\left(\sqrt{n^2+1} + n\right)}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0.$$

8.2.17 Because $\lim\limits_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$, $\lim\limits_{n\to\infty} \frac{\tan^{-1} n}{n} = 0$.

8.2.18 Let $y = n^{2/n}$. Then $\ln y = \frac{2\ln n}{n}$. By L'Hôpital's rule we have $\lim\limits_{x\to\infty} \frac{2\ln x}{x} = \lim\limits_{x\to\infty} \frac{2}{x} = 0$, so $\lim\limits_{n\to\infty} n^{2/n} = e^0 = 1$.

8.2.19 Find the limit of the logarithm of the expression, which is $n\ln\left(1+\frac{2}{n}\right)$. Using L'Hôpital's rule:

$$\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right)=\lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{1/n}=\lim_{n\to\infty}\frac{\frac{1}{1+(2/n)}\left(\frac{-2}{n^2}\right)}{-1/n^2}=\lim_{n\to\infty}\frac{2}{1+(2/n)}=2.$$

Thus the limit of the original expression is e^2.

8.2.20 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+5}\right)=\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+5}\right)}{1/n}=\lim_{n\to\infty}\frac{\frac{n+5}{n}\cdot\frac{5}{(n+5)^2}}{-1/n^2}=\lim_{n\to\infty}\frac{-5n}{n+5}=-5.$$

Thus the original limit is e^{-5}.

8.2.21 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n\to\infty}\frac{n}{2}\ln\left(1+\frac{1}{2n}\right)=\lim_{n\to\infty}\frac{\ln(1+(1/2n))}{2/n}=\lim_{n\to\infty}\frac{\frac{1}{1+(1/2n)}\cdot\frac{-1}{2n^2}}{-2/n^2}=\lim_{n\to\infty}\frac{1}{4(1+(1/2n))}=\frac{1}{4}.$$

Thus the original limit is $e^{1/4}$.

8.2.22 Find the limit of the logarithm of the expression, which is $3n\ln\left(1+\frac{4}{n}\right)$. Using L'Hôpital's rule:

$$\lim_{n\to\infty} 3n\ln\left(1+\frac{4}{n}\right)=\lim_{n\to\infty}\frac{3\ln\left(1+\frac{4}{n}\right)}{1/n}=\lim_{n\to\infty}\frac{\frac{1}{1+(4/n)}\left(\frac{-12}{n^2}\right)}{-1/n^2}=\lim_{n\to\infty}\frac{12}{1+(4/n)}=12.$$

Thus the limit of the original expression is e^{12}.

8.2.23 Using L'Hôpital's rule: $\lim_{n\to\infty}\frac{n}{e^n+3n}=\lim_{n\to\infty}\frac{1}{e^n+3}=0$.

8.2.24 $\ln\frac{1}{n}=-\ln n$, so this is $-\lim_{n\to\infty}\frac{\ln n}{n}$. By L'Hôpital's rule, we have $-\lim_{n\to\infty}\frac{\ln n}{n}=-\lim_{n\to\infty}\frac{1}{n}=0$.

8.2.25 Taking logs, we have $\lim_{n\to\infty}\frac{1}{n}\ln(1/n)=\lim_{n\to\infty}-\frac{\ln n}{n}=\lim_{n\to\infty}\frac{-1}{n}=0$ by L'Hôpital's rule. Thus the original sequence has limit $e^0=1$.

8.2.26 Find the limit of the logarithm of the expression, which is $n\ln\left(1-\frac{4}{n}\right)$, using L'Hôpital's rule: $\lim_{n\to\infty}n\ln\left(1-\frac{4}{n}\right)=\lim_{n\to\infty}\frac{\ln\left(1-\frac{4}{n}\right)}{1/n}=\lim_{n\to\infty}\frac{\frac{1}{1-(4/n)}\left(\frac{4}{n^2}\right)}{-1/n^2}=\lim_{n\to\infty}\frac{-4}{1-(4/n)}=-4$. Thus the limit of the original expression is e^{-4}.

8.2.27 Except for a finite number of terms, this sequence is just $a_n=ne^{-n}$, so it has the same limit as this sequence. Note that $\lim_{n\to\infty}\frac{n}{e^n}=\lim_{n\to\infty}\frac{1}{e^n}=0$, by L'Hôpital's rule.

8.2.28 $\ln(n^3+1)-\ln(3n^3+10n)=\ln\left(\frac{n^3+1}{3n^3+10n}\right)=\ln\left(\frac{1+n^{-3}}{3+10n^{-2}}\right)$, so the limit is $\ln(1/3)=-\ln 3$.

8.2.29 $\ln(\sin(1/n))+\ln n=\ln(n\sin(1/n))=\ln\left(\frac{\sin(1/n)}{1/n}\right)$. As $n\to\infty$, $\sin(1/n)/(1/n)\to 1$, so the limit of the original sequence is $\ln 1=0$.

8.2.30 Using L'Hôpital's rule:

$$\lim_{n\to\infty}n(1-\cos(1/n))=\lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n}=\lim_{n\to\infty}\frac{-\sin(1/n)(-1/n^2)}{-1/n^2}=-\sin(0)=0.$$

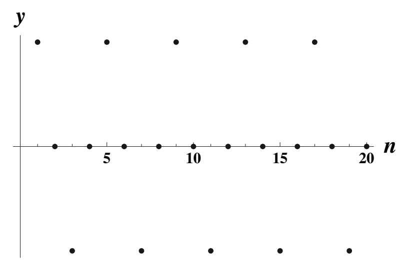
8.2.31 $\lim_{n\to\infty}n\sin(6/n)=\lim_{n\to\infty}\frac{\sin(6/n)}{1/n}=\lim_{n\to\infty}\frac{\frac{-6\cos(6/n)}{n^2}}{(-1/n^2)}=\lim_{n\to\infty}6\cos(6/n)=6\cdot\cos 0=6$.

8.2.32 Because $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$, and because both $-\frac{1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \to \infty$, the limit of the given sequence is also 0 by the Squeeze Theorem.

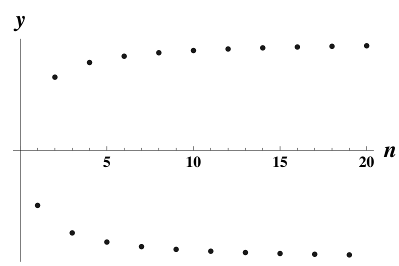
8.2.33 The terms with odd-numbered subscripts have the form $-\frac{n}{n+1}$, so they approach -1, while the terms with even-numbered subscripts have the form $\frac{n}{n+1}$ so they approach 1. Thus, the sequence has no limit.

8.2.34 Because $\frac{-n^2}{2n^3+n} \leq \frac{(-1)^{n+1}n^2}{2n^3+n} \leq \frac{n^2}{2n^3+n}$, and because both $\frac{-n^2}{2n^3+n}$ and $\frac{n^2}{2n^3+n}$ have limit 0 as $n \to \infty$, the limit of the given sequence is also 0 by the Squeeze Theorem. Note that $\lim_{n\to\infty} \frac{n^2}{2n^3+n} = \lim_{n\to\infty} \frac{1/n}{2+1/n^2} = \frac{0}{2} = 0$.

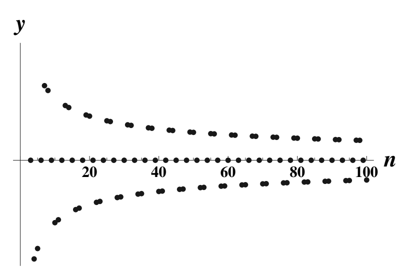
8.2.35 When n is an integer, $\sin\left(\frac{n\pi}{2}\right)$ oscillates between the values ± 1 and 0, so this sequence does not converge.



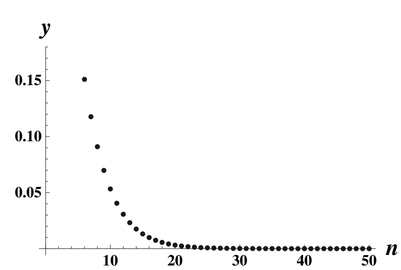
8.2.36 The even terms form a sequence $b_{2n} = \frac{2n}{2n+1}$, which converges to 1 (e.g. by L'Hôpital's rule); the odd terms form the sequence $b_{2n+1} = -\frac{n}{n+1}$, which converges to -1. Thus the sequence as a whole does not converge.



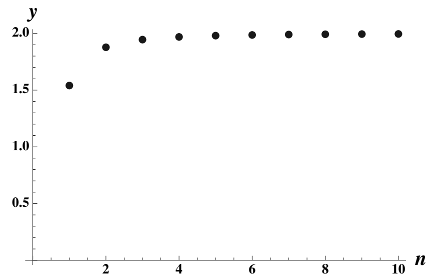
8.2.37 The numerator is bounded in absolute value by 1, while the denominator goes to ∞, so the limit of this sequence is 0.



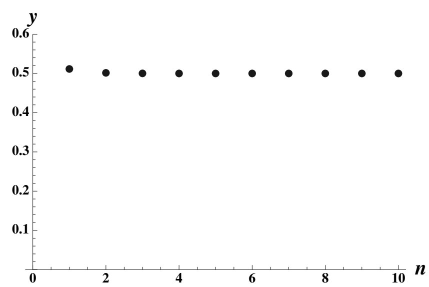
8.2.38 The reciprocal of this sequence is $b_n = \frac{1}{a_n} = 1 + \left(\frac{4}{3}\right)^n$, which increases without bound as $n \to \infty$. Thus a_n converges to zero.



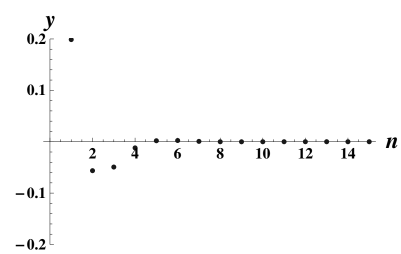
8.2.39 $\lim_{n\to\infty}(1+\cos(1/n)) = 1+\cos(0) = 2.$



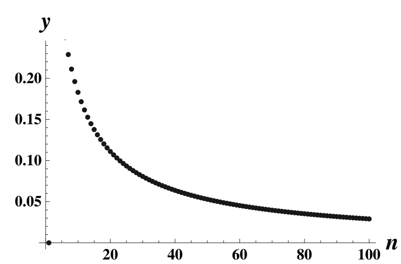
8.2.40 By L'Hôpital's rule we have: $\lim_{n\to\infty}\frac{e^{-n}}{2\sin(e^{-n})} = \lim_{n\to\infty}\frac{-e^{-n}}{2\cos(e^{-n})(-e^{-n})} = \frac{1}{2\cos 0} = \frac{1}{2}.$



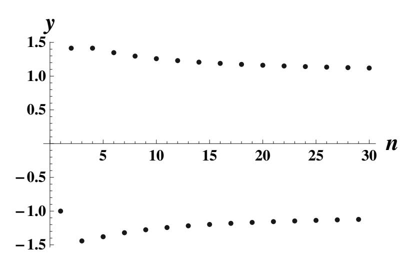
8.2.41 This is the sequence $\frac{\cos n}{e^n}$; the numerator is bounded in absolute value by 1 and the denominator increases without bound, so the limit is zero.



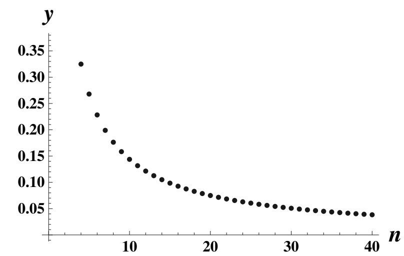
8.2.42 Using L'Hôpital's rule, we have $\lim_{n\to\infty}\frac{\ln n}{n^{1.1}} = \lim_{n\to\infty}\frac{1/n}{(1.1)n^{.1}} = \lim_{n\to\infty}\frac{1}{(1.1)n^{1.1}} = 0.$



8.2.43 Ignoring the factor of $(-1)^n$ for the moment, we see, taking logs, that $\lim_{n\to\infty}\frac{\ln n}{n} = 0$, so that $\lim_{n\to\infty}\sqrt[n]{n} = e^0 = 1$. Taking the sign into account, the odd terms converge to -1 while the even terms converge to 1. Thus the sequence does not converge.



8.2.44 $\lim_{n\to\infty} \frac{n\pi}{2n+2} = \frac{\pi}{2}$, using L'Hôpital's rule. Thus the sequence converges to $\cot(\pi/2) = 0$.



8.2.45 Because $0.2 < 1$, this sequence converges to 0. Because $0.2 > 0$, the convergence is monotone.

8.2.46 Because $1.2 > 1$, this sequence diverges monotonically to ∞.

8.2.47 Because $|-0.7| < 1$, the sequence converges to 0; because $-0.7 < 0$, it does not do so monotonically. The sequence converges by oscillation.

8.2.48 Because $|-1.01| > 1$, the sequence diverges; because $-1.01 < 0$, the divergence is not monotone.

8.2.49 Because $1.00001 > 1$, the sequence diverges; because $1.00001 > 0$, the divergence is monotone.

8.2.50 This is the sequence

$$\frac{2^{n+1}}{3^n} = 2 \cdot \left(\frac{2}{3}\right)^n;$$

because $0 < \frac{2}{3} < 1$, the sequence converges monotonically to zero.

8.2.51 Because $|-2.5| > 1$, the sequence diverges; because $-2.5 < 0$, the divergence is not monotone. The sequence diverges by oscillation.

8.2.52 $|-0.003| < 1$, so the sequence converges to zero; because $-.003 < 0$, the convergence is not monotone.

8.2.53 Because $-1 \le \cos n \le 1$, we have $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$. Because both $\frac{-1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

8.2.54 Because $-1 \le \sin 6n \le 1$, we have $-\frac{1}{5n} \le \frac{\sin 6n}{5n} \le \frac{1}{5n}$. Because both $-\frac{1}{5n}$ and $\frac{1}{5n}$ have limit 0 as $n \to \infty$, the given sequence does as well.

8.2.55 Because $-1 \le \sin n \le 1$ for all n, the given sequence satisfies $-\frac{1}{2^n} \le \frac{\sin n}{2^n} \le \frac{1}{2^n}$, and because both $\pm\frac{1}{2^n} \to 0$ as $n \to \infty$, the given sequence converges to zero as well by the Squeeze Theorem.

8.2.56 Because $-1 \le \cos(n\pi/2) \le 1$ for all n, we have $\frac{-1}{\sqrt{n}} \le \frac{\cos(n\pi/2)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ and because both $\pm\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, the given sequence converges to 0 as well by the Squeeze Theorem.

8.2.57 The inverse tangent function takes values between $-\pi/2$ and $\pi/2$, so the numerator is always between $-\pi$ and π. Thus $\frac{-\pi}{n^3+4} \le \frac{2\tan^{-1} n}{n^3+4} \le \frac{\pi}{n^3+4}$, and by the Squeeze Theorem, the given sequence converges to zero.

8.2.58 This sequence diverges. To see this, call the given sequence a_n, and assume it converges to limit L. Then because the sequence $b_n = \frac{n}{n+1}$ converges to 1, the sequence $c_n = \frac{a_n}{b_n}$ would converge to L as well. But $c_n = \sin^3 \frac{\pi n}{2}$ doesn't converge (because it is $1, -1, 1, -1 \cdots$), so the given sequence doesn't converge either.

8.2.59

a. After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5 \cdot d_{n-1} + 80$, because the half-life is one day. The initial condition is $d_1 = 80$.

b. The limit of this sequence is 160 mg.

c. Let $L = \lim_{n\to\infty} d_n$. Then from the recurrence relation, we have $d_n = 0.5 \cdot d_{n-1} + 80$, and thus $\lim_{n\to\infty} d_n = 0.5 \cdot \lim_{n\to\infty} d_{n-1} + 80$, so $L = 0.5 \cdot L + 80$, and therefore $L = 160$.

8.2.60

a.

$$\begin{aligned}
B_0 &= \$20,000\\
B_1 &= 1.005 \cdot B_0 - \$200 = \$19,900\\
B_2 &= 1.005 \cdot B_1 - \$200 = \$19,799.50\\
B_3 &= 1.005 \cdot B_2 - \$200 = \$19,698.50\\
B_4 &= 1.005 \cdot B_3 - \$200 = \$19,596.99\\
B_5 &= 1.005 \cdot B_4 - \$200 = \$19,494.97
\end{aligned}$$

b. $B_n = 1.005 \cdot B_{n-1} - \$200$

c. Using a calculator or computer program, B_n becomes negative after the 139^{th} payment, so 139 months or almost 11 years.

8.2.61

a.

$$\begin{aligned}
B_0 &= 0\\
B_1 &= 1.0075 \cdot B_0 + \$100 = \$100\\
B_2 &= 1.0075 \cdot B_1 + \$100 = \$200.75\\
B_3 &= 1.0075 \cdot B_2 + \$100 = \$302.26\\
B_4 &= 1.0075 \cdot B_3 + \$100 = \$404.52\\
B_5 &= 1.0075 \cdot B_4 + \$100 = \$507.56
\end{aligned}$$

b. $B_n = 1.0075 \cdot B_{n-1} + \$100$.

c. Using a calculator or computer program, $B_n > \$5,000$ during the 43^{rd} month.

8.2.62

a. Let D_n be the *total number* of liters of alcohol in the mixture after the n^{th} replacement. At the next step, 2 liters of the 100 liters is removed, thus leaving $0.98 \cdot D_n$ liters of alcohol, and then $0.1 \cdot 2 = 0.2$ liters of alcohol are added. Thus $D_n = 0.98 \cdot D_{n-1} + 0.2$. Now, $C_n = D_n/100$, so we obtain a recurrence relation for C_n by dividing this equation by 100: $C_n = 0.98 \cdot C_{n-1} + 0.002$.

$$\begin{aligned}
C_0 &= 0.4\\
C_1 &= 0.98 \cdot 0.4 + 0.002 = 0.394\\
C_2 &= 0.98 \cdot C_1 + 0.002 = 0.38812\\
C_3 &= 0.98 \cdot C_2 + 0.002 = 0.38236\\
C_4 &= 0.98 \cdot C_3 + 0.002 = 0.37671\\
C_5 &= 0.98 \cdot C_4 + 0.002 = 0.37118
\end{aligned}$$

The rounding is done to five decimal places.

b. Using a calculator or a computer program, $C_n < 0.15$ after the 89^{th} replacement.

c. If the limit of C_n is L, then taking the limit of both sides of the recurrence equation yields $L = 0.98L + 0.002$, so $.02L = .002$, and $L = .1 = 10\%$.

8.2.63 Because $n! \ll n^n$ by Theorem 8.6, we have $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

8.2.64 $\{3^n\} \ll \{n!\}$ because $\{b^n\} \ll \{n!\}$ in Theorem 8.6. Thus, $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

8.2.65 Theorem 8.6 indicates that $\ln^q n \ll n^p$, so $\ln^{20} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{20} n} = \infty$.

8.2.66 Theorem 8.6 indicates that $\ln^q n \ll n^p$, so $\ln^{1000} n \ll n^{10}$, so $\lim_{n\to\infty} \frac{n^{10}}{\ln^{1000} n} = \infty$.

8.2.67 By Theorem 8.6, $n^p \ll b^n$, so $n^{1000} \ll 2^n$, and thus $\lim_{n\to\infty} \frac{n^{1000}}{2^n} = 0$.

8.2.68 Note that $e^{1/10} = \sqrt[10]{e} \approx 1.1$. Let $r = \frac{e^{1/10}}{2}$ and note that $0 < r < 1$. Thus $\lim_{n\to\infty} \frac{e^{n/10}}{2^n} = \lim_{n\to\infty} r^n = 0$.

8.2.69 Let $\varepsilon > 0$ be given and let N be an integer with $N > \frac{1}{\varepsilon}$. Then if $n > N$, we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon$.

8.2.70 Let $\varepsilon > 0$ be given. We wish to find N such that $|(1/n^2) - 0| < \varepsilon$ if $n > N$. This means that $\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \varepsilon$. So choose N such that $\frac{1}{N^2} < \varepsilon$, so that $N^2 > \frac{1}{\varepsilon}$, and then $N > \frac{1}{\sqrt{\varepsilon}}$. This shows that such an N always exists for each ε and thus that the limit is zero.

8.2.71 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{3n^2}{4n^2+1} - \frac{3}{4}\right| = \left|\frac{-3}{4(4n^2+1)}\right| = \frac{3}{4(4n^2+1)} < \varepsilon$. But this means that $3 < 4\varepsilon(4n^2+1)$, or $16\varepsilon n^2 + (4\varepsilon - 3) > 0$. Solving the quadratic, we get $n > \frac{1}{4}\sqrt{\frac{3}{\varepsilon} - 4}$, provided $\varepsilon < 3/4$. So let $N = \frac{1}{4}\sqrt{\frac{3}{\varepsilon}}$ if $\epsilon < 3/4$ and let $N = 1$ otherwise.

8.2.72 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $|b^{-n} - 0| = b^{-n} < \varepsilon$, so that $-n \ln b < \ln \varepsilon$. So choose N to be any integer greater than $-\frac{\ln \varepsilon}{\ln b}$.

8.2.73 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{cn}{bn+1} - \frac{c}{b}\right| = \left|\frac{-c}{b(bn+1)}\right| = \frac{c}{b(bn+1)} < \varepsilon$. But this means that $\varepsilon b^2 n + (b\varepsilon - c) > 0$, so that $N > \frac{c}{b^2\varepsilon}$ will work.

8.2.74 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \varepsilon$. Thus we want $n < \varepsilon(n^2+1)$, or $\varepsilon n^2 - n + \varepsilon > 0$. Whenever n is larger than the larger of the two roots of this quadratic, the desired inequality will hold. The roots of the quadratic are $\frac{1 \pm \sqrt{1-4\varepsilon^2}}{2\varepsilon}$, so we choose N to be any integer greater than $\frac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon}$.

8.2.75

a. True. See Theorem 8.2 part 4.

b. False. For example, if $a_n = 1/n$ and $b_n = e^n$, then $\lim_{n\to\infty} a_n b_n = \infty$.

c. True. The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L. Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.

d. True. Note that a_n converges to zero. Intuitively, the nonzero terms of b_n are those of a_n, which converge to zero. More formally, given ϵ, choose N_1 such that for $n > N_1$, $a_n < \epsilon$. Let $N = 2N_1 + 1$. Then for $n > N$, consider b_n. If n is even, then $b_n = 0$ so certainly $b_n < \epsilon$. If n is odd, then $b_n = a_{(n-1)/2}$, and $(n-1)/2 > ((2N_1+1)-1)/2 = N_1$ so that $a_{(n-1)/2} < \epsilon$. Thus b_n converges to zero as well.

e. False. If $\{a_n\}$ happens to converge to zero, the statement is true. But consider for example $a_n = 2 + \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 2$, but $(-1)^n a_n$ does not converge (it oscillates between positive and negative values increasingly close to ± 2).

f. True. Suppose $\{0.000001a_n\}$ converged to L, and let $\epsilon > 0$ be given. Choose N such that for $n > N$, $|0.000001a_n - L| < \epsilon \cdot 0.000001$. Dividing through by 0.000001, we get that for $n > N$, $|a_n - 1000000L| < \epsilon$, so that a_n converges as well (to $1000000L$).

8.2.76 $\{2n-3\}_{n=3}^{\infty}$.

8.2.77 $\{(n-2)^2 + 6(n-2) - 9\}_{n=3}^{\infty} = \{n^2 + 2n - 17\}_{n=3}^{\infty}$.

8.2.78 If $f(t) = \int_1^t x^{-2}\,dx$, then $\lim_{t\to\infty} f(t) = \lim_{n\to\infty} a_n$. But

$$\lim_{t\to\infty} f(t) = \int_1^{\infty} x^{-2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\bigg|_1^b\right] = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$$

8.2.79 Evaluate the limit of each term separately: $\lim_{n\to\infty} \frac{75^{n-1}}{99^n} = \frac{1}{99}\lim_{n\to\infty}\left(\frac{75}{99}\right)^{n-1} = 0$, while $\frac{-5^n}{8^n} \le \frac{5^n \sin n}{8^n} \le \frac{5^n}{8^n}$, so by the Squeeze Theorem, this second term converges to 0 as well. Thus the sum of the terms converges to zero.

8.2.80 Because $\lim_{n\to\infty} \frac{10n}{10n+4} = 1$, and because the inverse tangent function is continuous, the given sequence has limit $\tan^{-1} 1 = \pi/4$.

8.2.81 Because $\lim_{n\to\infty} 0.99^n = 0$, and because cosine is continuous, the first term converges to $\cos 0 = 1$. The limit of the second term is $\lim_{n\to\infty} \frac{7^n+9^n}{63^n} = \lim_{n\to\infty}\left(\frac{7}{63}\right)^n + \lim_{n\to\infty}\left(\frac{9}{63}\right)^n = 0$. Thus the sum converges to 1.

8.2.82 Dividing the numerator and denominator by $n!$ gives $a_n = \frac{(4^n/n!)+5}{1+(2^n/n!)}$. By Theorem 8.6, we have $4^n \ll n!$ and $2^n \ll n!$. Thus, $\lim_{n\to\infty} a_n = \frac{0+5}{1+0} = 5$.

8.2.83 Dividing the numerator and denominator by 6^n gives $a_n = \frac{1+(1/2)^n}{1+(n^{100}/6^n)}$. By Theorem 8.6, $n^{100} \ll 6^n$. Thus $\lim_{n\to\infty} a_n = \frac{1+0}{1+0} = 1$.

8.2.84 Dividing the numerator and denominator by n^8 gives $a_n = \frac{1+(1/n)}{(1/n)+\ln n}$. Because $1 + (1/n) \to 1$ as $n \to \infty$ and $(1/n) + \ln n \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty} a_n = 0$.

8.2.85 We can write $a_n = \frac{(7/5)^n}{n^7}$. Theorem 8.6 indicates that $n^7 \ll b^n$ for $b > 1$, so $\lim_{n\to\infty} a_n = \infty$.

8.2.86 A graph shows that the sequence appears to converge. Assuming that it does, let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\lim_{n\to\infty} a_n + 2$, so $L = \frac{1}{2}L + 2$, and thus $\frac{1}{2}L = 2$, so $L = 4$.

8.2.87 A graph shows that the sequence appears to converge. Let its supposed limit be L, then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(2a_n(1-a_n)) = 2(\lim_{n\to\infty} a_n)(1-\lim_{n\to\infty} a_n)$, so $L = 2L(1-L) = 2L - 2L^2$, and thus $2L^2 - L = 0$, so $L = 0, \frac{1}{2}$. Thus the limit appears to be either 0 or 1/2; with the given initial condition, doing a few iterations by hand confirms that the sequence converges to 1/2: $a_0 = 0.3$; $a_1 = 2\cdot 0.3\cdot 0.7 = .42$; $a_2 = 2 \cdot 0.42 \cdot 0.58 = 0.4872$.

8.2.88 A graph shows that the sequence appears to converge, and to a value other than zero; let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + \frac{2}{a_n}) = \frac{1}{2}\lim_{n\to\infty} a_n + \frac{1}{\lim_{n\to\infty} a_n}$, so $L = \frac{1}{2}L + \frac{1}{L}$, and therefore $L^2 = \frac{1}{2}L^2 + 1$. So $L^2 = 2$, and thus $L = \sqrt{2}$.

8.2.89 Computing three terms gives $a_0 = 0.5, a_1 = 4 \cdot 0.5 \cdot 0.5 = 1, a_2 = 4 \cdot 1 \cdot (1-1) = 0$. All successive terms are obviously zero, so the sequence converges to 0.

8.2.90 A graph shows that the sequence appears to converge. Let its limit be L. Then $\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$, so $L = \sqrt{2+L}$. Thus we have $L^2 = 2 + L$, so $L^2 - L - 2 = 0$, and thus $L = -1, 2$. A square root can never be negative, so this sequence must converge to 2.

8.2.91 For $b = 2$, $2^3 > 3!$ but $16 = 2^4 < 4! = 24$, so the crossover point is $n = 4$. For e, $e^5 \approx 148.41 > 5! = 120$ while $e^6 \approx 403.4 < 6! = 720$, so the crossover point is $n = 6$. For 10, $24! \approx 6.2 \times 10^{23} < 10^{24}$, while $25! \approx 1.55 \times 10^{25} > 10^{25}$, so the crossover point is $n = 25$.

8.2.92

a. Rounded to the nearest fish, the populations are

$$\begin{aligned}
F_0 &= 4000 \\
F_1 &= 1.015F_0 - 80 = 3980 \\
F_2 &= 1.015F_1 - 80 \approx 3960 \\
F_3 &= 1.015F_2 - 80 \approx 3939 \\
F_4 &= 1.015F_3 - 80 \approx 3918 \\
F_5 &= 1.015F_4 - 80 \approx 3897
\end{aligned}$$

b. $F_n = 1.015F_{n-1} - 80$

c. The population decreases and eventually reaches zero.

d. With an initial population of 5500 fish, the population increases without bound.

e. If the initial population is less than 5333 fish, the population will decline to zero. This is essentially because for a population of less than 5333, the natural increase of 1.5% does not make up for the loss of 80 fish.

8.2.93

a. The profits for each of the first ten days, in dollars are:

n	0	1	2	3	4	5	6	7	8	9	10
h_n	130.00	130.75	131.40	131.95	132.40	132.75	133.00	133.15	133.20	133.15	133.00

b. The profit on an item is revenue minus cost. The total cost of keeping the heifer for n days is $.45n$, and the revenue for selling the heifer on the n^{th} day is $(200 + 5n) \cdot (.65 - .01n)$, because the heifer gains 5 pounds per day but is worth a penny less per pound each day. Thus the total profit on the n^{th} day is $h_n = (200 + 5n) \cdot (.65 - .01n) - .45n = 130 + 0.8n - 0.05n^2$. The maximum profit occurs when $-.1n + .8 = 0$, which occurs when $n = 8$. The maximum profit is achieved by selling the heifer on the 8^{th} day.

8.2.94

a. $x_0 = 7$, $x_1 = 6$, $x_2 = 6.5 = \frac{13}{2}$, $x_3 = 6.25$, $x_4 = 6.375 = \frac{51}{8}$, $x_5 = 6.3125 = \frac{101}{16}$, $x_6 = 6.34375 = \frac{203}{32}$.

b. For the formula given in the problem, we have $x_0 = \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^0 = 7$, $x_1 = \frac{19}{3} + \frac{2}{3} \cdot \frac{-1}{2} = \frac{19}{3} - \frac{1}{3} = 6$, so that the formula holds for $n = 0, 1$. Now assume the formula holds for all integers $\le k$; then

$$\begin{aligned} x_{k+1} = \frac{1}{2}(x_k + x_{k-1}) &= \frac{1}{2}\left(\frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^k + \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\right) \\ &= \frac{1}{2}\left(\frac{38}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k-1}\left(-\frac{1}{2}+1\right)\right) \\ &= \frac{1}{2}\left(\frac{38}{3} + 4 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1} \cdot \frac{1}{2}\right) \\ &= \frac{1}{2}\left(\frac{38}{3} + 2 \cdot \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}\right) \\ &= \frac{19}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^{k+1}. \end{aligned}$$

c. As $n \to \infty$, $(-1/2)^n \to 0$, so that the limit is 19/3, or 6 1/3.

8.2.95 The approximate first few values of this sequence are:

n	0	1	2	3	4	5	6
c_n	.7071	.6325	.6136	.6088	.6076	.6074	.6073

The value of the constant appears to be around 0.607.

8.2.96 We first prove that d_n is bounded by 200. If $d_n \le 200$, then $d_{n+1} = 0.5 \cdot d_n + 100 \le 0.5 \cdot 200 + 100 \le 200$. Because $d_0 = 100 < 200$, all d_n are at most 200. Thus the sequence is bounded. To see that it is monotone, look at

$$d_n - d_{n-1} = 0.5 \cdot d_{n-1} + 100 - d_{n-1} = 100 - 0.5 d_{n-1}.$$

But we know that $d_{n-1} \le 200$, so that $100 - 0.5 d_{n-1} \ge 0$. Thus $d_n \ge d_{n-1}$ and the sequence is nondecreasing.

8.2.97

a. If we "cut off" the expression after n square roots, we get a_n from the recurrence given. We can thus *define* the infinite expression to be the limit of a_n as $n \to \infty$.

b. $a_0 = 1$, $a_1 = \sqrt{2}$, $a_2 = \sqrt{1 + \sqrt{2}} \approx 1.5538$, $a_3 \approx 1.5981$, $a_4 \approx 1.6118$, and $a_5 \approx 1.6161$.

c. $a_{10} \approx 1.618$, which differs from $\frac{1+\sqrt{5}}{2} \approx 1.61803394$ by less than .001.

d. Assume $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1 + a_n} = \sqrt{1 + \lim_{n\to\infty} a_n}$, so $L = \sqrt{1 + L}$, and thus $L^2 = 1 + L$. Therefore we have $L^2 - L - 1 = 0$, so $L = \frac{1 \pm \sqrt{5}}{2}$.

Because clearly the limit is positive, it must be the positive square root.

e. Letting $a_{n+1} = \sqrt{p + \sqrt{a_n}}$ with $a_0 = p$ and assuming a limit exists we have $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{p + a_n}$ $= \sqrt{p + \lim_{n\to\infty} a_n}$, so $L = \sqrt{p + L}$, and thus $L^2 = p + L$. Therefore, $L^2 - L - p = 0$, so $L = \frac{1 \pm \sqrt{1+4p}}{2}$, and because we know that L is positive, we have $L = \frac{1+\sqrt{4p+1}}{2}$. The limit exists for all positive p.

8.2.98 Note that $1 - \frac{1}{i} = \frac{i-1}{i}$, so that the product is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots$, so that $a_n = \frac{1}{n}$ for $n \ge 2$. The sequence $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ has limit zero.

8.2.99

a. Define a_n as given in the problem statement. Then we can *define* the value of the continued fraction to be $\lim_{n\to\infty} a_n$.

b. $a_0 = 1$, $a_1 = 1 + \frac{1}{a_0} = 2$, $a_2 = 1 + \frac{1}{a_1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{a_2} = \frac{5}{3} \approx 1.667$, $a_4 = 1 + \frac{1}{a_3} = \frac{8}{5} = 1.6$, $a_5 = 1 + \frac{1}{a_4} = \frac{13}{8} = 1.625$.

c. From the list above, the values of the sequence alternately decrease and increase, so we would expect that the limit is somewhere between 1.6 and 1.625.

d. Assume that the limit is equal to L. Then from $a_{n+1} = 1 + \frac{1}{a_n}$, we have $\lim_{n\to\infty} a_{n+1} = 1 + \frac{1}{\lim_{n\to\infty} a_n}$, so $L = 1 + \frac{1}{L}$, and thus $L^2 - L - 1 = 0$. Therefore, $L = \frac{1\pm\sqrt{5}}{2}$, and because L is clearly positive, it must be equal to $\frac{1+\sqrt{5}}{2} \approx 1.618$.

e. Here $a_0 = a$ and $a_{n+1} = a + \frac{b}{a_n}$. Assuming that $\lim_{n\to\infty} a_n = L$ we have $L = a + \frac{b}{L}$, so $L^2 = aL + b$, and thus $L^2 - aL - b = 0$. Therefore, $L = \frac{a\pm\sqrt{a^2+4b}}{2}$, and because $L > 0$ we have $L = \frac{a+\sqrt{a^2+4b}}{2}$.

8.2.100

a. With $p = 0.5$ we have for $a_{n+1} = a_n^p$:

n	1	2	3	4	5	6	7
a_n	0.707	0.841	0.971	0.958	0.979	0.989	0.995

Experimenting with recurrence (1) one sees that for $0 < p \leq 1$ the sequence converges to 1, while for $p > 1$ the sequence diverges to ∞.

b. With $p = 1.2$ and $a_n = p^{a_{n-1}}$ we obtain

n	1	2	3	4	5	6	7	8	9	10
a_n	1.2	1.2446	1.2547	1.2570	1.2577	1.2577	1.2577	1.2577	1.2577	1.2577

With recurrence (2), in addition to converging for $p < 1$ it also converges for values of p less than approximately 1.444. Here is a table of approximate values for different values of p:

p	1.1	1.2	1.3	1.4	1.44	1.444	1.445
$\lim_{n\to\infty} a_n$	1.1118	1.25776	1.471	1.887	2.39385	2.587	Diverges

It appears that the upper limit of convergence is about 1.444.

8.2.101

a. $f_0 = f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, f_7 = 21, f_8 = 34, f_9 = 55, f_{10} = 89$.

b. The sequence is clearly not bounded.

c. $\frac{f_{10}}{f_9} \approx 1.61818$

d. We use induction. Note that $\frac{1}{\sqrt{5}}\left(\varphi+\frac{1}{\varphi}\right)=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}+\frac{2}{1+\sqrt{5}}\right)=\frac{1}{\sqrt{5}}\left(\frac{1+2\sqrt{5}+5+4}{2(1+\sqrt{5})}\right)=1=f_1$. Also note that $\frac{1}{\sqrt{5}}\left(\varphi^2-\frac{1}{\varphi^2}\right)=\frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}-\frac{2}{3+\sqrt{5}}\right)=\frac{1}{\sqrt{5}}\left(\frac{9+6\sqrt{5}+5-4}{2(3+\sqrt{5})}\right)=1=f_2$. Now note that

$$\begin{aligned}f_{n-1}+f_{n-2}&=\frac{1}{\sqrt{5}}(\varphi^{n-1}-(-1)^{n-1}\varphi^{1-n}+\varphi^{n-2}-(-1)^{n-2}\varphi^{2-n})\\&=\frac{1}{\sqrt{5}}((\varphi^{n-1}+\varphi^{n-2})-(-1)^n(\varphi^{2-n}-\varphi^{1-n})).\end{aligned}$$

Now, note that $\varphi-1=\frac{1}{\varphi}$, so that

$$\varphi^{n-1}+\varphi^{n-2}=\varphi^{n-1}\left(1+\frac{1}{\varphi}\right)=\varphi^{n-1}\cdot\varphi=\varphi^n$$

and

$$\varphi^{2-n}-\varphi^{1-n}=\varphi^{-n}(\varphi^2-\varphi)=\varphi^{-n}(\varphi(\varphi-1))=\varphi^{-n}.$$

Making these substitutions, we get

$$f_n=f_{n-1}+f_{n-2}=\frac{1}{\sqrt{5}}(\varphi^n-(-1)^n\varphi^{-n})$$

8.2.102

a. We show that the arithmetic mean of any two positive numbers exceeds their geometric mean. Let a, $b>0$; then $\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}(a-2\sqrt{ab}+b)=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2\geq 0$. Because in addition $a_0>b_0$, we have $a_n>b_n$ for all n.

b. To see that $\{a_n\}$ is decreasing, note that

$$a_{n+1}=\frac{a_n+b_n}{2}<\frac{a_n+a_n}{2}=a_n.$$

Similarly,

$$b_{n+1}=\sqrt{a_nb_n}>\sqrt{b_nb_n}=b_n,$$

so that $\{b_n\}$ is increasing.

c. $\{a_n\}$ is monotone and nonincreasing by part (b), and bounded below by part (a) (it is bounded below by any of the b_n), so it converges by the monotone convergence theorem. Similarly, $\{b_n\}$ is monotone and nondecreasing by part (b) and bounded above by part (a), so it too converges.

d.

$$a_{n+1}-b_{n+1}=\frac{a_n+b_n}{2}-\sqrt{a_nb_n}=\frac{1}{2}(a_n-2\sqrt{a_nb_n}+b_n)<\frac{1}{2}(a_n-2\sqrt{b_n^2}+b_n)=\frac{1}{2}(a_n-b_n).$$

Thus the difference between a_{n+1} and b_{n+1} is less than half the difference between a_n and b_n, so that difference goes to zero and the two limits are the same.

e. The AGM of 12 and 20 is approximately 15.745; Gauss' constant is $\frac{1}{\mathrm{AGM}(1,\sqrt{2})}\approx 0.8346$.

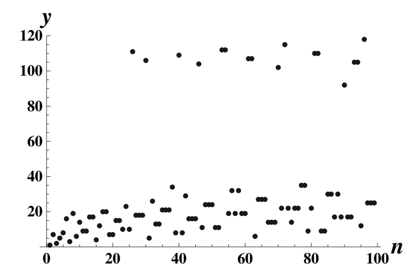
8.2.103

a.

$$\begin{aligned}
2 &: \quad 1 \\
3 &: \quad 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\
4 &: \quad 2,\ 1 \\
5 &: \quad 16,\ 8,\ 4,\ 2,\ 1 \\
6 &: \quad 3,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\
7 &: \quad 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\
8 &: \quad 4,\ 2,\ 1 \\
9 &: \quad 28,\ 14,\ 7,\ 22,\ 11,\ 34,\ 17,\ 52,\ 26,\ 13,\ 40,\ 20,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1 \\
10 &: \quad 5,\ 16,\ 8,\ 4,\ 2,\ 1
\end{aligned}$$

b. From the above, $H_2 = 1, H_3 = 7$, and $H_4 = 2$.

c. This plot is for $1 \le n \le 100$. Like hailstones, the numbers in the sequence a_n rise and fall but eventually crash to the earth. The conjecture appears to be true.



8.2.104 $\{a_n\} \ll \{b_n\}$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. But $\lim_{n\to\infty} \frac{ca_n}{db_n} = \frac{c}{d} \lim_{n\to\infty} \frac{a_n}{b_n} = 0$, so that $\{ca_n\} \ll \{db_n\}$.

8.2.105

a. Note that $a_2 = \sqrt{3a_1} = \sqrt{3\sqrt{3}} > \sqrt{3} = a_1$. Now assume that $\sqrt{3} = a_1 < a_2 < \ldots a_{k-1} < a_k$. Then

$$a_{k+1} = \sqrt{3a_k} > \sqrt{3a_{k-1}} = a_k.$$

Thus $\{a_n\}$ is increasing.

b. Clearly because $a_1 = \sqrt{3} > 0$ and $\{a_n\}$ is increasing, the sequence is bounded below by $\sqrt{3} > 0$. Further, $a_1 = \sqrt{3} < 3$; assume that $a_k < 3$. Then $a_{k+1} = \sqrt{3a_k} < \sqrt{3 \cdot 3} = 3$, so that $a_{k+1} < 3$. So by induction, $\{a_k\}$ is bounded above by 3.

c. Because $\{a_n\}$ is bounded and monotonically increasing, $\lim_{n\to\infty} a_n$ exists by Theorem 8.5.

d. Because the limit exists, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n} = \sqrt{3} \lim_{n\to\infty} \sqrt{a_n} = \sqrt{3}\sqrt{\lim_{n\to\infty} a_n}.$$

Let $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$; then $L = \sqrt{3}\sqrt{L}$, so that $L = 3$.

8.2.106 By Theorem 8.6,

$$\lim_{n\to\infty} \frac{2\ln n}{\sqrt{n}} = 2 \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = 0,$$

so that $\sqrt{n}$ has the larger growth rate. Using computational software, we see that $\sqrt{74} \approx 8.60233 < 2\ln 74 \approx 8.60813$, while $\sqrt{75} \approx 8.66025 > 2\ln 75 \approx 8.63493$.

8.2.107 By Theorem 8.6,
$$\lim_{n\to\infty} \frac{n^5}{e^{n/2}} = 2^5 \lim_{n\to\infty} \frac{(n/2)^5}{e^{n/2}} = 0,$$
so that $e^{n/2}$ has the larger growth rate. Using computational software we see that $e^{35/2} \approx 3.982 \times 10^7 < 35^5 \approx 5.252 \times 10^7$, while $e^{36/2} \approx 6.566 \times 10^7 > 36^5 \approx 6.047 \times 10^7$.

8.2.108 By Theorem 8.6, $\ln n^{10} \ll n^{1.001}$, so that $n^{1.001}$ has the larger growth rate. Using computational software we see that $35^{1.001} \approx 35.1247 < \ln 35^{10} \approx 35.5535$ while $36^{1.001} \approx 36.1292 > \ln 36^{10} \approx 35.8352$.

8.2.109 Experiment with a few widely separated values of n:

n	$n!$	$n^{0.7n}$
1	1	1
10	3.63×10^6	10^7
100	9.33×10^{157}	10^{140}
1000	4.02×10^{2567}	10^{2100}

It appears that $n^{0.7n}$ starts out larger, but is overtaken by the factorial somewhere between $n = 10$ and $n = 100$, and that the gap grows wider as n increases. Looking between $n = 10$ and $n = 100$ revels that for $n = 18$, we have $n! \approx 6.402 \times 10^{15} < n^{0.7n} \approx 6.553 \times 10^{15}$ while for $n = 19$ we have $n! \approx 1.216 \times 10^{17} > n^{0.7n} \approx 1.017 \times 10^{17}$.

8.2.110 By Theorem 8.6,
$$\lim_{n\to\infty} \frac{n^9 \ln^3 n}{n^{10}} = \lim_{n\to\infty} \frac{\ln^3 n}{n} = 0,$$
so that n^{10} has a larger growth rate. Using computational software we see that $93^{10} \approx 4.840 \times 10^{19} < 93^9 \ln^3 93 \approx 4.846 \times 10^{19}$ while $94^{10} \approx 5.386 \times 10^{19} > 94^9 \ln^3 94 \approx 5.374 \times 10^{19}$.

8.2.111 First note that for $a = 1$ we already know that $\{n^n\}$ grows fast than $\{n!\}$. So if $a > 1$, then $n^{an} \geq n^n$, so that $\{n^{an}\}$ grows faster than $\{n!\}$ for $a > 1$ as well. To settle the case $a < 1$, recall Stirling's formula which states that for large values of n,
$$n! \sim \sqrt{2\pi n} n^n e^{-n}.$$
Thus
$$\begin{aligned}
\lim_{n\to\infty} \frac{n!}{n^{an}} &= \lim_{n\to\infty} \frac{\sqrt{2\pi n} n^n e^{-n}}{n^{an}} \\
&= \sqrt{2\pi} \lim_{n\to\infty} n^{\frac{1}{2}+(1-a)n} e^{-n} \\
&\geq \sqrt{2\pi} \lim_{n\to\infty} n^{(1-a)n} e^{-n} \\
&= \sqrt{2\pi} \lim_{n\to\infty} e^{(1-a)n \ln n} e^{-n} \\
&= \sqrt{2\pi} \lim_{n\to\infty} e^{((1-a)\ln n - 1)n}.
\end{aligned}$$
If $a < 1$ then $(1 - a)\ln n - 1 > 0$ for large values of n because $1 - a > 0$, so that this limit is infinite. Hence $\{n!\}$ grows faster than $\{n^{an}\}$ exactly when $a < 1$.

