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ORDINARY DIFFERENTIAL EQUATIONS
Solution Manual

August 15, 2009

Springer
Berlin Heidelberg New York
Hong Kong London
Milan Paris Tokyo

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Solutions

Section 1.1

1. The rate of change in the population $P(t)$ is the derivative $P'(t)$. The Malthusian Growth Law states that the rate of change in the population is proportional to $P(t)$. Thus $P'(t) = kP(t)$, where k is the proportionality constant. Without reference to the t variable, the differential equation becomes $P' = kP$

2. a. This statement mathematically is $b(t) = b_0 P(t)$ where we have used b_0 to represent the proportionality constant.
b. This statement translates as $d(t) = d_0 P^2(t)$ where we have used d_0 to represent the proportionality constant.
c. The overall growth rate is $P'(t)$. Thus the Logistic Growth Law is

$$\begin{aligned} P'(t) &= b(t) - d(t) \\ &= b_0 P(t) - d_0 P^2(t) \\ &= (b_0 - d_0 P(t))P(t). \end{aligned}$$

3. Torricelli's law states that the change in height, $h'(t)$ is proportional to the square root of the height, $\sqrt{h(t)}$. Thus $h'(t) = \lambda\sqrt{h(t)}$, where λ is the proportionality constant.

4. The highest order derivative is y' so the order is 1 and the standard form is $y' = t^3/y^2$.

5. The highest order derivative is y'' so the order is 2. The standard form is $y'' = t^3/y'$.

6. The highest order derivative is y' so the order is 1 and the standard form is $y' = (e^t - ty)/t^2$.

7. The highest order derivative is y'' so the order is 2. The standard form is $y'' = (3y + ty')/t^2$.

8. The highest order derivative is y'' so the order is 2 and the standard form is $y'' = t^2 - 3y' - 2y$.

9. The highest order derivative is $y^{(4)}$ so the order is 4. Solving for $y^{(4)}$ gives the standard form: $y^{(4)} = \sqrt[3]{(1 - (y''')^4)/t}$.

10. The highest order derivative is y' so the order is 1 and the standard form is $y' = ty^4 - t^2 y$.

11. The highest order derivative is y''' so the order is 3. Solving for y''' gives the standard form: $y''' = 2y'' - 3y' + y$.

12. The following table summarizes the needed calculations:

Function	$y'(t)$	$2y(t)$
$y_1(t) = 0$	$y_1'(t) = 0$	$2y_1(t) = 0$
$y_2(t) = t^2$	$y_2'(t) = 2t$	$2y_2(t) = 2t^2$
$y_3(t) = 3e^{2t}$	$y_3'(t) = 6e^{2t}$	$2y_3(t) = 6e^{2t}$
$y_4(t) = 2e^{3t}$	$y_4'(t) = 6e^{3t}$	$2y_4(t) = 4e^{3t}$

Thus y_1 and y_3 are the only solutions.

13. The following table summarizes the needed calculations:

Function	$ty'(t)$	$y(t)$
$y_1(t) = 0$	$ty_1'(t) = 0$	$y_1(t) = 0$
$y_2(t) = 3t$	$ty_2'(t) = 3t$	$y_2(t) = 3t$
$y_3(t) = -5t$	$ty_3'(t) = -5t$	$y_3(t) = -5t$
$y_4(t) = t^3$	$ty_4'(t) = 3t^3$	$y_4(t) = t^3$

Thus y_1, y_2, and y_3 are solutions.

14. We first write the differential equation in standard form: $y'' = -4y$. The following table summarizes the needed calculations:

Function	$y''(t)$	$-4y(t)$
$y_1(t) = e^{2t}$	$y_1''(t) = 4e^{2t}$	$-4y_1(t) = -4e^{2t}$
$y_2(t) = \sin 2t$	$y_2''(t) = -4\sin 2t$	$-4y_2(t) = -4\sin 2t$
$y_3(t) = \cos(2t - 1)$	$y_3''(t) - 4\cos(2t - 1)$	$-4y_3(t) = -4\cos(2t - 1)$
$y_4(t) = t^2$	$y_4''(t) = 2$	$-4y_4(t) = -4t^2$

Thus y_2 and y_3 are solutions.

15. The following table summarizes the needed calculations:

Function	$y'(t)$	$2y(t)(y(t)-1)$
$y_1(t)=0$	$y_1'(t)=0$	$2y_1(t)(y_1(t)-1)=2\cdot 0\cdot(-1)=0$
$y_2(t)=1$	$y_2'(t)=0$	$2y_2(t)(y_2(t)-1)=2\cdot 1\cdot 0=0$
$y_3(t)=2$	$y_3'(t)=0$	$2y_3(t)(y_3(t)-1)=2\cdot 2\cdot 1=4$
$y_4(t)=\frac{1}{1-e^{2t}}$	$y_4'(t)=\frac{2e^{2t}}{(1-e^{2t})^2}$	$2y_4(t)(y_4(t)-1)=2\frac{1}{1-e^{2t}}\left(\frac{1}{1-e^{2t}}-1\right)$ $=2\frac{1}{1-e^{2t}}\frac{e^{2t}}{1-e^{2t}}=\frac{2e^{2t}}{(1-e^{2t})^2}$

Thus y_1, y_2, and y_4 are solutions.

16. The following table summarizes the needed calculations:

Function	$2y(t)y'(t)$	1
$y_1(t)=1$	$2y_1(t)y_1'(t)=0$	1
$y_2(t)=t$	$2y_2(t)y_2'(t)=2t$	1
$y_3(t)=\ln t$	$2y_3(t)y_3'(t)=2\frac{1}{t}\ln t=\frac{2\ln t}{t}$	1
$y_4(t)=\sqrt{t-4}$	$2y_4(t)y_4'(t)=2\sqrt{t-4}\frac{1}{2\sqrt{t-4}}=1$	1

Thus y_4 is the only solution.