8.3 Infinite Series

8.3.1 A geometric series is a series in which the ratio of successive terms in the underlying sequence is a constant. Thus a geometric series has the form $\sum ar^k$ where r is the constant. One example is $3 + 6 + 12 + 24 + 48 + \cdots$ in which $a = 3$ and $r = 2$.

8.3.2 A geometric sum is the sum of a finite number of terms which have a constant ratio; a geometric series is the sum of an infinite number of such terms.

8.3.3 The ratio is the common ratio between successive terms in the sum.

8.3.4 Yes, because there are only a finite number of terms.

8.3.5 No. For example, the geometric series with $a_n = 3 \cdot 2^n$ does not have a finite sum.

8.3.6 The series converges if and only if $|r| < 1$.

8.3.7 $S = 1 \cdot \dfrac{1-3^9}{1-3} = \dfrac{19682}{2} = 9841.$

8.3.8 $S = 1 \cdot \dfrac{1-(1/4)^{11}}{1-(1/4)} = \dfrac{4^{11}-1}{3 \cdot 4^{10}} = \dfrac{4194303}{3 \cdot 1048576} = \dfrac{1398101}{1048576} \approx 1.333.$

8.3.9 $S = 1 \cdot \dfrac{1-(4/25)^{21}}{1-4/25} = \dfrac{25^{21}-4^{21}}{25^{21}-4 \cdot 25^{20}} \approx 1.1905.$

8.3.10 $S = 16 \cdot \dfrac{1-2^9}{1-2} = 511 \cdot 16 = 8176.$

8.3.11 $S = 1 \cdot \dfrac{1-(-3/4)^{10}}{1+3/4} = \dfrac{4^{10}-3^{10}}{4^{10}+3 \cdot 4^9} = \dfrac{141361}{262144} \approx 0.5392.$

8.3.12 $S = (-2.5) \cdot \dfrac{1-(-2.5)^5}{1+2.5} = -70.46875.$

8.3.13 $S = 1 \cdot \dfrac{1-\pi^7}{1-\pi} = \dfrac{\pi^7-1}{\pi-1} \approx 1409.84.$

8.3.14 $S = \dfrac{4}{7} \cdot \dfrac{1-(4/7)^{10}}{3/7} = \dfrac{375235564}{282475249} \approx 1.328.$

8.3.15 $S = 1 \cdot \dfrac{1-(-1)^{21}}{2} = 1.$

8.3.16 $\dfrac{65}{27}.$

8.3.17 $\dfrac{1093}{2916}.$

8.3.18 $\dfrac{1}{5}\left(\dfrac{1-(3/5)^6}{1-3/5}\right) = \dfrac{7448}{15625}.$

8.3.19 $\dfrac{1}{1-1/4} = \dfrac{4}{3}.$

8.3.20 $\dfrac{1}{1-3/5} = \dfrac{5}{2}.$

8.3.21 $\dfrac{1}{1-0.9} = 10.$

8.3.22 $\dfrac{1}{1-2/7} = \dfrac{7}{5}.$

8.3.23 Divergent, because $r > 1$.

8.3.24 $\dfrac{1}{1-1/\pi} = \dfrac{\pi}{\pi-1}.$

8.3.25 $\dfrac{e^{-2}}{1-e^{-2}} = \dfrac{1}{e^2-1}.$

8.3.26 $\dfrac{5/4}{1-1/2} = \dfrac{5}{2}.$

8.3.27 $\dfrac{2^{-3}}{1-2^{-3}} = \dfrac{1}{7}.$

8.3.28 $\dfrac{3\cdot 4^3/7^3}{1-4/7}=\dfrac{64}{49}$.

8.3.29 $\dfrac{1/625}{1-1/5}=\dfrac{1}{500}$.

8.3.30 Note that this is the same as $\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^k$. Then $S=\dfrac{1}{1-3/4}=4$.

8.3.31 $\dfrac{1}{1-e/\pi}=\dfrac{\pi}{\pi-e}$. (Note that $e<\pi$, so $r<1$ for this series.)

8.3.32 $\dfrac{1/16}{1-3/4}=\dfrac{1}{4}$.

8.3.33 $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k 5^{3-k}=5^3\sum_{k=0}^{\infty}\left(\frac{1}{20}\right)^k=5^3\cdot\frac{1}{1-1/20}=\frac{5^3\cdot 20}{19}=\frac{2500}{19}$.

8.3.34 $\dfrac{3^6/8^6}{1-(3/8)^3}=\dfrac{729}{248320}$

8.3.35 $\dfrac{1}{1+9/10}=\dfrac{10}{19}$.

8.3.36 $-\dfrac{2/3}{1+2/3}=-\dfrac{2}{5}$.

8.3.37 $3\cdot\dfrac{1}{1+1/\pi}=\dfrac{3\pi}{\pi+1}$.

8.3.38 $\displaystyle\sum_{k=1}^{\infty}\left(-\frac{1}{e}\right)^k=-\frac{1/e}{1+1/e}=-\frac{1}{e+1}$.

8.3.39 $\dfrac{0.15^2}{1.15}=\dfrac{9}{460}\approx 0.0196$.

8.3.40 $-\dfrac{3/8^3}{1+1/8^3}=-\dfrac{1}{171}$.

8.3.41

a. $0.\overline{3}=0.333\ldots=\sum_{k=1}^{\infty}3(0.1)^k$.

b. The limit of the sequence of partial sums is 1/3.

8.3.42

a. $0.\overline{6}=0.666\ldots=\sum_{k=1}^{\infty}6(0.1)^k$.

b. The limit of the sequence of partial sums is 2/3.

8.3.43

a. $0.\overline{1}=0.111\ldots=\sum_{k=1}^{\infty}(0.1)^k$.

b. The limit of the sequence of partial sums is 1/9.

8.3.44

a. $0.\overline{5}=0.555\ldots=\sum_{k=1}^{\infty}5(0.1)^k$.

b. The limit of the sequence of partial sums is 5/9.

8.3.45

a. $0.\overline{09}=0.0909\ldots=\sum_{k=1}^{\infty}9(0.01)^k$.

b. The limit of the sequence of partial sums is 1/11.

8.3.46

a. $0.\overline{27}=0.272727\ldots=\sum_{k=1}^{\infty}27(0.01)^k$.

b. The limit of the sequence of partial sums is 3/11.

8.3.47

a. $0.\overline{037}=0.037037037\ldots=\sum_{k=1}^{\infty}37(0.001)^k$.

b. The limit of the sequence of partial sums is $37/999=1/27$.

8.3.48

a. $0.\overline{027}=0.027027027\ldots=\sum_{k=1}^{\infty}27(0.001)^k$

b. The limit of the sequence of partial sums is $27/999=1/37$.

8.3.49 $0.\overline{12}=0.121212\ldots=\displaystyle\sum_{k=0}^{\infty}.12\cdot 10^{-2k}=\frac{.12}{1-1/100}=\frac{12}{99}=\frac{4}{33}$.

8.3.50 $1.\overline{25}=1.252525\ldots=1+\displaystyle\sum_{k=0}^{\infty}.25\cdot 10^{-2k}=1+\frac{.25}{1-1/100}=1+\frac{25}{99}=\frac{124}{99}$.

8.3.51 $0.\overline{456} = 0.456456456\ldots = \displaystyle\sum_{k=0}^{\infty} .456 \cdot 10^{-3k} = \frac{.456}{1-1/1000} = \frac{456}{999} = \frac{152}{333}$.

8.3.52 $1.00\overline{39} = 1.00393939\ldots = 1+\displaystyle\sum_{k=0}^{\infty} .0039 \cdot 10^{-2k} = 1+\frac{.0039}{1-1/100} = 1+\frac{.39}{99} = 1+\frac{39}{9900} = \frac{9939}{9900} = \frac{3313}{3300}$.

8.3.53 $0.00\overline{952} = 0.00952952\ldots = \displaystyle\sum_{k=0}^{\infty} .00952 \cdot 10^{-3k} = \frac{.00952}{1-1/1000} = \frac{9.52}{999} = \frac{952}{99900} = \frac{238}{24975}$.

8.3.54 $5.12\overline{83} = 5.12838383\ldots = 5.12+\displaystyle\sum_{k=0}^{\infty} .0083 \cdot 10^{-2k} = 5.12+\frac{.0083}{1-1/100} = \frac{512}{100}+\frac{.83}{99} = \frac{128}{25}+\frac{83}{9900} = \frac{50771}{9900}$.

8.3.55 The second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+1} - \frac{1}{n+2} = \frac{n}{2n+4}$, and $\lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2}$.

8.3.56 The second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+2} - \frac{1}{n+3} = \frac{n}{3n+6}$, and $\lim_{n\to\infty} \frac{n}{3n+9} = \frac{1}{3}$.

8.3.57 $\dfrac{1}{(k+6)(k+7)} = \dfrac{1}{k+6} - \dfrac{1}{k+7}$, so the series given is the same as $\sum_{k=1}^{\infty}\left(\frac{1}{k+6} - \frac{1}{k+7}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+6} - \frac{1}{n+7}$. Thus $\lim_{n\to\infty} S_n = \frac{1}{7}$.

8.3.58 $\dfrac{1}{(3k+1)(3k+4)} = \dfrac{1}{3}\left(\dfrac{1}{3k+1} - \dfrac{1}{3k+4}\right)$, so the series given can be written $\dfrac{1}{3}\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{3k+1} - \frac{1}{3k+4}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $3(k+1)+1 = 3k+4$), so we are left with $S_n = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{3n+4}\right) = \frac{n+1}{3n+4}$ and $\lim_{n\to\infty} \frac{n+1}{3n+4} = \frac{1}{3}$.

8.3.59 Note that $\frac{4}{(4k-3)(4k+1)} = \frac{1}{4k-3} - \frac{1}{4k+1}$. Thus the given series is the same as $\displaystyle\sum_{k=3}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k+1}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $4(k+1)-3 = 4k+1$), so we have $S_n = \frac{1}{9} - \frac{1}{4n+1}$, and thus $\lim_{n\to\infty} S_n = \dfrac{1}{9}$.

8.3.60 Note that $\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$. Thus the given series is the same as $\displaystyle\sum_{k=3}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$. In that series, the second part of each term cancels with the first part of the succeeding term (because $2(k+1)-1 = 2k+1$), so we have $S_n = \frac{1}{5} - \frac{1}{2n+1}$. Thus, $\lim_{n\to\infty} S_n = \dfrac{1}{5}$.

8.3.61 $\ln\left(\dfrac{k+1}{k}\right) = \ln(k+1) - \ln k$, so the series given is the same as $\sum_{k=1}^{\infty}(\ln(k+1) - \ln k)$, in which the first part of each term cancels with the second part of the next term, so we have $S_n = \ln(n+1) - \ln 1 = \ln(n+1)$, and thus the series diverges.

8.3.62 Note that $S_n = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + (\sqrt{n+1}-\sqrt{n})$. The second part of each term cancels with the first part of the previous term. Thus, $S_n = \sqrt{n+1} - 1$. and because $\lim_{n\to\infty} \sqrt{n+1} - 1 = \infty$, the series diverges.

8.3.63 $\dfrac{1}{(k+p)(k+p+1)} = \dfrac{1}{k+p} - \dfrac{1}{k+p+1}$, so that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k+p} - \frac{1}{k+p+1}\right)$ and this series telescopes to give $S_n = \frac{1}{p+1} - \frac{1}{n+p+1} = \frac{n}{n(p+1)+(p+1)^2}$ so that $\lim\limits_{n\to\infty} S_n = \frac{1}{p+1}$.

8.3.64 $\dfrac{1}{(ak+1)(ak+a+1)} = \dfrac{1}{a}\left(\dfrac{1}{ak+1} - \dfrac{1}{ak+a+1}\right)$, so that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(ak+1)(ak+a+1)} = \frac{1}{a}\sum_{k=1}^{\infty}\left(\frac{1}{ak+1} - \frac{1}{ak+a+1}\right)$. This series telescopes - the second term of each summand cancels with the first term of the succeeding summage – so that $S_n = \frac{1}{a}\left(\frac{1}{a+1} - \frac{1}{an+a+1}\right)$, and thus the limit of the sequence is $\frac{1}{a(a+1)}$.

8.3.65 Let $a_n = \dfrac{1}{\sqrt{n+1}} - \dfrac{1}{\sqrt{n+3}}$. Then the second term of a_n cancels with the first term of a_{n+2}, so the series telescopes and $S_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{n-1+3}} - \frac{1}{\sqrt{n+3}}$ and thus the sum of the series is the limit of S_n, which is $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}}$.

8.3.66 The first term of the k^{th} summand is $\sin(\frac{(k+1)\pi}{2k+1})$; the second term of the $(k+1)^{\text{st}}$ summand is $-\sin(\frac{(k+1)\pi}{2(k+1)-1})$; these two are equal except for sign, so they cancel. Thus $S_n = -\sin 0 + \sin(\frac{(n+1)\pi}{2n+1}) = \sin(\frac{(n+1)\pi}{2n+1})$. Because $\frac{(n+1)\pi}{2n+1}$ has limit $\pi/2$ as $n \to \infty$, and because the sine function is continuous, it follows that $\lim\limits_{n\to\infty} S_n$ is $\sin(\frac{\pi}{2}) = 1$.

8.3.67 $16k^2 + 8k - 3 = (4k+3)(4k-1)$, so $\frac{1}{16k^2+8k-3} = \frac{1}{(4k+3)(4k-1)} = \frac{1}{4}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$. Thus the series given is equal to $\displaystyle\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$. This series telescopes, so $S_n = \frac{1}{4}\left(-1 - \frac{1}{4n+3}\right)$, so the sum of the series is equal to $\lim\limits_{n\to\infty} S_n = -\frac{1}{4}$.

8.3.68 This series clearly telescopes to give $S_n = -\tan^{-1}(1) + \tan^{-1}(n) = \tan^{-1}(n) - \frac{\pi}{4}$. Then because $\lim\limits_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$, the sum of the series is equal to $\lim\limits_{n\to\infty} S_n = \frac{\pi}{4}$.

8.3.69

a. True. $\left(\dfrac{\pi}{e}\right)^{-k} = \left(\dfrac{e}{\pi}\right)^{k}$; because $e < \pi$, this is a geometric series with ratio less than 1.

b. True. If $\displaystyle\sum_{k=12}^{\infty} a^k = L$, then $\displaystyle\sum_{k=0}^{\infty} a^k = \left(\sum_{k=0}^{11} a^k\right) + L$.

c. False. For example, let $0 < a < 1$ and $b > 1$.

d. True. Suppose $a > \frac{1}{2}$. Then we want $a = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Solving for r gives $r = 1 - \frac{1}{a}$. Because $a > 0$ we have $r < 1$; because $a > \frac{1}{2}$ we have $r > 1 - \frac{1}{1/2} = -1$. Thus $|r| < 1$ so that $\sum_{k=0}^{\infty} r^k$ converges, and it converges to a.

e. True. Suppose $a > -\frac{1}{2}$. Then we want $a = \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$. Solving for r gives $r = \frac{a}{a+1}$. For $a \ge 0$, clearly $0 \le r < 1$ so that $\sum_{k=1}^{\infty} r^k$ converges to a. For $-\frac{1}{2} < a < 0$, clearly $r < 0$, but $|a| < |a+1|$, so that $|r| < 1$. Thus in this case $\sum_{k=1}^{\infty} r^k$ also converges to a.

8.3.70 We have

$$S_n = \left(\sin^{-1} 1 - \sin^{-1}\frac{1}{2}\right) + \left(\sin^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{3}\right) + \cdots + \left(\sin^{-1}\frac{1}{n} - \sin^{-1}\frac{1}{n+1}\right).$$

Note that the first part of each term cancels the second part of the previous term, so the nth partial sum telescopes to be $\sin^{-1} 1 - \sin^{-1} \frac{1}{n+1}$. Because $\sin^{-1} 1 = \frac{\pi}{2}$ and $\lim_{n\to\infty} \sin^{-1} \dfrac{1}{n+1} = \sin^{-1} 0 = 0$, we have $\lim_{n\to\infty} S_n = \dfrac{\pi}{2}$.

8.3.71 This can be written as $\dfrac{1}{3}\sum_{k=1}^{\infty}\left(-\dfrac{2}{3}\right)^k$. This is a geometric series with ratio $r = -\frac{2}{3}$ so the sum is $\frac{1}{3} \cdot \frac{-2/3}{1-(-2/3)} = \frac{1}{3} \cdot \left(-\frac{2}{5}\right) = -\frac{2}{15}$.

8.3.72 This can be written as $\dfrac{1}{e}\sum_{k=1}^{\infty}\left(\dfrac{\pi}{e}\right)^k$. This is a geometric series with $r = \frac{\pi}{e} > 1$, so the series diverges.

8.3.73 Note that

$$\frac{\ln((k+1)k^{-1})}{(\ln k)\ln(k+1)} = \frac{\ln(k+1)}{(\ln k)\ln(k+1)} - \frac{\ln k}{(\ln k)\ln(k+1)} = \frac{1}{\ln k} - \frac{1}{\ln(k+1)}.$$

In the partial sum S_n, the first part of each term cancels the second part of the preceding term, so we have $S_n = \frac{1}{\ln 2} - \frac{1}{\ln(n+1)}$. Thus we have $\lim_{n\to\infty} S_n = \dfrac{1}{\ln 2}$.

8.3.74

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{1}{2}$.

b. Note that $\frac{1}{2^k} - \frac{1}{2^{k+1}} = \frac{2^{k+1}-2^k}{2^k 2^{k+1}} = \frac{1}{2^{k+1}}$. Thus, the original series can be written as $\sum_{k=1}^{\infty} \dfrac{1}{2^{k+1}}$ which is geometric with $r = 1/2$ and $a = 1/4$, so the sum is $\frac{1/4}{1-1/2} = \frac{1}{2}$.

8.3.75

a. Because the first part of each term cancels the second part of the previous term, the nth partial sum telescopes to be $S_n = \frac{4}{3} - \frac{4}{3^{n+1}}$. Thus, the sum of the series is $\lim_{n\to\infty} S_n = \dfrac{4}{3}$.

b. Note that $\frac{4}{3^k} - \frac{4}{3^{k+1}} = \frac{4\cdot 3^{k+1}-4\cdot 3^k}{3^k 3^{k+1}} = \frac{8}{3^{k+1}}$. Thus, the original series can be written as $\sum_{k=1}^{\infty} \dfrac{8}{3^{k+1}}$ which is geometric with $r = 1/3$ and $a = 8/9$, so the sum is $\frac{8/9}{1-1/3} = \frac{8}{9} \cdot \frac{3}{2} = \frac{4}{3}$.

8.3.76 It will take Achilles 1/5 hour to cover the first mile. At this time, the tortoise has gone 1/5 mile more, and it will take Achilles 1/25 hour to reach this new point. At that time, the tortoise has gone another 1/25 of a mile, and it will take Achilles 1/125 hour to reach this point. Adding the times up, we have

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1/5}{1-1/5} = \frac{1}{4},$$

so it will take Achilles 1/4 of an hour (15 minutes) to catch the tortoise.

8.3.77 At the n^{th} stage, there are 2^{n-1} triangles of area $A_n = \frac{1}{8}A_{n-1} = \frac{1}{8^{n-1}}A_1$, so the total area of the triangles formed at the n^{th} stage is $\dfrac{2^{n-1}}{8^{n-1}}A_1 = \left(\dfrac{1}{4}\right)^{n-1} A_1$. Thus the total area under the parabola is

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} A_1 = A_1\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} = A_1\frac{1}{1-1/4} = \frac{4}{3}A_1.$$

8.3.78

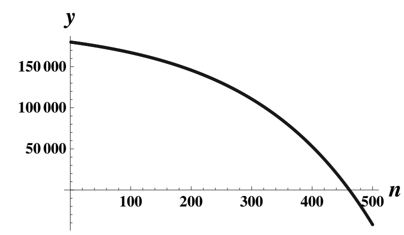
a. Note that $\frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\cdot\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right)$. Then

$$\sum_{k=1}^{\infty} \frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1}\right).$$

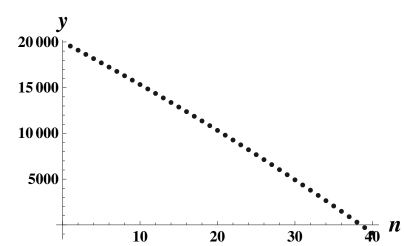
This series telescopes to give $S_n = \frac{1}{2}\left(\frac{1}{3-1} - \frac{1}{3^{n+1}-1}\right)$, so that the sum of the series is $\lim_{n\to\infty} S_n = \frac{1}{4}$.

b. We mimic the above computations. First, $\frac{a^k}{(a^{k+1}-1)(a^k-1)} = \frac{1}{a-1}\cdot\left(\frac{1}{a^k-1} - \frac{1}{a^{k+1}-1}\right)$, so we see that we cannot have $a = 1$, because the fraction would then be undefined. Continuing, we obtain $S_n = \frac{1}{a-1}\left(\frac{1}{a-1} - \frac{1}{a^{n+1}-1}\right)$. Now, $\lim_{n\to\infty} \frac{1}{a^{n+1}-1}$ converges if and only if the denominator grows without bound; this happens if and only if $|a| > 1$. Thus, the original series converges for $|a| > 1$, when it converges to $\frac{1}{(a-1)^2}$. Note that this is valid even for a negative.

8.3.79 It appears that the loan is paid off after about 470 months. Let B_n be the loan balance after n months. Then $B_0 = 180000$ and $B_n = 1.005 \cdot B_{n-1} - 1000$. Then $B_n = 1.005 \cdot B_{n-1} - 1000 = 1.005(1.005 \cdot B_{n-2} - 1000) - 1000 = (1.005)^2 \cdot B_{n-2} - 1000(1 + 1.005) = (1.005)^2 \cdot (1.005 \cdot B_{n-3} - 1000) - 1000(1+1.005) = (1.005)^3 \cdot B_{n-3} - 1000(1 + 1.005 + (1.005)^2) = \cdots = (1.005)^n B_0 - 1000(1+1.005+(1.005)^2+\cdots+(1.005)^{n-1}) = (1.005)^n \cdot 180000 - 1000\left(\frac{(1.005)^n-1}{1.005-1}\right)$. Solving this equation for $B_n = 0$ gives $n \approx 461.667$ months, so the loan is paid off after 462 months.



8.3.80 It appears that the loan is paid off after about 38 months. Let B_n be the loan balance after n months. Then $B_0 = 20000$ and $B_n = 1.0075 \cdot B_{n-1} - 60$. Then $B_n = 1.0075 \cdot B_{n-1} - 600 = 1.0075(1.0075 \cdot B_{n-2} - 600) - 600 = (1.0075)^2 \cdot B_{n-2} - 600(1 + 1.0075) = (1.0075)^2(1.0075 \cdot B_{n-3} - 600) - 600(1 + 1.0075) = (1.0075)^3 \cdot B_{n-3} - 600(1 + 1.0075 + (1.0075)^2) = \cdots = (1.0075)^n B_0 - 600(1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1}) = (1.0075)^n \cdot 20000 - 600\left(\frac{(1.0075)^n-1}{1.0075-1}\right)$. Solving this equation for $B_n = 0$ gives $n \approx 38.501$ months, so the loan is paid off after 39 months.



8.3.81 $F_n = (1.015)F_{n-1} - 120 = (1.015)((1.015)F_{n-2} - 120) - 120 = (1.015)((1.015)((1.015)F_{n-3} - 120) - 120) - 120 = \cdots = (1.015)^n(4000) - 120(1 + (1.015) + (1.015)^2 + \cdots + (1.015)^{n-1})$. This is equal to

$$(1.015)^n(4000) - 120\left(\frac{(1.015)^n - 1}{1.015 - 1}\right) = (-4000)(1.015)^n + 8000.$$

The long term population of the fish is 0.

8.3.82 Let A_n be the amount of antibiotic in your blood after n 6-hour periods. Then $A_0 = 200, A_n = 0.5A_{n-1} + 200$. We have $A_n = .5A_{n-1} + 200 = .5(.5A_{n-2} + 200) + 200 = .5(.5(.5A_{n-3} + 200) + 200) + 200 = \cdots = .5^n(200) + 200(1 + .5 + .5^2 + \cdots + .5^{n-1})$. This is equal to

$$.5^n(200) + 200\left(\frac{.5^n - 1}{.5 - 1}\right) = (.5^n)(200 - 400) + 400 = (-200)(.5^n) + 400.$$

The limit of this expression as $n \to \infty$ is 400, so the steady-state amount of antibiotic in your blood is 400 mg.

8.3.83 Under the one-child policy, each couple will have one child. Under the one-son policy, we compute the expected number of children as follows: with probability 1/2 the first child will be a son; with probability $(1/2)^2$, the first child will be a daughter and the second child will be a son; in general, with probability $(1/2)^n$, the first $n - 1$ children will be girls and the n^{th} a boy. Thus the expected number of children is the sum $\displaystyle\sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i$. To evaluate this series, use the following "trick": Let $f(x) = \displaystyle\sum_{i=1}^{\infty} ix^i$. Then $f(x) + \displaystyle\sum_{i=1}^{\infty} x^i = \sum_{i=1}^{\infty}(i + 1)x^i$. Now, let

$$g(x) = \sum_{i=1}^{\infty} x^{i+1} = -1 - x + \sum_{i=0}^{\infty} x^i = -1 - x + \frac{1}{1 - x}$$

and

$$g'(x) = f(x) + \sum_{i=1}^{\infty} x^i = f(x) - 1 + \sum_{i=0}^{\infty} x^i = f(x) - 1 + \frac{1}{1 - x}.$$

Evaluate $g'(x) = -1 - \frac{1}{(1-x)^2}$; then

$$f(x) = 1 - \frac{1}{1 - x} - 1 - \frac{1}{(1 - x)^2} = \frac{-1 + x + 1}{(1 - x)^2} = \frac{x}{(1 - x)^2}$$

Finally, evaluate at $x = \frac{1}{2}$ to get $f\left(\frac{1}{2}\right) = \sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^i = \frac{1/2}{(1-1/2)^2} = 2$. There will thus be twice as many children under the one-son policy as under the one-child policy.

8.3.84 Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane. Then the amount of light that was available before the beam went through the pane was $\frac{L_n}{1-p}$, so $\frac{pL_n}{1-p}$ is reflected back to the first pane, and $\frac{p^2L_n}{1-p}$ is then reflected back to the second pane. Of that, a fraction equal to $1 - p$ is transmitted through the window. Thus

$$L_{n+1} = (1 - p)\frac{p^2L_n}{1 - p} = p^2L_n.$$

The amount of light transmitted through the window the first time is $(1 - p)^2$. Thus the total amount is

$$\sum_{i=0}^{\infty} p^{2n}(1 - p)^2 = \frac{(1 - p)^2}{1 - p^2} = \frac{1 - p}{1 + p}.$$

8.3.85 Ignoring the initial drop for the moment, the height after the n^{th} bounce is $10p^n$, so the total time spent in that bounce is $2 \cdot \sqrt{2 \cdot 10p^n/g}$ seconds. The total time before the ball comes to rest (now including the time for the initial drop) is then $\sqrt{20/g} + \sum_{i=1}^{\infty} 2 \cdot \sqrt{2 \cdot 10p^n/g} = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}} \sum_{i=1}^{\infty}(\sqrt{p})^n = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}} \frac{\sqrt{p}}{1-\sqrt{p}} = \sqrt{\frac{20}{g}}\left(1 + \frac{2\sqrt{p}}{1-\sqrt{p}}\right) = \sqrt{\frac{20}{g}}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right)$ seconds.

8.3.86

a. The fraction of available wealth spent each month is $1-p$, so the amount spent in the n^{th} month is $W(1-p)^n$. The total amount spent is then $\sum_{n=1}^{\infty} W(1-p)^n = \frac{W(1-p)}{1-(1-p)} = W\left(\frac{1-p}{p}\right)$ dollars.

b. As $p \to 1$, the total amount spent approaches 0. This makes sense, because in the limit, if everyone saves all of the money, none will be spent. As $p \to 0$, the total amount spent gets larger and larger. This also makes sense, because almost all of the available money is being respent each month.

8.3.87

a. I_{n+1} is obtained by I_n by dividing each edge into three equal parts, removing the middle part, and adding two parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1} = \frac{4}{3}L_n$. This is a geometric sequence with a ratio greater than 1, so the n^{th} term grows without bound.

b. As the result of part (a), I_n has $3\cdot 4^n$ sides of length $\frac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of side length 3^{-n-1}. Thus the added area in I_{n+1} consists of $3\cdot 4^n$ equilateral triangles with side 3^{-n-1}. The area of an equilateral triangle with side x is $\dfrac{x^2\sqrt{3}}{4}$. Thus $A_{n+1} = A_n + 3\cdot 4^n \cdot \frac{3^{-2n-2}\sqrt{3}}{4} = A_n + \frac{\sqrt{3}}{12}\cdot\left(\frac{4}{9}\right)^n$, and $A_0 = \frac{\sqrt{3}}{4}$. Thus $A_{n+1} = A_0 + \sum_{i=0}^{n} \frac{\sqrt{3}}{12}\cdot\left(\frac{4}{9}\right)^i$, so that

$$A_\infty = A_0 + \frac{\sqrt{3}}{12}\sum_{i=0}^{\infty}\left(\frac{4}{9}\right)^i = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\frac{1}{1-4/9} = \frac{\sqrt{3}}{4}(1+\frac{3}{5}) = \frac{2}{5}\sqrt{3}.$$

8.3.88

a. $5\sum_{i=1}^{\infty} 10^{-k} = 5\sum_{i=1}^{\infty}\left(\frac{1}{10}\right)^k = 5\left(\frac{1/10}{9/10}\right) = \frac{5}{9}$.

b. $54\sum_{i=1}^{\infty} 10^{-2k} = 54\sum_{i=1}^{\infty}\left(\frac{1}{100}\right)^k = 54\left(\frac{1/100}{99/100}\right) = \frac{54}{99}$.

c. Suppose $x = 0.n_1n_2\ldots n_pn_1n_2\ldots$. Then we can write this decimal as $n_1n_2\ldots n_p\sum_{i=1}^{\infty}10^{-ip} = n_1n_2\ldots n_p\sum_{i=1}^{\infty}\left(\frac{1}{10^p}\right)^i = n_1n_2\ldots n_p\frac{1/10^p}{(10^p-1)/10^p} = \frac{n_1n_2\ldots n_p}{999\ldots 9}$, where here $n_1n_2\ldots n_p$ does not mean multiplication but rather the digits in a decimal number, and where there are p 9's in the denominator.

d. According to part (c), $0.12345678912345678912\ldots = \frac{123456789}{999999999}$

e. Again using part (c), $0.\bar{9} = \frac{9}{9} = 1$.

8.3.89 $|S - S_n| = \left|\sum_{i=n}^{\infty} r^k\right| = \left|\dfrac{r^n}{1-r}\right|$ because the latter sum is simply a geometric series with first term r^n and ratio r.

8.3.90

a. Solve $\frac{0.6^n}{0.4} < 10^{-6}$ for n to get $n = 29$.

b. Solve $\frac{0.15^n}{0.85} < 10^{-6}$ for n to get $n = 8$.

8.3.91

a. Solve $\left|\frac{(-0.8)^n}{1.8}\right| = \frac{0.8^n}{1.8} < 10^{-6}$ for n to get $n = 60$.

b. Solve $\frac{0.2^n}{0.8} < 10^{-6}$ for n to get $n = 9$.

8.3.92

a. Solve $\frac{0.72^n}{0.28} < 10^{-6}$ for n to get $n = 46$.

b. Solve $\left|\frac{(-0.25)^n}{1.25}\right| = \frac{0.25^n}{1.25} < 10^{-6}$ for n to get $n = 10$.

8.3.93

a. Solve $\frac{1/\pi^n}{1-1/\pi} < 10^{-6}$ for n to get $n = 13$.

b. Solve $\frac{1/e^n}{1-1/e} < 10^{-6}$ for n to get $n = 15$.

8.3.94

a. $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$; because f is represented by a geometric series, $f(x)$ exists only for $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{0.8} = 1.25$, $f(0.5) = \frac{1}{1-0.5} = 2$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

8.3.95

a. $f(x) = \sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$; because f is a geometric series, $f(x)$ exists only when the ratio, $-x$, is such that $|-x| = |x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1.2} = \frac{5}{6}$, $f(0.5) = \frac{1}{1+.05} = \frac{2}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

8.3.96

a. $f(x) = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$. f is a geometric series, so $f(x)$ is defined only when the ratio, x^2, is less than 1, which means $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1-.04} = \frac{25}{24}$, $f(0.5) = \frac{1}{1-0.25} = \frac{4}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

8.3.97 $f(x)$ is a geometric series with ratio $\frac{1}{1+x}$; thus $f(x)$ converges when $\left|\frac{1}{1+x}\right| < 1$. For $x > -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{1+x}$ and $\dfrac{1}{1+x} < 1$ when $1 < 1+x$, $x > 0$. For $x < -1$, $\left|\dfrac{1}{1+x}\right| = \dfrac{1}{-1-x}$, and this is less than 1 when $1 < -1-x$, i.e. $x < -2$. So $f(x)$ converges for $x > 0$ and for $x < -2$. When $f(x)$ converges, its value is $\frac{1}{1-\frac{1}{1+x}} = \frac{1+x}{x}$, so $f(x) = 3$ when $1 + x = 3x$, $x = \frac{1}{2}$.

8.3.98

a. Clearly for $k < n$, h_k is a leg of a right triangle whose hypotenuse is r_k and whose other leg is formed where the vertical line (in the picture) meets a diameter of the next smaller sphere; thus the other leg of the triangle is r_{k+1}. The Pythagorean theorem then implies that $h_k^2 = r_k^2 - r_{k+1}^2$.

b. The height is $H_n = \sum_{i=1}^{n} h_i = r_n + \sum_{i=1}^{n-1}\sqrt{r_i^2 - r_{i+1}^2}$ by part (a).

c. From part (b), because $r_i = a^{i-1}$,

$$\begin{aligned} H_n &= r_n + \sum_{i=1}^{n-1}\sqrt{r_i^2 - r_{i+1}^2} = a^{n-1} + \sum_{i=1}^{n-1}\sqrt{a^{2i-2} - a^{2i}} \\ &= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1}\sqrt{1-a^2} = a^{n-1} + \sqrt{1-a^2}\sum_{i=1}^{n-1} a^{i-1} \\ &= a^{n-1} + \sqrt{1-a^2}\left(\frac{1-a^{n-1}}{1-a}\right) \end{aligned}$$

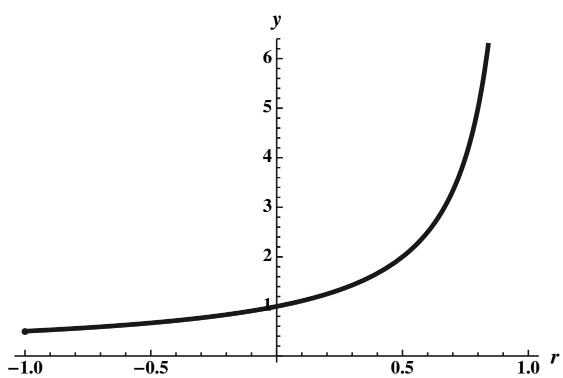
d. $\lim_{n\to\infty} H_n = \lim_{n\to\infty} a^{n-1} + \sqrt{1-a^2}\lim_{n\to\infty} \frac{1-a^{n-1}}{1-a} = 0 + \sqrt{1-a^2}\left(\frac{1}{1-a}\right) = \sqrt{\frac{1-a^2}{(1-a)(1-a)}} = \sqrt{\frac{1+a}{1-a}}.$

8.3.99

a. Using Theorem 8.7 in each case except for $r = 0$ gives

r	$f(r)$
-0.9	0.526
-0.7	0.588
-0.5	0.667
-0.2	0.833
0	1
0.2	1.250
0.5	2
0.7	3.333
0.9	10

b. A plot of f is



c. For $-1 < r < 1$ we have $f(r) = \frac{1}{1-r}$, so that

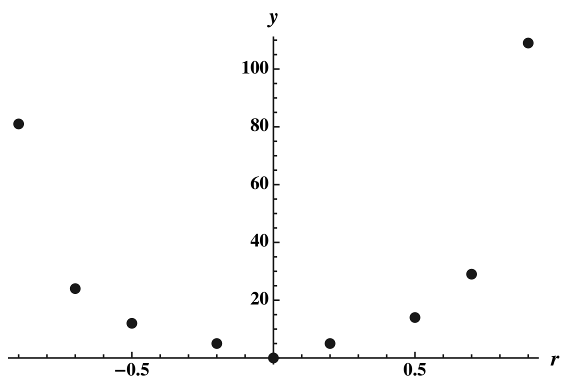
$$\lim_{r\to -1^+} f(r) = \lim_{r\to -1^+} \frac{1}{1-r} = \frac{1}{2}, \qquad \lim_{r\to 1^-} f(r) = \lim_{r\to 1^-} \frac{1}{1-r} = \infty.$$

8.3.100

a. In each case (except for $r = 0$ where $N(r)$ is clearly 0), compute $|S - S_n|$ for various values of n gives the following results:

r	$N(r)$	$\lvert S - S_{N(r)-1}\rvert$	$\lvert S - S_{N(r)}\rvert$
-0.9	81	1.0×10^{-4}	9.3×10^{-5}
-0.7	24	1.1×10^{-4}	7.9×10^{-5}
-0.5	12	1.6×10^{-4}	8.1×10^{-5}
-0.2	5	2.7×10^{-4}	5.3×10^{-5}
0	0	—	0
0.2	5	4.0×10^{-4}	8.0×10^{-5}
0.5	14	1.2×10^{-4}	6.1×10^{-5}
0.7	29	1.1×10^{-4}	7.5×10^{-5}
0.9	109	1.0×10^{-4}	9.3×10^{-5}

b. A plot of r versus $N(r)$ for these values of r is



c. The rate of convergence is faster for r closer to 0, since $N(r)$ is smaller. The reason for this is that r^k gets smaller faster as k increases when $|r|$ is closer to zero than when it is closer to 1.

8.4 The Divergence and Integral Tests

8.4.1 If the sequence of terms has limit 1, then the corresponding series diverges. It is necessary (but not sufficient) that the sequence of terms has limit 0 in order for the corresponding series to be convergent.

8.4.2 No. For example, the harmonic serkes $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges although $\frac{1}{k} \to 0$ as $k \to \infty$.

8.4.3 Yes. Either the series and the integral both converge, or both diverge, if the terms are positive and decreasing.

8.4.4 It converges for $p > 1$, and diverges for all other values of p.

8.4.5 For the same values of p as in the previous problem – it converges for $p > 1$, and diverges for all other values of p.

8.4.6 Let S_n be the partial sums. Then $S_{n+1} - S_n = a_{n+1} > 0$ because $a_{n+1} > 0$. Thus the sequence of partial sums is increasing.

8.4.7 The remainder of an infinite series is the error in approximating a convergent infinite series by a finite number of terms.

8.4.8 Yes. Suppose $\sum a_k$ converges to S, and let the sequence of partial sums be $\{S_n\}$. Then for any $\epsilon > 0$ there is some N such that for any $n > N$, $|S - S_n| < \epsilon$. But $|S - S_n|$ is simply the remainder R_n when the series is approximated to n terms. Thus $R_n \to 0$ as $n \to \infty$.

8.4.9 $a_k = \frac{k}{2k+1}$ and $\lim\limits_{k\to\infty} a_k = \frac{1}{2}$, so the series diverges.

8.4.10 $a_k = \frac{k}{k^2+1}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

8.4.11 $a_k = \frac{k}{\ln k}$ and $\lim\limits_{k\to\infty} a_k = \infty$, so the series diverges.

8.4.12 $a_k = \frac{k^2}{2^k}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

8.4.13 $a_k = \frac{1}{1000+k}$ and $\lim\limits_{k\to\infty} a_k = 0$, so the divergence test is inconclusive.

8.4.14 $a_k = \frac{k^3}{k^3+1}$ and $\lim\limits_{k\to\infty} a_k = 1$, so the series diverges.

8.4.15 $a_k = \frac{\sqrt{k}}{\ln^{10} k}$ and $\lim\limits_{k\to\infty} a_k = \infty$, so the series diverges.

8.4.16 $a_k = \frac{\sqrt{k^2+1}}{k}$ and $\lim\limits_{k\to\infty} a_k = 1$, so the series diverges.

8.4.17 $a_k = k^{1/k}$. In order to compute $\lim_{k\to\infty} a_k$, we let $y_k = \ln a_k = \frac{\ln k}{k}$. By Theorem 9.6, (or by L'Hôpital's rule), $\lim_{k\to\infty} y_k = 0$, so $\lim_{k\to\infty} a_k = e^0 = 1$. The given series thus diverges.

8.4.18 By Theorem 9.6 $k^3 \ll k!$, so $\lim_{k\to\infty} \frac{k^3}{k!} = 0$. The divergence test is inconclusive.

8.4.19 Clearly $\frac{1}{e^x} = e^{-x}$ is continuous, positive, and decreasing for $x \geq 2$ (in fact, for all x), so the integral test applies. Because

$$\int_2^\infty e^{-x}\,dx = \lim_{c\to\infty} \int_2^c e^{-x}\,dx = \lim_{c\to\infty} (-e^{-x})\Big|_2^c = \lim_{c\to\infty} (e^{-2} - e^{-c}) = e^{-2},$$

the Integral Test tells us that the original series converges as well.

8.4.20 Let $f(x) = \frac{x}{\sqrt{x^2+4}}$. $f(x)$ is continuous for $x \geq 1$. Note that $f'(x) = \frac{4}{(\sqrt{x^2+4})^3} > 0$. Thus f is increasing, and the conditions of the Integral Test aren't satisfied. The given series diverges by the Divergence Test.

8.4.21 Let $f(x) = x \cdot e^{-2x^2}$. This function is continuous for $x \geq 1$. Its derivative is $e^{-2x^2}(1 - 4x^2) < 0$ for $x \geq 1$, so $f(x)$ is decreasing. Because $\int_1^\infty x \cdot e^{-2x^2}\,dx = \frac{1}{4e^2}$, the series converges.

8.4.22 Let $f(x) = \frac{1}{\sqrt[3]{x+10}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because $\int_1^\infty \frac{1}{\sqrt[3]{x+10}}\,dx = \infty$, the series diverges.

8.4.23 Let $f(x) = \frac{1}{\sqrt{x+8}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because $\int_1^\infty \frac{1}{\sqrt{x+8}}\,dx = \infty$, the series diverges.

8.4.24 Let $f(x) = \frac{1}{x(\ln x)^2}$. $f(x)$ is continuous and decreasing for $x \geq 2$. Because $\int_2^\infty f(x)\,dx = \frac{1}{\ln 2}$ the series converges.

8.4.25 Let $f(x) = \frac{x}{e^x}$. $f(x)$ is clearly continuous for $x > 1$, and its derivative, $f'(x) = \frac{e^x - xe^x}{e^{2x}} = (1-x)\frac{e^x}{e^{2x}}$, is negative for $x > 1$ so that $f(x)$ is decreasing. Because $\int_1^\infty f(x)\,dx = 2e^{-1}$, the series converges.

8.4.26 Let $f(x) = \frac{1}{x \cdot \ln x \cdot \ln\ln x}$. $f(x)$ is continuous and decreasing for $x > 3$, and $\int_3^\infty \frac{1}{x\cdot\ln x\cdot\ln\ln x}\,dx = \infty$. The given series therefore diverges.

8.4.27 The integral test does not apply, because the sequence of terms is not decreasing.

8.4.28 $f(x) = \frac{x}{(x^2+1)^3}$ is decreasing and continuous, and $\int_1^\infty \frac{x}{(x^2+1)^3}\,dx = \frac{1}{16}$. Thus, the given series converges.

8.4.29 This is a p-series with $p = 10$, so this series converges.

8.4.30 $\sum_{k=2}^\infty \frac{k^e}{k^\pi} = \sum_{k=2}^\infty \frac{1}{k^{\pi-e}}$. Note that $\pi - e \approx 3.1416 - 2.71828 < 1$, so this series diverges.

8.4.31 $\sum_{k=3}^\infty \frac{1}{(k-2)^4} = \sum_{k=1}^\infty \frac{1}{k^4}$, which is a p-series with $p = 4$, thus convergent.

8.4.32 $\sum_{k=1}^\infty 2k^{-3/2} = 2\sum_{k=1}^\infty \frac{1}{k^{3/2}}$ is a p-series with $p = 3/2$, thus convergent.

8.4.33 $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^\infty \frac{1}{k^{1/3}}$ is a p-series with $p = 1/3$, thus divergent.

8.4.34 $\sum_{k=1}^\infty \frac{1}{\sqrt[3]{27k^2}} = \frac{1}{3}\sum_{k=1}^\infty \frac{1}{k^{2/3}}$ is a p-series with $p = 2/3$, thus divergent.

8.4.35

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^6}\,dx = \frac{1}{5n^5}$.

b. We solve $\frac{1}{5n^5} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5(n+1)^5}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^6}\,dx = S_n + \frac{1}{5n^5}$.

d. $S_{10} \approx 1.017341512$, so $L_{10} \approx 1.017341512 + \frac{1}{5\cdot 11^5} \approx 1.017342754$, and $U_{10} \approx 1.017341512 + \frac{1}{5\cdot 10^5} \approx 1.017343512$.

8.4.36

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^8}\,dx = \frac{1}{7n^7}$.

b. We solve $\frac{1}{7n^7} < 10^{-3}$ to obtain $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^8}\,dx = S_n + \frac{1}{7(n+1)^7}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^8}\,dx = S_n + \frac{1}{7n^7}$.

d. $S_{10} \approx 1.004077346$, so $L_{10} \approx 1.004077346 + \frac{1}{7\cdot 11^7} \approx 1.004077353$, and $U_{10} \approx 1.004077346 + \frac{1}{7\cdot 10^7} \approx 1.004077360$.

8.4.37

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{3^x}\,dx = \frac{1}{3^n \ln 3}$.

b. We solve $\frac{1}{3^n \ln 3} < 10^{-3}$ to obtain $n = 7$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^{n+1}\ln 3}$, and $U_n = S_n + \int_n^\infty \frac{1}{3^x}\,dx = S_n + \frac{1}{3^n \ln 3}$.

d. $S_{10} \approx 0.4999915325$, so $L_{10} \approx 0.4999915325 + \frac{1}{3^{11}\ln 3} \approx 0.4999966708$, and $U_{10} \approx 0.4999915325 + \frac{1}{3^{10}\ln 3} \approx 0.5000069475$.