17. The following table summarizes the needed calculations:

Function	$2y(t)y'(t)$	y^2+t-1
$y_1(t)=\sqrt{-t}$	$2\sqrt{-t}\dfrac{-1}{2\sqrt{-t}}=-1$	$(\sqrt{-t})^2+t-1=-1$
$y_2(t)=-\sqrt{e^t-t}$	$-2\sqrt{e^t-t}\dfrac{-(e^t-1)}{2\sqrt{e^t-t}}=e^t-1$	$(-\sqrt{e^t-t})^2+t-1=e^t-1$
$y_3(t)=\sqrt{t}$	$2\sqrt{t}\dfrac{1}{2\sqrt{t}}=1$	$(\sqrt{t})^2+t-1=2t-1$
$y_4(t)=-\sqrt{-t}$	$2(-\sqrt{-t})\dfrac{1}{2\sqrt{-t}}=-1$	$(-\sqrt{-t}))^2+y-1=-1$

Thus y_1, y_2, and y_4 are solutions.

18. The following table summarizes the needed calculations for the first three functions:

Function	$y'(t)$	$\dfrac{y^2(t) - 4y(t)t + 6t^2}{t^2}$
$y_1(t) = t$	1	$\dfrac{t^2 - 4t^2 + 6t^2}{t^2} = 3$
$y_2(t) = 2t$	2	$\dfrac{4t^2 - 8t^2 + 6t^2}{t^2} = 2$
$y_3(t) = 3t$	3	$\dfrac{9t^2 - 12t^2 + 6t^2}{t^2} = 3$

For $y_4(t) = \dfrac{3t + 2t^2}{1 + t} = \dfrac{t(3 + 2t)}{1 + t}$ the quotient rule and simplifying gives $y_4'(t) = \dfrac{2t^2 + 4t + 3}{(1 + t)^2}$. On the other hand,

$$\begin{aligned}\frac{y_4^2(t) - 4y_4(t)t + 6t^2}{t^2} &= \frac{\dfrac{t^2(3+2t)^2}{(1+t)^2} - \dfrac{4t^2(3+2t)}{(1+t)} + 6t^2}{t^2} \\ &= \frac{(3+2t)^2 - 4(3+2t)(1+t) + 6(1+t)^2}{(1+t)^2} \\ &= \frac{2t^2 + 4t + 3}{(1+t)^2}.\end{aligned}$$

It follows that y_2, y_3, and y_4 are solutions.

19.

$$\begin{aligned}y'(t) &= 3ce^{3t} \\ 3y + 12 &= 3(ce^{3t} - 4) + 12 = 3ce^{3t} - 12 + 12 = 3ce^{3t}.\end{aligned}$$

Note that $y(t)$ is defined for all $t \in \mathbb{R}$.

20.

$$\begin{aligned}y'(t) &= -ce^{-t} + 3 \\ -y(t) + 3t &= -ce^{-t} - 3t + 3 + 3t = -ce^{-t} + 3.\end{aligned}$$

Note that $y(t)$ is defined for all $t \in \mathbb{R}$.

21.

$$\begin{aligned}y'(t) &= \frac{ce^t}{(1 - ce^t)^2} \\ y^2(t) - y(t) &= \frac{1}{(1 - ce^t)^2} - \frac{1}{1 - ce^t} = \frac{1 - (1 - ce^t)}{(1 - ce^t)^2} = \frac{ce^t}{(1 - ce^t)^2}.\end{aligned}$$

If $c \leq 0$ then the denominator $1 - ce^t > 0$ and $y(t)$ has domain $\mathbb{R}$. If $c > 0$ then $1 - ce^t = 0$ if $t = \ln\frac{1}{c} = -\ln c$. Thus $y(t)$ is defined either on the interval $(-\infty, -\ln c)$ or $(-\ln c, \infty)$.

22.

$$y'(t) = ce^{t^2} 2t = 2cte^{t^2}$$
$$2ty(t) = 2tce^{t^2}.$$

23.

$$y'(t) = \frac{-ce^t}{ce^t - 1}$$
$$-e^y - 1 = -e^{-\ln(ce^t-1)} - 1 = \frac{-1}{ce^t - 1} - 1 = \frac{-ce^t}{ce^t - 1}.$$

24. We first calculate $y'(t) = -c(t+1)^{-2}$ so

$$(t+1)y'(t) + y(t) = (t+1)\frac{-c}{(t+1)^2} + \frac{c}{t+1} = \frac{-c}{t+1} + \frac{c}{t+1} = 0.$$

Observe that $y(t)$ is not defined at $t = -1$ so the two intervals where y is defined are $(-\infty, -1)$ and $(-1, \infty)$.

25.

$$y'(t) = -(c-t)^{-2}(-1) = \frac{1}{(c-t)^2}$$
$$y^2(t) = \frac{1}{(c-t)^2}.$$

The denominator of $y(t)$ is 0 when $t = c$. Thus the two intervals where $y(t)$ is defined are $(-\infty, c)$ and (c, ∞).

26. This is a differential equation we can solve by simple integration: We get $y(t) = \frac{t^2}{2} + 3t + c$.

27. Integration gives $y(t) = \frac{e^{2t}}{2} - t + c$.

28. Integration (by parts) gives $y(t) = -te^{-t} - e^{-t} + c$.

29. Observe that $\frac{t+1}{t} = 1 + \frac{1}{t}$. Integration gives $y(t) = t + \ln|t| + c$.

30. We integrate two times. First, $y'(t) = t^2 + t + c_1$. Second, $y(t) = \frac{t^3}{3} + \frac{t^2}{2} + c_1 t + c_2$.

31. We integrate two times. First, $y'(t) = -2\cos 3t + c_1$. Second, $y(t) = \frac{-2}{3}\sin 3t + c_1 t + c_2$.

32. From Problem 19 the general solution is $y(t) = ce^{3t} - 4$. At $t = 0$ we get $-2 = y(0) = ce^0 - 4 = c - 4$. It follows that $c = 2$ and $y(t) = 2e^{3t} - 4$.

33. From Problem 20 the general solution is $y(t) = ce^{-t} + 3t - 3$. At $t = 0$ we get $0 = y(0) = ce^0 + 3(0) - 3 = c - 3$. It follows that $c = 3$ and $y(t) = 3e^{-t} + 3t - 3$.

34. From Problem 21 the general solution is $y(t) = 1/(1 - ce^t)$. At $t = 0$ we get $1/2 = y(0) = \frac{1}{1-c}$. It follows that $c = -1$ and $y(t) = 1/(1 + e^t)$.

35. From Problem 24 the general solution is $y(t) = c(t + 1)^{-1}$. At $t = 1$ we get $-9 = y(1) = c(1 + 1)^{-1} = c/2$. It follows that $c = -18$ and $y(t) = -18(t + 1)^{-1}$.

36. From Problem 27 the general solution is $y(t) = e^{2t}/2 - t + c$. Evaluation at $t = 0$ gives $4 = e^0/2 - 0 + c = 1/2 + c$. Hence $c = 7/2$ and

37. From Problem 28 the general solution is $y(t) = -te^{-t} - e^{-t} + c$. Evaluation at $t = 0$ gives $-1 = y(0) = -1 + c$ so $c = 0$. Hence $y(t) = -te^{-t} - e^{-t}$.

38. From Problem 31 the general solution is $y(t) = \frac{-2}{3} \sin 3t + c_1 t + c_2$ and a $y'(t) = -2 \cos 3t + c_1$. Evaluation at $t = 0$ gives $1 = y(0) = c_2$ and $2 = y'(0) = -2 + c_1$. If follows that $c_1 = 4$ and $c_2 = 1$. Thus $y(t) = \frac{-2}{3} \sin 3t + 4t$.

39. Implicit differentiation with respect to t gives $6t + 8yy' = 0$.

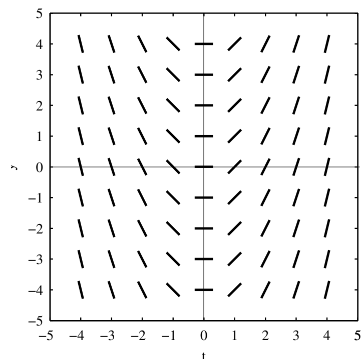
40. Implicit differentiation with respect to t gives $2yy' - 2t - 3t^2 = 0$.

41. Differentiation gives $y' = 2ce^{2t} + 1$. However, from the given function we have $ce^{2t} = y - t$. Substitution gives $y' = 2(y - t) + 1 = 2y - 2t + 1$.

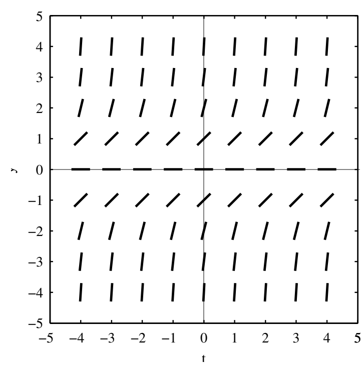
42. Differentiation gives $y' = 3ct^2 + 2t$. However, from the given function we have $ct^3 = y - t^2$ and hence $ct^2 = \frac{y-t^2}{t}$. Substitution gives $y' = 3\frac{y-t^2}{t} + 2t = \frac{3y}{t} - t$.

Section 1.2

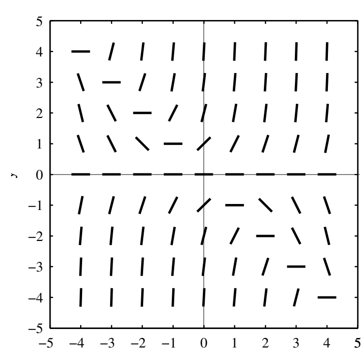
1. $$y' = t$$



2. $$y' = y^2$$



3. $$y' = y(t + t)$$



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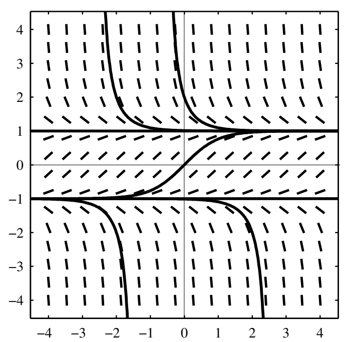
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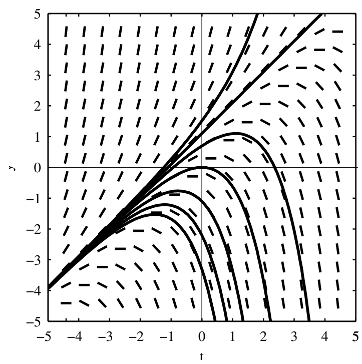
5.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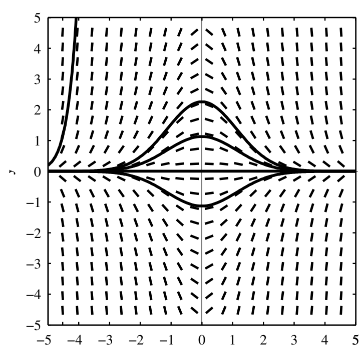
−5 −4 −3 −2 −1 0 1 2 3 4 5
t

6.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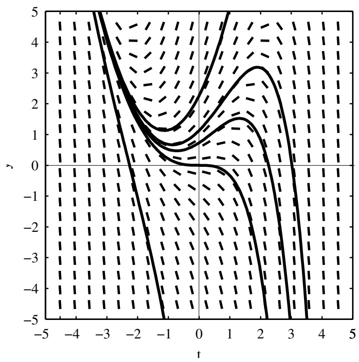
t

7.