8.4.38

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x\ln^2 x}\,dx = \frac{1}{\ln n}$.

b. We solve $\frac{1}{\ln n} < 10^{-3}$ to get $n = e^{1000} \approx 10^{434}$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln(n+1)}$, and $U_n = S_n + \int_n^\infty \frac{1}{x\ln^2 x}\,dx = S_n + \frac{1}{\ln n}$.

d. $S_{11} = \sum_{k=2}^{11} \frac{1}{k\ln^2 k} \approx 1.700396385$, so $L_{11} \approx 1.700396385 + \frac{1}{\ln 12} \approx 2.102825989$, and $U_{11} \approx 1.700396385 + \frac{1}{\ln 11} \approx 2.117428776$.

8.4.39

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^{3/2}}\,dx = 2n^{-1/2}$.

b. We solve $2n^{-1/2} < 10^{-3}$ to get $n > 4\times 10^6$, so let $n = 4\times 10^6 + 1$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^{3/2}}\,dx = S_n + 2(n+1)^{-1/2}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^{3/2}}\,dx = S_n + 2n^{-1/2}$.

d. $S_{10} = \sum_{k=1}^{10} \frac{1}{k^{3/2}} \approx 1.995336493$, so $L_{10} \approx 1.995336493 + 2\cdot 11^{-1/2} \approx 2.598359182$, and $U_{10} \approx 1.995336493 + 2\cdot 10^{-1/2} \approx 2.627792025$.

8.4.40

a. The remainder R_n is bounded by $\int_n^\infty e^{-x}\,dx = e^{-n}$.

b. We solve $e^{-n} < 10^{-3}$ to get $n = 7$.

c. $L_n = S_n + \int_{n+1}^\infty e^{-x}\,dx = S_n + e^{-(n+1)}$, and $U_n = S_n + \int_n^\infty e^{-x}\,dx = S_n + e^{-n}$.

d. $S_{10} = \sum_{k=1}^{10} e^{-k} \approx 0.5819502852$, so $L_{10} \approx 0.5819502852 + e^{-11} \approx 0.5819669869$, and $U_{10} \approx 0.5819502852 + e^{-10} \approx 0.5819956851$.

8.4.41

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$.

b. We solve $\frac{1}{2n^2} < 10^{-3}$ to get $n = 23$.

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^3}\,dx = S_n + \frac{1}{2(n+1)^2}$, and $U_n = S_n + \int_n^\infty \frac{1}{x^3}\,dx = S_n + \frac{1}{2n^2}$.

d. $S_{10} \approx 1.197531986$, so $L_{10} \approx 1.197531986 + \frac{1}{2\cdot 11^2} \approx 1.201664217$, and $U_{10} \approx 1.197531986 + \frac{1}{2\cdot 10^2} \approx 1.202531986$.

8.4.42

a. The remainder R_n is bounded by $\int_n^\infty xe^{-x^2}\,dx = \frac{1}{2e^{n^2}}$.

b. We solve $\frac{1}{2e^{n^2}} < 10^{-3}$ to get $n = 3$.

c. $L_n = S_n + \int_{n+1}^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{(n+1)^2}}$, and $U_n = S_n + \int_n^\infty xe^{-x^2}\,dx = S_n + \frac{1}{2e^{n^2}}$.

d. $S_{10} \approx 0.4048813986$, so $L_{10} \approx 0.4048813986 + \frac{1}{2e^{11^2}} \approx 0.4048813986$, and $U_{10} \approx 0.4048813986 + \frac{1}{2e^{10^2}} \approx 0.4048813986$.

8.4.43 This is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{12}$, so $\sum_{k=1}^\infty \frac{4}{12^k} = \frac{1/3}{1-1/12} = \frac{1/3}{11/12} = \frac{4}{11}$.

8.4.44 This is a geometric series with $a = 3/e^2$ and $r = 1/e$, so $\sum_{k=2}^\infty 3e^{-k} = \frac{3/e^2}{1-(1/e)} = \frac{3/e^2}{(e-1)/e} = \frac{3}{e(e-1)}$.

8.4.45 $$\sum_{k=0}^\infty \left(3\left(\frac{2}{5}\right)^k - 2\left(\frac{5}{7}\right)^k\right) = 3\sum_{k=0}^\infty \left(\frac{2}{5}\right)^k - 2\sum_{k=0}^\infty \left(\frac{5}{7}\right)^k = 3\left(\frac{1}{3/5}\right) - 2\left(\frac{1}{2/7}\right) = 5 - 7 = -2.$$

8.4.46 $$\sum_{k=1}^\infty \left(2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right) = 2\sum_{k=1}^\infty \left(\frac{3}{5}\right)^k + 3\sum_{k=1}^\infty \left(\frac{4}{9}\right)^k = 2\left(\frac{3/5}{2/5}\right) + 3\left(\frac{4/9}{5/9}\right) = 3 + \frac{12}{5} = \frac{27}{5}.$$

8.4.47 $$\sum_{k=1}^\infty \left(\frac{1}{3}\left(\frac{5}{6}\right)^k + \frac{3}{5}\left(\frac{7}{9}\right)^k\right) = \frac{1}{3}\sum_{k=1}^\infty \left(\frac{5}{6}\right)^k + \frac{3}{5}\sum_{k=1}^\infty \left(\frac{7}{9}\right)^k = \frac{1}{3}\left(\frac{5/6}{1/6}\right) + \frac{3}{5}\left(\frac{7/9}{2/9}\right) = \frac{5}{3} + \frac{21}{10} = \frac{113}{30}.$$

8.4.48 $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{2}(0.2)^k+\frac{3}{2}(0.8)^k\right)=\frac{1}{2}\sum_{k=0}^{\infty}(0.2)^k+\frac{3}{2}\sum_{k=0}^{\infty}(0.8)^k=\frac{1}{2}\left(\frac{1}{0.8}\right)+\frac{3}{2}\left(\frac{1}{0.2}\right)=\frac{5}{8}+\frac{15}{2}=\frac{65}{8}.$

8.4.49 $\displaystyle\sum_{k=1}^{\infty}\left(\left(\frac{1}{6}\right)^k+\left(\frac{1}{3}\right)^{k-1}\right)=\sum_{k=1}^{\infty}\left(\frac{1}{6}\right)^k+\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k-1}=\frac{1/6}{5/6}+\frac{1}{2/3}=\frac{17}{10}.$

8.4.50 $\displaystyle\sum_{k=0}^{\infty}\frac{2-3^k}{6^k}=\sum_{k=0}^{\infty}\left(\frac{2}{6^k}-\frac{3^k}{6^k}\right)=2\sum_{k=0}^{\infty}\left(\frac{1}{6}\right)^k-\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k=2\left(\frac{1}{5/6}\right)-\frac{1}{1/2}=\frac{2}{5}.$

8.4.51

a. True. The two series differ by a finite amount ($\sum_{k=1}^{9}a_k$), so if one converges, so does the other.

b. True. The same argument applies as in part (a).

c. False. If $\sum a_k$ converges, then $a_k\to 0$ as $k\to\infty$, so that $a_k+0.0001\to 0.0001$ as $k\to\infty$, so that $\sum(a_k+0.0001)$ cannot converge.

d. False. Suppose $p=-1.0001$. Then $\sum p^k$ diverges but $p+0.001=-0.9991$ so that $\sum(p+.0001)^k$ converges.

e. False. Let $p=1.0005$; then $-p+.001=-(p-.001)=-.9995$, so that $\sum k^{-p}$ converges (p-series) but $\sum k^{-p+.001}$ diverges.

f. False. Let $a_k=\frac{1}{k}$, the harmonic series.

8.4.52 Diverges by the Divergence Test because $\displaystyle\lim_{k\to\infty}a_k=\lim_{k\to\infty}\sqrt{\frac{k+1}{k}}=1\neq 0.$

8.4.53 Converges by the Integral Test because $\displaystyle\int_1^{\infty}\frac{1}{(3x+1)(3x+4)}\,dx=\int_1^{\infty}\frac{1}{3(3x+1)}-\frac{1}{3(3x+4)}\,dx=$ $\displaystyle\lim_{b\to\infty}\int_1^b\left(\frac{1}{3(3x+1)}-\frac{1}{3(3x+4)}\right)dx=\lim_{b\to\infty}\frac{1}{9}\left(\ln\left(\frac{3x+1}{3x+4}\right)\right)\Big|_1^b=\lim_{b\to\infty}=-\frac{1}{9}\cdot\ln(4/7)\approx 0.06217<\infty.$

Alternatively, this is a telescoping series with nth partial sum equal to $S_n=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3n+4}\right)$ which converges to $\frac{1}{12}$.

8.4.54 Converges by the Integral Test because $\displaystyle\int_0^{\infty}\frac{10}{x^2+9}\,dx=\frac{10}{3}\lim_{b\to\infty}\left(\tan^{-1}(x/3)\Big|_0^b\right)=\frac{10}{3}\frac{\pi}{2}\approx 5.236<\infty.$

8.4.55 Diverges by the Divergence Test because $\displaystyle\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{k}{\sqrt{k^2+1}}=1\neq 0.$

8.4.56 Converges because it is the sum of two geometric series. In fact, $\sum_{k=1}^{\infty}\frac{2^k+3^k}{4^k}=\sum_{k=1}^{\infty}(2/4)^k+\sum_{k=1}^{\infty}(3/4)^k=\frac{1/2}{1-(1/2)}+\frac{3/4}{1-(3/4)}=1+3=4.$

8.4.57 Converges by the Integral Test because $\displaystyle\int_2^{\infty}\frac{4}{x\ln^2 x}\,dx=\lim_{b\to\infty}\left(\frac{-4}{\ln x}\Big|_2^b\right)=\frac{4}{\ln 2}<\infty.$

8.4.58

a. In order for the series to converge, the integral $\int_2^{\infty}\frac{1}{x(\ln x)^p}\,dx$ must exist. But

$$\int\frac{1}{x(\ln x)^p}\,dx=\frac{1}{1-p}(\ln x)^{1-p},$$

so in order for this improper integral to exist, we must have that $1-p<0$ or $p>1$.

b. The series converges faster for $p = 3$ because the terms of the series get smaller faster.

8.4.59

a. Note that $\int \frac{1}{x \ln x (\ln \ln x)^p}\, dx = \frac{1}{1-p}(\ln \ln x)^{1-p}$, and thus the improper integral with bounds n and ∞ exists only if $p > 1$ because $\ln \ln x > 0$ for $x > e$. So this series converges for $p > 1$.

b. For large values of z, clearly $\sqrt{z} > \ln z$, so that $z > (\ln z)^2$. Write $z = \ln x$; then for large x, $\ln x > (\ln \ln x)^2$; multiplying both sides by $x \ln x$ we have that $x \ln^2 x > x \ln x (\ln \ln x)^2$, so that the first series converges faster because the terms get smaller faster.

8.4.60

a. $\sum \frac{1}{k^{2.5}}$.

b. $\sum \frac{1}{k^{0.75}}$.

c. $\sum \frac{1}{k^{3/2}}$.

8.4.61 Let $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Then this looks like a left Riemann sum for the function $y = \frac{1}{\sqrt{x}}$ on $[1, n+1]$. Because each rectangle lies above the curve itself, we see that S_n is bounded below by the integral of $\frac{1}{\sqrt{x}}$ on $[1, n+1]$. Now,

$$\int_1^{n+1} \frac{1}{\sqrt{x}}\, dx = \int_1^{n+1} x^{-1/2}\, dx = 2\sqrt{x}\bigg|_1^{n+1} = 2\sqrt{n+1} - 2.$$

This integral diverges as $n \to \infty$, so the series does as well by the bound above.

8.4.62 $\sum_{k=1}^{\infty}(a_k \pm b_k) = \lim_{n\to\infty} \sum_{k=1}^{n}(a_k \pm b_k) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \pm \lim_{n\to\infty} \sum_{k=1}^{n} b_k = A \pm B$.

8.4.63 $\sum_{k=1}^{\infty} ca_k = \lim\limits_{n\to\infty} \sum_{k=1}^{n} ca_k = \lim\limits_{n\to\infty} c \sum_{k=1}^{n} a_k = c \lim\limits_{n\to\infty} \sum_{k=1}^{n} a_k$, so that one sum diverges if and only if the other one does.

8.4.64 $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges by the Integral Test, because $\int_2^{\infty} \frac{1}{x \ln x} = \lim_{b\to\infty}\left(\ln \ln x|_2^b\right) = \infty$.

8.4.65 To approximate the sequence for $\zeta(m)$, note that the remainder R_n after n terms is bounded by

$$\int_n^{\infty} \frac{1}{x^m}\, dx = \frac{1}{m-1} n^{1-m}.$$

For $m = 3$, if we wish to approximate the value to within 10^{-3}, we must solve $\dfrac{1}{2} n^{-2} < 10^{-3}$, so that $n = 23$, and $\displaystyle\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151926$. The true value is ≈ 1.202056903.

For $m = 5$, if we wish to approximate the value to within 10^{-3}, we must solve $\dfrac{1}{4} n^{-4} < 10^{-3}$, so that $n = 4$, and $\displaystyle\sum_{k=1}^{4} \frac{1}{k^5} \approx 1.036341789$. The true value is ≈ 1.036927755.

8.4.66

a. Starting with $\cot^2 x < \dfrac{1}{x^2} < 1 + \cot^2 x$, substitute $k\theta$ for x:

$$\cot^2(k\theta) < \frac{1}{k^2\theta^2} < 1 + \cot^2(k\theta),$$
$$\sum_{k=1}^{n} \cot^2(k\theta) < \sum_{k=1}^{n} \frac{1}{k^2\theta^2} < \sum_{k=1}^{n}(1 + \cot^2(k\theta)),$$
$$\sum_{k=1}^{n} \cot^2(k\theta) < \frac{1}{\theta^2}\sum_{k=1}^{n} \frac{1}{k^2} < n + \sum_{k=1}^{n} \cot^2(k\theta).$$

Note that the identity is valid because we are only summing for k up to n, so that $k\theta < \frac{\pi}{2}$.

b. Substitute $\dfrac{n(2n-1)}{3}$ for the sum, using the identity:

$$\frac{n(2n-1)}{3} < \frac{1}{\theta^2}\sum_{k=1}^{n} \frac{1}{k^2} < n + \frac{n(2n-1)}{3},$$
$$\theta^2\frac{n(2n-1)}{3} < \sum_{k=1}^{n} \frac{1}{k^2} < \theta^2\frac{n(2n+2)}{3},$$
$$\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}.$$

c. By the Squeeze Theorem, if the expressions on either end have equal limits as $n \to \infty$, the expression in the middle does as well, and its limit is the same. The expression on the left is

$$\pi^2\frac{2n^2 - n}{12n^2 + 12n + 3} = \pi^2\frac{2 - n^{-1}}{12 + 12n^{-1} + 3n^{-2}},$$

which has a limit of $\dfrac{\pi^2}{6}$ as $n \to \infty$. The expression on the right is

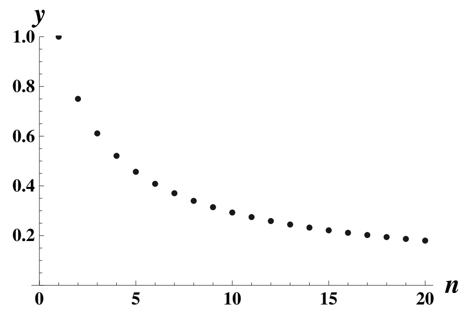
$$\pi^2\frac{2n^2 + 2n}{12n^2 + 12n + 3} = \pi^2\frac{2 + 2n^{-1}}{12 + 12n^{-1} + 3n^{-3}},$$

which has the same limit. Thus $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

8.4.67 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, splitting the series into even and odd terms. But $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2}$. Thus $\frac{\pi^2}{6} = \frac{1}{4}\frac{\pi^2}{6} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, so that the sum in question is $\frac{3\pi^2}{24} = \frac{\pi^2}{8}$.

8.4.68

a. $\{F_n\}$ is a decreasing sequence because each term in F_n is smaller than the corresponding term in F_{n-1} and thus the sum of terms in F_n is smaller than the sum of terms in F_{n-1}.

b. 

c. It appears that $\lim_{n\to\infty} F_n = 0$.

8.4.69

a. $x_1 = \sum_{k=2}^{2} \frac{1}{k} = \frac{1}{2}$, $x_2 = \sum_{k=3}^{4} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, $x_3 = \sum_{k=4}^{6} \frac{1}{k} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60}$.

b. x_n has n terms. Each term is bounded below by $\frac{1}{2n}$ and bounded above by $\frac{1}{n+1}$. Thus $x_n \ge n \cdot \frac{1}{2n} = \frac{1}{2}$, and $x_n \le n \cdot \frac{1}{n+1} < n \cdot \frac{1}{n} = 1$.

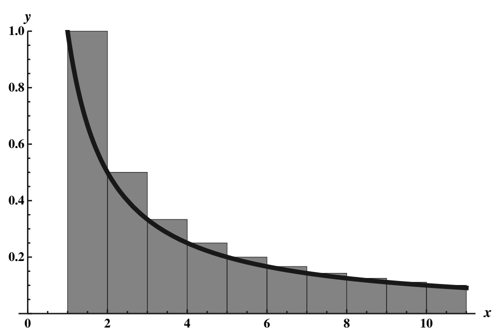
c. The right Riemann sum for $\int_1^2 \frac{dx}{x}$ using n subintervals has n rectangles of width $\frac{1}{n}$; the right edges of those rectangles are at $1 + \frac{i}{n} = \frac{n+i}{n}$ for $i = 1, 2, \ldots, n$. The height of such a rectangle is the value of $\frac{1}{x}$ at the right endpoint, which is $\frac{n}{n+i}$. Thus the area of the rectangle is $\frac{1}{n} \cdot \frac{n}{n+i} = \frac{1}{n+i}$. Adding up over all the rectangles gives x_n.

d. The limit $\lim_{n\to\infty} x_n$ is the limit of the right Riemann sum as the width of the rectangles approaches zero. This is precisely $\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2$.

8.4.70

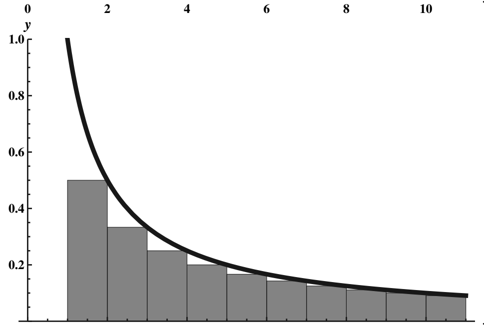
a. The first diagram is a left Riemann sum for $f(x) = \frac{1}{x}$ on the interval $[1, 11]$ (we assume $n = 10$ for purposes of drawing a graph). The area under the curve is $\int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1)$, and the sum of the areas of the rectangles is obviously $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Thus

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$



The second diagram is a right Riemann sum for the same function on the same interval. Considering only $[1, n]$, we see that, comparing the area under the curve and the sum of the areas of the rectangles, that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n.$$



Adding 1 to both sides gives the desired inequality.

b. According to part (a), $\ln(n+1) < S_n$ for $n = 1, 2, 3, \ldots,$, so that $E_n = S_n - \ln(n+1) > 0$.

c. Using the second figure above and assuming $n = 9$, the final rectangle corresponds to $\frac{1}{n+1}$, and the area under the curve between $n+1$ and $n+2$ is clearly $\ln(n+2) - \ln(n+1)$.

d. $E_{n+1} - E_n = S_{n+1} - \ln(n+2) - (S_n - \ln(n+1)) = \frac{1}{n+1} - (\ln(n+2) - \ln(n+1))$. But this is positive because of the bound established in part (c).

e. Using part (a), $E_n = S_n - \ln(n+1) < 1 + \ln n - \ln(n+1) < 1$.

f. E_n is a monotone (increasing) sequence that is bounded, so it has a limit.

g. The first ten values (E_1 through E_{10}) are

$$.3068528194, .401387711, .447038972, .473895421, .491573864,$$
$$.504089851, .513415601, .520632566, .526383161, .531072981.$$

$E_{1000} \approx 0.576716082$.

h. For $S_n > 10$ we need $10 - 0.5772 = 9.4228 > \ln(n+1)$. Solving for n gives $n \approx 12366.16$, so $n = 12367$.

8.4.71

a. Note that the center of gravity of any stack of dominoes is the average of the locations of their centers. Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x-coordinates). Let $m(n)$ be the x-coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n-1$, which is $\frac{1}{n}\sum_{i=0}^{n-1} m(i)$, so that $m(n) = 1 + \frac{1}{n}\sum_{i=0}^{n-1} m(i)$. We claim that in fact $m(n) = \sum_{k=1}^{n} \frac{1}{k}$. Use induction. This is certainly true for $n = 1$. Note first that $m(0) = 0$, so we can start the sum at 1 rather than at 0. Now, $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1} m(i) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=1}^{i}\frac{1}{j}$. Now, 1 appears $n-1$ times in the double sum, 2 appears $n-2$ times, and so forth, so we can rewrite this sum as $m(n) = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\frac{n-i}{i} = 1 + \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{n}{i} - 1\right) = 1 + \frac{1}{n}\left(n\sum_{i=1}^{n-1}\frac{1}{i} - (n-1)\right) = \sum_{i=1}^{n-1}\frac{1}{i} + 1 - \frac{n-1}{n} = \sum_{i=1}^{n}\frac{1}{i}$, and we are done by induction (noting that the statement is clearly true for $n = 0$, $n = 1$). Thus the maximum overhang is $\sum_{k=2}^{n}\frac{1}{k}$.

b. For an infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite.

8.4.72

a. The circumference of the kth layer is $2\pi \cdot \frac{1}{k}$, so its area is $2\pi \cdot \frac{1}{k}$ and thus the total vertical surface area $\sum_{k=1}^{\infty} 2\pi \cdot \frac{1}{k} = 2\pi\sum_{k=1}^{\infty}\frac{1}{k} = \infty$. The horizontal surface area, however, is π, since looking at the cake from above, the horizontal surface covers the circle of radius 1, which has area $\pi \cdot 1^2 = \pi$.

b. The volume of a cylinder of radius r and height h is $\pi r^2 h$, so the volume of the kth layer is $\pi \cdot \frac{1}{k^2} \cdot 1 = \frac{\pi}{k^2}$. Thus the volume of the cake is

$$\sum_{k=1}^{\infty}\frac{\pi}{k^2} = \pi\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^3}{6} \approx 5.168.$$

c. This cake has infinite surface area, yet it has finite volume!

8.4.73

a. Dividing both sides of the recurrence equation by f_n gives $\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n}$. Let the limit of the ratio of successive terms be L. Taking the limit of the previous equation gives $L = 1 + \frac{1}{L}$. Thus $L^2 = L + 1$, so $L^2 - L - 1 = 0$. The quadratic formula gives $L = \frac{1 \pm \sqrt{1 - 4\cdot(-1)}}{2}$, but we know that all the terms are positive, so we must have $L = \frac{1+\sqrt{5}}{2} = \phi \approx 1.618$.

b. Write the recurrence in the form $f_{n-1} = f_{n+1} - f_n$ and divide both sides by f_{n+1}. Then we have $\frac{f_{n-1}}{f_{n+1}} = 1 - \frac{f_n}{f_{n+1}}$. Taking the limit gives $1 - \frac{1}{\phi}$ on the right-hand side.

c. Consider the harmonic series with the given groupings, and compare it with the sum of $\frac{f_{k-1}}{f_{k+1}}$ as shown. The first three terms match exactly. The sum of the next two are $\frac{1}{4}+\frac{1}{5}>\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$. The sum of the next three are $\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$. The sum of the next five are $\frac{1}{9}+\cdots+\frac{1}{13}>5\cdot\frac{1}{13}=\frac{5}{13}$. Thus the harmonic series is bounded below by the series $\sum_{k=1}^{\infty}\frac{f_{k-1}}{f_{k+1}}$.

d. The result above implies that the harmonic series diverges, because the series $\sum_{k=1}^{\infty}\frac{f_{k-1}}{f_{k+1}}$ diverges, since its general term has limit $1-\frac{1}{\phi}\neq 0$.

8.5 The Ratio, Root, and Comparison Tests

8.5.1 Given a series $\sum a_k$ of positive terms, compute $\lim_{k\to\infty}\frac{a_{k+1}}{a_k}$ and call it r. If $0\le r<1$, the given series converges. If $r>1$ (including $r=\infty$), the given series diverges. If $r=1$, the test is inconclusive.

8.5.2 Given a series $\sum a_k$ of positive terms, compute $\lim_{k\to\infty}\sqrt[k]{a_k}$ and call it r. If $0\le r<1$, the given series converges. If $r>1$ (including $r=\infty$), the given series diverges. If $r=1$, the test is inconclusive.

8.5.3 Given a series of positive terms $\sum a_k$ that you suspect converges, find a series $\sum b_k$ that you know converges, for which $\lim_{k\to\infty}\frac{a_k}{b_k}=L$ where $L\ge 0$ is a finite number. If you are successful, you will have shown that the series $\sum a_k$ converges.

Given a series of positive terms $\sum a_k$ that you suspect diverges, find a series $\sum b_k$ that you know diverges, for which $\lim_{k\to\infty}\frac{a_k}{b_k}=L$ where $L>0$ (including the case $L=\infty$). If you are successful, you will have shown that $\sum a_k$ diverges.

8.5.4 The Divergence Test.

8.5.5 The Ratio Test.

8.5.6 The Comparison Test or the Limit Comparison Test.

8.5.7 The difference between successive partial sums is a term in the sequence. Because the terms are positive, differences between successive partial sums are as well, so the sequence of partial sums is increasing.

8.5.8 No. They all determine convergence or divergence by approximating or bounding the series by some other series known to converge or diverge; thus, the actual value of the series cannot be determined.

8.5.9 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{1}{(k+1)!}\cdot\frac{(k)!}{1}=\frac{1}{k+1}$, which goes to zero as $k\to\infty$, so the given series converges by the Ratio Test.

8.5.10 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{2^{k+1}}{(k+1)!}\cdot\frac{(k)!}{2^k}=\frac{2}{k+1}$; the limit of this ratio is zero, so the given series converges by the Ratio Test.

8.5.11 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{(k+1)^2}{4^(k+1)}\cdot\frac{4^k}{(k)^2}=\frac{1}{4}\left(\frac{k+1}{k}\right)^2$. The limit is $1/4$ as $k\to\infty$, so the given series converges by the Ratio Test.

8.5.12 The ratio between successive terms is

$$\frac{a_{k+1}}{a_k}=\frac{(k+1)^{(k+1)}}{2^{(k+1)}}\cdot\frac{2^k}{k^k}=\frac{k+1}{2}\left(\frac{k+1}{k}\right)^k.$$

Note that $\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k=e$, but $\lim_{k\to\infty}\frac{k+1}{2}=\infty$, so the given series diverges by the Ratio Test.

8.5.13 The ratio between successive terms is $\frac{a_{k+1}}{a_k}=\frac{(k+1)e^{-(k+1)}}{(k)e^{-(k)}}=\frac{k+1}{(k)e}$. The limit of this ratio as $k\to\infty$ is $1/e<1$, so the given series converges by the Ratio Test.

8.5.14 The ratio between successive terms is $\frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \left(\frac{k+1}{k}\right)^k$. This has limit e as $k \to \infty$, so the limit of the ratio of successive terms is $e > 1$, so the given series diverges by the Ratio Test.

8.5.15 The ratio between successive terms is $\frac{2^{k+1}}{(k+1)^{99}} \cdot \frac{(k)^{99}}{2^k} = 2\left(\frac{k}{k+1}\right)^{99}$; the limit as $k \to \infty$ is 2, so the given series diverges by the Ratio Test.

8.5.16 The ratio between successive terms is $\frac{(k+1)^6}{(k+1)!} \cdot \frac{(k)!}{(k)^6} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^6$; the limit as $k \to \infty$ is zero, so the given series converges by the Ratio Test.

8.5.17 The ratio between successive terms is $\frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{((k)!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)}$; the limit as $k \to \infty$ is $1/4$, so the given series converges by the Ratio Test.

8.5.18 Note that this series is $\sum_{k=1}^{\infty} \frac{2^k}{k^4}$. The ratio between successive terms is $\frac{2^{k+1}k^4}{2^k(k+1)^4} = 2\left(\frac{k}{k+1}\right)^4 \to 2$ as $k \to \infty$. So the given series diverges by the ratio test.

8.5.19 The kth root of the kth term is $\frac{10k^3+3}{9k^3+k+1}$. The limit of this as $k \to \infty$ is $\frac{10}{9} > 1$, so the given series diverges by the Root Test.

8.5.20 The kth root of the kth term is $\frac{2k}{k+1}$. The limit of this as $k \to \infty$ is $2 > 1$, so the given series diverges by the Root Test.

8.5.21 The kth root of the kth term is $\frac{k^{2/k}}{2}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

8.5.22 The kth root of the kth term is $\left(1 + \frac{3}{k}\right)^k$. The limit of this as $k \to \infty$ is $= e^3 > 1$, so the given series diverges by the Root Test.

8.5.23 The kth root of the kth term is $\left(\frac{k}{k+1}\right)^{2k}$. The limit of this as $k \to \infty$ is $e^{-2} < 1$, so the given series converges by the Root Test.

8.5.24 The kth root of the kth term is $\frac{1}{\ln(k+1)}$. The limit of this as $k \to \infty$ is 0, so the given series converges by the Root Test.

8.5.25 The kth root of the kth term is$\left(\frac{1}{k^k}\right)$. The limit of this as $k \to \infty$ is 0, so the given series converges by the Root Test.

8.5.26 The kth root of the kth term is $\frac{k^{1/k}}{e}$. The limit of this as $k \to \infty$ is $\frac{1}{e} < 1$, so the given series converges by the Root Test.

8.5.27 $\frac{1}{k^2+4} < \frac{1}{k^2}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges as well, by the Comparison Test.

8.5.28 Use the Limit Comparison Test with $\left\{\frac{1}{k^2}\right\}$. The ratio of the terms of the two series is $\frac{k^4+k^3-k^2}{k^4+4k^2-3}$ which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

8.5.29 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{k^3-k}{k^3+4}$ which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.30 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{0.0001k}{k+4}$ which has limit 0.0001 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.31 For all k, $\frac{1}{k^{3/2}+1} < \frac{1}{k^{3/2}}$. The series whose terms are $\frac{1}{k^{3/2}}$ is a p-series which converges, so the given series converges as well by the Comparison Test.

8.5.32 Use the Limit Comparison Test with $\{1/k\}$. The ratio of the terms of the two series is $k\sqrt{\frac{k}{k^3+1}} = \sqrt{\frac{k^3}{k^3+1}}$, which has limit 1 as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.33 $\sin(1/k) > 0$ for $k \geq 1$, so we can apply the Comparison Test with $1/k^2$. $\sin(1/k) < 1$, so $\frac{\sin(1/k)}{k^2} < \frac{1}{k^2}$. Because the comparison series converges, the given series converges as well.

8.5.34 Use the Limit Comparison Test with $\{1/3^k\}$. The ratio of the terms of the two series is $\frac{3^k}{3^k - 2^k} = \frac{1}{1-\left(\frac{2^k}{3^k}\right)}$, which has limit 1 as $k \to \infty$. Because the comparison series converges, the given series does as well.

8.5.35 Use the Limit Comparison Test with $\{1/k\}$. The ratio of the terms of the two series is $\frac{k}{2k-\sqrt{k}} = \frac{1}{2-1/\sqrt{k}}$, which has limit $1/2$ as $k \to \infty$. Because the comparison series diverges, the given series does as well.

8.5.36 $\frac{1}{k\sqrt{k+2}} < \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}}$. Because the series whose terms are $\frac{1}{k^{3/2}}$ is a $p-$series with $p > 1$, it converges. Because the comparison series converges, the given series converges as well.

8.5.37 Use the Limit Comparison Test with $\frac{k^{2/3}}{k^{3/2}}$. The ratio of corresponding terms of the two series is $\frac{\sqrt[3]{k^2+1}}{\sqrt{k^3+1}} \cdot \frac{k^{3/2}}{k^{2/3}} = \frac{\sqrt[3]{k^2+1}}{\sqrt[3]{k^2}} \cdot \frac{\sqrt{k^3}}{\sqrt{k^3+1}}$, which has limit 1 as $k \to \infty$. The comparison series is the series whose terms are $k^{2/3-3/2} = k^{-5/6}$, which is a p-series with $p < 1$, so it, and the given series, both diverge.

8.5.38 For all k, $\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ converges, the given series converges as well.

8.5.39

a. False. For example, let $\{a_k\}$ be all zeros, and $\{b_k\}$ be all 1's.

b. True. This is a result of the Comparison Test.

c. True. Both of these statements follow from the Comparison Test.

d. True. The limit of the ratio is always 1 in the case, so the test is inconclusive.

8.5.40 Use the Divergence Test: $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \left(1 - \frac{1}{k}\right)^k = \frac{1}{e} \neq 0$, so the given series diverges.

8.5.41 Use the Divergence Test: $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \left(1 + \frac{2}{k}\right)^k = e^2 \neq 0$, so the given series diverges.

8.5.42 Use the Root Test: The kth root of the kth term is $\frac{k^2}{2k^2+1}$. The limit of this as $k \to \infty$ is $\frac{1}{2} < 1$, so the given series converges by the Root Test.

8.5.43 Use the Ratio Test: the ratio of successive terms is $\frac{(k+1)^{100}}{(k+2)!} \cdot \frac{(k+1)!}{k^{100}} = \left(\frac{k+1}{k}\right)^{100} \cdot \frac{1}{k+2}$. This has limit $1^{100} \cdot 0 = 0$ as $k \to \infty$, so the given series converges by the Ratio Test.

8.5.44 Use the Comparison Test. Note that $\sin^2 k \leq 1$ for all k, so $\frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}$ for all k. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does the given series.

8.5.45 Use the Root Test. The kth root of the kth term is $(k^{1/k} - 1)^2$, which has limit 0 as $k \to \infty$, so the given series converges by the Root Test.

8.5.46 Use the Limit Comparison Test with the series whose kth term is $\left(\frac{2}{e}\right)^k$. Note that $\lim_{k\to\infty} \frac{2^k}{e^k-1} \cdot \frac{e^k}{2^k} = \lim_{k\to\infty} \frac{e^k}{e^k-1} = 1$. The given series thus converges because $\sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$ converges (because it is a geometric series with $r = \frac{2}{e} < 1$). Note that it is also possible to show convergence with the Ratio Test.

8.5.47 Use the Divergence Test: $\lim_{k\to\infty} \frac{k^2+2k+1}{3k^2+1} = \frac{1}{3} \neq 0$, so the given series diverges.

8.5.48 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{5^k}$. Note that $\lim_{k\to\infty} \frac{1}{5^k-1} \cdot \frac{5^k}{1} = 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges because it is a geometric series with $r = \frac{1}{5}$. Thus, the given series also converges.

8.5.49 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k\to\infty} \frac{\frac{1}{\ln k}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{\ln k} = \infty$, and because the harmonic series diverges, the given series does as well.

8.5.50 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{5^k}$. Note that $\lim_{k\to\infty} \frac{1}{5^k-3^k} \cdot \frac{5^k}{1} = \lim_{k\to\infty} \frac{1}{1-(3/5)^k} = 1$, and the series $\sum_{k=3}^{\infty} \frac{1}{5^k}$ converges because it is a geometric series with $r = \frac{1}{5}$. Thus, the given series also converges.

8.5.51 Use the Limit Comparison Test with the series whose kth term is $\frac{1}{k^{3/2}}$. Note that $\lim_{k\to\infty} \frac{1}{\sqrt{k^3-k+1}} \cdot \frac{\sqrt{k^3}}{1} = \lim_{k\to\infty} \sqrt{\frac{k^3}{k^3-k+1}} = \sqrt{1} = 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges because it is a p-series with $p = \frac{3}{2}$. Thus, the given series also converges.

8.5.52 Use the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^3}{(3k+3)!} \cdot \frac{(3k)!}{(k!)^3} = \frac{(k+1)^3}{(3k+1)(3k+2)(3k+3)}$, which has limit $1/27$ as $k \to \infty$. Thus the given series converges.

8.5.53 Use the Comparison Test. Each term $\frac{1}{k} + 2^{-k} > \frac{1}{k}$. Because the harmonic series diverges, so does this series.

8.5.54 Use the Comparison Test with $\{5/k\}$. Note that $\frac{5\ln k}{k} > \frac{5}{k}$ for $k > 1$. Because the series whose terms are $5/k$ diverges, the given series diverges as well.

8.5.55 Use the Ratio Test. $\frac{a_{k+1}}{a_k} = \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{(k)^k}{2^k(k)!} = 2\left(\frac{k}{k+1}\right)^k$, which has limit $\frac{2}{e}$ as $k \to \infty$, so the given series converges.

8.5.56 Use the Root Test. $\lim\limits_{k\to\infty} \left(1 - \frac{1}{k}\right)^k = e^{-1} < 1$, so the given series converges.

8.5.57 Use the Limit Comparison Test with $\{1/k^3\}$. The ratio of corresponding terms is $\frac{k^{11}}{k^{11}+3}$, which has limit 1 as $k \to \infty$. Because the comparison series converges, so does the given series.

8.5.58 Use the Root Test. $\lim\limits_{k\to\infty} \frac{1}{1+p} = \frac{1}{1+p} < 1$ because $p > 0$, so the given series converges.

8.5.59 This is a p-series with exponent greater than 1, so it converges.

8.5.60 Use the Comparison Test: $\frac{1}{k^2 \ln k} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ is a convergent $p-$series, the given series converges as well.

8.5.61 $\ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$, so this series telescopes. We get $\sum_{k=1}^{n} \ln\left(\frac{k+2}{k+1}\right) = \ln(n+2) - \ln 2$. Because $\lim_{n\to\infty} \ln(n+2) - \ln 2 = \infty$, the sequence of partial sums diverges, so the given series is divergent.

8.5.62 Use the Divergence Test. Note that $\lim_{k\to\infty} k^{-1/k} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1 \neq 0$, so the given series diverges.

8.5.63 For $k > 7$, $\ln k > 2$ so note that $\frac{1}{k^{\ln k}} < \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series converges as well.

8.5.64 Use the Limit Comparison Test with $\{1/k^2\}$. Note that $\frac{\sin^2(1/k)}{1/k^2} = \left(\frac{\sin(1/k)}{1/k}\right)^2$. Because $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, the limit of this expression is $1^2 = 1$ as $k \to \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series does as well.

8.5.65 Use the Limit Comparison Test with the harmonic series. $\frac{\tan(1/k)}{1/k}$ has limit 1 as $k \to \infty$ because $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$. Thus the original series diverges.

8.5.66 Use the Root Test. $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{100} \cdot \frac{1}{k} = 0$, so the given series converges.

8.5.67 Note that $\dfrac{1}{(2k+1)\cdot(2k+3)} = \dfrac{1}{2}\left(\dfrac{1}{2k+1} - \dfrac{1}{2k+3}\right)$. Thus this series telescopes.

$$\sum_{k=0}^{n} \frac{1}{(2k+1)(2k+3)} = \frac{1}{2}\sum_{k=0}^{n}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) = \frac{1}{2}\left(-\frac{1}{2n+3} + 1\right),$$

so the given series converges to $1/2$, because that is the limit of the sequence of partial sums.

8.5.68 This series is $\sum_{k=1}^{\infty} \frac{k-1}{k^2} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k^2}\right)$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, if the original series also converged, we would have that $\sum_{k=1}^{\infty} \frac{1}{k}$ converged, which is false. Thus the original series diverges.

8.5.69 This series is $\sum_{k=1}^{\infty} \frac{k^2}{k!}$. By the Ratio Test, $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(k+1)!} \cdot \frac{k!}{k^2} = \frac{1}{k+1}\left(\frac{k+1}{k}\right)^2$, which has limit 0 as $k \to \infty$, so the given series converges.

8.5.70 For any p, if k is sufficently large then $k^{1/p} > \ln k$ because powers grow faster than logs, so that $k > (\ln k)^p$ and thus $1/k < 1/(\ln k)^p$. Because $\sum 1/k$ diverges, we see that the original series diverges for all p.

8.5.71 For $p \le 1$ and $k > e$, $\frac{\ln k}{k^p} > \frac{1}{k^p}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges, so the given series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k, $\ln k < k^q$, so that by the Comparison Test, $\frac{\ln k}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so that $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ is a convergent p-series. Thus the original series is convergent precisely when $p > 1$.

8.5.72 For $p \ne 1$,

$$\int_2^{\infty} \frac{dx}{x \ln x (\ln\ln x)^p} = \lim_{b\to\infty}\left(\left.\frac{(\ln\ln x)^{1-p}}{1-p}\right|_2^b\right).$$

This improper integral converges if and only $p > 1$. If $p = 1$, we have

$$\int_2^{\infty} \frac{dx}{x(\ln x)\ln\ln x} = \lim_{b\to\infty} \ln\ln\ln x\Big|_2^b = \infty.$$

Thus the original series converges for $p > 1$.

8.5.73 For $p \le 1$, $\frac{(\ln k)^p}{k^p} > \frac{1}{k^p}$ for $k \ge 3$, and $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p \le 1$, so the original series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k, $(\ln k)^p < k^q$. Note that $\frac{(\ln k)^p}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ converges, so the given series converges. Thus, the given series converges exactly for $p > 1$.

8.5.74 Using the Ratio Test, $\frac{a_{k+1}}{a_k} = \frac{(k+1)!p^{k+1}}{(k+2)^{k+1}} \cdot \frac{(k+1)^k}{(k)!p^k} = \frac{(k+1)p(k+1)^k}{(k+2)^{k+1}} = p\left(\frac{k+1}{k+2}\right)^{k+1} = p \cdot \left(\frac{1}{1+\frac{1}{k+1}}\right)^{k+1}$, which has limit pe^{-1}. The series converges if the ratio limit is less than 1, so if $p < e$. If $p > e$, the given series diverges by the Ratio Test. If $p = e$, the given series diverges by the Divergence Test.

8.5.75 Use the Ratio Test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)p^{k+1}}{k+2} \cdot \frac{k+1}{kp^k} = p,$$

so the given series converges for $p < 1$ and diverges for $p > 1$. For $p = 1$ the given series diverges by limit comparison with the harmonic series.

8.5.76 $\ln\left(\frac{k}{k+1}\right)^p = p(\ln(k) - \ln(k+1))$, so

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)^p = p\sum_{k=1}^{\infty}(\ln(k) - \ln(k+1))$$

which telescopes, and the n^{th} partial sum is $-p\ln(n+1)$, and $\lim_{n\to\infty} -p\ln(n+1)$ is not a finite number for any value of p other than 0. The given series diverges for all values of p other than $p = 0$.

8.5.77 $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \left(1 - \frac{p}{k}\right)^k = e^{-p} \neq 0$, so this sequence diverges for all p by the Divergence Test.

8.5.78 Use the Limit Comparison Test: $\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$, because $\sum a_k$ converges. By the Limit Comparison Test, the series $\sum a_k^2$ must converge as well.

8.5.79 These tests apply only for series with positive terms, so assume $r > 0$. Clearly the series do not converge for $r = 1$, so we assume $r \neq 1$ in what follows. Using the Integral Test, $\sum r^k$ converges if and only if $\int_1^\infty r^x \, dx$ converges. This improper integral has value $\lim_{b\to\infty} \left.\frac{r^x}{\ln r}\right|_1^b$, which converges only when $\lim_{b\to\infty} r^b$ exists, which occurs only for $r < 1$. Using the Ratio Test, $\frac{a_{k+1}}{a_k} = \frac{r^{k+1}}{r^k} = r$, so by the Ratio Test, the series converges if and only if $r < 1$. Using the Root Test, $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{r^k} = \lim_{k\to\infty} r = r$, so again we have convergence if and only if $r < 1$. By the Divergence Test, we know that a geometric series diverges if $|r| \geq 1$.

8.5.80

a. Use the Limit Comparison Test with the divergent harmonic series. Note that $\lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, because $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Because the comparison series diverges, the given series does as well.

b. We use the Limit Comparison Test with the convergent series $\sum \frac{1}{k^2}$. Note that $\lim_{k\to\infty} \frac{(1/k)\sin(1/k)}{1/k^2} = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series converges.

8.5.81 To prove case (2), assume $L = 0$ and that $\sum b_k$ converges. Because $L = 0$, for every $\varepsilon > 0$, there is some N such that for all $n > N$, $|\frac{a_k}{b_k}| < \varepsilon$. Take $\varepsilon = 1$; this then says that there is some N such that for all $n > N$, $0 < a_k < b_k$. By the Comparison Test, because $\sum b_k$ converges, so does $\sum a_k$. To prove case (3), because $L = \infty$, then $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, so by the argument above, we have $0 < b_k < a_k$ for sufficient large k. But $\sum b_k$ diverges, so by the Comparison Test, $\sum a_k$ does as well.

8.5.82 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$. This has limit 0 as $k \to \infty$ for any value of x, so the series converges for all $x \geq 0$.

8.5.83 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{x^k} = x$. This has limit x as $k \to \infty$, so the series converges for $x < 1$. It clearly does not converge for $x = 1$. So the series converges for $x \in [0, 1)$.

8.5.84 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} = x \cdot \frac{k}{k+1}$, which has limit x as $k \to \infty$. Thus this series converges for $x < 1$; additionally, for $x = 1$ (where the Ratio Test is inconclusive), the series is the harmonic series which diverges. So the series converges for $x \in [0, 1)$.

8.5.85 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} = x\left(\frac{k}{k+1}\right)^2$, which has limit x as $k \to \infty$. Thus the series converges for $x < 1$. When $x = 1$, the series is $\frac{1}{k^2}$, which converges. Thus the original series converges for $0 \leq x \leq 1$.

8.5.86 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\frac{a_{k+1}}{a_k} = \frac{x^{2k+2}}{(k+1)^2} \cdot \frac{k^2}{x^{2k}} = x^2\left(\frac{k}{k+1}\right)^2$, which has limit x^2 as $k \to \infty$, so the series converges for $x < 1$. When $x = 1$, the series is $\frac{1}{k^2}$, which converges. Thus this series converges for $0 \leq x \leq 1$.

8.5.87 The series clearly converges for $x = 0$. For $x \neq 0$, we have $\dfrac{a_{k+1}}{a_k} = \dfrac{x^{k+1}}{2^{k+1}} \cdot \dfrac{2^k}{x^k} = \dfrac{x}{2}$, which has limit $x/2$ as $k \to \infty$. Thus the series converges for $0 \le x < 2$. For $x = 2$, it is obviously divergent.