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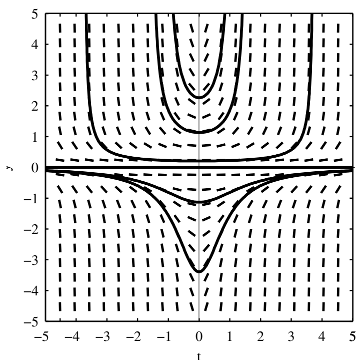

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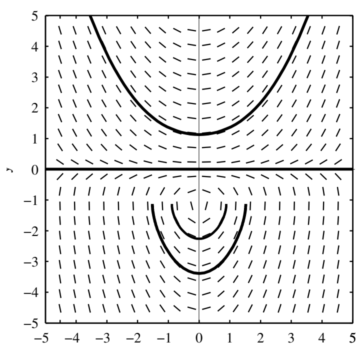

10. We set $y^2 = 0$ and see that $y = 0$ is the only constant (= equilibrium) solution.

11. We set $y(y + t) = 0$. We look for constant solutions to $y(y + t) = 0$, and we see that $y = 0$ is the only constant (= equilibrium) solution.

12. The equation $y - t = 0$ has no constant solution. Thus, there are no equilibrium solutions.

13. The equation $1 - y^2 = 0$ has two constant solutions: $y = 1$ and $y = -1$

14. We substitute $y = at + b$ into $y' = y - t$ to get $a = (a-1)t + b$. Equality for all t forces $a - 1 = 0$ and $a = b$. Thus $a = 1$ and $b = 1$ and the only linear solution is $y = t + 1$.

15. We substitute $y = at + b$ into $y' = \cos(t + y)$ to get $a = \cos((a+1)t + b)$. Equality for all t means that $\cos((a+1)t+b)$ must be a constant function, which can occur only if the coefficient of t is 0. This forces $a = -1$ leaving us with the equation $-1 = \cos b$. This implies $b = (2n+1)\pi$, where n is an integer. Hence $y = -t + (2n + 1)\pi$, $n \in \mathbb{Z}$ is a family of linear solutions.

Section 1.3

1. separable; $h(t) = 1$ and $g(y) = 2y(5 - y)$

2. In standard form we get $y' = (1 - y)/y$. This is separable; $h(t) = 1$ and $g(y) = (1 - y)/y$.

3. First write in standard form: $y' = \frac{1-2ty}{t^2}$. We cannot write $\frac{1-2ty}{t^2}$ as a product of a function of t and a function of y. It is not separable.

4. In standard form we get $y' = y(y - t)$. We cannot write $y(y - t)$ as a product of a function of t and a function of y. It is not separable.

5. Write in standard form to get: $y' = (y - 2yt)/y$. Here we can write $(y - 2ty)/y = 1 - 2t$. It is separable; $h(t) = 1 - 2t$ and $g(y) = 1$.

6. We can factor to get $y' = y^2(t-1) + t - 1 = (y^2+1)(t-1)$. It is separable; $h(t) = t - 1$ and $g(y) = y^2 + 1$.

7. In standard form we get $y' = \frac{-2ty}{t^2+3y^2}$. We cannot write $y' = \frac{-2ty}{t^2+3y^2}$ as a product of a function of t and a function of y. It is not separable

8. It is not separable as $t^2 + y^2$ cannot be written as a product of a function of t and a function of y.

9. In standard form we get: $y' = e^{-t}(y^3 - y)$ It is separable; $h(t) = e^{-t}$ and $g(y) = y^3 - y$

10. The variables are already separated, so integrate both sides of the equation to get $y^2/2 = t^2/2 + c$, which we can rewrite as $y^2 - t^2 = k$ where $k = 2c \in \mathbb{R}$ is a constant. Since $y(2) = -1$, substitute $t = 2$ and $y = -1$ to get that $k = (-1)^2 - 2^2 = -3$. Thus the solution is given implicitly by the equation $y^2 - t^2 = -3$ or we can solve explicitly to get $y = -\sqrt{t^2 - 3}$, where the negative square root is used since $y(2) = -1 < 0$.

11. In standard form we get $y' = \frac{1-y^2}{ty}$. Clearly, $y = \pm 1$ are equilibrium solutions. Separating the variables gives

$$\frac{y}{1-y^2}\,dy = \frac{1}{t}\,dt.$$

Integrating both sides of this equation (using the substitution $u = 1 - y^2$, $du = -2y\,dy$ for the integral on the left) gives

$$-\frac{1}{2}\ln|1-y^2| = \ln|t| + c.$$

Multiplying by -2, taking the exponential of both sides, and removing the absolute values gives $1 - y^2 = kt^{-2}$ where k is a nonzero constant. However, when $k = 0$ the equation becomes $1 - y^2 = 0$ and hence $y = \pm 1$. By considering an arbitrary constant (which we will call c), the implicit equation $t^2(1 - y^2) = c$ includes the two equilibrium solutions for $c = 0$.