8.5.88

a. Let P_n be the n^{th} partial product of the a_k: $P_n = \prod_{k=1}^n a_k$. Then $\sum_{k=1}^n \ln a_k = \ln \prod_{k=1}^n a_k = \ln P_n$. If $\sum \ln a_k$ is a convergent series, then $\sum_{k=1}^\infty \ln a_k = \lim_{n\to\infty} \ln P_n = L < \infty$. But then $e^L = \lim_{n\to\infty} e^{\ln P_n} = \lim_{n\to\infty} P_n$, so that the infinite product converges.

b.

n	2	3	4	5	6	7	8
P_n	3/4	2/3	5/8	3/5	7/12	4/7	9/16

It appears that $P_n = \frac{n+1}{2n}$, so that $\lim_{n\to\infty} P_n = \frac{1}{2}$.

c. Because $\lim_{n\to\infty} \prod_{k=2}^n \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$, taking logs and using part (a) we see that $\lim_{n\to\infty} \sum_{k=1}^n \ln\left(1 - \frac{1}{k^2}\right) = \ln \frac{1}{2} = -\ln 2$.

8.5.89

a. $\ln \prod_{k=0}^\infty e^{1/2^k} = \sum_{k=0}^\infty \frac{1}{2^k} = 2$, so that the original product converges to e^2.

b. $\ln \prod_{k=2}^\infty \left(1 - \frac{1}{k}\right) = \ln \prod_{k=2}^\infty \frac{k-1}{k} = \sum_{k=2}^\infty \ln \frac{k-1}{k} = \sum_{k=2}^\infty (\ln(k-1) - \ln(k))$. This series telescopes to give $S_n = -\ln(n)$, so the original series has limit $\lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{-\ln(n)} = 0$.

8.5.90 The sum on the left is simply the left Riemann sum over n equal intervals between 0 and 1 for $f(x) = x^p$. The limit of the sum is thus $\int_0^1 x^p dx = \frac{1}{p+1} x^{p+1}\Big|_0^1 = \frac{1}{p+1}$, because p is positive.

8.5.91

a. Use the Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{p^{k+1}(k+1)!} \cdot \frac{p^k (k)!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} = \frac{(2k+1)}{(k+1)p}$$

and this expression has limit $\frac{2}{p}$ as $k \to \infty$. Thus the series converges for $p > 2$.

b. Following the hint, when $p = 2$ we have $\displaystyle\sum_{k=1}^\infty \frac{(2k)!}{2^k k!(2 \cdot 4 \cdot 6 \cdots 2k)} = \sum_{k=1}^\infty \frac{(2k)!}{(2^k)^2 (k!)^2}$. Using Stirling's formula, the numerator is asymptotic to $2\sqrt{\pi}\sqrt{k}(2k)^{2k}e^{-2k} = 2\sqrt{\pi}\sqrt{k}(2^k)^2(k^k)^2 e^{-2k}$ while the denominator is asymptotic to $(2^k)^2 2\pi k (k^k)^2 e^{-2k}$, so the quotient is asymptotic to $\frac{1}{\sqrt{\pi}\sqrt{k}}$. Thus the original series diverges for $p = 2$ by the Limit Comparison Test with the divergent p-series $\sum_{k=1}^\infty \frac{1}{k^{1/2}}$.

8.6 Alternating Series

8.6.1 Because $S_{n+1} - S_n = (-1)^n a_{n+1}$ alternates signs.

8.6.2 Check that the terms of the series are nonincreasing in magnitude after some finite number of terms, and that $\lim_{k\to\infty} a_k = 0$.

8.6.3 We have

$$S = S_{2n+1} + (a_{2n} - a_{2n+1}) + (a_{2n+2} - a_{2n+3}) + \cdots$$

and each term of the form $a_{2k} - a_{2k+1} > 0$, so that $S_{2n+1} < S$. Also

$$S = S_{2n} + (-a_{2n+1} + a_{2n+2}) + (-a_{2n+3} + a_{2n+4}) + \cdots$$

and each term of the form $-a_{2k+1} + a_{2k+2} < 0$, so that $S < S_{2n}$. Thus the sum of the series is trapped between the odd partial sums and the even partial sums.

8.6.4 The difference between L and S_n is bounded in magnitude by a_{n+1}.

8.6.5 The remainder is less than the first neglected term because

$$S - S_n = (-1)^{n+1}(a_{n+1} + (-a_{n+2} + a_{n+3}) + \cdots)$$

so that the sum of the series *after* the first disregarded term has the opposite sign from the first disregarded term.

8.6.6 The alternating harmonic series $\sum(-1)^k \frac{1}{k}$ converges, but not absolutely.

8.6.7 No. If the terms are positive, then the absolute value of each term is the term itself, so convergence and absolute convergence would mean the same thing in this context.

8.6.8 The idea of the proof is to note that $0 \le |a_k| + a_k \le 2\,|a_k|$ and apply the Comparison Test to conclude that if $\sum |a_k|$ converges, then so does $\sum 2\,|a_k|$, and thus so must $\sum(|a_k| + a_k)$, and then conclude that $\sum a_k$ must converge as well.

8.6.9 Yes. For example, $\sum \frac{(-1)^k}{k^3}$ converges absolutely and thus not conditionally (see the definition).

8.6.10 The alternating harmonic series $\sum(-1)^k \frac{1}{k}$ converges conditionally, but not absolutely.

8.6.11 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{1}{2k+1} = 0$, so the given series converges.

8.6.12 The terms of the series decrease in magnitude, and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so the given series converges.

8.6.13 $\lim_{k\to\infty} \frac{k}{3k+2} = \frac{1}{3} \neq 0$, so the given series diverges.

8.6.14 $\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \neq 0$, so the given series diverges.

8.6.15 The terms of the series decrease in magnitude, and $\lim\limits_{k\to\infty} \frac{1}{k^3} = 0$, so the given series converges.

8.6.16 The terms of the series decrease in magnitude, and $\lim\limits_{k\to\infty} \frac{1}{k^2+10} = 0$, so the given series converges.

8.6.17 The terms of the series decrease in magnitude, and $\lim\limits_{k\to\infty} \frac{k^2}{k^3+1} = \lim\limits_{k\to\infty} \frac{1/k}{1+1/k^3} = 0$, so the given series converges.

8.6.18 The terms of the series eventually decrease in magnitude, because if $f(x) = \frac{\ln x}{x^2}$, then $f'(x) = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$, which is negative for large enough x. Further, $\lim\limits_{k\to\infty} \frac{\ln k}{k^2} = \lim\limits_{k\to\infty} \frac{1/k}{2k} = \lim\limits_{k\to\infty} \frac{1}{2k^2} = 0$. Thus the given series converges.

8.6.19 $\lim\limits_{k\to\infty} \frac{k^2-1}{k^2+3} = 1$, so the terms of the series do not tend to zero and thus the given series diverges.

8.6.20 $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{5}\right)^k$. $(1/5)^k$ is decreasing, and tends to zero as $k \to \infty$, so the given series converges.

8.6.21 $\lim_{k\to\infty} \left(1+\frac{1}{k}\right) = 1$, so the given series diverges.

8.6.22 Note that $\cos(\pi k) = (-1)^k$, and so the given series is alternating. Because $\lim_{k\to\infty} \frac{1}{k^2} = 0$ and $\frac{1}{k^2}$ is decreasing, the given series is convergent.

8.6.23 The derivative of $f(k) = \frac{k^{10}+2k^5+1}{k(k^{10}+1)}$ is $f'(k) = \frac{-(k^{20}+2k^{10}+12k^{15}-8k^5+1)}{k^2(k^{10}+1)^2}$. The numerator is negative for large enough values of k, and the denominator is always positive, so the derivative is negative for large enough k. Also, $\lim_{k\to\infty} \frac{k^{10}+2k^5+1}{k(k^{10}+1)} = \lim_{k\to\infty} \frac{1+2k^{-5}+k^{-10}}{k+k^{-9}} = 0$. Thus the given series converges.

8.6.24 Clearly $\frac{1}{k\ln^2 k}$ is nonincreasing, and $\lim_{k\to\infty} \frac{1}{k\ln^2 k} = 0$, so the given series converges.

8.6.25 $\lim_{k\to\infty} k^{1/k} = 1$ (for example, take logs and apply L'Hôpital's rule), so the given series diverges by the Divergence Test.

8.6.26 $a_{k+1} < a_k$ because $\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} = \left(\frac{k}{k+1}\right)^k < 1$. Additionally, $\frac{k!}{k^k} \to 0$ as $k \to \infty$, so the given series converges.

8.6.27 $\frac{1}{\sqrt{k^2+4}}$ is decreasing and tends to zero as $k \to \infty$, so the given series converges.

8.6.28 $\lim_{k\to\infty} k\sin(1/k) = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series diverges.

8.6.29 We want $\frac{1}{n+1} < 10^{-4}$, or $n+1 > 10^4$, so $n = 10^4$.

8.6.30 The series starts with $k = 0$, so we want $\frac{1}{n!} < 10^{-4}$, or $n! > 10^4 = 10000$. This happens for $n = 8$.

8.6.31 The series starts with $k = 0$, so we want $\frac{1}{2n+1} < 10^{-4}$, or $2n+1 > 10^4$, $n = 5000$.

8.6.32 We want $\frac{1}{(n+1)^2} < 10^{-4}$, or $(n+1)^2 > 10^4$, so $n = 100$.

8.6.33 We want $\frac{1}{(n+1)^4} < 10^{-4}$, or $(n+1)^4 > 10^4$, so $n = 10$.

8.6.34 The series starts with $k = 0$, so we want $\frac{1}{(2n+1)^3} < 10^{-4}$, or $2n+1 > 10^{4/3}$, so $n = 11$.

8.6.35 The series starts with $k = 0$, so we want $\frac{1}{3n+1} < 10^{-4}$, or $3n+1 > 10^4$, $n = 3334$.

8.6.36 We want $\frac{1}{(n+1)^6} < 10^{-4}$, or $(n+1)^6 > 10^4 = 10000$, so $n = 4$.

8.6.37 The series starts with $k = 0$, so we want $\frac{1}{4^n}\left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right) < 10^{-4}$, or $\frac{4^n(4n+1)(4n+2)(4n+3)}{4(20n^2+21n+5)} > 10000$, which occurs first for $n = 6$.

8.6.38 The series starts with $k = 0$, so we want $\frac{1}{3n+2} < 10^{-4}$, so $3n+2 > 10000$, $n = 3333$.

8.6.39 To figure out how many terms we need to sum, we must find n such that $\frac{1}{(n+1)^5} < 10^{-3}$, so that $(n+1)^5 > 1000$; this occurs first for $n = 3$. Thus $\frac{-1}{1} + \frac{1}{2^5} - \frac{1}{3^5} \approx -0.973$.

8.6.40 To figure out how many terms we need to sum, we must find n such that $\frac{1}{(2(n+1)+1)^3} < 10^{-3}$, or $(2n+3)^3 > 10^3$, so $2n+3 > 10$ and $n = 4$. Thus the approximation is $\sum_{k=1}^{4} \frac{(-1)^n}{(2n+1)^3} \approx -0.306$.

8.6.41 To figure out how many terms we need to sum, we must find n so that $\frac{n+1}{(n+1)^2+1} < 10^{-3}$, so that $\frac{(n+1)^2+1}{n+1} = n+1+\frac{1}{n+1} > 1000$. This occurs first for $n = 999$. We have $\sum_{k=1}^{999} \frac{(-1)^k k}{k^2+1} \approx -0.269$.

8.6.42 To figure out how many terms we need to sum, we must find n such that $\frac{n+1}{(n+1)^4+1} < 10^{-3}$, so that $\frac{(n+1)^4+1}{n+1} = (n+1)^3 + \frac{1}{n+1} > 1000$, which occurs for $n = 9$. We have $\sum_{k=1}^{9} \frac{(-1)^k k}{k^4+1} \approx -0.409$.

8.6.43 To figure how many terms we need to sum, we must find n such that $\frac{1}{(n+1)^{n+1}} < 10^{-3}$, or $(n+1)^{n+1} > 1000$, so $n = 4$ ($5^5 = 3125$). Thus the approximation is $\sum_{k=1}^{4} \frac{(-1)^n}{n^n} \approx -.783$.

8.6.44 To figure how many terms we need to sum, we must find n such that $\frac{1}{(2(n+1)+1)!} < 10^{-3}$, or $(2n+3)! > 1000$, so $2n+3 \geq 7$ and $n = 2$. The approximation is $\sum_{k=1}^{2} \frac{(-1)^{n+1}}{(2n+1)!} \approx 0.158$

8.6.45 The series of absolute values is a p-series with $p = 2/3$, so it diverges. The given alternating series does converge, though, by the Alternating Series Test. Thus, the given series is conditionally convergent.

8.6.46 The series of absolute values is a p-series with $p = 1/2$, so it diverges. The given alternating series does converge, though, by the Alternating Series Test. Thus, the given series is conditionally convergent.

8.6.47 The series of absolute values is a p-series with $p = 3/2$, so it converges absolutely.

8.6.48 The series of absolute values is $\sum \frac{1}{3^k}$, which converges, so the series converges absolutely.

8.6.49 The series of absolute values is $\sum \frac{|\cos(k)|}{k^3}$, which converges by the Comparison Test because $\frac{|\cos(k)|}{k^3} \leq \frac{1}{k^3}$. Thus the series converges absolutely.

8.6.50 The series of absolute values is $\sum \frac{k^2}{\sqrt{k^6+1}}$. The limit comparison test with $\frac{1}{k}$ gives $\lim_{k\to\infty} \frac{k^3}{\sqrt{k^6+1}} = \lim_{k\to\infty} \sqrt{\frac{k^6}{k^6+1}} = 1$. Because the comparison series diverges, so does the series of absolute values. The original series converges conditionally, however, because the terms are nonincreasing and $\lim_{k\to\infty} \frac{k^2}{\sqrt{k^6+1}} = \lim_{k\to\infty} \sqrt{\frac{k^4}{k^6+1}} = 0$.

8.6.51 The absolute value of the kth term of this series has limit $\pi/2$ as $k \to \infty$, so the given series is divergent by the Divergence Test.

8.6.52 The series of absolute values is a geometric series with $r = \frac{1}{e}$ and $|r| < 1$, so the given series converges absolutely

8.6.53 The series of absolute values is $\sum \frac{k}{2k+1}$, but $\lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$, so by the Divergence Test, this series diverges. The original series does not converge conditionally, either, because $\lim_{k\to\infty} a_k = \frac{1}{2} \neq 0$.

8.6.54 The series of absolute values is $\sum \frac{1}{\ln k}$, which diverges, so the series does not converge absolutely. However, because $\lim_{k\to\infty} \frac{1}{\ln k} \to 0$ and the terms are nonincreasing, the series does converge conditionally.

8.6.55 The series of absolute values is $\sum \frac{\tan^{-1}(k)}{k^3}$, which converges by the Comparison Test because $\frac{\tan^{-1}(k)}{k^3} < \frac{\pi}{2}\frac{1}{k^3}$, and $\sum \frac{\pi}{2}\frac{1}{k^3}$ converges because it is a constant multiple of a convergent $p-$series. So the original series converges absolutely.

8.6.56 The series of absolute values is $\sum \frac{e^k}{(k+1)!}$. Using the ratio test, $\frac{a_{k+1}}{a_k} = \frac{e^{k+1}}{(k+2)!} \cdot \frac{(k+1)!}{e^k} = \frac{e}{k+2}$, which tends to zero as $k \to \infty$, so the original series converges absolutely.

8.6.57

a. False. For example, consider the alternating harmonic series.

b. True. This is part of Theorem 8.21.

c. True. This statement is simply saying that a convergent series converges.

d. True. This is part of Theorem 8.21.

e. False. Let $a_k = \frac{1}{k}$.

f. True. Use the Comparison Test: $\lim_{k\to\infty} \frac{a_k^2}{a_k} = \lim_{k\to\infty} a_k = 0$ because $\sum a_k$ converges, so $\sum a_k^2$ and $\sum a_k$ converge or diverge together. Because the latter converges, so does the former.

g. True, by definition. If $\sum |a_k|$ converged, the original series would converge absolutely, not conditionally.

8.6.58 Neither condition is satisfied. $\frac{a_{k+1}}{a_k} = \frac{(k+1)(2k+1)}{(2k+3)k} = \frac{2k^2+3k+1}{2k^2+3k} > 1$, and $\lim_{k\to\infty} a_k = \frac{1}{2}$.

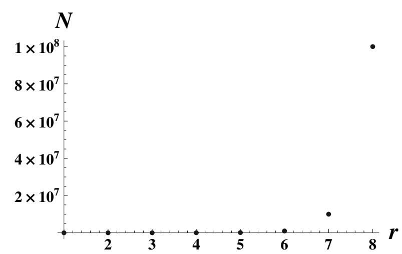
8.6.59 $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = 2 \cdot \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6} - \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12}$.

8.6.60 $\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = 2 \cdot \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4}$, and thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = \frac{\pi^4}{90} - \frac{1}{8} \cdot \frac{\pi^4}{90} = \frac{7\pi^4}{720}$.

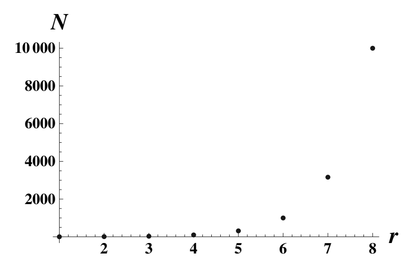
8.6.61 Write $r = -s$; then $0 < s < 1$ and $\sum r^k = \sum (-1)^k s^k$. Because $|s| < 1$, the terms s^k are nonincreasing and tend to zero, so by the Alternating Series Test, the series $\sum (-1)^k s^k = \sum r^k$ converges.

8.6.62

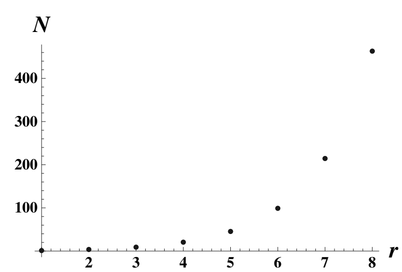
a. As p gets larger, fewer terms are needed to achieve a particular level of accuracy; this means that for larger p, the series converge faster.



$p = 1$

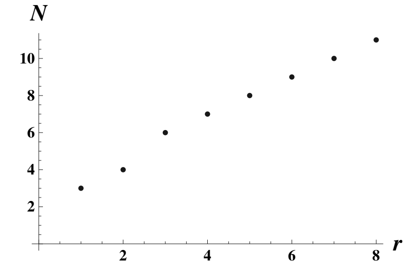


$p = 2$



$p = 3$

b. This graph shows that $\sum \frac{1}{k!}$ converges much faster than any of the powers of k.



8.6.63 Let $S = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$. Then

$$\begin{array}{rcccccccccc} S &=& \left(1-\frac{1}{2}\right) &+& \left(\frac{1}{3}-\frac{1}{4}\right) &+& \left(\frac{1}{5}-\frac{1}{6}\right) &+& \left(\frac{1}{7}-\frac{1}{8}\right) &+& \ldots \\ \frac{1}{2}S &=& \frac{1}{2} &-& \frac{1}{4} &+& \frac{1}{6} &-& \frac{1}{8} &+& \ldots \end{array}$$

Add these two series together to get

$$\frac{3}{2}S = \frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \cdots$$

To see that the results are as desired, consider a collection of four terms:

$$\begin{array}{ccccc} \cdots + & \left(\frac{1}{4k+1} - \frac{1}{4k+2}\right) & + & \left(\frac{1}{4k+3} - \frac{1}{4k+4}\right) & + \quad \ldots \\ \cdots & + \quad \frac{1}{4k+2} & - & \frac{1}{4k+4} & + \quad \cdots \end{array}$$

Adding these results in the desired sign pattern. This repeats for each group of four elements.

8.6.64

a. Note that we can write

$$S_n = -\frac{a_1}{2} + \frac{1}{2}\left(\sum_{k=1}^{n-1}(-1)^k(a_i - a_{i+1})\right) + \frac{(-1)^n a_n}{2},$$

so that

$$S_n + \frac{(-1)^{n+1}a_{n+1}}{2} = -\frac{a_1}{2} + \frac{1}{2}\left(\sum_{k=1}^{n}(-1)^k d_i\right)$$

where $d_i = a_i - a_{i+1}$. Now consider the expression on the right-hand side of this last equation as the nth partial sum of a series which converges to S. Because the d_i's are decreasing and positive, the error made by stopping the sum after n terms is less than the absolute value of the first omitted term, which would be $\frac{1}{2}|d_{n+1}| = \frac{1}{2}|a_{n+1} - a_{n+2}|$. The method in the text for approximating the error simply takes the absolute value of the first unused term as an approximation of $|S - S_n|$. Here, S_n is modified by adding half the next term. Because the terms are decreasing in magnitude, this should be a better approximation to S than just S_n itself; the right side shows that this intuition is correct, because $\frac{1}{2}|a_{n+1} - a_{n+2}|$ is at most a_{n+1} and is generally less than that (because generally $a_{n+2} < a_{n+1}$).

b. i. Using the method from the text, we need n such that $\frac{1}{n+1} < 10^{-6}$, i.e. $n > 10^6 - 1$. Using the modified method from this problem, we want $\frac{1}{2}|a_{n+1} - a_{n+2}| < 10^{-6}$, so

$$\frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2(n+1)(n+2)} < 10^{-6}$$

This is true when $10^6 < 2(n+1)(n+2)$, which requires $n > 705.6$, so $n \geq 706$.

ii. Using the method from the book, we need n such that $k \ln k > 10^6$, which means $k \geq 87848$. Using the method of this problem, we want

$$\frac{1}{2}\left|\left(\frac{1}{k\ln k} - \frac{1}{(k+1)\ln(k+1)}\right)\right| = \left|\frac{(k+1)\ln(k+1) - k\ln k}{2k(k+1)\ln k \ln(k+1)}\right| < 10^{-6},$$

so that $|2k(k+1)\ln k\ln(k+1)| > |10^6(k\ln k - (k+1)\ln(k+1))|$, which means $k \geq 319$.

iii. Using the method from the book, we need k such that $\sqrt{k} > 10^6$, so $k > 10^{12}$. Using the method of this problem, we want

$$\frac{1}{2}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \frac{\sqrt{k+1} - \sqrt{k}}{2\sqrt{k(k+1)}} < 10^{-6}$$

which means that $k > 3968.002$ so that $k \geq 3969$.

8.6.65 Both series diverge, so comparisons of their values are not meaningful.

8.6.66

a. The first ten terms are

$$(2-1) + \left(1 - \frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{2}{5} - \frac{1}{5}\right)$$

Suppose that $k = 2i$ is even (and so $k-1 = 2i-1$ is odd). Then the sum of the $(k-1)$st term and the kth term is $\frac{4}{k} - \frac{2}{k} = \frac{2}{k} = \frac{1}{i}$. Then the sum of the first $2n$ terms of the given series is $\sum_{i=1}^{n} \frac{1}{i}$.

b. Note that $\lim_{k\to\infty} \frac{4}{k+1} = \lim_{k\to\infty} \frac{2}{k} = 0$. Thus given $\epsilon > 0$ there exists N_1 so that for $k > N_1$, we have $\frac{4}{k+1} < \epsilon$. Also, there exist N_2 so that for $k > N_2$, $\frac{2}{k} < \epsilon$. Let N be the larger of N_1 or N_2. Then for $k > N$, we have $a_k < \epsilon$, as desired.

c. The series can be seen to diverge because the even partial sums have limit ∞. This does not contradict the alternating series test because the terms a_k are not nonincreasing.

Chapter Eight Review

1

a. False. Let $a_n = 1 - \frac{1}{n}$. This sequence has limit 1.

b. False. The terms of a sequence tending to zero is necessary but not sufficient for convergence of the series.

c. True. This is the definition of convergence of a series.

d. False. If a series converges absolutely, the definition says that it does not converge conditionally.

e. True. It has limit 1 as $n \to \infty$.

f. False. The subsequence of the even terms has limit 1 and the subsequence of odd terms has limit -1, so the sequence does not have a limit.

g. False. It diverges by the Divergence Test because $\lim_{k\to\infty} \frac{k^2}{k^2+1} = 1 \neq 0$.

h. True. The given series converges by the Limit Comparison Test with the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus its sequence of partial sums converges.

2 $\lim_{n\to\infty} \frac{n^2+4}{\sqrt{4n^4+1}} = \lim_{n\to\infty} \frac{1+4n^{-2}}{\sqrt{4+n^{-4}}} = \frac{1}{2}$.

3 $\lim_{n\to\infty} \frac{8^n}{n!} = 0$ because exponentials grow more slowly than factorials.

4 After taking logs, we want to compute

$$\lim_{n\to\infty} 2n\ln(1+3/n) = \lim_{n\to\infty} \frac{\ln(1+3/n)}{1/(2n)}.$$

By L'Hôpital's rule, this is $\lim_{n\to\infty} \frac{6n}{n+3}$ (after some algebraic manipulations), which is 6. Thus the original limit is e^6.

5 Take logs and compute $\lim_{n\to\infty}(1/n)\ln n = \lim_{n\to\infty}(\ln n)/n = \lim_{n\to\infty}\frac{1}{n} = 0$ by L'Hôpital's rule. Thus the original limit is $e^0 = 1$.

6 $\lim_{n\to\infty}(n-\sqrt{n^2-1}) = \lim_{n\to\infty} \frac{n-\sqrt{n^2-1}}{1}\cdot\frac{n+\sqrt{n^2-1}}{n+\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{1}{n+\sqrt{n^2+1}} = 0$.

7 Take logs, and then evaluate $\lim_{n\to\infty}\frac{1}{\ln n}\ln(1/n) = \lim_{n\to\infty}(-1) = -1$, so the original limit is e^{-1}.

8 This series oscillates among the values $\pm 1/2, \pm\sqrt{3}/2, \pm 1$, and 0, so it has no limit.

9 $a_n = (-1/0.9)^n = (-10/9)^n$. The terms grow without bound so the sequence does not converge.

10 $\lim_{n\to\infty}\tan^{-1} n = \lim_{x\to\infty}\tan^{-1} x = \frac{\pi}{2}$.

11

a. $S_1 = \frac{1}{3}$, $S_2 = \frac{11}{24}$, $S_3 = \frac{21}{40}$, $S_4 = \frac{17}{30}$.

b. $S_n = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$, because the series telescopes.

c. From part (b), $\lim_{n\to\infty} S_n = \frac{3}{4}$, which is the sum of the series.

12 This is a geometric series with ratio 9/10, so the sum is $\frac{9/10}{1-9/10} = 9$.

13 $\sum_{k=1}^{\infty} 3(1.001)^k = 3\sum_{k=1}^{\infty}(1.001)^k$. This is a geometric series with ratio greater than 1, so it diverges.

14 This is a geometric series with ratio $-1/5$, so the sum is $\frac{1}{1+1/5} = \frac{5}{6}$

15 $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, so the series telescopes, and $S_n = 1 - \frac{1}{n+1}$. Thus $\lim_{n\to\infty} S_n = 1$, which is the value of the series.

16 This series clearly telescopes, and $S_n = \frac{1}{\sqrt{n}} - 1$, so $\lim_{n\to\infty} S_n = -1$.

17 This series telescopes. $S_n = 3 - \frac{3}{3n+1}$, so that $\lim_{n\to\infty} S_n = 3$, which is the value of the series.

18 $\sum_{k=1}^{\infty} 4^{-3k} = \sum_{k=1}^{\infty}(1/64)^k$. This is a geometric series with ratio 1/64, so its sum is $\frac{1/64}{1-1/64} = \frac{1}{63}$.

19 $\sum_{k=1}^{\infty}\frac{2^k}{3^{k+2}} = \frac{1}{9}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{9}\cdot\frac{2/3}{1-2/3} = \frac{2}{9}$.

20 This is the difference of two convergent geometric series (because both have ratios less than 1). Thus the sum of the series is equal to

$$\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{k}-\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k+1}=\frac{1}{1-1/3}-\frac{2/3}{1-2/3}=\frac{3}{2}-2=-\frac{1}{2}.$$

21

a. It appears that the series converges, because the sequence of partial sums appears to converge to 1.5.

b. The convergence is uncertain.

c. This series clearly appears to diverge, because the partial sums seem to be growing without bound.

22 This is p-series with $p = 3/2 > 1$, so this series is convergent.

23 The series can be written $\sum \frac{1}{k^{2/3}}$, which is a p-series with $p = 2/3 < 1$, so this series diverges.

24 $a_k = \frac{2k^2+1}{\sqrt{k^3+2}} = \sqrt{\frac{4k^4+4k^2+1}{k^3+2}}$, so the sequence of terms diverges. By the Divergence Test, the given series diverges as well.

25 This is a geometric series with ratio $2/e < 1$, so the series converges.

26 Note that $\frac{1}{a_k} = \left(\left(1+\frac{3}{k}\right)^k\right)^2$, so $\lim_{k\to\infty}\frac{1}{a_k} = \lim_{k\to\infty}\left(\left(1+\frac{3}{k}\right)^k\right)^2 = (e^3)^2$, so $\lim_{k\to\infty} a_k = \frac{1}{e^6} \neq 0$, so the given series diverges by the Divergence Test.

27 Applying the Ratio Test:

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{2^{k+1}(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^k}{2^k k!}=\lim_{k\to\infty}2\left(\frac{k}{k+1}\right)^k=\frac{2}{e}<1,$$

so the given series converges.

28 Use the Limit Comparison Test with $\frac{1}{k}$:

$$\frac{1}{\sqrt{k^2+k}}\Big/\frac{1}{k}=\frac{k}{\sqrt{k^2+k}}=\sqrt{\frac{k^2}{k^2+k}},$$

which has limit 1 as $k\to\infty$. Because $\sum 1/k$ diverges, the original series does as well.

29 Use the Comparison Test: $\frac{3}{2+e^k} < \frac{3}{e^k}$, but $\sum \frac{3}{e^k}$ converges because it is a geometric series with ratio $\frac{1}{e} < 1$. Thus the original series converges as well.

30 $\lim_{k\to\infty} a_k = \lim_{k\to\infty} k\sin(1/k) = \lim_{k\to\infty}\frac{\sin(1/k)}{1/k} = 1$, so the given series diverges by the Divergence Test.

31 $a_k = \frac{k^{1/k}}{k^3} = \frac{1}{k^{3-1/k}}$. For $k \geq 2$, then, $a_k < \frac{1}{k^2}$. Because $\sum \frac{1}{k^2}$ converges, the given series also converges, by the Comparison Test.

32 Use the Comparison Test: $\frac{1}{1+\ln k} > \frac{1}{k}$ for $k > 1$. Because $\sum \frac{1}{k}$ diverges, the given series does as well.

33 Use the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{(k+1)^5}{e^{k+1}}\cdot\frac{e^k}{k^5} = \frac{1}{e}\cdot\left(\frac{k+1}{k}\right)^5$, which has limit $1/e < 1$ as $k\to\infty$. Thus the given series converges.

34 For $k > 5$, we have $k^2 - 10 > (k-1)^2$, so that $a_k = \frac{2}{k^2-10} < \frac{2}{(k-1)^2}$. Because $\sum \frac{2}{(k-1)^2}$ converges, the original series does as well.

35 Use the Comparison Test. Because $\lim_{k\to\infty} \dfrac{\ln k}{k^{1/2}} = 0$, we have that for sufficiently large k, $\ln k < k^{1/2}$, so that $a_k = \frac{2\ln k}{k^2} < \frac{2k^{1/2}}{k^2} = \frac{2}{k^{3/2}}$. Now $\sum \frac{2}{k^{3/2}}$ is convergent, because it is a p-series with $p = 3/2 > 1$. Thus the original series is convergent.

36 By the Ratio Test: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k+1}{e^{k+1}} \cdot \frac{e^k}{k} = \lim_{k\to\infty} \frac{1}{e} \cdot \frac{k+1}{k} = \frac{1}{e} < 1$. Thus the given series converges.

37 Use the Ratio Test. The ratio of successive terms is $\frac{2\cdot 4^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{2\cdot 4^k} = \frac{4}{(2k+3)(2k+2)}$. This has limit 0 as $k \to \infty$, so the given series converges.

38 Use the Ratio Test. The ratio of successive term is $\frac{9^{k+1}}{(2k+2)!} \cdot \frac{(2k)!}{9^k} = \frac{9}{(2k+2)(2k+1)}$. This has limit 0 as $k \to \infty$, so the given series converges.

39 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k\to\infty} \dfrac{\coth k}{k} \cdot \dfrac{k}{1} = \lim_{k\to\infty} \coth k = 1$. Because the harmonic series diverges, the given series does as well.

40 Use the Limit Comparison Test with the convergent geometric series whose kth term is $\frac{1}{e^k}$. We have $\lim_{k\to\infty} \frac{1}{\sinh k} \cdot \frac{e^k}{1} = \lim_{k\to\infty} \frac{2e^k}{e^k - e^{-k}} = 2\lim_{k\to\infty} \frac{1}{1-e^{-2k}} = 2$. The given series is therefore convergent.

41 Use the Divergence Test. $\lim_{k\to\infty} \tanh k = \lim_{k\to\infty} \frac{e^k + e^{-k}}{e^k - e^{-k}} = 1 \neq 0$, so the given series diverges.

42 Use the Limit Comparison Test with the convergent geometric series whose kth term is $\frac{1}{e^k}$. We have $\lim_{k\to\infty} \frac{1}{\cosh k} \cdot \frac{e^k}{1} = \lim_{k\to\infty} \frac{2e^k}{e^k + e^{-k}} = 2\lim_{k\to\infty} \frac{1}{1+e^{-2k}} = 2$. The given series is therefore convergent.

43 $|a_k| = \frac{1}{k^2-1}$. Use the Limit Comparison Test with the convergent series $\sum \frac{1}{k^2}$. Because $\lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1$, the given series converges absolutely.

44 This series does not converge, because $\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \frac{k^2+4}{2k^2+1} = \frac{1}{2}$.

45 Use the Ratio Test on the absolute values of the sequence of terms: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+1}{e^{k+1}} \cdot \frac{e^k}{k} = \lim_{k\to\infty} \frac{1}{e} \cdot \frac{k+1}{k} = \frac{1}{e} < 1$. Thus, the original series is absolutely convergent.

46 Using the Limit Comparison Test with the harmonic series, we consider $\lim_{k\to\infty} a_k/(1/k) = \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}}$ $= \lim_{k\to\infty} \sqrt{\frac{k^2}{k^2+1}} = 1$; because the comparison series diverges, so does the original series. Thus the series is not absolutely convergent. However, the terms are clearly decreasing to zero, so it is conditionally convergent.

47 Use the Ratio Test on the absolute values of the sequence of terms: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10}{k+1} = 0$, so the series converges absolutely.

48 $\sum \frac{1}{k\ln k}$ does not converge because $\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \ln(\ln x)\Big|_2^\infty = \infty$, so the improper integral diverges. Thus the given series does not converge absolutely. However, it does converge conditionally because the terms are decreasing and approach zero.

49 Because $k^2 \ll 2^k$, $\lim_{k\to\infty} \frac{-2\cdot(-2)^k}{k^2} \neq 0$. The given series thus diverges by the Divergence Test.

50 The series of absolute values converges, by the Limit Comparison Test with the convergent geometric series whose kth term is $\frac{1}{e^k}$. This follows because $\lim_{k\to\infty} \frac{1}{e^k + e^{-k}} \cdot \frac{e^k}{1} = \lim_{k\to\infty} \frac{1}{1+e^{-2k}} = 1$.

51

a. For $|x| < 1$, $\lim\limits_{k\to\infty} x^k = 0$, so this limit is zero.

b. This is a geometric series with ratio $-4/5$, so the sum is $\frac{1}{1+4/5} = \frac{5}{9}$.

52

a. $\lim\limits_{k\to\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim\limits_{k\to\infty}\frac{1}{k(k+1)} = 0$.

b. This series telescopes, and $S_n = 1 - \frac{1}{n+1}$, so $\lim\limits_{n\to\infty} S_n = 1$, which is the sum of the series.

53 Consider the constant sequence with $a_k = 1$ for all k. The sequence $\{a_k\}$ converges to 1, but the corresponding series $\sum a_k$ diverges by the divergence test.

54 This is not possible. If the series $\sum_{k=1}^{\infty} a_k$ converges, then we must have $\lim_{k\to\infty} a_k = 0$.

55

a. This sequence converges because $\lim_{k\to\infty}\frac{k}{k+1} = \lim_{k\to\infty}\frac{1}{1+\frac{1}{k}} = \frac{1}{1+0} = 1$.

b. Because the sequence of terms has limit 1 (which means its limit isn't zero) this series diverges by the divergence test.

56 No. The geometric sequence converges for $-1 < r \le 1$, while the geometric series converges for $-1 < r < 1$. So the geometric sequence converges for $r = 1$ but the geometric series does not.

57 Because the series converges, we must have $\lim\limits_{k\to\infty} a_k = 0$. Because it converges to 8, the partial sums converge to 8, so that $\lim\limits_{k\to\infty} S_k = 8$.

58 R_n is given by

$$R_n \le \int_n^{\infty} \frac{1}{x^5}\,dx = \lim_{b\to\infty}\left(-\frac{1}{4x^4}\bigg|_n^b\right) = \frac{1}{4n^4}.$$

Thus to approximate the sum to within 10^{-4}, we need $\frac{1}{4n^4} < 10^{-4}$, so $4n^4 > 10^4$ and $n = 8$.

59 The series converges absolutely for $p > 1$, conditionally for $0 < p \le 1$ in which case $\{k^{-p}\}$ is decreasing to zero.

60 By the Integral Test, the series converges if and only if the following integral converges:

$$\int_2^{\infty} \frac{1}{x\ln^p(x)}\,dx = \lim_{b\to\infty}\left(\frac{1}{1-p}\ln^{(1-p)}(x)\bigg|_2^b\right) = \lim_{b\to\infty}\frac{1}{1-p}\ln^{(1-p)}(b) - \left(\frac{1}{1-p}\right)\cdot\ln^{(1-p)}(2).$$

This limit exists only if $1 - p < 0$, i.e. $p > 1$. Note that the above calculation is for the case $p \ne 1$. In the case $p = 1$, the integral also diverges.

61 The sum is 0.2500000000 to ten decimal places. The maximum error is

$$\int_{20}^{\infty} \frac{1}{5^x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{5^x\ln 5}\bigg|_{20}^b\right) = \frac{1}{5^{20}\ln 5} \approx 6.5\times 10^{-15}.$$

62 The sum is 1.037. The maximum error is

$$\int_{20}^{\infty} \frac{1}{x^5}\,dx = \lim_{b\to\infty}\left(-\frac{1}{4x^4}\bigg|_{20}^b\right) = \frac{1}{4\cdot 20^4} \approx 1.6\times 10^{-6}.$$

63 The maximum error is a_{n+1}, so we want $a_{n+1} = \frac{1}{(k+1)^4} < 10^{-8}$, or $(k+1)^4 > 10^8$, so $k = 100$.

64

a. $\sum_{k=0}^{\infty} e^{kx} = \sum_{k=0}^{\infty} (e^x)^k = \frac{1}{1-e^x} = 2$, so $1 - e^x = 1/2$. Thus $e^x = 1/2$ and $x = -\ln(2)$.

b. $\sum_{k=0}^{\infty} (3x)^k = \frac{1}{1-3x} = 4$, so that $1 - 3x = \frac{1}{4}$, $x = \frac{1}{4}$.

c. The x's cancel, so the equation reads $\sum_{k=0}^{\infty} \left(\frac{1}{k-1/2} - \frac{1}{k+1/2}\right) = 6$. The series telescopes, so that the left side, up to n, is

$$\sum_{k=0}^{n} \left(\frac{1}{k-1/2} - \frac{1}{k+1/2}\right) = \frac{1}{-1/2} - \frac{1}{n+1/2} = -2 - \frac{1}{n+1/2}$$

and in the limit the equation then reads $-2 = 6$, so that there is no solution.

65

a. Let T_n be the amount of additional tunnel dug during week n. Then $T_0 = 100$ and $T_n = .95 \cdot T_{n-1} = (.95)^n T_0 = 100(0.95)^n$, so the total distance dug in N weeks is

$$S_N = 100 \sum_{k=0}^{N-1} (0.95)^k = 100 \left(\frac{1-(0.95)^N}{1-0.95}\right) = 2000(1 - 0.95^N).$$

Then $S_{10} \approx 802.5$ meters and $S_{20} \approx 1283.03$ meters.

b. The longest possible tunnel is $S_\infty = 100 \sum_{k=0}^{\infty} (0.95)^k = \frac{100}{1-.95} = 2000$ meters.

66 Let t_n be the time required to dig meters $(n-1) \cdot 100$ through $n \cdot 100$, so that $t_1 = 1$ week. Then $t_n = 1.1 \cdot t_{n-1} = (1.1)^{n-1} t_1 = (1.1)^{n-1}$ weeks. The time required to dig 1500 meters is then

$$\sum_{k=1}^{15} t_k = \sum_{k=1}^{15} (1.1)^{k-1} \approx 31.77 \text{ weeks.}$$

So it is not possible.

67

a. The area of a circle of radius r is πr^2. For $r = 2^{1-n}$, this is $2^{2-2n}\pi$. There are 2^{n-1} circles on the n^{th} page, so the total area of circles on the n^{th} page is $2^{n-1} \cdot \pi 2^{2-2n} = 2^{1-n}\pi$.

b. The sum of the areas on all pages is $\sum_{k=1}^{\infty} 2^{1-k}\pi = 2\pi \sum_{k=1}^{\infty} 2^{-k} = 2\pi \cdot \frac{1/2}{1/2} = 2\pi$.

68 $x_0 = 1$, $x_1 \approx 1.540302$, $x_2 \approx 1.57079$, $x_3 \approx 1.570796327$, which is $\frac{\pi}{2}$ to nine decimal places. Thus $p = 2$.

69

a. $B_n = 1.0025 B_{n-1} + 100$ and $B_0 = 100$.

b. $B_n = 100 \cdot 1.0025^n + 100 \cdot \frac{1-1.0025^n}{1-1.0025} = 100 \cdot 1.0025^n - 40000(1 - 1.0025^n) = 40000(1.0025^{n+1} - 1)$.

70

a. $a_n = \displaystyle\int_0^1 x^n \, dx = \frac{1}{n+1} x^{n+1}\Big|_0^1 = \frac{1}{n+1}$, so $\displaystyle\lim_{n\to\infty} a_n = 0$.

b. $b_n = \displaystyle\int_1^n \frac{1}{x^p} \, dx = \frac{1}{1-p} x^{1-p}\Big|_1^n = \frac{1}{1-p}(n^{1-p} - 1)$. Because $p > 1$, $n^{1-p} \to 0$ as $n \to \infty$, so that $\displaystyle\lim_{n\to\infty} b_n = \frac{1}{p-1}$.

71

a. $T_1 = \frac{\sqrt{3}}{16}$ and $T_2 = \frac{7\sqrt{3}}{64}$.

b. At stage n, 3^{n-1} triangles of side length $1/2^n$ are removed. Each of those triangles has an area of $\dfrac{\sqrt{3}}{4 \cdot 4^n} = \dfrac{\sqrt{3}}{4^{n+1}}$, so a total of

$$3^{n-1} \cdot \frac{\sqrt{3}}{4^{n+1}} = \frac{\sqrt{3}}{16} \cdot \left(\frac{3}{4}\right)^{n-1}$$

is removed at each stage. Thus

$$T_n = \frac{\sqrt{3}}{16} \sum_{k=1}^{n} \left(\frac{3}{4}\right)^{k-1} = \frac{\sqrt{3}}{16} \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^{k} = \frac{\sqrt{3}}{4} \left(1 - \left(\frac{3}{4}\right)^{n}\right).$$

c. $\lim_{n\to\infty} T_n = \frac{\sqrt{3}}{4}$ because $\left(\frac{3}{4}\right)^n \to 0$ as $n \to \infty$.

d. The area of the triangle was originally $\frac{\sqrt{3}}{4}$, so none of the original area is left.

72 Because the given sequence is non-decreasing and bounded above by 1, it must have a limit. A reasonable conjecture is that the limit is 1.

Chapter 9

Power Series

9.1 Approximating Functions With Polynomials

9.1.1 Let the polynomial be $p(x)$. Then $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

9.1.2 It generally increases, because the more derivatives of f are taken into consideration, the better "fit" the polynomial will provide to f.

9.1.3 The approximations are $p_0(0.1) = 1$, $p_1(0.1) = 1 + \frac{0.1}{2} = 1.05$, and $p_2(0.1) = 1 + \frac{0.1}{2} - \frac{.01}{8} = 1.04875$.

9.1.4 The first three terms: $f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

9.1.5 The remainder is the difference between the value of the Taylor polynomial at a point and the true value of the function at that point, $R_n(x) = f(x) - p_n(x)$.

9.1.6 This is explained in Theorem 9.2. The idea is that the error when using an nth order Taylor polynomial centered at a is $|R_n(x)| \le M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$ where M is an upper bound for the $(n+1)$st derivative of f for values between a and x.

9.1.7

a. Note that $f(1) = 8$, and $f'(x) = 12\sqrt{x}$, so $f'(1) = 12$. Thus, $p_1(x) = 8 + 12(x-1)$.

b. $f''(x) = 6/\sqrt{x}$, so $f''(1) = 6$. Thus $p_2(x) = 8 + 12(x-2) + 3(x-1)^2$.

c. $p_1(1.1) = 12 \cdot 0.1 + 8 = 9.2$. $p_2(1.1) = 3(.1)^2 + 12 \cdot 0.1 + 8 = 9.23$.

9.1.8

a. Note that $f(1) = 1$, and that $f'(x) = -1/x^2$, so $f'(1) = -1$. Thus, $p_1(x) = 1 - (x-1) = -x + 2$.

b. $f''(x) = 2/x^3$, so $f''(1) = 2$. Thus, $p_2(x) = 2 - x + (x-1)^2$.

c. $p_1(1.05) = 0.95$. $p_2(1.05) = (0.05)^2 - 0.05 + 2 = .953$.

9.1.9

a. $f'(x) = -e^{-x}$, so $p_1(x) = f(0) + f'(0)x = 1 - x$.

b. $f''(x) = e^{-x}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 - x + \frac{1}{2}x^2$.

c. $p_1(0.2) = 0.8$, and $p_2(0.2) = 1 - 0.2 + \frac{1}{2}(0.04) = 0.82$.