12. The variables are already separated, so integrate both sides to get $y^4/4 = t^2/2 + c$, c a real constant. This can be simplified to $y^4 = 2t^2 + c$. (where we replace $4c$ by c) We leave the answer in implicit form.

13. The variables are already separated, so integrate both sides to get $y^5/5 = t^2/2 + 2t + c$, c a real constant. Simplifying gives $y^5 = \frac{5}{2}t^2 + 10t + c$. We leave the answer in implicit form

14. There is an equilibrium solution $y = 0$. Separating variables give $y^{-2}y' = t$ and integrating gives $-y^{-1} = t^2/2 + c$. Thus $y = -2/(t^2 + 2c)$, c a real constant. This is equivalent to writing $y = -2/(t^2 + c)$, c a real constant, since twice an arbitrary constant is still an arbitrary constant.

15. In standard form we get $y' = (1 - y)\tan t$ so $y = 1$ is a solution. Separating variables gives $\frac{dy}{1-y} = \tan t\; dt$. The function $\tan t$ is continuous on the interval $(-\pi/2, \pi/2)$ and so has an antiderivative. Integration gives $-\ln|1 - y| = -\ln|\cos t| + k_1$. Multiplying by -1 and exponentiating gives $|1 - y| = k_2\,|\cos t|$ where k_2 is a positive constant. Removing the absolute value signs gives $1 - y = k_3 \cos t$, with $k_3 \neq 0$. If we allow $k_3 = 0$ we get the equilibrium solution $y = 1$. Thus the solution can be written $y = 1 - c\cos t$, c any real constant.

16. An equilibrium solution is $y = 0$. Separating variables gives $y^{-n}\,dy = t^m\,dt$ and integrating gives $\frac{y^{1-n}}{1-n} = \frac{t^{m+1}}{m+1} + c$, c a real constant. Simplifying gives $y^{1-n} = \frac{1-n}{m+1}t^{m+1} + c$, and the equilibrium solution $y = 0$.

17. There are two equilibrium solutions; $y = 0$ and $y = 4$. Separating variables and using partial fractions gives $\frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right) dy = dt$. Integrating and simplifying gives $\ln\left|\frac{y}{4-y}\right| = 4t + k_1$ which is equivalent to $\frac{y}{4-y} = ce^{4t}$, c a nonzero constant. Solving for y gives $y = \frac{4ce^{4t}}{1+ce^{4t}}$. When $c = 0$ we get the equilibrium solution $y = 0$. However, there is no c which gives the other equilibrium solution $y = 4$.

18. There are no equilibrium solutions. Separating variables gives $\frac{y}{y^2+1}\,dy = dt$ and integrating gives $\frac{1}{2}\ln(y^2 + 1) = t + k$. Solving for y^2 gives $y^2 = ce^{2t} - 1$, where $c > 0$.

19. Separating variables gives $\frac{dy}{y^2+1} = dt$ and integrating gives $\tan^{-1} y = t+c$. Thus $y = \tan(t + c)$, c a real constant.

20. Separating variables gives $y\,dy = \left(\frac{-1}{t} - t\right) dt$ and integrating gives $\frac{y^2}{2} = -\ln|t| - \frac{t^2}{2} + c$. Simplifying gives $y^2 + t^2 + \ln t^2 = c$, c a real constant.

21. In standard form we get $y' = \frac{-(y+1)}{y-1}\frac{1}{1+t^2}$ from which we see that $y = -1$ is an equilibrium solution. Separating variables and simplifying gives $\left(\frac{2}{y+1} - 1\right) dy = \frac{dt}{t^2+1}$. Integrating and simplifying gives $\ln(y + 1)^2 - y = \tan^{-1} t + c$.

22. Separating variables gives $2y\,dy = e^t\,dt$ and integrating gives $y^2 = e^t + c$, c a constant.

23. The equilibrium solution is $y = 0$. Separating variables gives $y^{-2}\,dy = \frac{dt}{1-t}$. Integrating and simplifying gives $y = \frac{1}{\ln|1-t|+c}$, c real constant.

24. In standard form we get $y' = y(y+1)$ from which we see $y = 0$ and $y = -1$ are equilibrium solutions. The equilibrium solution $y(t) = 0$ satisfies the initial condition $y(0) = 0$ so $y(t) = 0$ is the required solution.

25. $y = 0$ is the only equilibrium solution. The equilibrium solution $y(t) = 0$ satisfies the initial condition $y(1) = 0$ so $y(t) = 0$ is the required solution.

26. Rewriting we get $y' = \frac{dy}{dx} = \frac{x+2}{x}y$ from which we see that $y = 0$ is an equilibrium solution. Separating variables gives $\frac{dy}{y} = \left(1 + \frac{2}{x}\right) dx$ and integrating gives $\ln|y| = x + \ln x^2 + k$, k a constant. Solving for y by taking the exponential of both sides gives $y = cx^2e^x$, and allowing $c = 0$ gives the equilibrium solution. The initial condition gives $e = y(1) = ce$ so $c = 1$. Thus $y = x^2e^x$.

27. In standard form we get $y' = -2ty$ so $y = 0$ is a solution. Separating variables and integrating gives $\ln|y| = -t^2 + k$. Solving for y gives $y = ce^{-t^2}$

and allowing $c = 0$ gives the equilibrium solution. The initial condition implies $4 = y(0) = ce^0 = c$. Thus $y = 4e^{-t^2}$.