9.1.10

a. $f'(x) = \frac{1}{2}x^{-1/2}$, so $p_1(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$.

b. $f''(x) = -\frac{1}{4}x^{-3/2}$, so $p_2(x) = f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$.

c. $p_1(3.9) = 2 + \frac{1}{4}(-0.1) = 2 - 0.025 = 1.975$, and $p_2(3.9) = 2 - 0.025 - \frac{1}{64}(0.001) = 1.975$.

9.1.11

a. $f'(x) = -\frac{1}{(x+1)^2}$, so $p_1(x) = f(0) + f'(0)x = 1 - x$.

b. $f''(x) = \frac{2}{(x+1)^3}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 - x + x^2$.

c. $p_1(0.05) = 0.95$, and $p_2(0.05) = 1 - 0.05 + 0.0025 = 0.953$.

9.1.12

a. $f'(x) = -\sin x$, so $p_1(x) = \cos(\pi/4) - \sin(\pi/4)(x - \pi/4) = \frac{\sqrt{2}}{2}(1 - (x - \pi/4))$.

b. $f''(x) = -\cos x$, so

$$\begin{aligned} p_2(x) &= \cos(\pi/4) - \sin(\pi/4)(x-\pi/4) - \frac{1}{2}\cos(\pi/4)(x-\pi/4)^2 \\ &= \frac{\sqrt{2}}{2}\left(1 - (x-\pi/4) - \frac{1}{2}(x-\pi/4)^2\right). \end{aligned}$$

c. $p_1(0.24\pi) \approx 0.729$, $p_2(0.24\pi) \approx 0.729$.

9.1.13

a. $f'(x) = (1/3)x^{-2/3}$, so $p_1(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8)$.

b. $f''(x) = (-2/9)x^{-5/3}$, so $p_2(x) = f(8) + f'(8)(x-8) + \frac{1}{2}f''(8)(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

c. $p_1(7.5) \approx 1.958$, $p_2(7.5) \approx 1.957$.

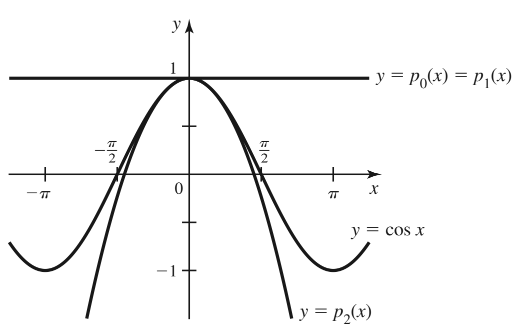
9.1.14

a. $f'(x) = \frac{1}{1+x^2}$, so $p_1(x) = f(0) + f'(0)x = x$.

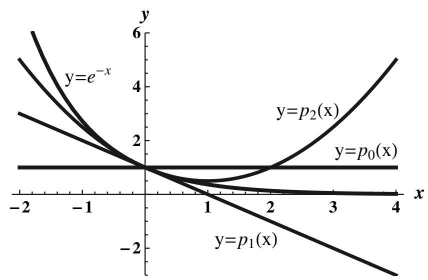
b. $f''(x) = -\frac{2x}{(1+x^2)^2}$, so $p_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$.

c. $p_1(0.1) = p_2(0.1) = 0.1$.

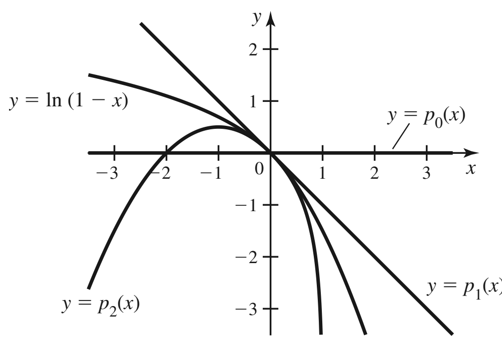
9.1.15 $f(0) = 1, f'(0) = -\sin 0 = 0, f''(0) = -\cos 0 = -1$, so that $p_0(x) = 1$, $p_1(x) = 1$, $p_2(x) = 1 - \frac{1}{2}x^2$.



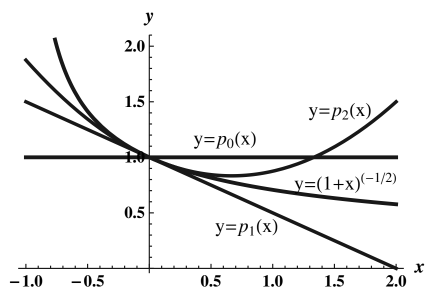
9.1.16 $f(0) = 1$, $f'(0) = -e^0 = -1$, $f''(0) = e^0 = 1$, so that $p_0(x) = 1$, $p_1(x) = 1 - x$, $p_2(x) = 1 - x + \frac{x^2}{2}$.



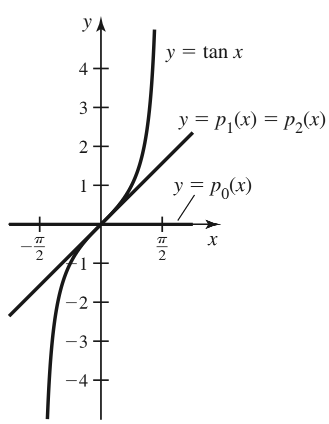
9.1.17 $f(0) = 0$, $f'(0) = -\frac{1}{1-0} = -1, f''(0) = -\frac{1}{(1-0)^2} = -1$, so that $p_0(x) = 0$, $p_1(x) = -x$, $p_2(x) = -x - \frac{1}{2}x^2$.



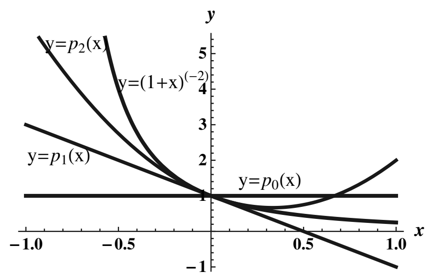
9.1.18 $f(0) = 1$, $f'(0) = (-1/2)(0+1)^{-3/2} = -1/2$, $f''(0) = (3/4)(0+1)^{-5/2} = 3/4$, so that $p_0(x) = 1$, $p_1(x) = 1 - \frac{x}{2}$, $p_2(x) = 1 - \frac{x}{2} + \frac{3}{8}x^2$.



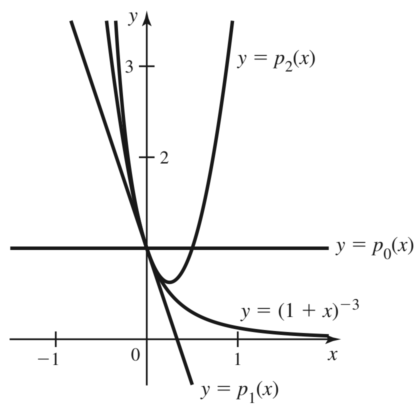
9.1.19 $f(0) = 0$. $f'(x) = \sec^2 x$, $f''(x) = 2\tan x \sec^2 x$, so that $f'(0) = 1$, $f''(0) = 0$. Thus $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x$.



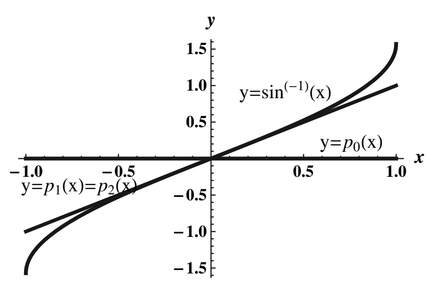
9.1.20 $f(0) = 1$, $f'(0) = (-2)(1+0)^{-3} = -2$, $f''(0) = 6(1+0)^{-4} = 6$. Thus $p_0(x) = 1$, $p_1(x) = 1 - 2x$, $p_2(x) = 1 - 2x + 3x^2$.



9.1.21 $f(0) = 1$, $f'(0) = -3(1+0)^{-4} = -3$, $f''(0) = 12(1+0)^{-5} = 12$, so that $p_0(x) = 1$, $p_1(x) = 1 - 3x$, $p_2(x) = 1 - 3x + 6x^2$.



9.1.22 $f(0) = 0$, $f'(x) = \frac{1}{\sqrt{1-x^2}}$, $f''(x) = \frac{x}{(1-x^2)^{3/2}}$, so that $f'(0) = 1$, $f''(0) = 0$. Thus $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x$.



9.1.23

a. $p_2(0.05) \approx 1.025$.

b. The absolute error is $\sqrt{1.05} - p_2(0.05) \approx 7.68 \times 10^{-6}$.

9.1.24

a. $p_2(0.1) \approx 1.032$.

b. The absolute error is $1.1^{1/3} - p_2(0.1) \approx 5.8 \times 10^{-5}$.

9.1.25

a. $p_2(0.08) \approx 0.962$.

b. The absolute error is $p_2(0.08) - \frac{1}{\sqrt{1.08}} \approx 1.5 \times 10^{-4}$.

9.1.26

a. $p_2(0.06) = 0.058$.

b. The absolute error is $\ln 1.06 - p_2(0.06) \approx 6.9 \times 10^{-5}$.

9.1.27

a. $p_2(0.15) \approx 0.861$.

b. The absolute error is $p_2(0.15) - e^{-0.15} \approx 5.4 \times 10^{-4}$.

9.1.28

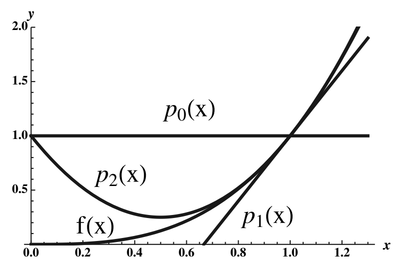
a. $p_2(0.12) \approx 0.726$.

b. The absolute error is $p_2(0.12) = \frac{1}{1.12^3} \approx 1.5 \times 10^{-2}$.

9.1.29

a. Note that $f(1) = 1$, $f'(1) = 3$, and $f''(1) = 6$. Thus, $p_0(x) = 1$, $p_1(x) = 1 + 3(x-1)$, and $p_2(x) = 1 + 3(x-1) + 3(x-1)^2$.

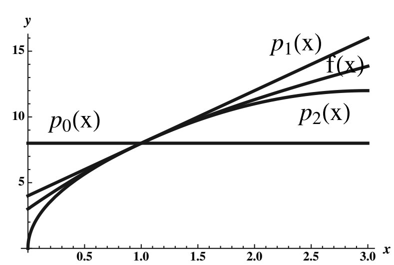
b.



9.1.30

a. Note that $f(1) = 8$, $f'(1) = \frac{4}{\sqrt{1}} = 4$, and $f''(1) = \frac{-2}{(1)^{3/2}} = -2$ Thus, $p_0(x) = 8$, $p_1(x) = 8 + 4(x-1)$, $p_2(x) = 8 + 4(x-1) - (x-1)^2$.

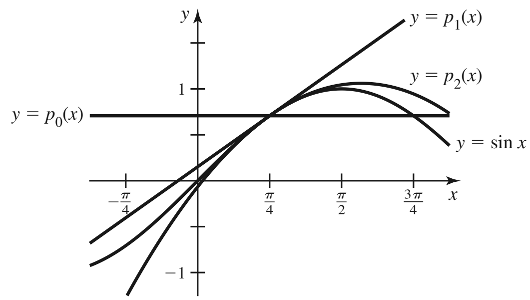
b.



9.1.31

a. $p_0(x) = \frac{\sqrt{2}}{2}$, $p_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$, $p_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2$.

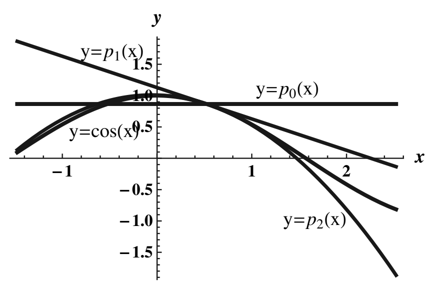
b.



9.1.32

a. $p_0(x) = \frac{\sqrt{3}}{2}$, $p_1(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$, $p_2(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2$.

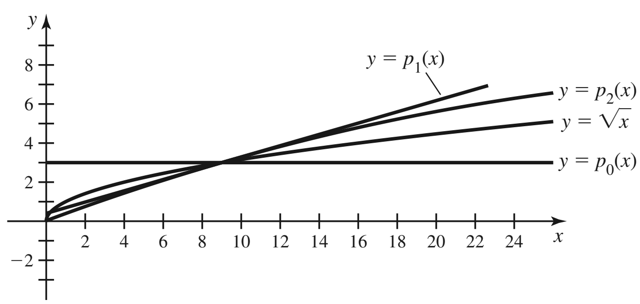
b.



9.1.33

a. $p_0(x) = 3$, $p_1(x) = 3 + \frac{1}{6}(x-9)$, $p_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$.

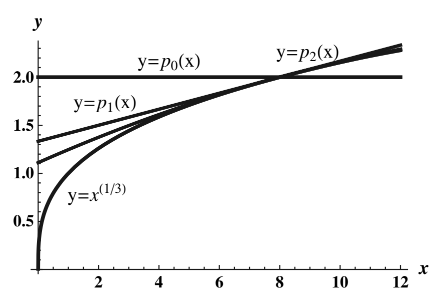
b.



9.1.34

a. $p_0(x) = 2$, $p_1(x) = 2 + \frac{1}{12}(x-8)$, $p_2(x) = x + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$.

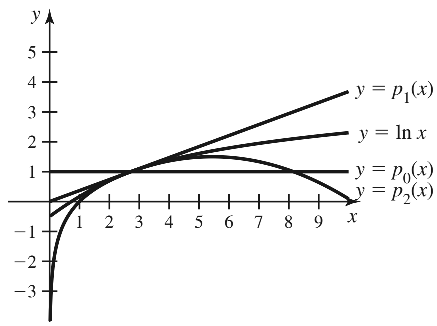
b.



9.1.35

a. $p_0(x) = 1$, $p_1(x) = 1 + \frac{1}{e}(x - e)$, $p_2(x) = 1 + \frac{1}{e}(x - e) - \frac{1}{2e^2}(x - e)^2$.

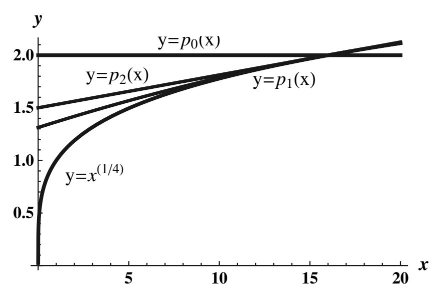
b.



9.1.36

a. $p_0(x) = 2$, $p_1(x) = 2 + \frac{1}{32}(x - 16)$, $p_2(x) = 2 + \frac{1}{32}(x - 16) - \frac{3}{4096}(x - 16)^2$.

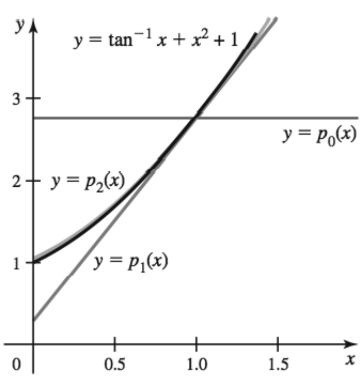
b.



9.1.37

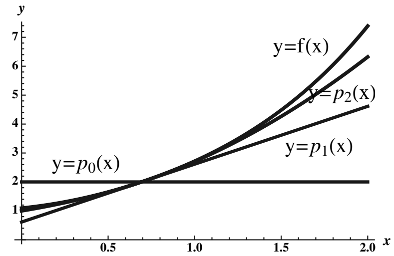
a. $f(1) = \frac{\pi}{4} + 2$, $f'(1) = \frac{1}{2} + 2 = \frac{5}{2}$. $f''(1) = -\frac{1}{2} + 2 = \frac{3}{2}$. $p_0(x) = 2 + \frac{\pi}{4}$, $p_1(x) = 2 + \frac{\pi}{4} + \frac{5}{2}(x - 1)$, $p_2(x) = 2 + \frac{\pi}{4} + \frac{5}{2}(x - 1) + \frac{3}{4}(x - 1)^2$.

b.



9.1.38

a. $f(\ln 2) = 2$, $f'(\ln 2) = 2$, $f''(\ln 2) = 2$. So $p_0(x) = 2$, $p_1(x) = 2 + 2(x - \ln 2)$, $p_2(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2$.

b. 

9.1.39

a. Ue the Taylor polynomial centered at 0 with $f(x) = e^x$. We have $p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$. $p_3(0.12) \approx 1.127$.

b. $|f(0.12) - p_3(0.12)| \approx 8.9 \times 10^{-6}$.

9.1.40

a. Use the Taylor polynomial centered at 0 with $f(x) = \cos(x)$. We have $p_3(x) = 1 - \frac{1}{2}x^2$. $p_3(-0.2) = 0.98$.

b. $|f(0.12) - p_3(0.12)| \approx 6.7 \times 10^{-5}$.

9.1.41

a. Use the Taylor polynomial centered at 0 with $f(x) = \tan(x)$. We have $p_3(x) = x + \frac{1}{3}x^3$. $p_3(-0.1) \approx -0.100$.

b. $|p_3(-0.1) - f(-0.1)| \approx 1.3 \times 10^{-6}$.

9.1.42

a. Use the Taylor polynomial centered at 0 with $f(x) = \ln(1 + x)$. We have $p_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$. $p_3(0.05) \approx 0.0488$.

b. $|p_3(0.05) - f(0.05)| \approx 1.5 \times 10^{-6}$.

9.1.43

a. Use the Taylor polynomial centered at 0 with $f(x) = \sqrt{1 + x}$. We have $p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$. $p_3(0.06) \approx 1.030$.

b. $|f(0.06) - p_3(0.06)| \approx 4.9 \times 10^{-7}$.

9.1.44

a. Use the Taylor polynomial centered at 81 with $f(x) = \sqrt[4]{x}$. We have $p_3(x) = 3 + \frac{1}{108}(x-81) - \frac{1}{23328}(x-81)^2 + \frac{7}{22674816}(x - 81)^3$. $p_3(79) \approx 2.981$.

b. $|p_3(79) - f(79)| \approx 4.3 \times 10^{-8}$.

9.1.45

a. Use the Taylor polynomial centered at 100 with $f(x) = \sqrt{x}$. We have $p_3(x) = 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2 + \frac{1}{1600000}(x - 100)^3$. $p_3(101) \approx 10.050$.

b. $|p_3(101) - f(101)| \approx 3.9 \times 10^{-9}$.

9.1.46

a. Use the Taylor polynomial centered at 125 with $f(x) = \sqrt[3]{x}$. We have $p_3(x) = 5 + \frac{1}{75}(x - 125) - \frac{1}{28125}(x - 125)^2 + \frac{1}{6328125}(x - 125)^3$. $p_3(125) \approx 5.013$.

b. $|p_3(126) - f(126)| \approx 8.4 \times 10^{-10}$.

9.1.47

a. Use the Taylor polynomial centered at 0 with $f(x) = \sinh(x)$. Note that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = 1$. Then we have $p_3(x) = x + x^3/6$, so $\sinh(.5) \approx (.5)^3/6 + .5 \approx 0.521$.

b. $|p_3(.5) - \sinh(.5)| \approx 2.6 \times 10^{-4}$.

9.1.48

a. Use the Taylor polynomial centered at 0 with $f(x) = \tanh(x)$, Note that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -2$. Then we have $p_3(x) = -x^3/3 + x$, so $\tanh(.5) \approx -(.5)^2/3 + .5 \approx 0.449$.

b. $|p_3(x) - \tanh(.5)| \approx 3.8 \times 10^{-3}$.

9.1.49 With $f(x) = \sin x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for c between 0 and x.

9.1.50 With $f(x) = \cos 2x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for c between 0 and x.

9.1.51 With $f(x) = e^{-x}$ we have $f^{(n+1)}(x) = (-1)^{n+1}e^{-x}$, so that $R_n(x) = \dfrac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for c between 0 and x.

9.1.52 With $f(x) = \cos x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}\left(x - \dfrac{\pi}{2}\right)^{n+1}$ for c between $\frac{\pi}{2}$ and x.

9.1.53 With $f(x) = \sin x$ we have $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}\left(x - \dfrac{\pi}{2}\right)^{n+1}$ for c between $\frac{\pi}{2}$ and x.

9.1.54 With $f(x) = \frac{1}{1-x}$ we have $f^{(n+1)}(x) = (-1)^{n+1}\frac{1}{(1-x)^{n+2}}$ so that $R_n(x) = \dfrac{(-1)^{n+1}}{(1-c)^{n+2}}\left(x^{n+1}\right)$ for c between 0 and x.

9.1.55 $f(x) = \sin x$, so $f^{(5)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_4(x)| \le \frac{0.3^5}{5!} \approx 2.0 \times 10^{-5}$.

9.1.56 $f(x) = \cos x$, so $f^{(4)}(x) = \cos x$. Because $\cos x$ is bounded in magnitude by 1, the remainder is bounded by $|R_3(x)| \le \frac{0.45^4}{4!} \approx 1.7 \times 10^{-3}$.

9.1.57 $f(x) = e^x$, so $f^{(5)}(x) = e^x$. Because $e^{0.25}$ is bounded by 2, $|R_4(x)| \le 2 \cdot \frac{0.25^5}{5!} \approx 1.63 \times 10^{-5}$.

9.1.58 $f(x) = \tan x$, so $f^{(3)}(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$. Now, since both $\tan x$ and $\sec x$ are increasing on $[0, \pi/2]$, and $0.3 < \frac{\pi}{6} \approx 0.524$, we can get an upper bound on $f^{(3)}(x)$ on $[0, 0.3]$ by evaluating at $\frac{\pi}{6}$; this gives $f^{(3)}(x) < \frac{16}{3}$ on $[0, 0.3]$. Thus $|R_2(x)| \le \frac{16}{3} \cdot \frac{0.3^3}{3!} = 2.4 \times 10^{-2}$.

9.1.59 $f(x) = e^{-x}$, so $f^{(5)}(x) = -e^{-x}$. Because $f^{(5)}$ achieves its maximum magnitude in the range at $x = 0$, which has absolute value 1, $|R_4(x)| \le 1 \cdot \frac{0.5^5}{5!} \approx 2.6 \times 10^{-4}$.

9.1.60 $f(x) = \ln(1 + x)$, so $f^{(4)}(x) = -\frac{6}{(x+1)^4}$. On $[0, 0.4]$, the maximum magnitude is 6, so $|R_3(x)| \le 6 \cdot \frac{0.4^4}{4!} = 6.4 \times 10^{-3}$.

9.1.61 Here $n = 3$ or 4, so use $n = 4$, and $M = 1$ because $f^{(5)}(x) = \cos x$, so that $R_4(x) \le \frac{(\pi/4)^5}{5!} \approx 2.49 \times 10^{-3}$.

9.1.62 $n = 2$ or 3, so use $n = 3$, and $M = 1$ because $f^{(4)}(x) = \cos x$, so that $|R_3(x)| \le \frac{(\pi/4)^4}{4!} \approx 1.6 \times 10^{-2}$.

9.1.63 $n = 2$ and $M = e^{1/2} < 2$, so $|R_2(x)| \le 2 \cdot \frac{(1/2)^3}{3!} \approx 4.2 \times 10^{-2}$.

9.1.64 $n = 1$ or 2, so use 2, and $f^{(3)}(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$. On $[\frac{-\pi}{6}, \frac{\pi}{6}]$ this achieves its maximum value at $\pm\frac{\pi}{6}$; that value is $\frac{16}{3}$. Thus $|R_2(x)| \le \frac{16}{3} \cdot \frac{(\pi/6)^3}{3!} \approx 1.28 \times 10^{-1}$.

9.1.65 $n = 2$; $f^{(3)}(x) = \frac{2}{(1+x)^3}$, which achieves its maximum at $x = -0.2$: $|f^{(3)}(x)| = \frac{2}{0.8^3} < 4$. Then $|R_2(x)| \le 4 \cdot \frac{0.2^3}{3!} \approx 5.4 \times 10^{-3}$.

9.1.66 $n = 1$, $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, which achieves its maximum magnitude at $x = -0.1$, where it is less than 1/3. Thus $R_1(x) \le \frac{1}{3} \cdot \frac{0.1^2}{2!} \approx 1.7 \times 10^{-3}$.

9.1.67 Use the Taylor series for e^x at $x = 0$. The derivatives of e^x are e^x. On $[-0.5, 0]$, the maximum magnitude of any derivative is thus 1 at $x = 0$, so $|R_n(-0.5)| \le \frac{0.5^{n+1}}{(n+1)!}$, so for $R_n(-0.5) < 10^{-3}$ we need $n = 4$.

9.1.68 Use the Taylor series at $x = 0$ for $\sin x$. The magnitude of any derivative of $\sin x$ is bounded by 1, so $|R_n(0.2)| \le \frac{0.2^{n+1}}{(n+1)!}$, so for $R_n(0.2) < 10^{-3}$ we need $n = 3$.

9.1.69 Use the Taylor series for $\cos x$ at $x = 0$. The magnitude of any derivative of $\cos x$ is bounded by 1, so $|R_n(-0.25)| \le \frac{0.25^{n+1}}{(n+1)!}$, so for $|R_n(-0.25)| < 10^{-3}$ we need $n = 3$.

9.1.70 Use the Taylor series for $f(x) = \ln(1+x)$ at $x = 0$. Then $|f^{(n+1)}(x)| = \frac{n!}{(1+x)^{n+1}}$, which for $x \in [-0.15, 0]$ achieves its maximum at $x = -.15$. This maximum is less than $(1.2)^{n+1} \cdot n!$. Thus $|R_n(-0.15)| \le (1.2)^{n+1} \cdot n! \cdot \frac{.18^{n+1}}{(n+1)!} = \frac{1.2 \cdot (0.15)^{n+1}}{n}$, so for $|R_n(-0.15)| < 10^{-3}$ we need $n = 3$.

9.1.71 Use the Taylor series for $f(x) = \sqrt{x}$ at $x = 1$. Then $|f^{(n+1)}(x)| = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}} x^{-(2n+1)/2}$, which achieves its maximum on $[1, 1.06]$ at $x = 1$. Then

$$|R_n(1.06)| \le \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}} \cdot \frac{(1.06-1)^{n+1}}{(n+1)!},$$

and for $|R_n(0.06)| < 10^{-3}$ we need $n = 1$.

9.1.72 Use the Taylor series for $f(x) = \sqrt{1/(1-x)}$ at $x = 0$. Then $|f^{(n+1)}(x)| = \dfrac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}}(1 - x)^{(-3-2n)/2}$, which achieves its maximum on $[0, 0.15]$ at $x = 0.15$. Thus

$$\begin{aligned} |R_n(0.15)| &\le \frac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}} \cdot \left(\frac{1}{1-0.15}\right)^{(2n+3)/2} \cdot \frac{0.15^{n+1}}{(n+1)!} \\ &= \frac{1 \cdot 3 \cdot \cdots \cdot (2n+1)}{2^{n+1}(n+1)!} \cdot \left(\frac{0.15^{n+1}}{0.85^{(2n+3)/2}}\right), \end{aligned}$$

and for $|R_n(0.15)| < 10^{-3}$ we need $n = 3$.

9.1.73

a. False. If $f(x) = e^{-2x}$, then $f^{(n)}(x) = (-1)^n 2^n e^{-2x}$, so that $f^{(n)}(0) \ne 0$ and all powers of x are present in the Taylor series.

b. True. The constant term of the Taylor series is $f(0) = 1$. Higher-order terms all involve derivatives of $f(x) = x^5 - 1$ evaluated at $x = 0$; clearly for $n < 5$, $f^{(n)}(0) = 0$, and for $n > 5$, the derivative itself vanishes. Only for $n = 5$, where $f^{(5)}(x) = 5!$, is the derivative nonzero, so the coefficient of x^5 in the Taylor series is $f^{(5)}(0)/5! = 1$ and the Taylor polynomial of order 10 is in fact $x^5 - 1$. Note that this statement is true of any polynomial of degree at most 10.

c. True. The odd derivatives of $\sqrt{1+x^2}$ vanish at $x=0$, while the even ones do not.

d. True. Clearly the second-order Taylor polynomial for f at a has degree at most 2. However, the coefficient of $(x-a)^2$ is $\frac{1}{2}f''(a)$, which is zero because f has an inflection point at a.

9.1.74 Let $p(x) = \sum_{k=0}^{n} c_k(x-a)^k$ be the n^{th} polynomial for $f(x)$ at a. Because $f(a) = p(a)$, it follows that $c_0 = f(0)$. Now, the k^{th} derivative of $p(x)$, $1 \le k \le n$, is $p^{(k)}(x) = k!c_k +$ terms involving $(x-a)^i$, $i > 0$, so that $f^{(k)}(a) = p^{(k)}(a) = k! \cdot c_k$ so that $c_k = \frac{f^{(k)}(a)}{k!}$.

9.1.75

a. This matches (C) because for $f(x) = (1+2x)^{1/2}$, $f''(x) = -(1+2x)^{-3/2}$ so $\frac{f''(0)}{2!} = -\frac{1}{2}$.

b. This matches (E) because for $f(x) = (1+2x)^{-1/2}$, $f''(x) = 3(1+2x)^{-5/2}$, so $\frac{f''(0)}{2!} = \frac{3}{2}$.

c. This matches (A) because $f^{(n)}(x) = 2^n e^{2x}$, so that $f^{(n)}(0) = 2^n$, which is (A)'s pattern.

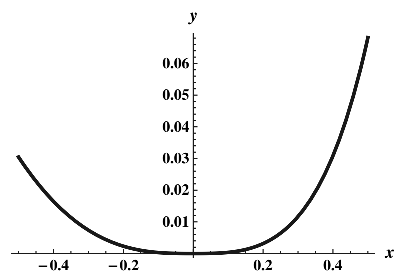
d. This matches (D) because $f''(x) = 8(1+2x)^{-3}$ and $f''(0) = 8$, so that $f''(0)/2! = 4$

e. This matches (B) because $f'(x) = -6(1+2x)^{-4}$ so that $f'(0) = -6$.

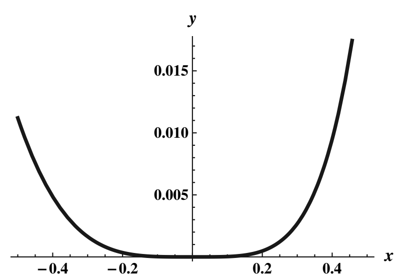
f. This matches (F) because $f^{(n)}(x) = (-2)^n e^{-2x}$, so $f^{(n)}(0) = (-2)^n$, which is (F)'s pattern.

9.1.76

a.



$|\ln(1-x) - p_2(x)|$



$|\ln(1-x) - p_3(x)|$

b. The error seems to be largest at $x = \frac{1}{2}$ and smallest at $x = 0$.

c. The error bound found in Example 7 for $|\ln(1-x) - p_3(x)|$ was 0.25. The actual error seems much less than that, about 0.02.

9.1.77

a. $p_2(0.1) = 0.1$. The maximum error in the approximation is $1 \cdot \frac{0.1^3}{3!} \approx 1.67 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error in the approximation is $1 \cdot \frac{0.2^3}{3!} \approx 1.33 \times 10^{-3}$.

9.1.78

a. $p_1(0.1) = 0.1$. $f''(x) = 2\tan x(1+\tan^2 x)$. Because $\tan(0.1) < 0.2$, $|f''(c)| \le 2(.2)(1+.2^2) = 0.416$. Thus the maximum error is $\frac{0.416}{2!} \cdot 0.1^2 \approx 2.1 \times 10^{-3}$.

b. $p_1(0.2) = 0.2$. The maximum error is $\frac{0.416}{2} \cdot 0.2^2 \approx 8.3 \times 10^{-3}$.

9.1.79

a. $p_3(0.1) = 1 - .01/2 = 0.995$. The maximum error is $1 \cdot \frac{0.1^4}{4!} \approx 4.2 \times 10^{-6}$.

b. $p_3(0.2) = 1 - .04/2 = 0.98$. The maximum error is $1 \cdot \frac{0.2^4}{4!} \approx 6.7 \times 10^{-5}$.

9.1.80

a. $p_2(0.1) = 0.1$ (we can take $n = 2$ because the coefficient of x^2 in $p_2(x)$ is 0). $f^{(3)}(x) = \frac{6x^2-2}{(x^2+1)^3}$ has a maximum magnitude value of 2, the maximum error is $2 \cdot \frac{0.1^3}{3!} \approx 3.3 \times 10^{-4}$.

b. $p_2(0.2) = 0.2$. The maximum error is $2 \cdot \frac{0.2^3}{3!} \approx 2.7 \times 10^{-3}$.

9.1.81

a. $p_1(0.1) = 1.05$. Because $|f''(x)| = \frac{1}{4}(1+x)^{-3/2}$ has a maximum value of $1/4$ at $x = 0$, the maximum error is $\frac{1}{4} \cdot \frac{0.1^2}{2} \approx 1.3 \times 10^{-3}$.

b. $p_1(0.2) = 1.1$. The maximum error is $\frac{1}{4} \cdot \frac{0.2^2}{2} = 5 \times 10^{-3}$.

9.1.82

a. $p_2(0.1) = 0.1 - 0.01/2 = 0.095$. Because $|f^{(3)}(x)| = \frac{2}{(x+1)^3}$ achieves a maximum of 2 at $x = 0$, the maximum error is $2 \cdot \frac{0.1^3}{3!} \approx 3.3 \times 10^{-4}$.

b. $p_2(0.2) = 0.2 - 0.04/2 = 0.18$. The maximum error is $2 \cdot \frac{0.2^3}{3!} \approx 2.7 \times 10^{-3}$.

9.1.83

a. $p_1(0.1) = 1.1$. Because $f''(x) = e^x$ is less than 2 on $[0, 0.1]$, the maximum error is less than $2 \cdot \frac{0.1^2}{2!} = 10^{-2}$.

b. $p_1(0.2) = 1.2$. The maximum error is less than $2 \cdot \frac{0.2^2}{2!} = .04 = 4 \times 10^{-2}$.

9.1.84

a. $p_1(0.1) = 0.1$. Because $f''(x) = \frac{x}{(1-x^2)^{3/2}}$ is less than 1 on $[0, 0.2]$, the maximum error is $1 \cdot \frac{0.1^3}{3!} \approx 1.7 \times 10^{-4}$.

b. $p_1(0.2) = 0.2$. The maximum error is $1 \cdot \frac{0.2^3}{3!} \approx 1.3 \times 10^{-3}$.

9.1.85

a.

	$\lvert\sec x - p_2(x)\rvert$	$\lvert\sec x - p_4(x)\rvert$
-0.2	3.4×10^{-4}	5.5×10^{-6}
-0.1	2.1×10^{-5}	8.5×10^{-8}
0.0	0	0
0.1	2.1×10^{-5}	8.5×10^{-8}
0.2	3.4×10^{-4}	5.5×10^{-6}

b. The errors are equal for positive and negative x. This makes sense, because $\sec(-x) = \sec x$ and $p_n(-x) = p_n(x)$ for $n = 2, 4$. The errors appear to get larger as x gets farther from zero.

9.1.86

a.

	$\lvert\cos x - p_2(x)\rvert$	$\lvert\cos x - p_4(x)\rvert$
-0.2	6.66×10^{-5}	8.88×10^{-8}
-0.1	4.17×10^{-6}	1.39×10^{-9}
0.0	0	0
0.1	4.17×10^{-6}	1.39×10^{-9}
0.2	6.66×10^{-5}	8.88×10^{-8}

b. The errors are equal for positive and negative x. This makes sense, because $\cos(-x) = \cos x$ and $p_n(-x) = p_n(x)$ for $n = 2, 4$. The errors appear to get larger as x gets farther from zero.

9.1.87

a.

	$\|e^{-x} - p_1(x)\|$	$\|e^{-x} - p_2(x)\|$
-0.2	2.14×10^{-2}	1.40×10^{-3}
-0.1	5.17×10^{-3}	1.71×10^{-4}
0.0	0	0
0.1	4.84×10^{-3}	1.63×10^{-4}
0.2	1.87×10^{-2}	1.27×10^{-3}

b. The errors are different for positive and negative displacements from zero, and appear to get larger as x gets farther from zero.

9.1.88

a.

	$\|f(x) - p_1(x)\|$	$\|f(x) - p_2(x)\|$
-0.2	2.31×10^{-2}	3.14×10^{-4}
-0.1	5.36×10^{-3}	3.61×10^{-4}
0.0	0	0
0.1	4.69×10^{-3}	3.10×10^{-4}
0.2	1.77×10^{-2}	2.32×10^{-3}

b. The errors are different for positive and negative displacements from zero, and appear to get larger as x gets farther from zero.

9.1.89

a.

	$\|\tan x - p_1(x)\|$	$\|\tan x - p_3(x)\|$
-0.2	2.71×10^{-3}	4.34×10^{-5}
-0.1	3.35×10^{-4}	1.34×10^{-6}
0.0	0	0
0.1	3.35×10^{-4}	1.34×10^{-6}
0.2	2.71×10^{-3}	4.34×10^{-5}

b. The errors are equal for positive and negative x. This makes sense, because $\tan(-x) = -\tan x$ and $p_n(-x) = -p_n(x)$ for $n = 1, 3$. The errors appear to get larger as x gets farther from zero.

9.1.90 The true value of $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} \approx 0.966$. The 6^{th}-order Taylor polynomial for $\cos x$ centered at $x = 0$ is

$$p_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$\left\|p_n\left(\frac{\pi}{12}\right) - \cos\frac{\pi}{12}\right\|$
1	1.0000000000	3.41×10^{-2}
2	0.9657305403	1.95×10^{-4}
3	0.9657305403	1.95×10^{-4}
4	0.9659262729	4.47×10^{-7}
5	0.9659262729	4.47×10^{-7}
6	0.9659258257	5.47×10^{-10}

The 6^{th}-order Taylor polynomial for $\cos x$ centered at $x = \pi/6$ is

$$p_6(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{48}\left(x - \frac{\pi}{6}\right)^4 - \frac{1}{240}\left(x - \frac{\pi}{6}\right)^5 - \frac{\sqrt{3}}{1440}\left(x - \frac{\pi}{6}\right)^6.$$

Evaluating the polynomials at $x = \pi/12$ produces the following table:

n	$p_n\left(\frac{\pi}{12}\right)$	$\lvert p_n\left(\frac{\pi}{12}\right) - \cos\frac{\pi}{12}\rvert$
1	0.9969250977	3.10×10^{-2}
2	0.9672468750	1.32×10^{-3}
3	0.9657515877	1.74×10^{-4}
4	0.9659210972	4.73×10^{-6}
5	0.9659262214	3.95×10^{-7}
6	0.9659258342	7.88×10^{-9}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate when n is even while using the polynomial centered at $x = \pi/6$ is more accurate when n is odd. To see why, consider the remainder. Let $f(x) = \cos x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at 0 is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $0 < c < \frac{\pi}{12}$, while the magnitude of the remainder when approximating $f(\pi/12)$ by the polynomial p_n centered at $\pi/6$ is:

$$\left|R_n\left(\frac{\pi}{12}\right)\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}\left(\frac{\pi}{12}\right)^{n+1}$$

for some c with $\frac{\pi}{12} < c < \frac{\pi}{6}$. When n is odd, $|f^{(n+1)}(c)| = |\cos c|$. Because $\cos x$ is a positive and decreasing function over $[0, \pi/6]$, the magnitude of the remainder in using the polynomial centered at $\pi/6$ will be less than the remainder in using the polynomial centered at 0, and the former polynomial will be more accurate. When n is even, $|f^{(n+1)}(c)| = |\sin c|$. Because $\sin x$ is a positive and increasing function over $[0, \pi/6]$, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\pi/6$, and the former polynomial will be more accurate.

9.1.91 The true value of $e^{0.35} \approx 1.419067549$. The 6^{th}-order Taylor polynomial for e^x centered at $x = 0$ is

$$p_6(x) \quad = \quad 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$\lvert p_n(0.35) - e^{0.35}\rvert$
1	1.350000000	6.91×10^{-2}
2	1.411250000	7.82×10^{-3}
3	1.418395833	6.72×10^{-4}
4	1.419021094	4.65×10^{-5}
5	1.419064862	2.69×10^{-6}
6	1.419067415	1.33×10^{-7}

The 6^{th}-order Taylor polynomial for e^x centered at $x = \ln 2$ is

$$p_6(x) = 2 + 2(x - \ln 2) + (x - \ln 2)^2 + \frac{1}{3}(x - \ln 2)^3 + \frac{1}{12}(x - \ln 2)^4 + \frac{1}{60}(x - \ln 2)^5 + \frac{1}{360}(x - \ln 2)^6.$$

Evaluating the polynomials at $x = 0.35$ produces the following table:

n	$p_n(0.35)$	$\lvert p_n(0.35) - e^{0.35}\rvert$
1	1.313705639	1.05×10^{-1}
2	1.431455626	1.24×10^{-2}
3	1.417987101	1.08×10^{-3}
4	1.419142523	7.50×10^{-5}
5	1.419063227	4.32×10^{-6}
6	1.419067762	2.13×10^{-7}

Comparing the tables shows that using the polynomial centered at $x = 0$ is more accurate for all n. To see why, consider the remainder. Let $f(x) = e^x$. By Theorem 9.2, the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at 0 is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(0.35)^{n+1} = \frac{e^c}{(n+1)!}(0.35)^{n+1}$$

for some c with $0 < c < 0.35$ while the magnitude of the remainder when approximating $f(0.35)$ by the polynomial p_n centered at $\ln 2$ is:

$$|R_n(0.35)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|0.35 - \ln 2|^{n+1} = \frac{e^c}{(n+1)!}(\ln 2 - 0.35)^{n+1}$$

for some c with $0.35 < c < \ln 2$. Because $\ln 2 - 0.35 \approx 0.35$, the relative size of the magnitudes of the remainders is determined by e^c in each remainder. Because e^x is an increasing function, the remainder in using the polynomial centered at 0 will be less than the remainder in using the polynomial centered at $\ln 2$, and the former polynomial will be more accurate.

9.1.92

a. Let x be a point in the interval on which the derivatives of f are assumed continuous. Then f' is continuous on $[a, x]$, and the Fundamental Theorem of Calculus implies that because f is an antiderivative of f', then $\int_a^x f'(t)\,dt = f(x) - f(a)$, or $f(x) = f(a) + \int_a^x f'(t)\,dt$.

b. Using integration by parts with $u = f'(t)$ and $dv = dt$, note that we may choose any antiderivative of dv; we choose $t - x = -(x - t)$. Then

$$\begin{aligned} f(x) &= f(a) - f'(t)(x-t)\Big|_{t=a}^{x} + \int_a^x (x-t)f''(t)\,dt \\ &= f(a) - f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt. \end{aligned}$$

c. Integrate by parts again, using $u = f''(t)$, $dv = (x-t)\,dt$, so that $v = -\frac{(x-t)^2}{2}$:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt \\ &= f(a) + f'(a)(x-a) - \frac{(x-t)^2}{2}f''(t)\Big|_a^x + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt \\ &= f(a) + f'(a)(x-a) + \frac{f''(t)}{2}(x-a)^2 + \frac{1}{2}\int_a^x (x-t)^2 f'''(t)\,dt. \end{aligned}$$

It is clear that continuing this process will give the desired result, because successive integral of $x - t$ give $-\frac{1}{k!}(x-t)^k$.

d. **Lemma:** Let g and h be continuous functions on the interval $[a,b]$ with $g(t) \geq 0$. Then there is a number c in $[a,b]$ with

$$\int_a^b h(t)g(t)\,dt = h(c)\int_a^b g(t)\,dt.$$

Proof: We note first that if $g(t) = 0$ for all t in $[a,b]$, then the result is clearly true. We can thus assume that there is some t in $[a,b]$ for which $g(t) > 0$. Because g is continuous, there must be an interval about this t on which g is strictly positive, so we may assume that

$$\int_a^b g(t)\,dt > 0.$$

Because h is continuous on $[a,b]$, the Extreme Value Theorem shows that h has an absolute minimum value m and an absolute maximum value M on the interval $[a,b]$. Thus

$$m \leq h(t) \leq M$$

for all t in $[a,b]$, so

$$m\int_a^b g(t)\,dt \leq \int_a^b h(t)g(t)\,dt \leq M\int_a^b g(t)\,dt.$$

Because $\int_a^b g(t)\,dt > 0$, we have

$$m \leq \frac{\int_a^b h(t)g(t)\,dt}{\int_a^b g(t)\,dt} \leq M.$$

Now there are points in $[a,b]$ at which $h(t)$ equals m and M, so the Intermediate Value Theorem shows that there is a point c in $[a,b]$ at which

$$h(c) = \frac{\int_a^b h(t)g(t)\,dt}{\int_a^b g(t)\,dt}$$

or

$$\int_a^b h(t)g(t)\,dt = h(c)\int_a^b g(t)\,dt.$$

Applying the lemma with $h(t) = \frac{f^{(n+1)}(t)}{n!}$, $g(t) = (x-t)^n$, we see that $R_n(x) = \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n\,dt = \frac{f^{(n+1)}(c)}{n!}\cdot\frac{1}{n+1}(x-a)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c \in [a,b]$.

9.1.93

a. The slope of the tangent line to $f(x)$ at $x = a$ is by definition $f'(a)$; by the point-slope form for the equation of a line, we have $y - f(a) = f'(a)(x-a)$, or $y = f(a) + f'(a)(x-a)$.

b. The Taylor polynomial centered at a is $p_1(x) = f(a) + f'(a)(x-a)$, which is the tangent line at a.

9.1.94

a. $p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$, so that $p_2'(x) = f'(a) + f''(a)(x-a)$ and $p_2''(x) = f''(a)$. If f has a local maximum at a, then $f'(a) = 0, f''(a) \leq 0$, but then $p_2'(a) = 0$ and $p_2''(a) \leq 0$ by the above, so that $p_2(x)$ also has a local maximum at a.

b. Similarly, if f has a local minimum at a, then $f'(a) = 0, f''(a) \geq 0$, but then $p_2'(a) = 0$ and $p_2''(a) \geq 0$ by the above, so that $p_2(x)$ also has a local minimum at a.

c. Recall that f has an inflection point at a if the second derivative of f changes sign at a. But $p_2''(x)$ is a constant, so p_2 does not have an inflection point at a (or anywhere else).

d. No. For example, let $f(x) = x^3$. Then $p_2(x) = 0$, so that the second-order Taylor polynomial has a local maximum at $x = 0$, but $f(x)$ does not. It also has a local minimum at $x = 0$, but $f(x)$ does not.