28. Since $\cot y = 0$ at $y = \frac{\pi}{2} + m\pi$ for all integers m we have equilibrium lines at $y = \frac{\pi}{2} + m\pi$, none of which satisfy the initial condition $y(1) = \frac{\pi}{4}$. Separating variables gives $\tan y\, dy = \frac{dt}{t}$ and integrating gives $-\ln|\cos y| = \ln t + c$. We can solve for c here using the initial condition: we get $c = -\ln\cos\frac{\pi}{4} = -\ln\left(\frac{\sqrt{2}}{2}\right)$. Solving for y gives $y = \cos^{-1}\frac{1}{\sqrt{2}t}$

29. Separating variables gives $\frac{dy}{y} = \frac{u}{u^2+1}\, du$ and integrating gives $\ln|y| = \ln\sqrt{u^2+1} + k$. Solving for y gives $y = c\sqrt{u^2+1}$, for $c \neq 0$. The initial condition gives $2 = y(0) = c$. So $y = 2\sqrt{u^2+1}$.

30. In standard form we get $y' = \frac{t}{t+2}y$ so $y = 0$ is an equilibrium solution. Separating variables gives $\frac{dy}{y} = \left(1 - \frac{2}{t+2}\right) dt$. Integrating we get $\ln|y| = t - 2\ln|t+2| + k$. Solving for y we get $y = c\frac{e^t}{(t+2)^2}$, for $c \neq 0$. However, allowing $c = 0$ gives the equilibrium solution.

31. We assume the decay model $N(t) = N(0)e^{-\lambda t}$. If t is the age of the bone then $N(t) = \frac{1}{3}N(0)$. Thus $\frac{1}{3} = e^{-\lambda t}$. Solving for t gives $t = \frac{\ln 3}{\lambda} = \frac{5730\ln 3}{\ln 2} \approx 9082$ years

32. Let m denote the number of Argon-40 atoms in the sample. Then $8m$ is the number of Potassium-40 atoms. Let t be the age of the rock. Then t years ago there were $m + 8m = 9m$ atoms of Potassium-40. Hence $N(0) = 9m$. On the other hand, $8m = N(t) = N(0)e^{-\lambda t} = 9me^{-\lambda t}$. This implies that $\frac{8}{9} = e^{-\lambda t}$ and hence $t = \frac{-\ln\frac{8}{9}}{\lambda} = \frac{-\tau}{\ln 2}\ln\frac{8}{9} \approx 212$ million years old.

33. We need only solve $.3N(0) = N(0)e^{-\lambda t}$ for t. We get $t = -\frac{\ln .3}{\lambda} = -\frac{5.27\ln .3}{\ln 2} = 9.15$ years.

34. The ambient temperature is $32°$ F, the temperature of the ice water. From Equation (12) we get $T(t) = 32 + ke^{rt}$. At $t = 0$ we get $70 = 32 + k$, so $k = 38$ and $T(t) = 32 + 38e^{rt}$. After 30 minutes we have $55 = T(30) = 32 + 38e^{30r}$ and solving for r gives $r = \frac{1}{30}\ln\frac{23}{38}$. To find the time t when $T(t) = 45$ we solve $45 = 32 + 38e^{rt}$, with r as above. We get $t = 30\frac{\ln 13 - \ln 38}{\ln 23 - \ln 38} \approx 64$ minutes.

35. The ambient temperature is $T_a = 70°$. Equation (12) gives $T(t) = 70 + ke^{rt}$ for the temperature of the coffee at time t. Since the initial temperature of the coffee is $T(0) = 180$ we get $180 = T(0) = 70 + k$. Thus $k = 110$. The constant r is determined from the temperature at a second time: $140 = T(3) = 70 + 110e^{3r}$ so $r = \frac{1}{3}\ln\frac{7}{11} \approx -.1507$. Thus

$T(t) = 70 + 110e^{rt}$, with r as calculated. The temperature requested is $T(5) = 70 + 110\left(\frac{7}{11}\right)^{\frac{5}{3}} \approx 121.8°$.

36. The ambient temperature is $T_a = 65°$. Equation (12) gives $T(t) = 65 + ke^{rt}$ for the temperature at time t. Since the initial temperature of the thermometer is $T(0) = 90$ we get $90 = T(0) = 65 + k$. Thus $k = 25$. The constant r is determined from the temperature at a second time: $85 = T(2) = 65 + 25e^{2r}$ so $r = \frac{1}{2}\ln\frac{4}{5}$. Thus $T(t) = 65 + 25e^{rt}$, with $r = \frac{1}{2}\ln\frac{4}{5}$. To answer the first question we solve the equation $75 = T(t) = 65 + 25e^{rt}$ for t. We get $t = 2\frac{\ln 2 - \ln 5}{\ln 4 - \ln 5} \approx 8.2$ minutes. The temperature at $t = 20$ is $T(20) = 65 + 25\left(\frac{4}{5}\right)^{10} \approx 67.7°$.

37. The ambient temperature is $T_a = 70°$. Equation (12) gives $T(t) = 70 + ke^{rt}$ for the temperature of the soda at time t. Since the initial temperature of the soda is $T(0) = 40$ we get $40 = T(0) = 70 + k$. Thus $k = -30$. The constant r is determined from the temperature at a second time: $60 = T(2) = 70 - 30e^{2r}$ so $r = \frac{1}{2}\ln\frac{1}{3}$. Thus $T(t) = 70 - 30e^{rt}$, with $r = \frac{1}{2}\ln\frac{1}{3}$. The temperature at $t = 1$ is $T(1) = 70 - 30e^{\frac{1}{2}\ln\frac{1}{3}} = 70 - \frac{30}{\sqrt{3}} \approx 52.7°$.