9.1.95

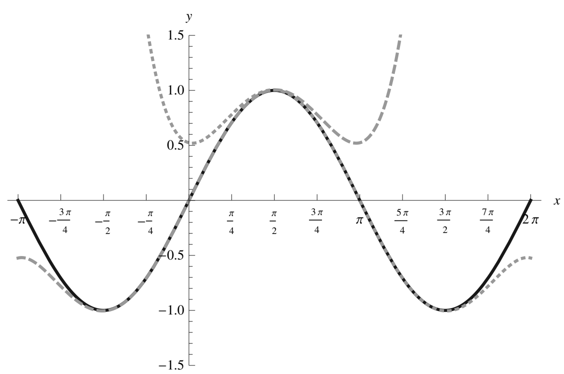
a. We have

$$\begin{aligned} f(0) &= f^{(4)}(0) = \sin 0 = 0 & f(\pi) &= f^{(4)}(\pi) = \sin\pi = 0 \\ f'(0) &= f^{(5)}(0) = \cos 0 = 1 & f'(\pi) &= f^{(5)}(0) = \cos\pi = -1 \\ f''(0) &= -\sin 0 = 0 & f''(\pi) &= -\sin\pi = 0 \\ f'''(0) &= -\cos 0 = -1 & f'''(\pi) &= -\cos\pi = 1. \end{aligned}$$

Thus

$$\begin{aligned} p_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ q_5(x) &= -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5. \end{aligned}$$

b. A plot of the three functions, with $\sin x$ the black solid line, $p_5(x)$ the dashed line, and $q_5(x)$ the dotted line is below.



$p_5(x)$ and $\sin x$ are almost indistinguishable on $[-\pi/2, \pi/2]$, after which $p_5(x)$ diverges pretty quickly from $\sin x$. $q_5(x)$ is reasonably close to $\sin x$ over the entire range, but the two are almost indistinguishable on $[\pi/2, 3\pi/2]$. $p_5(x)$ is a better approximation than $q_5(x)$ on about $[-\pi, \pi/2)$, while $q_5(x)$ is better on about $(\pi/2, 2\pi]$.

c. Evaluating the errors gives

x	$\lvert\sin x - p_5(x)\rvert$	$\lvert\sin x - q_5(x)\rvert$
$\frac{\pi}{4}$	3.6×10^{-5}	7.4×10^{-2}
$\frac{\pi}{2}$	4.5×10^{-3}	4.5×10^{-3}
$\frac{3\pi}{4}$	7.4×10^{-2}	3.6×10^{-5}
$\frac{5\pi}{4}$	2.3	3.6×10^{-5}
$\frac{7\pi}{4}$	20.4	7.4×10^{-2}

d. $p_5(x)$ is a better approximation than $q_5(x)$ only at $x = \frac{\pi}{4}$, in accordance with part (b). The two are equal at $x = \frac{\pi}{2}$, after which $q_5(x)$ is a substantially better approximation than $p_5(x)$.

9.1.96

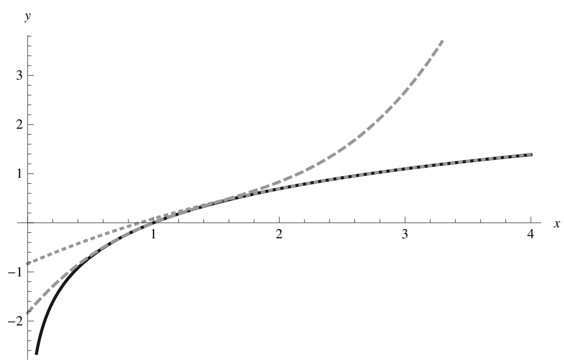
a. We have

$$f(1) = \ln 1 = 0 \qquad f(e) = \ln e = 1$$
$$f'(1) = 1 \qquad f'(e) = \frac{1}{3}$$
$$f''(1) = -1 \qquad f''(e) = -\frac{1}{e^2}$$
$$f'''(1) = 2 \qquad f'''(e) = \frac{2}{e^3}.$$

Thus

$$p_3(x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$
$$q_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3.$$

b. A plot of the three functions, with $\ln x$ the black solid line, $p_3(x)$ the dashed line, and $q_3(x)$ the dotted line is below.



c. Evaluating the errors gives

x	$\lvert\ln x - p_3(x)\rvert$	$\lvert\ln x - q_3(x)\rvert$
0.5	2.6×10^{-2}	3.6×10^{-1}
1.0	0	8.4×10^{-2}
1.5	1.1×10^{-2}	1.6×10^{-2}
2.0	1.4×10^{-1}	1.5×10^{-3}
2.5	5.8×10^{-1}	1.1×10^{-5}
3.0	1.6	2.7×10^{-5}
3.5	3.3	1.4×10^{-3}

d. $p_3(x)$ is a better approximation than $q_3(x)$ for $x = 0.5$, 1.0, and 1.5, and $q_3(x)$ is a better approximation for the other points. To see why this is true, note that on $[0.5, 4]$ that $f^{(4)}(x) = -\frac{6}{x^4}$ is bounded in magnitude by $\frac{6}{0.5^4} = 96$, so that (using P_3 for the error term for p_3 and Q_3 as the error term for q_3)

$$P_3(x) \le 96 \cdot \frac{|x-1|^4}{4!} = 4\,|x-1|^4, \qquad Q_3(x) \le 96 \cdot \frac{|x-e|^4}{4!} = 4\,|x-e|^4.$$

Thus the relative sizes of $P_3(x)$ and $Q_3(x)$ are governed by the distance of x from 1 and e. Looking at the different possibilities for x reveals why the results in part (c) hold.

9.1.97

a. We have

$$f(36) = \sqrt{36} = 6 \qquad f(49) = \sqrt{49} = 7$$
$$f'(36) = \frac{1}{2} \cdot \frac{1}{\sqrt{36}} = \frac{1}{12} \qquad f'(49) = \frac{1}{2} \cdot \frac{1}{\sqrt{49}} = \frac{1}{14}.$$

Thus

$$p_1(x) = 6 + \frac{1}{12}(x - 36) \qquad q_1(x) = 7 + \frac{1}{14}(x - 49).$$

b. Evaluating the errors gives

x	$\lvert\sqrt{x} - p_1(x)\rvert$	$\lvert\sqrt{x} - q_1(x)\rvert$
37	5.7×10^{-4}	6.0×10^{-2}
39	5.0×10^{-3}	4.1×10^{-2}
41	1.4×10^{-2}	2.5×10^{-2}
43	2.6×10^{-2}	1.4×10^{-2}
45	4.2×10^{-2}	6.1×10^{-3}
47	6.1×10^{-2}	1.5×10^{-3}

c. $p_1(x)$ is a better approximation than $q_1(x)$ for $x \le 41$, and $q_1(x)$ is a better approximation for $x \ge 43$. To see why this is true, note that $f''(x) = -\frac{1}{4}x^{-3/2}$, so that on $[36, 49]$ it is bounded in magnitude by $\frac{1}{4} \cdot 36^{-3/2} = \frac{1}{864}$. . Thus (using P_1 for the error term for p_1 and Q_1 for the error term for q_1)

$$P_1(x) \le \frac{1}{864} \cdot \frac{|x - 36|^2}{2!} = \frac{1}{1728}(x - 36)^2, \qquad Q_1(x) \le \frac{1}{864} \cdot \frac{|x - 49|^2}{2!} = \frac{1}{1728}(x - 49)^2.$$

It follows that the relative sizes of $P_1(x)$ and $Q_1(x)$ are governed by the distance of x from 36 and 49. Looking at the different possibilities for x reveals why the results in part (b) hold.

9.1.98

a. The quadratic Taylor polynomial for $\sin x$ centered at $\frac{\pi}{2}$ is

$$\begin{aligned} p_2(x) &= \sin\frac{\pi}{2} + \cos\frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right) - \frac{1}{2}\sin\frac{\pi}{2} \cdot \left(x - \frac{\pi}{2}\right)^2 \\ &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 \\ &= -\frac{1}{2}x^2 + \frac{\pi}{2}x + 1 - \frac{\pi^2}{8}. \end{aligned}$$

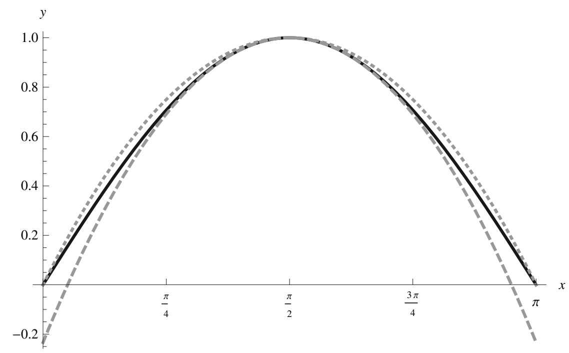
b. Let $q(x) = ax^2 + bx + c$. Because $q(0) = \sin 0 = 0$, we must have $c = 0$, so that $q(x) = ax^2 + bx$. Then the other two conditions give us a pair of linear equation in a and b:

$$\begin{aligned} \frac{\pi^2}{4}a + \frac{\pi}{2}b &= 1 \\ \pi^2 a + \pi b &= 0 \end{aligned}$$

where the first equation comes from the fact that $q(\pi/2) = \sin(\pi/2) = 1$ and the second from the fact that $q(\pi) = \sin\pi = 0$. Solving the linear system of equations gives $b = \frac{4}{\pi}$ and $a = -\frac{4}{\pi^2}$, so that

$$q(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x.$$

c. A plot of the three function, with $\sin x$ the black solid line, $p_2(x)$ the dashed line, and $q(x)$ the dotted line is below.



d. Evaluating the errors gives

x	$\lvert\sin x - p_2(x)\rvert$	$\lvert\sin x - q(x)\rvert$
$\frac{\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
$\frac{\pi}{2}$	0	0
$\frac{3\pi}{4}$	1.6×10^{-2}	4.3×10^{-2}
π	2.3×10^{-1}	0

e. q is a better approximation than p at $x = \pi$, and the two are equal at $x = \frac{\pi}{2}$. At the other two points, however, $p_2(x)$ is a better approximation than $q(x)$. Clearly $q(x)$ will be exact at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$, because it was chosen that way. Also clearly $p_2(x)$ will be exact at $x = \frac{\pi}{2}$ since it is the Taylor polynomial centered at $\frac{\pi}{2}$. The fact that $p_2(x)$ is a better approximation than $q(x)$ at the two intermediate points is a result of the way the polynomials were constructed: the goal of $p_2(x)$ was to be as good an approximation as possible near $x = \frac{\pi}{2}$, while the goal of $q(x)$ was to match $\sin x$ at three given points. Overall, it appears that $q(x)$ does a better job over the full range (the total area between $q(x)$ and $\sin x$ is certainly smaller than the total area between $p_2(x)$ and $\sin x$).

9.2 Properties of Power Series

9.2.1 $c_0 + c_1x + c_2x^2 + c_3x^3$.

9.2.2 $c_0 + c_1(x-3) + c_2(x-3)^2 + c_3(x-3)^3$.

9.2.3 Generally the Ratio Test or Root Test is used.

9.2.4 Theorem 9.3 says that on the interior of the interval of convergence, a power series centered at a converges absolutely, and that the interval of convergence is symmetric about a. So it makes sense to try to find this interval using the Ratio Test, and check the endpoints individually.

9.2.5 The radius of convergence does not change, but the interval of convergence may change at the endpoints.

9.2.6 $2R$, because for $|x| < 2R$ we have $|x/2| < R$ so that $\sum c_k(x/2)^k$ converges.

9.2.7 $|x| < \frac{1}{4}$.

9.2.8 $(-1)^k c_k x^k = c_k(-x)^k$, so the two series have the same radius of convergence, because $|-x| = |x|$.

9.2.9 Using the Root Test: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}|2x| = |2x|$. So the radius of convergence is $\frac{1}{2}$. At $x = 1/2$ the series is $\sum 1$ which diverges, and at $x = -1/2$ the series is $\sum(-1)^k$ which also diverges. So the interval of convergence is $(-1/2, 1/2)$.

9.2.10 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\frac{(2x)^{k+1}}{(k+1)!}\cdot\frac{k!}{(2x)^k}\right| = \lim\limits_{k\to\infty}\left|\frac{2x}{k+1}\right| = 0$. So the radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$.

9.2.11 Using the Root Test, $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\frac{|x-1|}{k^{1/k}} = |x-1|$. So the radius of convergence is 1. At $x = 2$, we have the harmonic series (which diverges) and at $x = 0$ we have the alternating harmonic series (which converges). Thus the interval of convergence is $[0, 2)$.

9.2.12 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\frac{(x-1)^{k+1}}{(k+1)!}\cdot\frac{k!}{(x-1)^k}\right| = \lim\limits_{k\to\infty}\left|\frac{x-1}{k+1}\right| = 0$. Thus the radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$.

9.2.13 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\frac{(k+1)^{k+1}x^{k+1}}{k^kx^k}\right| = \lim\limits_{k\to\infty}(k+1)\left(\frac{k+1}{k}\right)^k|x| = \infty$ (for $x \neq 0$) because $\lim\limits_{k\to\infty}\left(\frac{k+1}{k}\right)^k = e$. Thus, the radius of convergence is 0, the series only converges at $x = 0$.

9.2.14 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\frac{(k+1)!(x-10)^{k+1}}{k!(x-10)^k}\right| = \lim\limits_{k\to\infty}(k+1)|x-10| = \infty$ (for $x \neq 10$). Thus, the radius of convergence is 0, the series only converges at $x = 10$.

9.2.15 Using the Root Test: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\sin(1/k)|x| = \sin(0)|x| = 0$. Thus, the radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$.

9.2.16 Using the Root Test: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\frac{2|x-3|}{k^{1/k}} = 2|x-3|$. Thus, the radius of convergence is $1/2$. When $x = 7/2$, we have the harmonic series (which diverges), and when $x = 5/2$, we have the alternating harmonic series which converges. The interval of convergence is thus $[5/2, 7/2)$.

9.2.17 Using the Root Test: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\frac{|x|}{3} = \frac{|x|}{3}$, so the radius of convergence is 3. At -3, the series is $\sum(-1)^k$, which diverges. At 3, the series is $\sum 1$, which diverges. So the interval of convergence is $(-3, 3)$.

9.2.18 Using the Root Test: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\frac{|x|}{5} = \frac{|x|}{5}$, so the radius of convergence is 5. At 5, we obtain $\sum(-1)^k$ which diverges. At -5, we have $\sum 1$, which also diverges. So the interval of convergence is $(-5, 5)$.

9.2.19 Using the Root Test: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}\frac{|x|}{k} = 0$, so the radius of convergence is infinite and the interval of convergence is $(-\infty,\infty)$.

9.2.20 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty}\left|\left(\frac{(k+1)(x-4)^{k+1}}{2^{k+1}}\cdot\frac{2^k}{k(x-4)^k}\right)\right| = \lim\limits_{k\to\infty}\left(\frac{k+1}{k}\cdot\frac{|x-4|}{2}\right) = \frac{|x-4|}{2}$, so that the radius of convergence is 2. The interval is $(2, 6)$, because at the left endpoint, the series becomes $\sum k$ (which diverges) and at the right endpoint, it becomes $\sum(-1)^k k$ (which diverges).

9.2.21 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{(k+1)^2x^{2k+2}}{(k+1)!}\cdot\frac{k!}{k^2x^{2k}}\right| = \lim\limits_{k\to\infty}\frac{k+1}{k^2}x^2 = 0$, so the radius of convergence is infinite, and the interval of convergence is $(-\infty,\infty)$.

9.2.22 Using the Root Test: $\lim\limits_{k\to\infty}\sqrt[k]{|a_k|} = \lim\limits_{k\to\infty}k^{1/k}|x-1| = |x-1|$. The radius of convergence is therefore 1. At both $x = 2$ and $x = 0$ the series diverges by the Divergence Test. The interval of convergence is therefore $(0, 2)$.

9.2.23 Using the Ratio Test: $\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{2k+3}}{3^k}\cdot\frac{3^{k-1}}{x^{2k+1}}\right| = \frac{x^2}{3}$ so that the radius of convergence is $\sqrt{3}$. At $x = \sqrt{3}$, the series is $\sum 3\sqrt{3}$, which diverges. At $x = -\sqrt{3}$, the series is $\sum(-3\sqrt{3})$, which also diverges, so the interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

9.2.24 $\sum \left(\frac{-x}{10}\right)^{2k} = \sum \left(\frac{x^2}{100}\right)^k$. Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{x^2}{100} = \frac{x^2}{100}$, so that the radius of convergence is 10. At $x = \pm 10$, the series is then $\sum 1$, which diverges, so the interval of convergence is $(-10, 10)$.

9.2.25 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{(|x-1|)k}{k+1} = |x-1|$, so the series converges when $|x-1| < 1$, so for $0 < x < 2$. The radius of convergence is 1. At $x = 2$, the series diverges by the Divergence Test. At $x = 0$, the series diverges as well by the Divergence Test. Thus the interval of convergence is $(0, 2)$.

9.2.26 Using the Ratio Test:

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \left|\frac{(-2)^{k+1}(x+3)^{k+1}}{3^{k+2}} \cdot \frac{3^{k+1}}{(-2)^k(x+3)^k}\right| = \frac{2}{3}|x+3|.$$

Thus the series converges when $\frac{2}{3}|x+3| < 1$, or $-\frac{9}{2} < x < -\frac{3}{2}$. At $x = -\frac{9}{2}$, the series diverges by the Divergence Test. At $x = -\frac{3}{2}$, the series diverges by the Divergence Test. Thus the interval of convergence is $\left(-\frac{9}{2}, -\frac{3}{2}\right)$.

9.2.27 Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^{20}x^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^k k^{20}}\right| = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{20} \frac{|x|}{(2k+2)(2k+3)} = 0$, so the radius of convergence is infinite, and the interval of convergence is $(-\infty, \infty)$.

9.2.28 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{|x^3|}{27} = \frac{|x^3|}{27}$, so the radius of convergence is 3. The series is divergent by the Divergence Test for $x = \pm 3$, so the interval of convergence is $(-3, 3)$.

9.2.29 $f(3x) = \frac{1}{1-3x} = \sum_{k=0}^{\infty} 3^k x^k$, which converges for $|x| < 1/3$, and diverges at the endpoints.

9.2.30 $g(x) = \frac{x^3}{1-x} = \sum_{k=0}^{\infty} x^{k+3}$, which converges for $|x| < 1$ and is divergent at the endpoints.

9.2.31 $h(x) = \frac{2x^3}{1-x} = \sum_{k=0}^{\infty} 2x^{k+3}$, which converges for $|x| < 1$ and is divergent at the endpoints.

9.2.32 $f(x^3) = \frac{1}{1-x^3} = \sum_{k=0}^{\infty} x^{3k}$. By the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = |x^3|$, so this series also converges for $|x| < 1$. It is divergent at the endpoints.

9.2.33 $p(x) = \frac{4x^{12}}{1-x} = \sum_{k=0}^{\infty} 4x^{k+12} = 4\sum_{k=0}^{\infty} x^{k+12}$, which converges for $|x| < 1$. It is divergent at the endpoints.

9.2.34 $f(-4x) = \frac{1}{1+4x} = \sum_{k=0}^{\infty}(-4x)^k = \sum_{k=0}^{\infty}(-1)^k 4^k x^k$, which converges for $|x| < 1/4$ and is divergent at the endpoints.

9.2.35 $f(3x) = \ln(1-3x) = -\sum_{k=1}^{\infty} \frac{(3x)^k}{k} = -\sum_{k=1}^{\infty} \frac{3^k}{k} x^k$. Using the Ratio Test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{3k}{k+1} |x| = 3|x|,$$

so the radius of convergence is $1/3$. The series diverges at $1/3$ (harmonic series), and converges at $-1/3$ (alternating harmonic series).

9.2.36 $g(x) = x^3 \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^{k+3}}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1} |x| = |x|$, so the radius of convergence is 1. The series diverges at 1 and converges at -1.

9.2.37 $h(x) = x\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^{k+1}}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1} |x| = |x|$, so the radius of convergence is 1, and the series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series).

9.2.38 $f(x^3) = \ln(1-x^3) = -\sum_{k=1}^{\infty} \frac{x^{3k}}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1}\left|x^3\right| = \left|x^3\right|$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series).

9.2.39 $p(x) = 2x^6\ln(1-x) = -2\sum_{k=1}^{\infty} \frac{x^{k+6}}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1}|x| = |x|$, so the radius of convergence is 1. The series diverges at 1 (harmonic series) but converges at -1 (alternating harmonic series).

9.2.40 $f(-4x) = \ln(1+4x) = -\sum_{k=1}^{\infty} \frac{(-4x)^k}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1}4|x| = 4|x|$, so the radius of convergence is 1/4. The series converges at 1/4 (alternating harmonic series) but diverges at $-1/4$ (harmonic series).

9.2.41 The power series for $f(x)$ is $\sum_{k=0}^{\infty}(2x)k$, convergent for $-1 < 2x < 1$, so for $-1/2 < x < 1/2$. The power series for $g(x) = f'(x)$ is $\sum_{k=1}^{\infty} k(2x)^{k-1}\cdot 2 = 2\sum_{k=1}^{\infty} k(2x)^{k-1}$, also convergent on $|x| < 1/2$.

9.2.42 The power series for $f(x)$ is $\sum_{k=0}^{\infty} x^k$, convergent for $-1 < x < 1$, so the power series for $g(x) = \frac{1}{2}f''(x)$ is $\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{1}{2}\sum_{k=0}^{\infty}(k+1)(k+2)x^k$, also convergent on $|x| < 1$.

9.2.43 The power series for $f(x)$ is $\sum_{k=0}^{\infty} x^k$, convergent for $-1 < x < 1$, so the power series for $g(x) = \frac{1}{6}f'''(x)$ is $\frac{1}{6}\sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \frac{1}{6}\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)x^k$, also convergent on $|x| < 1$.

9.2.44 The power series for $f(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, convergent on $|x| < 1$. Because $g(x) = -\frac{1}{2}f'(x)$, the power series for g is $-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k 2kx^{2k-1} = \sum_{k=1}^{\infty}(-1)^{k+1}kx^{2k-1}$, also convergent on $|x| < 1$.

9.2.45 The power series for $\frac{1}{1-3x}$ is $\sum_{k=0}^{\infty}(3x)^k$, convergent on $|x| < 1/3$. Because $g(x) = \ln(1-3x) = -3\int \frac{1}{1-3x}\,dx$ and because $g(0) = 0$, the power series for $g(x)$ is $-3\sum_{k=0}^{\infty} 3^k \frac{1}{k+1}x^{k+1} = -\sum_{k=1}^{\infty} \frac{3^k}{k}x^k$, also convergent on $[-1/3, 1/3)$.

9.2.46 The power series for $\frac{x}{1+x^2}$ is $x\sum_{k=0}^{\infty}(-1)^k x^{2k} = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$, convergent on $|x| < 1$. Because $g(x) = 2\int f(x)\,dx$, and because $g(0) = 0$, the power series for $g(x)$ is $2\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+2}x^{2k+2} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{k+1}x^{2k+2}$. This can be written as $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}x^{2k}$, which is convergent on $[-1, 1]$.

9.2.47 Start with $g(x) = \frac{1}{1+x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^k$. Because $f(x) = g(x^2)$, its power series is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$. The radius of convergence is still 1, and the series is divergent at both endpoints. The interval of convergence is $(-1, 1)$.

9.2.48 Start with $g(x) = \frac{1}{1-x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty} x^k$. Because $f(x) = g(x^4)$, its power series is $\sum_{k=0}^{\infty} x^{4k}$. The radius of convergence is still 1, and the series is divergent at both endpoints. The interval of convergence is $(-1, 1)$.

9.2.49 Note that $f(x) = \frac{3}{3+x} = \frac{1}{1+(1/3)x}$. Let $g(x) = \frac{1}{1+x}$. The power series for $g(x)$ is $\sum_{k=0}^{\infty}(-1)^k x^k$, so the power series for $f(x) = g((1/3)x)$ is $\sum_{k=0}^{\infty}(-1)^k 3^{-k}x^k = \sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k$. Using the Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{3^{-(k+1)}x^{k+1}}{3^{-k}x^k}\right| = \frac{|x|}{3}$, so the radius of convergence is 3. The series diverges at both endpoints. The interval of convergence is $(-3, 3)$.

9.2.50 Note that $f(x) = \frac{1}{2}\ln(1-x^2)$. The power series for $g(x) = \ln(1-x)$ is $-\sum_{k=1}^{\infty}\frac{1}{k}x^k$, so the power series for $f(x) = \frac{1}{2}g(x^2)$ is $\frac{-1}{2}\sum_{k=1}^{\infty}\frac{1}{k}x^{2k}$. The radius of convergence is still 1. The series diverges at both 1 and -1, its interval of convergence is $(-1, 1)$.

9.2.51 Note that $f(x) = \ln\sqrt{4-x^2} = \frac{1}{2}\ln(4-x^2) = \frac{1}{2}\left(\ln 4 + \ln\left(1-\frac{x^2}{4}\right)\right) = \ln 2 + \frac{1}{2}\ln\left(1-\frac{x^2}{4}\right)$. Now, the power series for $g(x) = \ln(1-x)$ is $-\sum_{k=1}^{\infty}\frac{1}{k}x^k$, so the power series for $f(x)$ is $\ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\frac{x^{2k}}{4^k} = \ln 2 - \sum_{k=1}^{\infty}\frac{x^{2k}}{k2^{2k+1}}$. Now, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}}{(k+1)2^{2k+3}}\cdot\frac{k2^{2k+1}}{x^{2k}}\right| = \lim_{k\to\infty}\frac{k}{4(k+1)}x^2 = \frac{x^2}{4}$, so that the radius of convergence is 2. The series diverges at both endpoints, so its interval of convergence is $(-2, 2)$.

9.2.52 By Example 5, the Taylor series for $g(x) = \tan^{-1} x$ is $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$, so that $f(x) = g((2x)^2)$ has Taylor series $\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{4k+2}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k+1}}{2k+1} x^{4k+2}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{4^{2k+3}x^{4k+6}}{2k+3} \cdot \frac{2k+1}{4^{2k+1}x^{4k+2}}\right| = \lim_{k\to\infty} \frac{16(2k+1)}{2k+3} x^4 = 16x^4$, so that the radius of convergence is $1/2$. The interval of convergence is $(-1/2, 1/2)$.

9.2.53

a. True. This power series is centered at $x = 3$, so its interval of convergence will be symmetric about 3.

b. True. Use the Root Test.

c. True. Substitute x^2 for x in the series.

d. True. Because the power series is zero on the interval, all its derivatives are as well, which implies (differentiating the power series) that all the c_k are zero.

9.2.54 Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k |x| = ex$. Thus, the radius of convergence is $\frac{1}{e}$.

9.2.55 Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)!\, x^{k+1}}{(k+1)^{k+1}} \cdot \frac{k^k}{k!\, x^k}\right| = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k |x| = \frac{1}{e}|x|$. The radius of convergence is therefore e.

9.2.56 $1 + \sum_{k=1}^{\infty} \frac{1}{2k} x^k$

9.2.57 $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} x^k$

9.2.58 $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(k+1)^2}$

9.2.59 $\sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$

9.2.60 The power series for $f(ax)$ is $\sum c_k (ax)^k$. Then $\sum c_k (ax)^k$ converges if and only if $|ax| < R$ (because $\sum c_k x^k$ converges for $|x| < R$), which happens if and only if $|x| < \frac{R}{|a|}$.

9.2.61 The power series for $f(x-a)$ is $\sum c_k (x-a)^k$. Then $\sum c_k (x-a)^k$ converges if and only if $|x - a| < R$, which happens if and only if $a - R < x < a + R$, so the radius of convergence is the same.

9.2.62 Let's first consider where this series converges. By the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} (x^2+1)^2 = (x^2+1)^2$, which is always greater than 1 for $x \neq 0$. This series also diverges when $x = 0$, because there we have the divergent series $\sum 1$. Because this series diverges everywhere, it doesn't represent any function, except perhaps the empty function.

9.2.63 This is a geometric series with ratio $\sqrt{x} - 2$, so its sum is $\frac{1}{1-(\sqrt{x}-2)} = \frac{1}{3-\sqrt{x}}$. Again using the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = |\sqrt{x} - 2|$, so the interval of convergence is given by $|\sqrt{x} - 2| < 1$, so $1 < \sqrt{x} < 3$ and $1 < x < 9$. The series diverges at both endpoints.

9.2.64 This series is $\frac{1}{4} \sum_{k=1}^{\infty} \frac{x^{2k}}{k}$. Because $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is the power series for $-\ln(1-x)$, the power series given is $-\frac{1}{4}\ln(1-x^2)$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \left|\lim_{k\to\infty} \frac{x^{2k+2}}{4k+4} \cdot \frac{4k}{x^{2k}}\right| = \lim_{k\to\infty} \frac{k}{k+1} x^2 = x^2$, so the radius of convergence is 1. The series diverges at both endpoints (it is a multiple of the harmonic series). The interval of convergence is $(-1, 1)$.

9.2.65 This is a geometric series with ratio e^{-x}, so its sum is $\frac{1}{1-e^{-x}}$. By the Root Test, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = e^{-x}$, so the power series converges for $x > 0$.

9.2.66 This is a geometric series with ratio $\frac{x-2}{9}$, so its sum is $\frac{(x-2)/9}{1-(x-2)/9} = \frac{x-2}{9-(x-2)} = \frac{x-2}{11-x}$. Using the Root Test: $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left|\frac{x-2}{9}\right| = \left|\frac{x-2}{9}\right|$, so the series converges for $|x-2| < 9$, or $-7 < x < 11$. It diverges at both endpoints.

9.2.67 This is a geometric series with ratio $(x^2-1)/3$, so its sum is $\frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{3-(x^2-1)} = \frac{3}{4-x^2}$. Using the Root Test, the series converges for $\left|x^2-1\right| < 3$, so that $-2 < x^2 < 4$ or $-2 < x < 2$. It diverges at both endpoints.

9.2.68 Replacing x by $x-1$ gives $\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}$. Using the Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(x-1)^{k+1}}{k+1} \cdot \frac{k}{(x-1)^k}\right| = \lim_{k\to\infty} \frac{k}{k+1}\,|x-1| = |x-1|$, so that the series converges for $|x-1| < 1$. Checking the endpoints, the interval of convergence is $(0, 2]$.

9.2.69 The power series for e^x is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Substitute $-x$ for x to get $e^{-x} = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k!}$. The series converges for all x.

9.2.70 Substitute $2x$ for x in the power series for e^x to get $e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$. The series converges for all x.

9.2.71 Substitute $-3x$ for x in the power series for e^x to get $e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \sum_{k=0}^{\infty}(-1)^k \frac{3^k}{k!} x^k$. The series converges for all x.

9.2.72 Multiply the power series for e^x by x^2 to get $x^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$, which converges for all x.

9.2.73 The power series for $x^m f(x)$ is $\sum c_k x^{k+m}$. The radius of convergence of this power series is determined by the limit

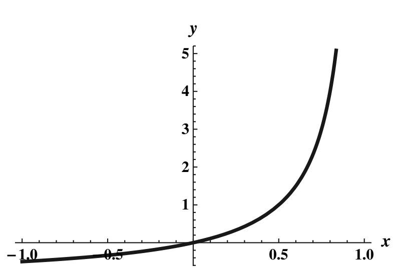
$$\lim_{k\to\infty} \left|\frac{c_{k+1}x^{k+1+m}}{c_k x^{k+m}}\right| = \lim_{k\to\infty} \left|\frac{c_{k+1}x^{k+1}}{c_k x^k}\right|,$$

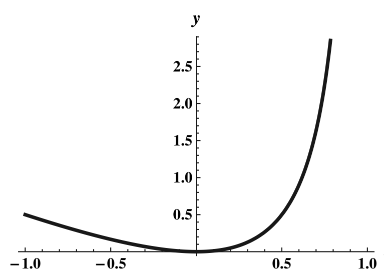
and the right-hand side is the limit used to determine the radius of convergence for the power series for $f(x)$. Thus the two have the same radius of convergence.

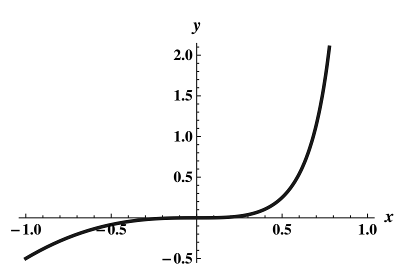
9.2.74

a. $R_n = f(x) - S_n(x) = \sum_{k=n}^{\infty} x^k$. This is a geometric series with ratio x. Its sum is then $R_n = \frac{x^n}{1-x}$ as desired.

b. $R_n(x)$ increases without bound as x approaches 1, and its absolute value smallest at $x = 0$ (where it is zero). In general, for $x > 0$, $R_n(x) < R_{n-1}(x)$, so the approximations get better the more terms of the series are included.

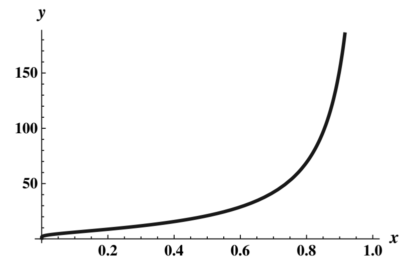






c. To minimize $|R_n(x)|$, set its derivative to zero. Assuming $n > 1$, we have $R_n'(x) = \frac{n(1-x)x^{n-1}+x^n}{(1-x)^2}$, which is zero for $x = 0$. There is a minimum at this critical point.

d. The following is a plot that shows, for each $x \in (0, 1)$, the n required so that $R_n(x) < 10^{-6}$. The closer x gets to 1, the more terms are required in order for the estimate given by the power series to be accurate. The number of terms increases rapidly as $x \to 1$.



9.2.75

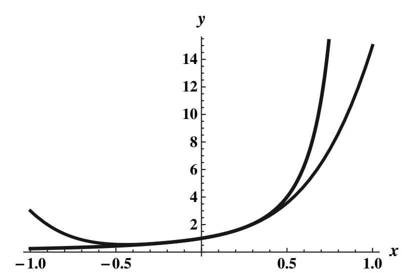
a. $f(x)g(x) = c_0d_0 + (c_0d_1 + c_1d_0)x + (c_0d_2 + c_1d_1 + c_2d_0)x^2 + \ldots$

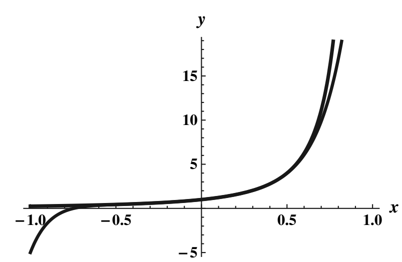
b. The coefficient of x^n in $f(x)g(x)$ is $\sum_{i=0}^{n} c_i d_{n-i}$.

9.2.76 The function $\frac{1}{\sqrt{1-x^2}}$ is the derivative of the inverse sine function, and $\sin^{-1}(0) = 0$, so the power series for $\sin^{-1} x$ is the integral of the given power series, or $x + \frac{1}{6}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots$. This can also be written $x + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k\cdot(2k+1)}x^{2k+1}$.

9.2.77

a. For both graphs, the difference between the true value and the estimate is greatest at the two ends of the range; the difference at 0.9 is greater than that at -0.9.





b. The difference between $f(x)$ and $S_n(x)$ is greatest for $x = 0.9$; at that point, $f(x) = \frac{1}{(1-0.9)^2} = 100$, so we want to find n such that $S_n(x)$ is within 0.01 of 100. We find that $S_{111} \approx 99.98991435$ and $S_{112} \approx 99.99084790$, so $n = 112$.

9.3 Taylor Series

9.3.1 The nth Taylor Polynomial is the nth sum of the corresponding Taylor Series.

9.3.2 In order to have a Taylor series centered at a, a function f must have derivatives of all orders on some interval containing a.

9.3.3 The n^{th} coefficient is $\frac{f^{(n)}(a)}{n!}$.

9.3.4 The interval of convergence is found in the same manner that it is found for a more general power series.

9.3.5 Substitute x^2 for x in the Taylor series. By theorems proved in the previous section about power series, the interval of convergence does not change except perhaps at the endpoints of the interval.

9.3.6 The Taylor series terminates if $f^{(n)}(0) = 0$ for $n > N$ for some N. For $(1+x)^p$, this occurs if and only if p is an integer ≥ 0.

9.3.7 It means that the limit of the remainder term is zero.

9.3.8 The Maclaurin series is $e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$. This is determined by substituting $2x$ for x in the Maclaurin series for e^x.

9.3.9

a. Note that $f(0) = 1$, $f'(0) = -1$, $f''(0) = 1$, and $f'''(0) = -1$. So the Maclaurin series is $1 - x + x^2/2 - x^3/6 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k!}$.

c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

9.3.10

a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -4$, $f'''(0) = 0$, $f^{(4)}(0) = 16$, ... Thus the Maclaurin series is $1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{(2x)^{2k}}{(2k)!}$

c. The series converges on $(-\infty, \infty)$, as can be seen from the Ratio Test.

9.3.11

a. Because the series for $\frac{1}{1+x}$ is $1 - x + x^2 - x^3 + \cdots$, the series for $\frac{1}{1+x^2}$ is $1 - x^2 + x^4 - x^6 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k x^{2k}$.

c. The absolute value of the ratio of consecutive terms is x^2, so by the Ratio Test, the radius of convergence is 1. The series diverges at the endpoints by the Divergence Test, so the interval of convergence is $(-1, 1)$.

9.3.12

a. Note that $f(0) = 0$, $f'(0) = 4$, $f''(0) = -16$, $f'''(0) = 128$, and $f''''(0) = -1526$. Thus, the series is given by $4x - \frac{16x^2}{2} + \frac{128x^3}{6} - \frac{1536x^4}{24} + \cdots$.

b. $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{(k-1)!(4x)^k}{k!} = \sum_{k=1}^{\infty}(-1)^{k+1} \frac{(4x)^k}{k}$.

c. The absolute value of the ratio of consecutive terms is $\frac{4|x|k}{k+1}$, which has limit $4|x|$ as $k \to \infty$, so the interval of convergence is $(-1/4, 1/4]$. Note that for $x = 1/4$ we have the alternating harmonic series, while for $x = -1/4$ we have negative 1 times the harmonic series, which diverges.

9.3.13

a. Note that $f(0) = 1$, and that $f^{(n)}(0) = 2^n$. Thus, the series is given by $1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$.

c. The absolute value of the ratio of consecutive terms is $\frac{2|x|}{n}$, which has limit 0 as $n \to \infty$. So by the Ratio Test, the interval of convergence is $(-\infty, \infty)$.

9.3.14

a. Substitute $2x$ for x in the Taylor series for $(1+x)^{-1}$, to obtain the series $1 - 2x + 4x^2 - 8x^3 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k(2x)^k$.

c. The Root Test shows that the series converges absolutely for $|2x| < 1$, or $|x| < 1/2$. The interval of convergence is $(-1/2, 1/2)$, because the series at both endpoints diverge by the Divergence Test.

9.3.15

a. By integrating the Taylor series for $\frac{1}{1+x^2}$ (which is the derivative of $\tan^{-1}(x)$), we obtain the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. Then by replacing x by $x/2$ we have $\frac{x}{2} - \frac{x^3}{3 \cdot 2^3} + \frac{x^5}{5 \cdot 2^5} - \frac{x^7}{7 \cdot 2^7} + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)\cdot 2^{2k+1}} x^{2k+1}$.

c. By the Ratio Test (the ratio of consecutive terms has limit $\frac{x^2}{4}$), the radius of convergence is $|x| < 2$. Also, at the endpoints we have convergence by the Alternating Series Test, so the interval of convergence is $[-2, 2]$.

9.3.16

a. Substitute $3x$ for x in the Taylor series for $\sin x$, to obtain the series $3x - \frac{9x^3}{2} + \frac{81x^5}{40} - \frac{243x^7}{560} + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{3^{2k+1}}{(2k+1)!} x^{2k+1}$.

c. The ratio of successive terms is $\frac{9}{2n(2n+1)}x^2$, which has limit zero as $n \to \infty$, so the interval of convergence is $(-\infty, \infty)$.

9.3.17

a. Note that $f(0) = 1$, $f'(0) = \ln 3$, $f''(0) = \ln^2 3$, $f''(0) = \ln^3 3$. So the first four terms of the desired series are $1 + (\ln 3)x + \frac{\ln^2 3}{2}x^2 + \frac{\ln^3 3}{6}x^3 + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{(\ln^k 3)x^k}{k!}$.

c. The ratio of successive terms is $\frac{(\ln^{k+1} 3)x^{k+1}}{(k+1)!} \cdot \frac{k!}{(\ln^k 3)x^k} = \frac{\ln 3}{k+1}x$, and the limit as $k \to \infty$ of this quantity is 0, so the interval of convergence is $(-\infty, \infty)$.

9.3.18

a. Note that $f(0) = 0$, $f'(0) = \frac{1}{\ln 3}$, $f''(0) = -\frac{1}{\ln 3}$, $f'''(0) = \frac{2}{ln 3}$, $f''''(0) = -\frac{6}{\ln 3}$. So the first terms of the desired series are $0 + \frac{x}{\ln 3} - \frac{x^2}{2\ln 3} + \frac{x^3}{3\ln 3} - \frac{x^4}{4\ln 3} + \cdots$.

b. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k \ln 3}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{x^{k+1}}{(k+1)\ln 3} \cdot \frac{k \ln 3}{x^k}\right| = \frac{k}{k+1}|x|$, which has limit $|x|$ as $k \to \infty$. Thus the radius of convergence is 1. At $x = -1$ we have a multiple of the harmonic series (which diverges) and at $x = 1$ we have a multiple of the alternating harmonic series (which converges) so the interval of convergence is $(-1, 1]$.

9.3.19

a. Note that $f(0) = 1$, $f'(0) = 0$, $f''(0) = 9$, $f'''(0) = 0$, etc. The first terms of the series are $1 + 9x^2/2 + 81x^4/4! + 3^6x^6/6! + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{(3x)^{2k}}{(2k)!}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{(3x)^{2k+2}}{(2k+2)!} \cdot \frac{(2k)!}{(3x)^{2k}}\right| = \frac{1}{(2k+2)(2k+1)} \cdot 9x^2$, which has limit 0 as $x \to \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

9.3.20

a. Note that $f(0) = 0$, $f'(0) = 2$, $f''(0) = 0$, $f'''(0) = 8$, etc. The first terms of the series are $2x + 8x^3/6 + 32x^5/5! + 128x^7/7! + \cdots$, or $2x + \frac{4x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{315} + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{2^{2k+1}x^{2k+1}}{(2k+1)!}$.

c. The absolute value of the ratio of successive terms is $\left|\frac{2^{2k+3}x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{2^{2k+1}x^{2k+1}}\right| = \frac{4}{(2k+3)(2k+2)}x^2$, which has limit 0 as $x \to \infty$. The interval of convergence is therefore $(-\infty, \infty)$.

9.3.21

a. Note that $f(\pi/2) = 1$, $f'(\pi/2) = \cos(\pi/2) = 0$, $f''(\pi/2) = -\sin(\pi/2) = -1$, $f'''(\pi/2) = -\cos(\pi/2) = 0$, and so on. Thus the series is given by $1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k}$.

9.3.22

a. Note that $f(\pi) = -1$, $f'(\pi) = -\sin\pi = 0$, $f''(\pi) = -\cos\pi = 1$, $f'''(\pi) = -\sin\pi = 0$, and so on. Thus the series is given by $-1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + \frac{1}{720}(x-\pi)^6 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k)!}(x-\pi)^{2k}$.

9.3.23

a. Note that $f^{(k)}(1) = (-1)^k \frac{k!}{1^{k+1}} = (-1)^k \cdot k!$. Thus the series is given by $1-(x-1)+(x-1)^2-(x-1)^3+\cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k(x-1)^k$.

9.3.24

a. Note that $f^{(k)}(2) = (-1)^k \frac{k!}{2^{k+1}}$. Thus the series is given by $\frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 + \cdots$.

b. $\sum_{k=0}^{\infty}(-1)^k\frac{1}{2^{k+1}}(x-2)^k$.

9.3.25

a. Note that $f^{(k)}(3) = (-1)^{k-1}\frac{(k-1)!}{3^k}$. Thus the series is given by $\ln(3)+\frac{x-3}{3}-\frac{1}{18}(x-3)^2+\frac{1}{81}(x-3)^3+\cdots$.

b. $\ln 3 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\cdot 3^k}(x-3)^k$.

9.3.26

a. Note that $f^{(k)}(\ln 2) = 2$. Thus the series is given by $2 + 2(x - \ln(2)) + (x-\ln(2))^2 + \frac{1}{3}(x-\ln(2))^3 + \frac{1}{12}(x-\ln(2))^4 + \cdots$.

b. $\sum_{k=0}^{\infty}\frac{2}{k!}(x-\ln(2))^k$.

9.3.27

a. Note that $f(1) = 2$, $f'(1) = 2\ln 2$, $f''(1) = 2\ln^2 2$, $f'''(1) = 2\ln^3 2$. The first terms of the series are $2 + (2\ln 2)(x-1) + (\ln^2 2)(x-1)^2 + \frac{(\ln^3 2)(x-1)^3}{3} + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{2(x-1)^k \ln^k 2}{k!}$.

9.3.28

a. Note that $f(2) = 100$, $f'(2) = 100\ln 10$, $f''(2) = 100\ln^2 10$, $f'''(2) = 100\ln^3 10$. The first terms of the series are $100 + 100(\ln 10)(x-2) + 50(\ln^2 10)(x-2)^2 + \frac{50}{3}(\ln^3 10)(x-2)^3 + \cdots$.

b. $\sum_{k=0}^{\infty} \frac{100(x-2)^k \ln^k 10}{k!}$.

9.3.29 Because the Taylor series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, the first four terms of the Taylor series for $\ln(1+x^2)$ are $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$, obtained by substituting x^2 for x.

9.3.30 Because the Taylor series for $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, the first four terms of the Taylor series for $\sin x^2$ are $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$, obtained by substituting x^2 for x.

9.3.31 Because the Taylor series for $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, the first four terms of the Taylor series for $\frac{1}{1-2x}$ are $1 + 2x + 4x^2 + 8x^3 + \cdots$ obtained by substituting $2x$ for x.

9.3.32 Because the Taylor series for $\ln(1+x)$ is $x - x^2/2 + x^3/3 - x^4/4 + \cdots$, the first four terms of the Taylor series for $2x - 2x^2 + 8x^3/3 - 4x^4 + \cdots$ obtained by substituting $2x$ for x.

9.3.33 The Taylor series for $e^x - 1$ is the Taylor series for e^x, less the constant term of 1, so it is $x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$. Thus, the first four terms of the Taylor series for $\frac{e^x-1}{x}$ are $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$, obtained by dividing the terms of the first series by x.