38. The ambient temperature is $T_a = 70°$. Equation (12) gives $T(t) = 70 + ke^{rt}$ for the temperature of the coffee at time t. We are asked to determine the initial temperature of the coffee so $T(0)$ is unknown. However, we have the equations

$$\begin{aligned} 150 &= T(5) = 70 + ke^{5r} \\ 142 &= T(6) = 70 + ke^{6r} \end{aligned}$$

or

$$\begin{aligned} 80 &= ke^{5r} \\ 72 &= ke^{4r}. \end{aligned}$$

Dividing the second equation by the first gives $\frac{72}{80} = e^r$ so $r = \ln 0.9$. From the first equation we get $k = 80e^{-5r} \approx 135.5$. We now calculate $T(0) = 70 + k \approx 205.5°$

39. The ambient temperature is $T_a = 40°$. Equation (12) gives $T(t) = 40 + ke^{rt}$ for the temperature of the beer at time t. Since the initial temperature of the beer is $T(0) = 80$ we get $80 = T(0) = 40 + k$. Thus $k = 40$. The constant r is determined from the temperature at a second time: $60 = T(1) = 40 + 40e^r$ so $r = -\ln 2$. Thus $T(t) = 40 + 40e^{rt}$, with $r = -\ln 2$. We now solve the equation $50 = T(t) = 40 + 40e^{rt}$ for t and get $t = \frac{-\ln 4}{-\ln 2} = 2$. She should therefore put the beer in the refrigerator at 2 p.m.

40. Let us start time $t = 0$ at 1980. Then $P(0) = 290$. The Malthusian growth model gives $P(t) = 290e^{rt}$. At $t = 10$ (1990) we have $370 = 290e^{10r}$

and hence $r = \frac{1}{10}\ln\frac{37}{29}$. At $t = 30$ (2010) we have $P(30) = 290e^{30r} = 290\left(\frac{37}{29}\right)^3 \approx 602$.

41. The initial population is $40 = P(0)$. Since the population doubles in 3 hours we have $P(3) = 80$ or $80 = 40e^{3r}$. Hence $r = \frac{\ln 2}{3}$. Now we can compute the population after 30 hours: $P(30) = 40e^{30r} = 40(2^{10}) = 40,960$.

42. We have $3P(0) = P(5) = P(0)e^{3r}$. So $r = \frac{\ln 3}{5}$. Now we solve the equation $2P(0) = P(t) = P(0)e^{rt}$ for t. We get $t = \frac{\ln 2}{r} = \frac{5\ln 2}{\ln 3} \approx 3.15$ years.

43. In the logistic growth equation $m = 800$ and $P(0) = 290$. Thus $P(t) = \frac{800\cdot 290}{290+510e^{-rt}}$. To determine r we use $P(10) = 370$ to get $370 = \frac{800\cdot 290}{290+510e^{-10r}}$. A simple calculation give $r = \frac{1}{10}\ln\frac{1887}{1247}$. Now the population in 2010 is $P(30) = \frac{800\cdot 290}{290+510\left(\frac{1247}{1887}\right)^3} \approx 530$

44. In the logistics equation $m = 5000$ and $P_0 = 2000$. Thus $P(t) = \frac{10{,}000{,}000}{2{,}000+3{,}000e^{-rt}} = \frac{10{,}000}{2+3e^{-rt}}$. Since $P(2) = 3000$ we get $3000 = \frac{10{,}000}{2+3e^{-rt}}$. Solving this equation for r gives $r = \ln\frac{3}{2}$. Now $P(4) = \frac{10{,}000}{2+3e^{-4r}} = \frac{10{,}000}{2+3\left(\frac{2}{3}\right)^4} \approx 3857$

45. Let $x = e^{-rt_0}$. Then $x^2 = e^{-2rt}$. The equation $P(t_0) = P_1$ implies that $x = \frac{P_0(m-P_1)}{P_1(m-P_0)}$. The equation $P(2t_0) = P_2$ implies $x^2 = \frac{P_0(m-P_2)}{P_2(m-P_0)}$. These equation together imply $\frac{P_0^2(m-P_1)^2}{P_1^2(m-P_0)^2} = \frac{P_0(m-P_2)}{P_2(m-P_0)}$. Cross multiplying and simplifying leads to $(P_0P_2-P_1^2)m+(P_1^2P_0+P_1^2P_2-2P_0P_1P_2) = 0$. Solving for m gives the result. Now replace the formula for m into $e^{-rt_0} = x = \frac{P_0(m-P_1)}{P_1(m-P_0)}$. Simplifying gives $e^{-rt_0} = \frac{P_0}{P_2}\frac{P_2-P_1}{P_1-P_0}$. The formula for r follows after taking the natural log of both sides.