9.3.34 Because the Taylor series for $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, the first four terms of the Taylor series for $\cos x^3$ are $1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$, obtained by substituting x^3 for x.

9.3.35 Because the Taylor series for $(1+x)^{-1}$ is $1 - x + x^2 - x^3 + \cdots$, if we substitute x^4 for x, we obtain $1 - x^4 + x^8 - x^{12} + \cdots$.

9.3.36 The Taylor series for $\tan^{-1} x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots$. Thus, the Taylor series for $\tan^{-1} x^2$ is $x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} - \cdots$ and, multiplying by x, the Taylor series for $x\tan^{-1} x^2$ is $x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \frac{x^{15}}{7} - \cdots$.

9.3.37 The Taylor series for $\sinh x$ is $x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$. Thus, the Taylor series for $\sinh x^2$ is $x^2 + \frac{x^6}{6} + \frac{x^{10}}{120} + \frac{x^{14}}{5040} + \cdots$ obtained by substituting x^2 for x.

9.3.38 The Taylor series for $\cosh x$ is $1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$. Thus, the Taylor series for $\cosh 3x$ is $1 + \frac{9x^2}{2} + \frac{81x^4}{24} + \frac{729x^6}{720} + \cdots$, obtained by substituting $3x$ for x.

9.3.39

a. The binomial coefficients are $\binom{-2}{0} = 1$, $\binom{-2}{1} = \frac{-2}{1!} = -2$, $\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$, $\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$.

Thus the first four terms of the series are $1 - 2x + 3x^2 - 4x^3 + \cdots$.

b. $1 - 2\cdot 0.1 + 3\cdot 0.01 - 4\cdot 0.001 = 0.826$

9.3.40

a. The binomial coefficients are $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2}$, $\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}$, so the first four terms of the series are $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$.

b. $1 + \frac{1}{2} \cdot .06 - \frac{1}{8} \cdot .06^2 + \frac{1}{16} \cdot .06^3 \approx 1.030$

9.3.41

a. The binomial coefficients are $\binom{1/4}{0} = 1$, $\binom{1/4}{1} = \frac{1/4}{1} = \frac{1}{4}$, $\binom{1/4}{2} = \frac{(1/4)(-3/4)}{2!} = -\frac{3}{32}$, $\binom{1/4}{3} = \frac{(1/4)(-3/4)(-7/4)}{3!} = \frac{7}{128}$, so the first four terms of the series are $1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + \cdots$.

b. Substitute $x = 0.12$ to get approximately 1.029.

9.3.42

a. The binomial coefficients are $\binom{-3}{0} = 1$, $\binom{-3}{1} = -3$, $\binom{-3}{2} = \frac{(-3)(-4)}{2!} = 6$, $\binom{-3}{3} = \frac{(-3)(-4)(-5)}{3!} = -10$, so the first four terms of the series are $1 - 3x + 6x^2 - 10x^3 + \cdots$.

b. Substitute $x = 0.1$ to get 0.750.

9.3.43

a. The binomial coefficients are $\binom{-2/3}{0} = 1$, $\binom{-2/3}{1} = -\frac{2}{3}$, $\binom{-2/3}{2} = \frac{(-2/3)(-5/3)}{2!} = \frac{5}{9}$, $\binom{-2/3}{3} = \frac{(-2/3)(-5/3)(-8/3)}{3!} = -\frac{40}{81}$, so the first four terms of the series are $1 - \frac{2}{3}x + \frac{5}{9}x^2 - \frac{40}{81}x^3 + \cdots$.

b. Substitute $x = 0.18$ to get 0.89512.

9.3.44

a. The binomial coefficients are $\binom{2/3}{0} = 1$, $\binom{2/3}{1} = \frac{2}{3}$, $\binom{2/3}{2} = \frac{(2/3)(-1/3)}{2!} = -\frac{1}{9}$, $\binom{2/3}{3} = \frac{(2/3)(-1/3)(-4/3)}{3!} = \frac{4}{81}$, so the first four terms of the series are $1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 + \cdots$

b. Substitute $x = 0.02$ to get ≈ 1.013289284.

9.3.45 $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \cdots$. By the Ratio Test, the radius of convergence is 1. At the endpoints, the series obtained are convergent by the Alternating Series Test. Thus, the interval of convergence is $[-1, 1]$.

9.3.46 $\sqrt{4+x} = 2\sqrt{1+x/4} = 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \cdots$. The interval of convergence is $(-4, 4]$.

9.3.47 $\sqrt{9-9x} = 3\sqrt{1-x} = 3 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{3}{16}x^3 - \cdots$. The interval of convergence is $[-1, 1]$.

9.3.48 $\sqrt{1-4x} = 1 - 2x - 2x^2 - 4x^3 - \cdots$, obtained by substituting $-4x$ for x in the original series. The interval of convergence of $[-1/4, 1/4)$.

9.3.49 $\sqrt{a^2+x^2} = a\sqrt{1+\frac{x^2}{a^2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \cdots$. The series converges when $\frac{x^2}{a^2}$ is less than 1 in magnitude, so the radius of convergence is a. The series given by the endpoints is convergent by the Alternating Series Test, so the interval of convergence is $[-a, a]$.

9.3.50 $\sqrt{4-16x^2} = 2\sqrt{1-(2x)^2} = 2 - 4x^2 - 4x^4 - 8x^6 - \cdots$. Because $2x$ was substituted for x to produce this series, this series converges when $-1 < 2x < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$. Because only even powers of x appear in the series, the series at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are identical, and are convergent. Thus the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

9.3.51 $(1+4x)^{-2} = 1 - 2(4x) + 3(4x)^2 - 4(4x)^3 + \cdots = 1 - 8x + 48x^2 - 256x^3 + \cdots$.

9.3.52 $\frac{1}{(1-4x)^2} = (1-4x)^{-2} = 1 - 2(-4x) + 3(-4x)^2 - 4(-4x)^3 + \cdots = 1 + 8x + 48x^2 + 256x^3$.

9.3.53 $\frac{1}{(4+x^2)^2} = (4+x^2)^{-2} = \frac{1}{16}(1+(x^2/4))^{-2} = \frac{1}{16}\left(1 - 2\cdot\frac{x^2}{4} + 3\cdot\frac{x^4}{16} - 4\cdot\frac{x^6}{64} + \cdots\right) = \frac{1}{16} - \frac{1}{32}x^2 + \frac{3}{256}x^4 - \frac{1}{256}x^6 + \cdots$

9.3.54 Note that $x^2-4x+5 = 1+(x-2)^2$, so $(1+(x-2)^2)^{-2} = 1-2(x-2)^2+3(x-2)^4-4(x-2)^6+\cdots$.

9.3.55 $(3+4x)^{-2} = \frac{1}{9}\left(1+\frac{4x}{3}\right)^{-2} = \frac{1}{9} - \frac{2}{9}\left(\frac{4x}{3}\right) + \frac{3}{9}\left(\frac{4x}{3}\right)^2 - \frac{4}{9}\left(\frac{4x}{3}\right)^3 + \cdots$.

9.3.56 $(1+4x^2)^{-2} = (1+(2x)^2)^{-2} = 1-2(2x)^2+3(2x)^4-4(2x)^6+\cdots = 1-8x^2+48x^4-256x^6+\cdots$.

9.3.57 The interval of convergence for the Taylor series for $f(x) = \sin x$ is $(-\infty,\infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c. Because $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$, we have

$$\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{1}{(n+1)!}\left|x^{n+1}\right| = 0$$

for any x.

9.3.58 The interval of convergence for the Taylor series for $f(x) = \cos 2x$ is $(-\infty,\infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c. The nth derivative of $\cos 2x$ is 2^n times either $\pm\sin x$ or $\pm\cos x$, so that $f^{(n+1)}$ is bounded by 2^{n+1} in magnitude. Thus $\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\left|x^{n+1}\right| = \lim_{n\to\infty}\frac{(2|x|)^{n+1}}{(n+1)!} = 0$ for any x.

9.3.59 The interval of convergence for the Taylor series for e^{-x} is $(-\infty,\infty)$. The remainder is $R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for some c. Thus $\lim_{n\to\infty}|R_n(x)| = 0$ for any x.

9.3.60 The interval of convergence for the Taylor series for $f(x) = \cos x$ is $(-\infty,\infty)$. The remainder is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-\pi/2)^{n+1}$ for some c. Because $f^{n+1}(x)$ is $\pm\cos x$ or $\pm\sin x$, we have

$$\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{1}{(n+1)!}\left|(x-\pi/2)^{n+1}\right| = 0$$

for any x.

9.3.61

a. False. Not all of its derivatives are defined at zero - in fact, none of them are.

b. True. The derivatives of $\csc x$ involve positive powers of $\csc x$ and $\cot x$, both of which are defined at $\pi/2$, so that $\csc x$ has continuous derivatives at $\pi/2$.

c. False. For example, the Taylor series for $f(x^2)$ doesn't converge at $x = 1.9$, because the Taylor series for $f(x)$ doesn't converge at $1.9^2 = 3.61$.

d. False. The Taylor series centered at 1 involves derivatives of f evaluated at 1, not at 0.

e. True. The follows because the Taylor series must itself be an even function.

9.3.62

a. The relevant Taylor series are: $\cos 2x = 1-2x^2+\frac{2}{3}x^4-\frac{4}{45}x^6+\cdots$, and $2\sin x = 2x-\frac{1}{3}x^3+\frac{1}{60}x^5-\cdots$. Thus, the first four terms of the resulting series are $\cos 2x + 2\sin x = 1+2x-2x^2-\frac{1}{3}x^3+\frac{2}{3}x^4+\cdots$.

b. Because each series converges (absolutely) on $(-\infty,\infty)$, so does their sum. The radius of convergence is ∞.

9.3.63

a. The relevant Taylor series are: $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots$ and $e^{-x} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots$. Thus the first four terms of the resulting series are $\frac{1}{2}\left(e^x+e^{-x}\right) = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$.

b. Because each series converges (absolutely) on $(-\infty,\infty)$, so does their sum. The radius of convergence is ∞.

9.3.64

a. The first four terms of the Taylor series for $\sin x$ are $x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}$, so the first four terms for $\frac{\sin x}{x}$ are $1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}$.

b. The radius of convergence is the same as that for $\sin x$, namely ∞.

9.3.65

a. Use the binomial theorem. The binomial coefficients are $\binom{-2/3}{0} = 1$, $\binom{-2/3}{1} = -\frac{2}{3}$, $\binom{-2/3}{2} = \frac{(-2/3)(-5/3)}{2!} = \frac{5}{9}$, $\binom{-2/3}{3} = \frac{(-2/3)(-5/3)(-8/3)}{3!} = -\frac{40}{81}$ and then, substituting x^2 for x, we obtain $1-\frac{2}{3}x^2+\frac{5}{9}x^4-\frac{40}{81}x^6+\cdots$.

b. From Theorem 9.6 the radius of convergence is determined from $\left|x^2\right| < 1$, so it is 1.

9.3.66

a. The first four terms of $\cos x$ are $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}$, so the first four terms of $\cos x^2$ are $1-\frac{x^4}{2}+\frac{x^8}{24}-\frac{x^{12}}{720}$, and thus the first four terms of $x^2\cos x^2$ are $x^2-\frac{x^6}{2}+\frac{x^{10}}{24}-\frac{x^{14}}{720}$.

b. The radius of convergence is ∞.

9.3.67

a. From the binomial formula, the Taylor series for $(1-x)^p$ is $\sum\binom{p}{k}(-1)^k x^k$, so the Taylor series for $(1-x^2)^p$ is $\sum\binom{p}{k}(-1)^k x^{2k}$. Here $p = 1/2$, and the binomial coefficients are $\binom{1/2}{0} = 1$, $\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2}$, $\binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}$, $\binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}$ so that $(1-x^2)^{1/2} = 1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6+\cdots$.

b. From Theorem 9.6 the radius of convergence is determined from $\left|x^2\right| < 1$, so it is 1.

9.3.68

a. Because $b^x = e^{x\ln b}$, the Taylor series is $1+x\ln b+\frac{1}{2!}(x\ln b)^2+\frac{1}{3!}(x\ln b)^3+\cdots$

b. Because the series for e^x converges on $(-\infty,\infty)$, the radius of convergence for the series in part a is ∞.

9.3.69

a. $f(x) = (1+x^2)^{-2}$; using the binomial series and substituting x^2 for x we obtain $1-2x^2+3x^4-4x^6+\cdots$.

b. From Theorem 9.6 the radius of convergence is determined from $\left|x^2\right| < 1$, so it is 1.

9.3.70 Because $f(36) = 6$, and $f'(x) = \frac{1}{2}x^{-1/2}$, $f'(36) = \frac{1}{12}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f''(36) = -\frac{1}{864}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, and $f'''(36) = \frac{3}{62208}$, the first four terms of the Taylor series are $6+\frac{1}{12}(x-36)-\frac{1}{864\cdot 2!}(x-36)^2+\frac{3}{62208\cdot 3!}(x-36)^3$. Evaluating at $x = 39$ we get 6.245008681.

9.3.71 Because $f(64) = 4$, and $f'(x) = \frac{1}{3}x^{-2/3}$, $f'(64) = \frac{1}{48}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, $f''(64) = -\frac{1}{4608}$, $f'''(x) = \frac{10}{27}x^{-8/3}$, and $f'''(64) = \frac{10}{1769472} = \frac{5}{884736}$, the first four terms of the Taylor series are $4+\frac{1}{48}(x-64)-\frac{1}{4608\cdot 2!}(x-64)^2+\frac{5}{884736\cdot 3!}(x-64)^3$. Evaluating at $x = 60$, we get 3.914870274.

9.3.72 Because $f(4)=\frac{1}{2}$, and $f'(x)=-\frac{1}{2}x^{-3/2}$, $f'(4)=-\frac{1}{16}$, $f''(x)=\frac{3}{4}x^{-5/2}$, $f''(4)=\frac{3}{128}$, $f'''(x)=-\frac{15}{8}x^{-7/2}$, and $f'''(4)=-\frac{15}{1024}$, the first four terms of the Taylor series are $\frac{1}{2}-\frac{1}{16}(x-4)+\frac{3}{128\cdot 2!}(x-4)^2-\frac{15}{1024\cdot 3!}(x-4)^3$. Evaluating at $x=3$, we get 0.5766601563.

9.3.73 Because $f(16)=2$, and $f'(x)=\frac{1}{4}x^{-3/4}$, $f'(16)=\frac{1}{32}$, $f''(x)=-\frac{3}{16}x^{-7/4}$, $f''(16)=-\frac{3}{2048}$, $f'''(x)=\frac{21}{64}x^{-11/4}$, and $f'''(16)=\frac{21}{131072}$, the first four terms of the Taylor series are $2+\frac{1}{32}(x-16)-\frac{3}{2048\cdot 2!}(x-16)^2+\frac{21}{131072\cdot 3!}(x-16)^3$. Evaluating at $x=13$, we get 1.898937225.

9.3.74 Evaluate the binomial coefficient $\binom{-1}{k}=\frac{(-1)(-2)\cdots(-1-k+1)}{k!}=(-1)^k$, so that the binomial expansion for $(1+x)^{-1}$ is $\sum_{k=0}^{\infty}(-1)^k x^k$. Substituting $-x$ for x, we obtain $(1-x)^{-1}=\sum_{k=0}^{\infty}(-1)^k(-x)^k=\sum_{k=0}^{\infty}x^k$.

9.3.75 Evaluate the binomial coefficient $\binom{1/2}{k}=\frac{(1/2)(-1/2)(-3/2)\cdots(1/2-k+1)}{k!}=\frac{(1/2)(-1/2)\cdots((3-2k)/2)}{k!}=(-1)^{k-1}2^{-k}\frac{1\cdot 3\cdots(2k-3)}{k!}=(-1)^{k-1}2^{-k}\frac{(2k-2)!}{2^{k-1}\cdot(k-1)!\cdot k!}=(-1)^{k-1}2^{1-2k}\cdot\frac{1}{k}\binom{2k-2}{k-1}$. This is the coefficient of x^k in the Taylor series for $\sqrt{1+x}$. Substituting $4x$ for x, the Taylor series becomes $\sum_{k=0}^{\infty}(-1)^{k-1}2^{1-2k}\cdot\frac{1}{k}\binom{2k-2}{k-1}(4x)^k=\sum_{k=0}^{\infty}(-1)^{k-1}\frac{2}{k}\binom{2k-2}{k-1}x^k$. If we can show that k divides $\binom{2k-2}{k-1}$, we will be done, for then the coefficient of x^k will be an integer. But $\binom{2k-2}{k-1}-\binom{2k-2}{k-2}=\frac{(2k-2)!}{(k-1)!(k-1)!}-\frac{(2k-2)!}{(k-2)!k!}=\frac{(2k-2)!}{(k-1)!(k-1)!}-\frac{(2k-2)!(k-1)}{(k-1)!(k-1)!k}=\frac{k(2k-2)!-(k-1)(2k-2)!}{k(k-1)!(k-1)!}=\frac{1}{k}\frac{(2k-2)!}{(k-1)!(k-1)!}=\frac{1}{k}\binom{2k-2}{k-1}$ and thus we have shown that k divides $\binom{2k-2}{k-1}$.

9.3.76 The two Taylor series are:

$$8+\frac{1}{16}(x-64)-\frac{1}{4096}(x-64)^2+\frac{1}{524288}(x-64)^3-\frac{5}{268435456}(x-64)^4+\cdots$$
$$9+\frac{1}{18}(x-81)-\frac{1}{5832}(x-81)^2+\frac{1}{944784}(x-81)^3-\frac{5}{612220032}(x-81)^4+\cdots.$$

Evaluating these Taylor series at $n=2,3,4$ (after the quadratic, cubic, and quartic terms) we obtain the errors:

n	64	81
2	9.064×10^{-4}	-8.297×10^{-4}
3	-7.019×10^{-5}	-5.813×10^{-5}
4	6.106×10^{-6}	-4.550×10^{-6}

The errors using the Taylor series centered at 81 are consistently smaller.

9.3.77

a. The Maclaurin series for $\sin x$ is $x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots$. Squaring the first four terms yields

$$\begin{aligned}&\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7\right)^2\\&=x^2-\frac{2}{3!}x^4+\left(\frac{2}{5!}+\frac{1}{3!3!}\right)x^6+\left(-2\cdot\frac{1}{7!}-2\cdot\frac{1}{3!5!}\right)x^8\\&=x^2-\frac{1}{3}x^4+\frac{2}{45}x^6-\frac{1}{315}x^8.\end{aligned}$$

b. The Maclaurin series for $\cos x$ is $1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\cdots$. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1-\cos 2x)/2$, we obtain

$$\begin{aligned}&(1-(1-\frac{1}{2}(2x)^2+\frac{1}{4!}(2x)^4-\frac{1}{6!}(2x)^6)+\frac{1}{8!}(2x)^8)/2\\&=(2x^2-\frac{2}{3}x^4+\frac{4}{45}x^6-\frac{2}{315}x^8)/2\\&=x^2-\frac{1}{3}x^4+\frac{2}{45}x^6-\frac{1}{315}x^8,\end{aligned}$$

and the two are the same.

c. If $f(x) = \sin^2 x$, then $f(0) = 0$, $f'(x) = \sin 2x$, so $f'(0) = 0$. $f''(x) = 2\cos 2x$, so $f''(x) = 2$, $f'''(x) = -4\sin 2x$, so $f'''(0) = 0$. Note that from this point $f^{(n)}(0) = 0$ if n is odd and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n. Thus, the series for $\sin^2 x$ is

$$2x^2/2 - 8x^4/4! + 32x^6/6! - 128x^8/8! + \cdots = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots.$$

9.3.78

a. The Maclaurin series for $\cos x$ is $1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$. Squaring the first four terms yields

$$\begin{aligned}(1-&\frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6)^2\\ &= 1 - (\frac{1}{2} + \frac{1}{2})x^2 + (\frac{1}{4!} + \frac{1}{4!} + \frac{1}{4})x^4 + (-\frac{1}{6!} - \frac{1}{6!} - \frac{1}{2\cdot 4!} - \frac{1}{2\cdot 4!})x^6\\ &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6.\end{aligned}$$

b. Substituting $2x$ for x in the Maclaurin series for $\cos x$ and then computing $(1 + \cos 2x)/2$, we obtain

$$\begin{aligned}(1 + 1-&\frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6)/2\\ &= (2 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6)/2\\ &= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6,\end{aligned}$$

and the two are the same.

c. If $f(x) = \cos^2 x$, then $f(0) = 1$. Also, $f'(x) = -2\cos x \sin x = -\sin 2x$. So $f'(0) = 0$. $f''(x) = -2\cos 2x$, so $f''(0) = -2$. $f'''(x) = 8\sin 2x$, so $f'''(0) = 0$. Note that from this point on, $f^{(n)}(0) = 0$ if n is odd, and $f^{(n)}(0) = \pm 2^{n-1}$ if n is even, with the signs alternating for every other even n. Thus, the series for $\cos^2 x$ is

$$1 - 2x^2/2 + 8x^4/4! - 32x^6/6! + \cdots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots.$$

9.3.79 There are many solutions. For example, first find a series that has $(-1, 1)$ as an interval of convergence, say $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Then the series $\frac{1}{1-x/2} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ has $(-2, 2)$ as its interval of convergence. Now shift the series up so that it is centered at 4. We have $\sum_{k=0}^{\infty} \left(\frac{x-4}{2}\right)^k$, which has interval of convergence $(2, 6)$.

9.3.80 $-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}x^4 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot10}x^5$.

9.3.81 $\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}x^4 - \frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}x^5$.

9.3.82

a. The Maclaurin series in question are

$$\begin{aligned}\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\\ e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,\end{aligned}$$

so substituting the series for $\sin x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\sin x} = 1 + (x - \frac{1}{3!}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

b. The Maclaurin series in question are

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots,$$

so substituting the series for $\tan x$ for x in the series for e^x (and considering only those terms that will give us an exponent at most 3), we obtain $e^{\tan x} = 1 + (x + \frac{1}{3}x^3) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots$.

c. The Maclaurin series in question are

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots,$$

so substituting the series for $\sin x$ for x in the series for $\sqrt{1+x^2}$ (and considering only those terms that will give us an exponent at most 4), we obtain $\sqrt{1+\sin^2 x} = 1 + \frac{1}{2}(x - \frac{1}{3!}x^3)^2 - \frac{1}{8}x^4 + \cdots = 1 + \frac{1}{2}x^2 - \frac{7}{24}x^4 + \cdots$.

9.3.83 Use the Taylor series for $\cos x$ centered at $\pi/4$: $\frac{\sqrt{2}}{2}(1 - (x-\pi/4) - \frac{1}{2}(x-\pi/4)^2 + \frac{1}{6}(x-\pi/4)^3 + \cdots)$. The remainder after n terms (because the derivatives of $\cos x$ are bounded by 1 in magnitude) is $|R_n(x)| \le \frac{1}{(n+1)!} \cdot \left(\frac{\pi}{4} - \frac{2\pi}{9}\right)^{n+1}$.

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 3$. Evaluating the first four terms (through $n = 3$) of the series we get 0.7660427050. The true value is ≈ 0.7660444431.

9.3.84 Use the Taylor series for $\sin x$ centered at π: $-(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \cdots$. The remainder after n terms (because the derivatives of $\sin x$ are bounded by 1 in magnitude) is $|R_n(x)| \le \frac{1}{(n+1)!} \cdot (\pi - 0.98\pi)^{n+1}$.

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 2$. Evaluating the first term of the series gives 0.06283185307. The true value is ≈ 0.06279051953.

9.3.85 Use the Taylor series for $f(x) = x^{1/3}$ centered at 64: $4 + \frac{1}{48}(x-64) - \frac{1}{9216}(x-64)^2 + \cdots$. Because we wish to evaluate this series at $x = 83$, $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(83-64)^{n+1}$. We compute that $|f^{(n+1)}(c)| = \frac{2\cdot 5\cdots(3n-1)}{3^{n+1}c^{(3n+2)/3}}$, which is maximized at $c = 64$. Thus

$$|R_n(x)| \le \frac{2\cdot 5\cdots(3n-1)}{3^{n+1}64^{(3n+2)/3}(n+1)!}19^{n+1}$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 5$. Evaluating the terms of the series through $n = 5$ gives 4.362122553. The true value is ≈ 4.362070671.

9.3.86 Use the Taylor series for $f(x) = x^{-1/4}$ centered at 16: $\frac{1}{2} - \frac{1}{128}(x-16) + \frac{5}{16384}(x-16)^2 + \cdots$. Because we wish to evaluate this series at $x = 17$, $|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(17-16)^{n+1}$. We compute that $|f^{(n+1)}(c)| = \frac{1\cdot 5\cdots(4n+1)}{4^{n+1}c^{(4n+5)/4}}$ which is maximized at $c = 16$. Thus

$$|R_n(x)| \le \frac{1\cdot 5\cdots(4n+1)}{4^{n+1}16^{(4n+5)/4}(n+1)!}1^{n+1}$$

Solving for $|R_n(x)| < 10^{-4}$, we obtain $n = 2$. Evaluating the terms of the series through $n = 2$ gives 0.4924926758. The true value is ≈ 0.4924790605.

9.3.87

a. Use the Taylor series for $(125 + x)^{1/3}$ centered at $x = 0$. Using the first four terms and evaluating at $x = 3$ gives a result (5.03968) accurate to within 10^{-4}.

b. Use the Taylor series for $x^{1/3}$ centered at $x = 125$. Note that this gives the identical Taylor series except that the exponential terms are $(x - 125)^n$ rather than x^n. Thus we need terms up through $(x - 125)^3$, just as before, evaluated at $x = 128$, and we obtain the identical result.

c. Because the two Taylor series are the same except for the shifting, the results are equivalent.

9.3.88 Suppose that f is differentiable.
Consider the remainder after the zeroth term of the Taylor series. Taylor's Theorem says that

$$R_0(x) = \frac{f'(c)}{1!}(x - a)^1 \quad \text{for some } c \text{ between } x \text{ and } a,$$

but $f(x) = f(a) + R_0(x)$, which gives $f(x) = f(a) + f'(c)(x - a)$. Rearranging, we obtain $f'(c) = \frac{f(x)-f(a)}{x-a}$ for some c between x and a, which is the conclusion of the Mean Value Theorem.

9.3.89 Consider the remainder after the first term of the Taylor series. Taylor's Theorem indicates that $R_1(x) = \frac{f''(c)}{2}(x - a)^2$ for some c between x and a, so that $f(x) = f(a) + f'(a)(x - a) + \frac{f''(c)}{2}(x - a)^2$. But $f'(a) = 0$, so that for every x in an interval containing a, there is a c between x and a such that $f(x) = f(a) + \frac{f''(c)}{2}(x - a)^2$.

a. If $f''(x) > 0$ on the interval containing a, then for every x in that interval, we have $f(x) = f(a) + \frac{f''(c)}{2}(x - a)^2$ for some c between x and a. But $f''(c) > 0$ and $(x - a)^2 > 0$, so that $f(x) > f(a)$ and a is a local minimum.

b. If $f''(x) < 0$ on the interval containing a, then for every x in that interval, we have $f(x) = f(a) + \frac{f''(c)}{2}(x - a)^2$ for some c between x and a. But $f''(c) < 0$ and $(x - a)^2 > 0$, so that $f(x) < f(a)$ and a is a local maximum.

9.3.90

a. To show that $f'(0) = 0$, we compute the limits of the left and right difference quotients and show that they are both zero:

$$\lim_{x\to 0^+} \frac{e^{-1/x^2} - 0}{x} = \lim_{x\to 0^+} \frac{e^{-1/x^2}}{x} \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{e^{-1/x^2} - 0}{x} = \lim_{x\to 0^-} \frac{e^{-1/x^2}}{x}.$$

For the limit from the right, use the substitution $x = \frac{1}{\sqrt{y}}$; then $y = x^2$ and the limit becomes

$$\lim_{y\to\infty} e^{-y}\sqrt{y} = \lim_{y\to\infty} \frac{\sqrt{y}}{e^y} = 0,$$

because exponentials dominate power functions. Similarly, for the limit from the left, use the substitution $x = -\frac{1}{\sqrt{y}}$; then again $y = x^2$ and the limit becomes

$$\lim_{y\to\infty} (-e^{-y}\sqrt{y}) = -\lim_{y\to\infty} \frac{\sqrt{y}}{e^y} = 0.$$

Since the left and right limits are both zero, it follows that f is differentiable at $x = 0$, and its derivative is zero.

b. Because $f^{(k)}(0) = 0$, the Taylor series centered at 0 has only one term:$f(x) = f(0) = 0$, so the Taylor series is zero.

c. It does not converge to $f(x)$ because $f(x) \neq 0$ for all $x \neq 0$.

9.4 Working with Taylor Series

9.4.1 Replace f and g by their Taylor series centered at a, and evaluate the limit.

9.4.2 Integrate the Taylor series for $f(x)$ centered at a, and evaluate it at the endpoints.

9.4.3 Substitute -0.6 for x in the Taylor series for e^x centered at 0. Note that this series is an alternating series, so the error can easily be estimated by looking at the magnitude of the first neglected term.

9.4.4 Take the Taylor series for $\sin^{-1}(x)$ centered at 0 and evaluate it at $x = 1$, then multiply the result by 2.

9.4.5 The series is $f'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}$, which converges for $|x| < b$.

9.4.6 It must have derivatives of all orders on some interval containing a.

9.4.7 Because $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$, we have $\frac{e^x-1}{x} = 1 + x/2! + \cdots$, so $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

9.4.8 Because $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, we have $\frac{\tan^{-1} x - x}{x^3} = \frac{-1}{3} + \frac{x^2}{5} - \cdots$.
So $\lim_{x\to 0} \frac{\tan^{-1} x - x}{3} = \frac{-1}{3}$.

9.4.9 Because $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, we have $\frac{-x-\ln(1-x)}{x^2} = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots$, so $\lim_{x\to 0} \frac{-x-\ln(1-x)}{x^2} = \frac{1}{2}$.

9.4.10 Because $\sin 2x = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} + \cdots$, we have $\frac{\sin 2x}{x} = 2 - \frac{4x^2}{3} + \frac{4x^4}{15} + \cdots$, so $\lim_{x\to 0} \frac{\sin 2x}{x} = 2$.

9.4.11 We compute that

$$\begin{aligned} \frac{e^x - e^{-x}}{x} &= \frac{1}{x}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)\right) \\ &= \frac{1}{x}\left(2x + \frac{x^3}{3} + \cdots\right) = 2 + \frac{x^2}{3} + \cdots \end{aligned}$$

so the limit of $\dfrac{e^x - e^{-x}}{x}$ as $x \to 0$ is 2.

9.4.12 Because $-e^x = -1 - x - x^2/2 - x^3/6 + \cdots$, we have $\frac{1+x-e^x}{4x^2} = -\frac{1}{8} - \frac{x}{24} + \cdots$, so $\lim_{x\to 0} \frac{1+x-e^x}{4x^2} = -\frac{1}{8}$.

9.4.13 We compute that

$$\begin{aligned} \frac{2\cos 2x - 2 + 4x^2}{2x^4} &= \frac{1}{2x^4}\left(2(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \frac{(2x)^6}{720} + \cdots) - 2 + 4x^2\right) \\ &= \frac{1}{2x^4}\left(\frac{(2x)^4}{12} - \frac{(2x)^6}{360} + \cdots\right) = \frac{2}{3} - \frac{4x^2}{45} + \cdots \end{aligned}$$

so the limit of $\dfrac{2\cos 2x - 2 + 4x^2}{2x^4}$ as $x \to 0$ is $\dfrac{2}{3}$.

9.4.14 We substitute $t = \dfrac{1}{x}$ and find $\lim\limits_{t\to 0} \dfrac{\sin t}{t}$. We compute that

$$\frac{\sin t}{t} = \frac{1}{t}\left(t - \frac{t^3}{6} + \cdots\right) = 1 - \frac{t^2}{6} + \cdots$$

so the limit of $x \sin\left(\dfrac{1}{x}\right)$ as $x \to \infty$ is 1.

9.4.15 We have $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$, so that

$$\frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{x^3/3 - x^4/4 + \cdots}{x^3} = \frac{1}{3} - \frac{x}{4} + \cdots$$

so that $\lim_{x\to 0} \frac{\ln(1+x) - x + x^2/2}{x^3} = \frac{1}{3}$.

9.4.16 The Taylor series for $\ln(x-3)$ centered at $x = 4$ is

$$(x-4) - \frac{1}{2}(x-4)^2 + \cdots .$$

We compute that

$$\begin{aligned}\frac{x^2-16}{\ln(x-3)} &= \frac{x^2-16}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} = \frac{(x-4)(x+4)}{(x-4) - \frac{1}{2}(x-4)^2 + \cdots} \\ &= \frac{x+4}{1 - \frac{1}{2}(x-4) + \cdots}\end{aligned}$$

so the limit of $\frac{x^2-16}{\ln(x-3)}$ as $x \to 4$ is 8.

9.4.17 We compute that

$$\begin{aligned}\frac{3\tan^{-1}x - 3x + x^3}{x^5} &= \frac{1}{x^5}\left(3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - 3x + x^3\right) \\ &= \frac{1}{x^5}\left(\frac{3x^5}{5} - \frac{3x^7}{7} + \cdots\right) = \frac{3}{5} - \frac{3x^2}{7} + \cdots\end{aligned}$$

so the limit of $\frac{3\tan^{-1}x - 3x + x^3}{x^5}$ as $x \to 0$ is $\frac{3}{5}$.

9.4.18 The Taylor series for $\sqrt{1+x}$ centered at 0 is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots .$$

We compute that

$$\begin{aligned}\frac{\sqrt{1+x} - 1 - (x/2)}{4x^2} &= \frac{1}{4x^2}\left(\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) - 1 - \frac{x}{2}\right) \\ &= \frac{1}{4x^2}\left(-\frac{x^2}{8} + \frac{x^3}{16} + \cdots\right) = -\frac{1}{32} + \frac{x}{64} + \cdots\end{aligned}$$

so the limit of $\frac{\sqrt{1+x} - 1 - (x/2)}{4x^2}$ as $x \to 0$ is $-\frac{1}{32}$.

9.4.19 The Taylor series for $\sin 2x$ centered at 0 is

$$\sin 2x = 2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \cdots = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \cdots .$$

Thus

$$\begin{aligned}\frac{12x - 8x^3 - 6\sin 2x}{x^5} &= \frac{12 - 8x^3 - (12x - 8x^3 + \frac{8}{5}x^5 - \frac{16}{105}x^7 + \cdots)}{x^5} \\ &= -\frac{8}{5} + \frac{16}{105}x^2 - \cdots ,\end{aligned}$$

so $\lim_{x\to 0} \frac{12x - 8x^3 - 6\sin 2x}{x^5} = -\frac{8}{5}$.

9.4.20 The Taylor series for $\ln x$ centered at 1 is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots .$$

We compute that

$$\frac{x-1}{\ln x} = \frac{x-1}{(x-1) - \frac{1}{2}(x-1)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x-1) + \cdots}$$

so the limit of $\frac{x-1}{\ln x}$ as $x \to 1$ is 1.

9.4.21 The Taylor series for $\ln(x-1)$ centered at 2 is

$$\ln(x-1) = (x-2) - \frac{1}{2}(x-2)^2 + \cdots .$$

We compute that

$$\frac{x-2}{\ln(x-1)} = \frac{x-2}{(x-2) - \frac{1}{2}(x-2)^2 + \cdots} = \frac{1}{1 - \frac{1}{2}(x-2) + \cdots}$$

so the limit of $\frac{x-2}{\ln(x-1)}$ as $x \to 2$ is 1.

9.4.22 Because $e^{1/x} = 1 + (1/x) + 1/(2x^2) + \cdots$, we have

$$x(e^{1/x} - 1) = 1 + 1/(2x) + \cdots .$$

Thus, $\lim_{x\to\infty} x(e^{1/x} - 1) = 1$.

9.4.23 Computing Taylor series centers at 0 gives

$$e^{-2x} = 1 - 2x + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \cdots = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots$$

$$e^{-x/2} = 1 - \frac{x}{2} + \frac{1}{2!}\left(-\frac{x}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x}{2}\right)^3 + \cdots = 1 - \frac{x}{2} + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \cdots .$$

Thus

$$\begin{aligned}\frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} &= \frac{1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots - (4 - 2x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \cdots) + 3}{2x^2} \\ &= \frac{\frac{3}{2}x^2 - \frac{5}{4}x^3 + \cdots}{2x^2} \\ &= \frac{3}{4} - \frac{5}{8}x + \cdots\end{aligned}$$

so $\lim_{x\to 0} \frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2} = \frac{3}{4}$.

9.4.24 The Taylor series for $(1-2x)^{-1/2}$ centered at 0 is

$$(1-2x)^{-1/2} = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots .$$

We compute that

$$\begin{aligned}\frac{(1-2x)^{-1/2} - e^x}{8x^2} &= \frac{1}{8x^2}\left(\left(1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \cdots\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\right) \\ &= \frac{1}{8x^2}\left(x^2 + \frac{7x^3}{3} + \cdots\right) = \frac{1}{8} + \frac{7x}{24} + \cdots\end{aligned}$$

so the limit of $\frac{(1-2x)^{-1/2} - e^x}{8x^2}$ as $x \to 0$ is $\frac{1}{8}$.

9.4.25

a. $f'(x) = \frac{d}{dx}(\sum_{k=0}^{\infty} \frac{x^k}{k!}) = \sum_{k=1}^{\infty} k\frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)$.

b. $f'(x) = e^x$ as well.

c. The series converges on $(-\infty, \infty)$.

9.4.26

a. $f'(x) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}) = \sum_{k=1}^{\infty}(-1)^k(2k)\frac{x^{2k-1}}{(2k)!} = \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!} = -\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

b. $f'(x) = -\sin x$.

c. The series converges on $(-\infty, \infty)$, because the series for $\cos x$ does.

9.4.27

a. $f'(x) = \frac{d}{dx}(\ln(1+x)) = \frac{d}{dx}(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}x^k) = \sum_{k=1}^{\infty}(-1)^{k+1}x^{k-1} = \sum_{k=0}^{\infty}(-1)^k x^k$.

b. This is the power series for $\frac{1}{1+x}$.

c. The Taylor series for $\ln(1+x)$ converges on $(-1, 1)$, as does the Taylor series for $\frac{1}{1+x}$.

9.4.28

a. $f'(x) = \frac{d}{dx}(\sin x^2) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}) = \sum_{k=0}^{\infty}(-1)^k \cdot 2(2k+1)\frac{x^{4k+1}}{(2k+1)!} = 2\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+1}}{(2k)!} = 2x\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k}}{(2k)!}$.

b. This is the power series for $2x\cos x^2$.

c. Because the Taylor series for $\sin x^2$ converges everywhere, the Taylor series for $2x\cos x^2$ does as well.

9.4.29

a.

$$f'(x) = \frac{d}{dx}(e^{-2x}) = \frac{d}{dx}(\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}) = \frac{d}{dx}(\sum_{k=0}^{\infty}(-2)^k\frac{x^k}{k!}) = -2\sum_{k=1}^{\infty}(-2)^{k-1}\frac{x^{k-1}}{(k-1)!} = -2\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}.$$

b. This is the Taylor series for $-2e^{-2x}$.

c. Because the Taylor series for e^{-2x} converges on $(-\infty, \infty)$, so does this one.

9.4.30

a. We have

$$f'(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty}x^k\right) = \frac{d}{dx}\left(1+\sum_{k=1}^{\infty}x^k\right) = \sum_{k=1}^{\infty}kx^{k-1} = \sum_{k=0}^{\infty}(k+1)x^k.$$

b. From the formula for $(1+x)^p$ in Table 9.5, we see that the Taylor series for $\frac{1}{(1-x)^2}$ is

$$\sum_{k=0}^{\infty}\frac{(-2)(-3)\cdots(-2-k+1)}{k!}(-x)^k = \sum_{k=0}^{\infty}(-1)^k(-1)^k\frac{(k+1)!}{k!}x^k = \sum_{k=0}^{\infty}(k+1)x^k,$$

so that $f'(x)$ is simply $\frac{1}{(1-x)^2}$ as expected.

c. Since the Taylor series for $\frac{1}{1-x}$ converges on $(-1,1)$, so does the series for $\frac{1}{(1-x)^2}$. Checking the endpoints, we see that the series diverges at both endpoints by the Divergence test, so that the interval of convergence for $f'(x)$ is also $(-1,1)$.

9.4.31

a. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, so $\frac{d}{dx}\tan^{-1} x^2 = 1 - x^2 + x^4 - x^6 + \cdots$.

b. This is the series for $\frac{1}{1+x^2}$.

c. Because the series for $\tan^{-1} x$ has a radius of convergence of 1, this series does too. Checking the endpoints shows that the interval of convergence is $(-1,1)$.

9.4.32

a. $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots$, so $\frac{d}{dx}[-\ln(1-x)] = 1 + x + x^2 + x^3 + \cdots$.

b. This is the series for $\frac{1}{1-x}$.

c. The interval of convergence for $\frac{1}{1-x}$ is $(-1,1)$.

9.4.33

a. Because $y(0) = 2$, we have $0 = y'(0) - y(0) = y'(0) - 2$ so that $y'(0) = 2$. Differentiating the equation gives $y''(0) = y'(0)$, so that $y''(0) = 2$. Successive derivatives also have the value 2 at 0, so the Taylor series is $2\sum_{k=0}^{\infty} \frac{t^k}{k!}$.

b. $2\sum_{k=0}^{\infty} \frac{t^k}{k!} = 2e^t$.

9.4.34

a. Because $y(0) = 0$, we see that $y'(0) = 8$. Differentiating the equation gives $y''(0) + 4y'(0) = 0$, so $y''(0) + 4\cdot 8 = 0$, $y''(0) = -4\cdot 8$. Continuing, $y'''(0) + 4\cdot(-4\cdot 8) = 0$, so $y'''(0) = 4\cdot 4\cdot 8$, and in general $y^{(k)}(0) = (-1)^{k+1}2\cdot 4^k$ for $k \geq 1$, so the Taylor series is $2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(4t)^k}{k!}$.

b. $2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(4t)^k}{k!} = 2(1 - e^{-4t})$.

9.4.35

a. $y(0) = 2$, so that $y'(0) = 16$. Differentiating, $y''(t) - 3y'(t) = 0$, so that $y''(0) = 48$, and in general $y^{(k)}(0) = 3y^{(k-1)}(0) = 3^{k-1}\cdot 16$. Thus the power series is $2 + \frac{16}{3}\sum_{k=1}^{\infty}\frac{(3t)^k}{k!} = 2 + \sum_{k=1}^{\infty}\frac{3^{k-1}16}{k!}t^k$.

b. $2 + \frac{16}{3}\sum_{k=1}^{\infty}\frac{(3t)^k}{k!} = 2 + \frac{16}{3}(e^{3t} - 1) = \frac{16}{3}e^{3t} - \frac{10}{3}$.

9.4.36

a. $y(0) = 2$, so $y'(0) = 12 + 9 = 21$. Differentiating, $y^{(n)}(0) = 6y^{(n-1)}(0)$ for $n > 1$, so that $y^{(n)}(0) = 6^{n-1}\cdot 21$ for $n \geq 1$. Thus the power series is $2 + \sum_{k=1}^{\infty} 21\cdot 6^{k-1}\frac{t^k}{k!} = 2 + \frac{7}{2}\sum_{k=1}^{\infty}\frac{(6t)^k}{k!}$.

b. $2 + \frac{7}{2}\sum_{k=1}^{\infty}\frac{(6t)^k}{k!} = 2 + \frac{7}{2}(e^{6t} - 1) = \frac{7}{2}e^{6t} - \frac{3}{2}$.

9.4.37 The Taylor series for e^{-x^2} is $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}$. Thus, the desired integral is $\int_0^{0.25}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)k!}\Big|_0^{0.25} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)k!4^{2k+1}}$. Because this is an alternating series, to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{1}{(2n+3)(n+1)!\cdot 4^{2n+3}} < 10^{-4}$. This occurs for $n = 1$, so $\sum_{k=0}^{1}(-1)^k\frac{1}{(2k+1)\cdot k!\cdot 4^{2k+1}} = \frac{1}{4} - \frac{1}{192} \approx 0.245$.

9.4.38 The Taylor series for $\sin x^2$ is $\sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}$. Thus the desired integral is

$$\int_0^{0.2} \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}\,dx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+3}}{(4k+3)(2k+1)!}\bigg|_0^{0.2} = \sum_{k=0}^{\infty}(-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}.$$

Because this is an alternating series, to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{0.2^{4n+7}}{(4n+7)(2n+3)!} < 10^{-4}$. This occurs first for $n = 0$, so we obtain $\frac{0.2^3}{3\cdot 1!} \approx 2.67 \times 10^{-3}$.

9.4.39 The Taylor series for $\cos 2x^2$ is $\sum_{k=0}^{\infty}(-1)^k \frac{(2x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k \frac{4^k x^{4k}}{(2k)!}$. Note that $\cos x$ is an even function, so we compute the integral from 0 to 0.35 and double it:

$$2\int_0^{0.35} \sum_{k=0}^{\infty}(-1)^k \frac{4^k x^{4k}}{(2k)!}\,dx = 2\left(\sum_{k=0}^{\infty}(-1)^k \frac{4^k x^{4k+1}}{(4k+1)(2k)!}\right)\bigg|_0^{0.35} = 2\left(\sum_{k=0}^{\infty}(-1)^k \frac{4^k (0.35)^{4k+1}}{(4k+1)(2k)!}\right).$$

Because this is an alternating series, to approximate it to within $\frac{1}{2}\cdot 10^{-4}$, we must find n such that $a_{n+1} < \frac{1}{2}\cdot 10^{-4}$, or $\frac{4^{n+1}(0.35)^{4n+5}}{(4n+3)(2n+2)!} < \frac{1}{2}\cdot 10^{-4}$. This occurs first for $n = 1$, and we have $2\left(.35 - \frac{4\cdot(0.35)^5}{5\cdot 2!}\right) \approx 0.696$.

9.4.40 The Taylor series for $(1 + x^4)^{1/2}$ is $\sum_{k=0}^{\infty}\binom{1/2}{k}x^{4k}$, so the desired integral is

$$\int_0^{0.2} \sum_{k=0}^{\infty}\binom{1/2}{k}x^{4k}\,dx = \sum_{k=0}^{\infty}\frac{1}{4k+1}\binom{1/2}{k}x^{4k+1}\bigg|_0^{0.2} = \sum_{k=0}^{\infty}\frac{1}{4k+1}\binom{1/2}{k}(0.2)^{4k+1}.$$

This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\left|\frac{1}{4n+5}\binom{1/2}{n+1}(0.2)^{4n+5}\right| < 10^{-4}$. This happens first for $n = 0$, so the approximation is $\binom{1/2}{0}\cdot 0.2 = 0.2$.

9.4.41 $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots$, so $\int \tan^{-1} x\,dx = \int (x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots)\,dx = C + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots$. Thus, $\int_0^{0.35} \tan^{-1} x\,dx = \frac{(0.35)^2}{2} - \frac{(0.35)^4}{12} + \frac{(0.35)^6}{30} - \frac{(0.35)^8}{56} + \cdots$. Note that this series is alternating, and $\frac{(0.35)^6}{30} < 10^{-4}$, so we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.060.

9.4.42 $\ln(1 + x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$, so $\int \ln(1 + x^2)\,dx = \int (x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots)\,dx = C + \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} - \frac{x^9}{36} + \frac{x^{11}}{55} + \cdots$. Thus, $\int_0^{0.4} \ln(1 + x^2)\,dx = \frac{(0.4)^3}{3} - \frac{(0.4)^5}{10} + \frac{(0.4)^7}{21} - \frac{(0.4)^9}{36} + \cdots$. Because $\frac{(0.4)^7}{21} < 10^{-4}$, we add the first two terms to approximate the integral to the desired accuracy. Calculating gives approximately 0.020.

9.4.43 The Taylor series for $(1+x^6)^{-1/2}$ is $\sum_{k=0}^{\infty}\binom{-1/2}{k}x^{6k}$, so the desired integral is $\int_0^{0.5}\sum_{k=0}^{\infty}\binom{-1/2}{k}x^{6k}\,dx = \sum_{k=0}^{\infty}\frac{1}{6k+1}\binom{-1/2}{k}x^{6k+1}\bigg|_0^{0.5} = \sum_{k=0}^{\infty}\frac{1}{6k+1}\binom{-1/2}{k}(0.5)^{6k+1}$. This is an alternating series because the binomial coefficients alternate in sign, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\left|\frac{1}{6n+7}\binom{-1/2}{n+1}(0.5)^{6n+7}\right| < 10^{-4}$. This occurs first for $n = 1$, so we have $\binom{-1/2}{0}0.5 + \frac{1}{7}\binom{-1/2}{1}(0.5)^7 \approx 0.499$.

9.4.44 The Taylor series for $\frac{\ln(1+t)}{t}$ centered at 0 is $\sum_{k=0}^{\infty}(-1)^k \frac{t^k}{k+1}$. The desired integral is thus $\int_0^{0.2}\sum_{k=0}^{\infty}(-1)^k \frac{t^k}{k+1}\,dt = \sum_{k=0}^{\infty}(-1)^k \frac{t^{k+1}}{(k+1)^2}\bigg|_0^{0.2} = \sum_{k=0}^{\infty}(-1)^k \frac{(0.2)^{k+1}}{(k+1)^2}$. This is an alternating series, so to approximate it to within 10^{-4}, we must find n such that $a_{n+1} < 10^{-4}$, or $\frac{(0.2)^{n+2}}{(n+2)^2} < 10^{-4}$. This occurs first for $n = 3$, so we have $\sum_{k=0}^{3}(-1)^k \frac{(0.2)^{k+1}}{(k+1)^2} \approx 0.191$.

9.4.45 Use the Taylor series for e^x at 0: $1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}$.

9.4.46 Use the Taylor series for e^x at 0: $1+\frac{1/2}{1!}+\frac{(1/2)^2}{2!}+\frac{(1/2)^3}{3!}=1+\frac{1}{2}+\frac{1}{8}+\frac{1}{8\cdot 3!}$.

9.4.47 Use the Taylor series for $\cos x$ at 0: $1-\frac{2^2}{2!}+\frac{2^4}{4!}-\frac{2^6}{6!}$

9.4.48 Use the Taylor series for $\sin x$ at 0: $1-\frac{1^3}{3!}+\frac{1^5}{5!}-\frac{1^7}{7!}=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}$.

9.4.49 Use the Taylor series for $\ln(1+x)$ evaluated at $x=1/2$: $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{4}\cdot\frac{1}{16}$.

9.4.50 Use the Taylor series for $\tan^{-1}x$ evaluated at 1/2: $\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{8}+\frac{1}{5}\cdot\frac{1}{32}-\frac{1}{7}\cdot\frac{1}{128}$.

9.4.51 The Taylor series for f centered at 0 is $\frac{-1+\sum_{k=0}^{\infty}\frac{x^k}{k!}}{x}=\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}=\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}$. Evaluating both sides at $x=1$, we have $e-1=\sum_{k=0}^{\infty}\frac{1}{(k+1)!}$.

9.4.52 The Taylor series for f centered at 0 is $\frac{-1+\sum_{k=0}^{\infty}\frac{x^k}{k!}}{x}=\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}=\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}$. Differentiating, the Taylor series for $f'(x)$ is $f'(x)=\frac{(x-1)e^x+1}{x^2}=\sum_{k=1}^{\infty}\frac{kx^{k-1}}{(k+1)!}$. Evaluating both sides at 2 gives $\frac{e^2+1}{4}=\sum_{k=1}^{\infty}\frac{k\cdot 2^{k-1}}{(k+1)!}$.

9.4.53 The Maclaurin series for $\ln(1+x)$ is $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{x^{k+1}k}{x^k(k+1)}\right|=|x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1, so the interval of convergence is $(-1,1]$. Evaluating at 1 gives $\ln 2=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$.

9.4.54 The Taylor series for $\ln(1+x)$ at 0 is $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{x^{k+1}k}{x^k(k+1)}\right|=|x|$, so the radius of convergence is 1. The series diverges at -1 and converges at 1, so the interval of convergence is $(-1,1]$. Evaluate both sides at $-1/2$ to get $f(\frac{-1}{2})=\ln(1/2)=-\ln 2=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(-1/2)^k}{k}=-\sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}$, so that $\ln 2=\sum_{k=1}^{\infty}\frac{1}{k\cdot 2^k}$.

9.4.55 $\sum_{k=0}^{\infty}\frac{x^k}{2^k}=\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k=\frac{1}{1-\frac{x}{2}}=\frac{2}{2-x}$.

9.4.56 $\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k=\frac{1}{1+\frac{x}{3}}=\frac{3}{3+x}$.

9.4.57 $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{4^k}=\sum_{k=0}^{\infty}\left(\frac{-x^2}{4}\right)^k=\frac{1}{1+\frac{x^2}{4}}=\frac{4}{4+x^2}$.

9.4.58 $\sum_{k=0}^{\infty}2^kx^{2k+1}=x\sum_{k=0}^{\infty}(2x^2)^k=\frac{x}{1-2x^2}$.

9.4.59 $\ln(1+x)=-\sum_{k=1}^{\infty}(-1)^k\frac{x^k}{k}$, so $\ln(1-x)=-\sum_{k=1}^{\infty}\frac{x^k}{k}$, and finally $-\ln(1-x)=\sum_{k=1}^{\infty}\frac{x^k}{k}$.

9.4.60 $\sum_{k=0}^{\infty}\frac{(-1)^kx^{k+1}}{4^k}=-4\sum_{k=0}^{\infty}\left(\frac{-x}{4}\right)^{k+1}=-4(-1+\sum_{k=0}^{\infty}\left(\frac{-x}{4}\right)^k)=4-\frac{4}{1+\frac{x}{4}}=4-\frac{16}{4+x}=\frac{4x}{4+x}$

9.4.61

$$\begin{aligned}\sum_{k=1}^{\infty}(-1)^k\frac{kx^{k+1}}{3^k} &= \sum_{k=1}^{\infty}(-1)^k\frac{k}{3^k}x^{k+1}=\sum_{k=1}^{\infty}k\left(-\frac{1}{3}\right)^k x^{k+1}\\ &= x^2\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k kx^{k-1}=x^2\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k\frac{d}{dx}(x^k)\\ &= x^2\frac{d}{dx}\left(\sum_{k=1}^{\infty}\left(-\frac{x}{3}\right)^k\right)=x^2\frac{d}{dx}\left(\frac{1}{1+\frac{x}{3}}\right)=-\frac{3x^2}{(x+3)^2}.\end{aligned}$$

9.4.62 By Exercise 53, $\sum_{k=1}^{\infty}\frac{x^k}{k}=-\ln(1-x)$, so $\sum_{k=1}^{\infty}\frac{x^{2k}}{k}=\sum_{k=1}^{\infty}\frac{(x^2)^k}{k}=-\ln(1-x^2)$.

9.4.63 $\sum_{k=2}^{\infty} \frac{k(k-1)x^k}{3^k} = x^2 \sum_{k=2}^{\infty} \frac{k(k-1)x^{k-2}}{3^k} = x^2 \frac{d^2}{dx^2} \left(\sum_{k=2}^{\infty} \frac{x^k}{3^k} \right)$
$= x^2 \frac{d^2}{dx^2} \left(\sum_{k=2}^{\infty} \left(\frac{x}{3}\right)^k \right) = x^2 \frac{d^2}{dx^2} \left(\frac{x^2}{9} \cdot \frac{1}{1-\frac{x}{3}} \right) = x^2 \frac{d^2}{dx^2} \left(\frac{x^2}{9-3x} \right) = x^2 \frac{-6}{(x-3)^3} = \frac{-6x^2}{(x-3)^3}$.

9.4.64 $\sum_{k=2}^{\infty} \frac{x^k}{k(k-1)} = \sum_{k=2}^{\infty} \frac{x^k}{k-1} - \sum_{k=2}^{\infty} \frac{x^k}{k} = x \sum_{k=1}^{\infty} \frac{x^k}{k} - \sum_{k=1}^{\infty} \frac{x^k}{k} + x, = -x\ln(1-x) + \ln(1-x) + x = x + (1-x)\ln(1-x)$.

9.4.65

a. False. This is because $\frac{1}{1-x}$ is not continuous at 1, which is in the interval of integration.

b. False. The Ratio Test shows that the radius of convergence for the Taylor series for $\tan^{-1} x$ centered at 0 is 1.

c. True. $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$. Substitute $x = \ln 2$.

9.4.66 The Taylor series for e^{ax} centered at 0 is

$$e^{ax} = 1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots .$$

We compute that

$$\begin{aligned} \frac{e^{ax}-1}{x} &= \frac{1}{x}\left(\left(1 + ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots\right) - 1\right) \\ &= \frac{1}{x}\left(ax + \frac{(ax)^2}{2} + \frac{(ax)^3}{6} + \cdots\right) = a + \frac{a^2x}{2} + \frac{a^3x^2}{6} + \cdots \end{aligned}$$

so the limit of $\dfrac{e^{ax}-1}{x}$ as $x \to 0$ is a.

9.4.67 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots .$$

We compute that

$$\begin{aligned} \frac{\sin ax}{\sin bx} &= \frac{ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots}{bx - \frac{(bx)^3}{6} + \frac{(bx)^5}{120} - \cdots} \\ &= \frac{a - \frac{a^3x^2}{6} + \frac{a^5x^4}{120} - \cdots}{b - \frac{b^3x^2}{6} + \frac{b^5x^4}{120} - \cdots} \end{aligned}$$

so the limit of $\dfrac{\sin ax}{\sin bx}$ as $x \to 0$ is $\dfrac{a}{b}$.

9.4.68 The Taylor series for $\sin ax$ centered at 0 is

$$\sin ax = ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots$$

and the Taylor series for $\tan^{-1} ax$ centered at 0 is

$$\tan^{-1} ax = ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \cdots .$$

We compute that

$$\begin{aligned} \frac{\sin ax - \tan^{-1} ax}{bx^3} &= \frac{1}{bx^3}\left(\left(ax - \frac{(ax)^3}{6} + \frac{(ax)^5}{120} - \cdots\right) - \left(ax - \frac{(ax)^3}{3} + \frac{(ax)^5}{5} - \cdots\right)\right) \\ &= \frac{1}{bx^3}\left(\frac{(ax)^3}{6} - \frac{23(ax)^5}{120} + \cdots\right) = \frac{a^3}{6b} - \frac{23a^5}{120b}x^2 + \cdots \end{aligned}$$

so the limit of $\dfrac{\sin ax - \tan^{-1} ax}{bx^3}$ as $x \to 0$ is $\dfrac{a^3}{6b}$.

9.4.69 Compute instead the limit of the log of this expression, $\lim_{x\to 0} \frac{\ln(\sin x/x)}{x^2}$. If the Taylor expansion of $\ln(\sin x/x)$ is $\sum_{k=0}^{\infty} c_k x^k$, then $\lim_{x\to 0} \frac{\ln(\sin x/x)}{x^2} = \lim_{x\to 0} \sum_{k=0}^{\infty} c_k x^{k-2} = \lim_{x\to 0} c_0 x^{-2} + c_1 x^{-1} + c_2$, because the higher-order terms have positive powers of x and thus approach zero as x does. So compute the terms of the Taylor series of $\ln\left(\frac{\sin x}{x}\right)$ up through the quadratic term. The relevant Taylor series are: $\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$ and we substitute the Taylor series for $\frac{\sin x}{x} - 1$ for x in the Taylor series for $\ln(1+x)$. Because the lowest power of x in the first Taylor series is 2, it follows that only the linear term in the series for $\ln(1+x)$ will give any powers of x that are at most quadratic. The only term that results is $-\frac{1}{6}x^2$. Thus $c_0 = c_1 = 0$ in the above, and $c_2 = -\frac{1}{6}$, so that $\lim_{x\to 0} \frac{\ln(\sin x/x)}{x^2} = -\frac{1}{6}$ and thus $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

9.4.70 We can find the Taylor series for $\ln(x + \sqrt{1+x^2})$ by substituting into $\ln(1+t)$ the Taylor series for $x + \sqrt{x^2+1} - 1$. The Taylor series in question are: $x + \sqrt{x^2+1} - 1 = x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \ldots$, $\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \frac{1}{6}t^6 + \frac{1}{7}t^7 - \ldots$. Substituting the former into the latter and simplifying (not a simple task!), we obtain $\ln(x+\sqrt{x^2+1}) = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$. Using the second definition, start with the Taylor series for $(1+t^2)^{-1/2}$, which is $1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \ldots$, and integrate it: $\int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \ldots\right) dt = \left(t - \frac{1}{6}t^3 + \frac{3}{40}t^5 - \frac{5}{112}t^7 + \ldots\right)\Big|_0^x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \ldots$.

9.4.71 The Taylor series we need are $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$, $e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \ldots$. We are looking for powers of x^3 and x^4 that occur when the first series is substituted for t in the second series. Clearly there will be no odd powers of x, because $\cos x$ has only even powers. Thus the coefficient of x^3 is zero, so that $f^{(3)}(0) = 0$. The coefficient of x^4 comes from the expansion of $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ in each term of e^t. Higher powers of x clearly cannot contribute to the coefficient of x^4. Thus consider $\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)^k$. The term $-\frac{1}{2}x^2$ generates $\binom{k}{2}$ terms of value $\frac{1}{4}x^4$ for $k \ge 2$, while the other term generates k terms of value $\frac{1}{24}x^4$ for $k \ge 1$. These terms all have to be divided by the $k!$ appearing in the series for e^t. So the total coefficient of x^4 is $\frac{1}{24}\sum_{k=1}^{\infty} \frac{k}{k!} + \frac{1}{4}\sum_{k=2}^{\infty} \binom{k}{2}\frac{1}{k!}, = \frac{1}{24}\sum_{k=1}^{\infty} \frac{1}{(k-1)!} + \frac{1}{4}\sum_{k=2}^{\infty} \frac{1}{2\cdot(k-2)!}, = \frac{1}{24}\sum_{k=0}^{\infty} \frac{1}{k!} + \frac{1}{8}\sum_{k=0}^{\infty} \frac{1}{k!}, = \frac{1}{24}e + \frac{1}{8}e = \frac{e}{6}$ Thus $f^{(4)}(0) = \frac{e}{6} \cdot 4! = 4e$.

9.4.72 The Taylor series for $(1+x)^{-1/3}$ is $(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \ldots$, so we want the coefficients of x^3 and x^4 in $(x^2+1)\left(1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4\right)$. The coefficient of x^3 is $-\frac{1}{3} - \frac{14}{81} = -\frac{41}{81}$, and the coefficient of x^4 is $\frac{2}{9} + \frac{35}{243} = \frac{89}{243}$. Thus $f^{(3)}(0) = 6 \cdot \frac{-41}{81} = \frac{-82}{27}$, and $f^{(4)}(0) = 24 \cdot \frac{89}{243} = \frac{712}{81}$.

9.4.73 The Taylor series for $\sin t^2$ is $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \ldots$, so that $\int_0^x \sin t^2\, dt = \frac{1}{3}t^3 - \frac{1}{7\cdot 3!}t^7 + \ldots\Big|_0^x = \frac{1}{3}x^3 - \frac{1}{7\cdot 3!}x^7 + \ldots$. Thus $f^{(3)}(0) = \frac{3!}{3} = 2$ and $f^{(4)}(0) = 0$.

9.4.74 $\frac{1}{1+t^4} = 1 - t^4 + t^8 + \ldots$, so that $\int_0^x \frac{1}{1+t^4} dt = t - \frac{1}{5}t^5 + \frac{1}{9}t^9 + \ldots\Big|_0^x = x - \frac{1}{5}x^5 + \ldots$ so that both $f^{(3)}(0)$ and $f^{(4)}(0)$ are zero.

9.4.75 Consider the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{x}\sum_{k=0}^{\infty} kx^k$ so that $\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$. Evaluate both sides at $x = 1/2$ to see that the sum of the series is $\frac{1/2}{(1-1/2)^2} = 2$. Thus the expected number of tosses is 2.

9.4.76

a. $\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k} = \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{25}{36}\right)^k = \frac{1}{6} \cdot \frac{1}{1-25/36} = \frac{6}{11}$.

b. Consider the series $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$. Differentiating both sides gives $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$ Evaluating at $x = 5/6$ and multiplying the result by $1/6$, we get $\frac{1}{6} \cdot \frac{1}{(1-5/6)^2} = 6$.

9.4.77

a. We look first for a Taylor series for $(1-k^2\sin^2\theta)^{-1/2}$. Because $(1-k^2x^2)^{-1/2} = (1-(kx)^2)^{-1/2} = \sum_{i=0}^{\infty}\binom{-1/2}{i}(kx)^{2i}$, and $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$, substituting the second series into the first gives $\frac{1}{\sqrt{1-k^2\sin^2\theta}} = 1 + \frac{1}{2}k^2\theta^2 + \left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\theta^4 + \left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\theta^6 +$ $\left(\frac{-1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\theta^8 + \dots$.

Integrating with respect to θ and evaluating at $\pi/2$ (the value of the antiderivative is 0 at 0) gives $\frac{1}{2}\pi + \frac{1}{48}k^2\pi^3 + \frac{1}{160}\left(-\frac{1}{6}k^2 + \frac{3}{8}k^4\right)\pi^5 + \frac{1}{896}\left(\frac{1}{45}k^2 - \frac{1}{4}k^4 + \frac{5}{16}k^6\right)\pi^7 + \frac{1}{4608}\left(-\frac{1}{630}k^2 + \frac{3}{40}k^4 - \frac{5}{16}k^6 + \frac{35}{128}k^8\right)\pi^9$. Evaluating these terms for $k = 0.1$ gives $F(0.1) \approx 1.574749680$. (The true value is approximately 1.574745562.)

b. The terms above, with coefficients of k^n converted to decimal approximations, is $1.5707 + .3918\cdot k^2 + .3597\cdot k^4 - .9682\cdot k^6 + 1.7689\cdot k^8$. The coefficients are all less than 2 and do not appear to be increasing very much if at all, so if we want the result to be accurate to within 10^{-3} we should probably take n such that $k^n < \frac{1}{2}\times 10^{-3} = .0005$, so $n = 4$ for this value of k.

c. By the above analysis, we would need a larger n because $0.2^n > 0.1^n$ for a given value of n.

9.4.78

a. $\frac{\sin t}{t} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$.

b. $\int_0^x \frac{\sin t}{t}\,dt = \sum_{k=0}^{\infty}\int_0^x(-1)^k\frac{t^{2k}}{(2k+1)!}\,dt = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)(2k+1)!}$.

c. This is an alternating series, so we want n such that $a_{n+1} < 10^{-3}$, or $\frac{0.5^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$ (resp. $\frac{1^{2n+3}}{(2n+3)(2n+3)!} < 10^{-3}$), which gives $n = 1$ (resp. $n = 2$). Thus $\text{Si}(0.5) \approx \frac{0.5}{1} - \frac{0.5^3}{3\cdot 3!} \approx 0.4930555556$, $\text{Si}(1.0) \approx 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} \approx 0.9461111111$.

9.4.79

a. By the Fundamental Theorem, $S'(x) = \sin x^2$, $C'(x) = \cos x^2$.

b. The relevant Taylor series are $\sin t^2 = t^2 - \frac{1}{3!}t^6 + \frac{1}{5!}t^{10} - \frac{1}{7!}t^{14} + \dots$, and $\cos t^2 = 1 - \frac{1}{2!}t^4 + \frac{1}{4!}t^8 - \frac{1}{6!}t^{12} + \dots$. Integrating, we have $S(x) = \frac{1}{3}x^3 - \frac{1}{7\cdot 3!}x^7 + \frac{1}{11\cdot 5!}x^{11} - \frac{1}{15\cdot 7!}x^{15} + \dots$, and $C(x) = x - \frac{1}{5\cdot 2!}x^5 + \frac{1}{9\cdot 4!}x^9 - \frac{1}{13\cdot 6!}x^{13} + \dots$.

c. $S(0.05) \approx \frac{1}{3}(0.05)^3 - \frac{1}{42}(0.05)^7 + \frac{1}{1320}(0.05)^{11} - \frac{1}{75600}(0.05)^{15} \approx 4.166664807\times 10^{-5}$. $C(-0.25) \approx (-0.25) - \frac{1}{10}(-0.25)^5 + \frac{1}{216}(-0.25)^9 - \frac{1}{9360}(-0.25)^{13} \approx -.2499023616$.

d. The series is alternating. Because $a_{n+1} = \frac{1}{(4n+7)(2n+3)!}(0.05)^{4n+7}$, and this is less than 10^{-4} for $n = 0$, only one term is required.

e. The series is alternating. Because $a_{n+1} = \frac{1}{(4n+5)(2n+2)!}(0.25)^{4n+5}$, and this is less than 10^{-6} for $n = 1$, two terms are required.

9.4.80

a. $\frac{d}{dx}\text{erf}(x) = \frac{2}{\sqrt{\pi}}(e^{-x^2})$.

b. $e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{k!}$, so that the Maclaurin series for the error function is $\text{erf(x)} = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \dots\right)$.

c. $\text{erf}(0.15) \approx \frac{2}{\sqrt{\pi}}\left(0.15 - \frac{0.15^3}{3} + \frac{0.15^5}{10} - \frac{0.15^7}{42}\right) \approx 0.1679959712$.

$\text{erf}(-0.09) \approx \frac{2}{\sqrt{\pi}}\left(-0.09 + \frac{0.09^3}{3} - \frac{0.09^5}{10} + \frac{0.09^7}{42}\right) \approx -.1012805939$.

d. The first omitted term in each case is $\frac{x^9}{9\cdot 5!} = \frac{x^9}{1080}$. For $x = 0.15$, this is $\approx 3.56 \times 10^{-11}$. For $x = -0.09$, this is (in absolute value) $\approx 3.59 \times 10^{-13}$.

9.4.81

a. $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{16\cdot 2!^2}x^4 - \frac{1}{2^6\cdot 3!^2}x^6 + \dots$.

b. Using the Ratio Test: $\left|\frac{a_{k+1}}{a_k}\right| = \frac{x^{2k+2}}{2^{2k+2}((k+1)!)^2} \cdot \frac{2^{2k}(k!)^2}{x^{2k}} = \frac{x^2}{4(k+1)^2}$, which has limit 0 as $k \to \infty$ for any x. Thus the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

c. Starting only with terms up through x^8, we have $J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 + \dots$, $J_0'(x) = -\frac{1}{2}x + \frac{1}{16}x^3 - \frac{1}{384}x^5 + \frac{1}{18432}x^7 + \dots$, $J_0''(x) = -\frac{1}{2} + \frac{3}{16}x^2 - \frac{5}{384}x^4 + \frac{7}{18432}x^6 + \dots$ so that $x^2J_0(x) = x^2 - \frac{1}{4}x^4 + \frac{1}{64}x^6 - \frac{1}{2304}x^8 + \frac{1}{147456}x^{10} + \dots$, $xJ_0'(x) = -\frac{1}{2}x^2 + \frac{1}{16}x^4 - \frac{1}{384}x^6 + \frac{1}{18432}x^8 + \dots$, $x^2J_0''(x) = -\frac{1}{2}x^2 + \frac{3}{16}x^4 - \frac{5}{384}x^6 + \frac{7}{18432}x^8 + \dots$, and $x^2J_0''(x) + xJ_0'(x) + x^2J_0(x) = 0$.

9.4.82 $\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots$

9.4.83

a. The power series for $\cos x$ has only even powers of x, so that the power series has the same value evaluated at $-x$ as it does at x.

b. The power series for $\sin x$ has only odd powers of x, so that evaluating it at $-x$ gives the opposite of its value at x.

9.4.84 Long division gives $\csc x = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \cdots$, so that $\csc x \approx \frac{1}{x} + \frac{1}{6}x$ as $x \to 0^+$.

9.4.85

a. Because $f(a) = g(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots} \\ &= \lim_{x\to a}\frac{f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots}{g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots} \\ &= \lim_{x\to a}\frac{f'(a) + \frac{1}{2}f''(a)(x-a) + \cdots}{g'(a) + \frac{1}{2}g''(a)(x-a) + \cdots} = \frac{f'(a)}{g'(a)}.\end{aligned}$$

Because $f'(x)$ and $g'(x)$ are assumed to be continuous at a and $g'(a) \neq 0$,

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

and we have that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

which is one form of L'Hôpital's Rule.

b. Because $f(a) = g(a) = f'(a) = g'(a) = 0$, we use the Taylor series for $f(x)$ and $g(x)$ centered at a to compute that

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots}{g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \frac{1}{6}g'''(a)(x-a)^3 + \cdots} \\ &= \lim_{x\to a}\frac{\frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \cdots}{\frac{1}{2}g''(a)(x-a)^2 + \frac{1}{6}g'''(a)(x-a)^3 + \cdots} \\ &= \lim_{x\to a}\frac{\frac{1}{2}f''(a) + \frac{1}{6}f'''(a)(x-a) + \cdots}{\frac{1}{2}g''(a) + \frac{1}{6}g'''(a)(x-a) + \cdots} = \frac{f''(a)}{g''(a)}.\end{aligned}$$

Because $f''(x)$ and $g''(x)$ are assumed to be continuous at a and $g''(a) \neq 0$,

$$\frac{f''(a)}{g''(a)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

and we have that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

which is consistent with two applications of L'Hôpital's Rule.

9.4.86

a. Clearly $x = \sin s$ because BE, of length x, is the side opposite the angle measured by s in a right triangle with unit length hypotenuse.

b. In the formula $\frac{1}{2}r^2\theta$ for the formula for the area of a circular sector, we have $r = 1$, and $\theta = s$, so that the area is in fact $\frac{s}{2}$. But the area can also be expressed as an integral as follows: the area of the sector is the area under the circle between P and F (i.e. the area of the region $PAEF$), minus the area of the right triangle PEF. The area of the right triangle is $\frac{1}{2}x\sqrt{1-x^2}$ by the Pythagorean theorem and the formula for the area of a triangle. Equating these two formulae for the area of the sector, we have $\frac{s}{2} = \int_0^x \sqrt{1-t^2}\,dt - \frac{1}{2}x\sqrt{1-x^2}$, so $s = 2\int_0^x \sqrt{1-t^2}\,dt - x\sqrt{1-x^2}$.

c. The Taylor series for $\sqrt{1-t^2}$ is $1 - \frac{1}{2}t^2 - \frac{1}{8}t^4 - \frac{1}{16}t^6 - \frac{5}{128}t^8 - \ldots$. Integrating and evaluating at x we have $s = \sin^{-1} x = 2\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \frac{5}{1152}x^9\right) - x\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right) + \cdots = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots$.

d. Suppose $x = \sin s = a_0 + a_1 s + a_2 s^2 + \ldots$. Then $x = \sin(\sin^{-1}(x)) = a_0 + a_1(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots) + a_2((x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots)^2 + \ldots$. Equating coefficients yields $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = \frac{-1}{6}$, and so on.

Chapter Nine Review

1

a. True. The approximations tend to get better as n increases in size, and also when the value being approximated is closer to the center of the series. Because 2.1 is closer to 2 than 2.2 is, and because $3 > 2$, we should have $|p_3(2.1) - f(2.1)| < |p_2(2.2) - f(2.2)|$.

b. False. The interval of convergence may or may not include the endpoints.

c. True. The interval of convergence is an interval centered at 0, and the endpoints may or may not be included.

d. True. Because $f(x)$ is a polynomial, all its derivatives vanish after a certain point (in this case, $f^{(12)}(x)$ is the last nonzero derivative).

2 $p_3(x) = 2x - \frac{(2x)^3}{3!}$.

3 $p_2(x) = 1$.

4 $p_2(x) = 1 - x + \frac{x^2}{2}$.

5 $p_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.

6 $p_2(x) = \frac{\sqrt{2}}{2}\left(1 - (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2\right)$.

7 $p_2(x) = x - 1 - \frac{1}{2}(x-1)^2$.

8 $p_4(x) = 8x^3/3! + 2x = 4x^3/3 + 2x$.

9 $p_3(x) = \frac{5}{4} + \frac{3(x-\ln 2)}{4} + \frac{5(x-\ln 2)^2}{8} + \frac{(x-\ln 2)^3}{8}$.

10

a. $p_0(x) = p_1(x) = 1$, and $p_2(x) = 1 - \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$\lvert p_n(-0.08) - \cos(-0.08)\rvert$
0	1	3.2×10^{-3}
1	1	3.2×10^{-3}
2	0.997	1.7×10^{-6}

11

a. $p_0(x) = 1$, $p_1(x) = 1 + x$, and $p_2(x) = 1 + x + \frac{x^2}{2}$.

b.

n	$p_n(-0.08)$	$\lvert p_n(-0.08) - e^{-0.08}\rvert$
0	1	7.7×10^{-2}
1	0.92	3.1×10^{-3}
2	0.923	8.4×10^{-5}

12

a. $p_0(x) = 1$, $p_1(x) = 1 + \frac{1}{2}x$, and $p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$.

b.

n	$p_n(0.08)$	$\lvert p_n(0.08) - \sqrt{1+0.08}\rvert$
0	1	3.9×10^{-2}
1	1.04	7.7×10^{-4}
2	1.039	3.0×10^{-5}

13

a. $p_0(x) = \frac{\sqrt{2}}{2}$, $p_1(x) = \frac{\sqrt{2}}{2}(1 + (x - \pi/4))$, and $p_2(x) = \frac{\sqrt{2}}{2}\left(1 + (x-\pi/4) - \frac{1}{2}(x-\pi/4)^2\right)$.

b.

n	$p_n(\pi/5)$	$\lvert p_n(\pi/5) - \sin(\pi/5)\rvert$
0	0.707	1.2×10^{-1}
1	0.596	8.2×10^{-3}
2	0.587	4.7×10^{-4}

14 The bound is $|R_n(x)| \le M\frac{|x|^{n+1}}{(n+1)!}$, where M is a bound for $|e^x|$ (because e^x is its own derivative) on $[-1, 1]$. Thus take $M = 3$ so that $|R_3(x)| \le \frac{3x^4}{4!} = \frac{x^4}{8}$. But $|x| < 1$, so this is at most $\frac{1}{8}$.

15 The derivatives of $\sin x$ are bounded in magnitude by 1, so $|R_n(x)| \le M\frac{|x|^{n+1}}{(n+1)!} \le \frac{|x|^{n+1}}{(n+1)!}$. But $|x| < \pi$, so $|R_3(x)| \le \frac{\pi^4}{24}$.

16 The third derivative of $\ln(1-x)$ is $\frac{-2}{(x-1)^3}$, which is bounded in magnitude by 16 on $|x| < 1/2$ (at $x = 1/2$). Thus $|R_3(x)| \le 16\frac{|x|^4}{4!} \le 16\frac{1}{2^4 4!} = \frac{1}{4!}$.

17 Using the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2x^{k+1}}{(k+1)!}\cdot\frac{k!}{k^2x^k}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^2\frac{|x|}{k+1} = 0$, so the interval of convergence is $(-\infty, \infty)$.

18 Using the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{4k+4}}{(k+1)^2}\cdot\frac{k^2}{x^{4k}}\right| = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 x^4 = x^4$, so that the radius of convergence is 1. Because $\sum\frac{1}{k^2}$ converges, the given power series converges at both endpoints, so its interval of convergence is $[-1, 1]$.

19 Using the Ratio Test, $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\left|\frac{(x+1)^{2k+2}}{(k+1)!}\cdot\frac{k!}{(x+1)^{2k}}\right| = \lim_{k\to\infty}\frac{1}{k+1}(x+1)^2 = 0$, so the interval of convergence is $(-\infty, \infty)$.

20 Using the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-1)^{k+1}}{(k+1)5^{k+1}}\cdot\frac{k5^k}{(x-1)^k}\right| = \lim_{k\to\infty}\frac{k}{5k+5}|x-1| = \frac{1}{5}(|x-1|)$, so the series converges when $|1/5(x-1)| < 1$, or $-5 < x-1 < 5$, so that $-4 < x < 6$. At $x = -4$, the series is the alternating harmonic series. At $x = 6$, it is the harmonic series, so the interval of convergence is $[-4, 6)$.

21 By the Root Test, $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\left(\frac{|x|}{9}\right)^3 = \frac{|x^3|}{729}$, so the series converges for $|x| < 9$. The series given by letting $x = \pm 9$ are both divergent by the Divergence Test. Thus, $(-9, 9)$ is the interval of convergence.

22 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x+2)^{k+1}}{\sqrt{k+1}}\cdot\frac{\sqrt{k}}{(x+2)^k}\right| = \lim_{k\to\infty}\sqrt{\frac{k}{k+1}}(|x+2|) = |x+2|$, so that the series converges for $|x+2| < 1$, so $-3 < x < -1$. At $x = -3$, we have a series which converges by the Alternating Series Test. At $x = -1$, we have the divergent $p-$series with $p = 1/2$. Thus, $[-3, -1)$ is the interval of convergence.

23 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{(x+2)^{k+1}}{2^{k+1}\ln(k+1)}\cdot\frac{2^k\ln k}{(x+2)^k}\right| = \lim_{k\to\infty}\frac{\ln k}{2\ln(k+1)}|x+2| = \frac{|x+2|}{2}$. The radius of convergence is thus 2, and a check of the endpoints gives the divergent series $\sum\frac{1}{\ln k}$ at $x = 0$ and the convergent alternating series $\sum\frac{(-1)^k}{\ln k}$ at $x = -4$. The interval of convergence is therefore $[-4, 0)$.

24 By the Ratio Test, $\lim_{k\to\infty}\left|\frac{x^{2k+3}}{2k+3}\cdot\frac{2k+1}{x^{2k+1}}\right| = x^2$. The radius of convergence is thus 1. At each endpoint we have a divergent series, so the interval of convergence is $(-1, 1)$.

25 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty} x^{2k}$. By the Root Test, this converges for $\left|x^2\right| < 1$, so $-1 < x < 1$. It diverges at both endpoints, so the interval of convergence is $(-1, 1)$.

26 The Maclaurin series for $f(x)$ is determined by replacing x by $(-x)^3$ in the power series for $\frac{1}{1-x}$, so it is $\sum_{k=0}^{\infty}(-1)^k x^{3k}$. The radius of convergence is still 1. The series diverges at both endpoints, so the interval of convergence is $(-1, 1)$.

27 The Maclaurin series for $f(x)$ is $\sum_{k=0}^{\infty}(-5x)^k = \sum_{k=0}^{\infty}(-5)^k x^k$. By the Root Test, this has radius of convergence 1/5. Checking the endpoints, we obtain an interval of convergence of $(-1/5, 1/5)$.

28 Replace x by $-x$ in the original power series, and multiply the result by $10x$, to get the Maclaurin series for $f(x)$, which is $\sum_{k=0}^{\infty}(-1)^k 10x^{k+1}$. By the Ratio Test, the radius of convergence is 1. Checking the endpoints, we obtain an interval of convergence of $(-1, 1)$.

29 Note that $\frac{1}{1-10x} = \sum_{k=0}^{\infty}(10x)^k$, so $\frac{1}{10}\cdot\frac{1}{1-10x} = \frac{1}{10}\sum_{k=0}^{\infty}(10x)^k$. Taking the derivative of $\frac{1}{10}\cdot\frac{1}{1-10x}$ gives $f(x)$. Thus, the Maclaurin series for $f(x)$ is $\frac{1}{10}\sum_{k=1}^{\infty}10k(10x)^{k-1} = \sum_{k=1}^{\infty}k(10x)^{k-1}$. Using the Ratio Test, we see that the radius of convergence is $1/10$, and checking endpoints we obtain an interval of convergence of $(-1/10, 1/10)$.

30 Integrating $\frac{1}{1-x}$ and then replacing x by $4x$ gives $-f(x)$, so the series for $f(x)$ is $-\sum_{k=0}^{\infty}\frac{1}{k+1}(4x)^{k+1}$. The Ratio Test shows that the series has a radius of convergence of $1/4$; checking the endpoints, we obtain an interval of convergence of $[-1/4, 1/4)$.

31 The first three terms are $1 + 3x + \frac{9x^2}{2}$. The series is $\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}$.

32 The first three terms are $1 - (x-1) + (x-1)^2$. The series is $\sum_{k=0}^{\infty}(-1)^k(x-1)^k$.

33 The first three terms are $-(x-\pi/2) + \frac{1}{6}(x-\pi/2)^3 - \frac{1}{120}(x-\pi/2)^5$. The series is

$$\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k+1)!}\left(x-\frac{\pi}{2}\right)^{2k+1}.$$

34 The first three terms for $\frac{1}{1+x}$ are $1 - x + x^2$, so the first three terms of $x^2\cdot\frac{1}{1+x}$ are $x^2 - x^3 + x^4$. The series is $\sum_{k=0}^{\infty}(-1)^k x^{k+2}$.

35 The first three terms are $4x - \frac{1}{3}(4x)^3 + \frac{1}{5}(4x)^5$. The series is $\sum_{k=0}^{\infty}(-1)^k\frac{(4x)^{2k+1}}{2k+1}$.

36 The nth derivative of $f(x) = \sin(2x)$ is $\pm 2^n$ times either $\sin 2x$ or $\cos 2x$. Evaluated at $-\frac{\pi}{2}$, the even derivatives are therefore zero, and the $(2n+1)^{\text{st}}$ derivative is $(-1)^{n+1}2^{2n+1}$. The Taylor series for $\sin 2x$ around $x = -\frac{\pi}{2}$ is thus $-2\left(x+\frac{\pi}{2}\right) + \frac{2^3}{3!}\left(x+\frac{\pi}{2}\right)^3 - \frac{2^5}{5!}\left(x+\frac{\pi}{2}\right)^5 + \cdots$, and the general series is $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{2^{2k+1}}{(2k+1)!}\left(x+\frac{\pi}{2}\right)^{2k+1}$.

37 The nth derivative of $\cosh 3x$ at $x = 0$ is 0 if n is odd and is 3^n if n is even. The first 3 terms of the series are thus $1 + \frac{9x^2}{2!} + \frac{81x^4}{4!}$. The whole series can be written as $\sum_{k=0}^{\infty}\frac{(3x)^{2k}}{(2k)!}$.

38 $f(0) = \frac{1}{4}$, $f'(x) = \frac{-2x}{(x^2+4)^2}$, so $f'(0) = 0$. $f''(x) = \frac{6x^2-8}{(x^2+4)^3}$, so $f''(0) = -\frac{1}{8}$. $f'''(0) = 0$, and $f''''(0) = \frac{3}{8}$. The first three terms are $\frac{1}{4} - \frac{x^2}{16} + \frac{x^4}{64}$. The series is given by $\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{4^{k+1}}$.

39 $f(x) = \binom{1/3}{0} + \binom{1/3}{1}x + \binom{1/3}{2}x^2 + \cdots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots$.

40 $f(x) = \binom{-1/2}{0} + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \cdots = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$.

41 $f(x) = \binom{-3}{0} + \binom{-3}{1}\frac{x}{2} + \binom{-3}{2}\frac{x^2}{4} + \cdots = 1 - \frac{3}{2}x + \frac{3}{2}x^2 + \cdots$.

42 $f(x) = \binom{-5}{0} + \binom{-5}{1}(2x) + \binom{-5}{2}(2x)^2 + \cdots = 1 - 10x + 60x^2 + \cdots$.

43 $R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$ for some c between 0 and x, and $\lim_{n\to\infty}|R_n(x)| \le e^{-|x|}\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$, because $n!$ grows faster than $|x|^n$ as $n\to\infty$ for all x.

44 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c between 0 and x. Because all derivatives of $\sin x$ are bounded in magnitude by 1, we have $\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$ because $n!$ grows faster than $|x|^n$ as $n\to\infty$ for all x.

45 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c in $(-1/2, 1/2)$. Now, $\left|f^{(n+1)}(c)\right| = \frac{n!}{(1+c)^{n+1}}$, so $\lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}(2|x|)^{n+1}\cdot\frac{1}{n+1} \le \lim_{n\to\infty}1^{n+1}\frac{1}{n+1} = 0$.

46 $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c in $(-1/2, 1/2)$. Now the $(n+1)^{\text{st}}$ derivative of $(\sqrt{1+x})$ is $\pm\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}(1+x)^{(2n+1)/2}}$, so for c in $(-1/2, 1/2)$, this is bounded in magnitude by $\frac{1\cdot3\cdot5\cdots(2n-1)}{2^{n+1}(1/2)^{(2n+1)/2}} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^{1/2}}$, and thus

$$\begin{aligned}\lim_{n\to\infty}|R_n(x)| &= \lim_{n\to\infty}\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \\ &\le \lim_{n\to\infty}\frac{1\cdot3\cdot5\cdot(2n-1)}{\sqrt{2}}\cdot\frac{1}{2^{n+1}\cdot(n+1)!} \\ &= \lim_{n\to\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{\sqrt{2}}\cdot\frac{1}{2\cdot4\cdot6\cdots(2n+2)} \\ &= \lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{1}{2n+2}\right) = 0.\end{aligned}$$

for x in $(-1/2, 1/2)$.

47 The Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots.$$

We compute that

$$\begin{aligned}\frac{x^2/2 - 1 + \cos x}{x^4} &= \frac{1}{x^4}\left(x^2/2 - 1 + \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)\right) \\ &= \frac{1}{x^4}\left(\frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) = \frac{1}{24} - \frac{x^2}{720} + \cdots\end{aligned}$$

so the limit of $\frac{x^2/2 - 1 + \cos x}{x^4}$ as $x \to 0$ is $\frac{1}{24}$.

48 The Taylor series for $\sin x$ centered at 0 is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

and the Taylor series for $\tan^{-1} x$ centered at 0 is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

We compute that

$$\begin{aligned}&\frac{2\sin x - \tan^{-1} x - x}{2x^5} \\ &= \frac{1}{2x^5}\left(2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - x\right) \\ &= \frac{1}{2x^5}\left(\frac{11x^5}{60} + \frac{359x^7}{2520} - \cdots\right) = -\frac{11}{120} + \frac{359x^2}{5040} - \cdots\end{aligned}$$

so the limit of $\frac{2\sin x - \tan^{-1} x - x}{2x^5}$ as $x \to 0$ is $-\frac{11}{120}$.

49 The Taylor series for $\ln(x-3)$ centered at 4 is

$$\ln(x-3) = (x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3 - \cdots.$$

We compute that

$$\begin{aligned}\frac{\ln(x-3)}{x^2-16} &= \frac{1}{(x-4)(x+4)}\left((x-4)-\frac{1}{2}(x-4)^2+\frac{1}{3}(x-4)^3-\cdots\right)\\ &= \frac{1}{(x-4)(x+4)}\left((x-4)\left(1-\frac{1}{2}(x-4)+\frac{1}{3}(x-4)^2-\cdots\right)\right)\\ &= \frac{1}{x+4}\left(1-\frac{1}{2}(x-4)+\frac{1}{3}(x-4)^2-\cdots\right)\end{aligned}$$

so the limit of $\frac{\ln(x-3)}{x^2-16}$ as $x \to 4$ is $\frac{1}{8}$.

50 The Taylor series for $\sqrt{1+2x}$ centered at 0 is

$$\sqrt{1+2x} = 1+x-\frac{x^2}{2}+\frac{x^3}{2}-\cdots.$$

We compute that

$$\begin{aligned}\frac{\sqrt{1+2x}-1-x}{x^2} &= \frac{1}{x^2}\left(\left(1+x-\frac{x^2}{2}+\frac{x^3}{2}-\cdots\right)-1-x\right)\\ &= \frac{1}{x^2}\left(-\frac{x^2}{2}+\frac{x^3}{2}-\cdots\right) = -\frac{1}{2}+\frac{x}{2}-\cdots\end{aligned}$$

so the limit of $\frac{\sqrt{1+2x}-1-x}{x^2}$ as $x \to 0$ is $-\frac{1}{2}$.

51 The Taylor series for $\sec x$ centered at 0 is

$$\sec x = 1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots$$

and the Taylor series for $\cos x$ centered at 0 is

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots.$$

We compute that

$$\begin{aligned}&\frac{\sec x-\cos x-x^2}{x^4}\\ &= \frac{1}{x^4}\left(\left(1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots\right)-\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)-x^2\right)\\ &= \frac{1}{x^4}\left(\frac{x^4}{6}+\frac{31x^6}{360}+\cdots\right) = \frac{1}{6}+\frac{31x^2}{360}+\cdots\end{aligned}$$

so the limit of $\frac{\sec x-\cos x-x^2}{x^4}$ as $x \to 0$ is $\frac{1}{6}$.

52 The Taylor series for $(1+x)^{-2}$ centered at 0 is

$$(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$$

and the Taylor series for $\sqrt[3]{1-6x}$ centered at 0 is

$$\sqrt[3]{1-6x} = 1-2x-4x^2-\frac{40x^3}{3}-\cdots.$$