46. We have $P(0) = P_0 = 400$, $P(3) = P_1 = 700$, and $P(6) = P_2 = 1000$. Using the result of the previous problem we get $m = \frac{700(700(400+1000)-2\cdot 400\cdot 1000)}{(700)^2-400\cdot 1000} = 1,400$

Section 1.4

1. This equation is already in standard form with $p(t) = 3$. An antiderivative of $p(t)$ is $P(t) = \int 3\,dt = 3t$ so the integrating factor is $\mu(t) = e^{3t}$. If we multiply the differential equation $y' + 3y = e^t$ by $\mu(t)$, we get the equation

$$e^{3t}y' + 3e^{3t}y = e^{4t},$$

and the left hand side of this equation is a perfect derivative, namely, $(e^{3t}y)'$. Thus, $(e^{3t}y)' = e^{4t}$. Now take antiderivatives of both sides and multiply by e^{-3t}. This gives

$$y = \frac{1}{4}e^t + ce^{-3t}$$

for the general solution of the equation. To find the constant c to satisfy the initial condition $y(0) = -2$, substitute $t = 0$ into the general solution to get $-2 = y(0) = \frac{1}{4} + c$. Hence $c = -\frac{9}{4}$, and the solution of the initial value problem is

$$y = \frac{1}{4}e^t - \frac{9}{4}e^{-3t}.$$

2. Divide by $\cos t$ to put the equation in the standard form

$$y' + (\tan t)y = \sec t.$$

In this case $p(t) = \tan t$, an antiderivative is $P(t) = \ln(\sec t)$, and the integrating factor is $\mu(t) = \sec t$. (We do not need $|\sec t|$ since we are working near $t = 0$ where $\sec t > 0$.) Now multiply by the integrating factor to get $(\sec t)y' + (\sec t \tan t)y = \sec^2 t$, the left hand side of which is a perfect derivative. Thus $((\sec t)y)' = \sec^2 t$ and taking antiderivatives of both sides gives $(\sec t)y = \tan t + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by $1/\sec t = \cos t$ to get $y = \sin t + c\cos t$ for the general solution. Letting $t = 0$ gives $5 = y(0) = \sin 0 + c\cos 0 = c$ so $c = 5$ and

$$y = \sin t + 5\cos t.$$

3. This equation is already in standard form. In this case $p(t) = -2$, an antiderivative is $P(t) = -2t$, and the integrating factor is $\mu(t) = e^{-2t}$. Now multiply by the integrating factor to get

$$e^{-2t}y' - 2e^{-2t}y = 1,$$

the left hand side of which is a perfect derivative $((e^{-2t})y)'$. Thus $((e^{-2t})y)' = 1$ and taking antiderivatives of both sides gives

$$(e^{-2t})y = t + c,$$

where $c \in \mathbb{R}$ is a constant. Now multiply by e^{2t} to get $y = te^{2t} + ce^{2t}$ for the general solution. Letting $t = 0$ gives $4 = y(0) = c$ so

$$y = te^{2t} + 4e^{2t}.$$

4. Divide by t to put the equation in the standard form

$$y' + \frac{1}{t}y = \frac{e^t}{t}$$

In this case $p(t) = 1/t$, an antiderivative is $P(t) = \ln t$, and the integrating factor is $\mu(t) = t$. Now multiply the standard form equation by the integrating factor to get $ty' + y = e^t$, the left hand side of which is a perfect derivative $(ty)'$. (Note that this is just the original left hand side of the equation. Thus if we had recognized that the left hand side was already a perfect derivative, the preliminary steps could have been skipped for this problem, and we could have proceeded directly to the next step.) Thus the equation can be written as $(ty)' = e^t$ and taking antiderivatives of both sides gives $ty = e^t + c$ where $c \in \mathbb{R}$ is a constant. Now divide by t to get

$$y = \frac{e^t}{t} + \frac{c}{t}$$

for the general solution.

5. The general solution from Problem 4 is $y = \frac{e^t}{t} + \frac{c}{t}$. Now let $t = 1$ to get $0 = e + c$. So $c = -e$ and $y = \frac{e^t}{t} - \frac{e}{t}$.

6. Divide by t to put the equation in the standard form

$$y' + \frac{m}{t}y = \ln t.$$

In this case $p(t) = \frac{m}{t}$, an antiderivative is $P(t) = m \ln t = \ln t^m$, and the integrating factor is $\mu(t) = t^m$. Now multiply the standard form equation by the integrating factor to get $t^m y' + mt^{m-1}y = t^m \ln t$, the left hand side of which is a perfect derivative $((t^m)y)'$. Thus $((t^m)y)' = t^m \ln t$. To integrate $t^m \ln t$ we consider the cases $m = -1$ and $m \neq -1$ separately.

Case $m = -1$**:** A simple substitution gives $\int t^{-1} \ln t \, dt = \frac{(\ln t)^2}{2} + c$. Hence, $t^{-1}y = \frac{(\ln t)^2}{2} + c$ and so $y = \frac{t(\ln t)^2}{2} + ct$

Case $m \neq -1$**:** Use integration by parts to get $\int t^m \ln t \, dt = \frac{t^{m+1} \ln t}{m+1} - \frac{t^{m+1}}{(m+1)^2} + c$. Then $y = \frac{t \ln t}{m+1} - \frac{t}{(m+1)^2} + \frac{c}{t^m}$.

7. We first put the equation in standard form and get

$$y' + \frac{1}{t}y = \cos(t^2).$$

In this case $p(t) = \frac{1}{t}$, an antiderivative is $P(t) = \ln t$, and the integrating factor is $\mu(t) = t$. Now multiply by the integrating factor to get

$$ty' + y = t\cos(t^2),$$

the left hand side of which is a perfect derivative $(ty)'$. Thus $(ty)' = t\cos(t^2)$ and taking antiderivatives of both sides gives $ty = \frac{1}{2}\sin(t^2) + c$ where $c \in \mathbb{R}$ is a constant. Now divide by t to get $y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$. for the general solution.

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8. In this case $p(t) = 2$ and the integrating factor is $e^{\int 2\,dt} = e^{2t}$. Now multiply to get $e^{2t}y' + 2e^{2t}y = e^{2t}\sin t$, which simplifies to $(e^{2t}y)' =$

$e^{2t}\sin t$. Now integrate both sides to get $e^{2t}y = \frac{1}{5}(-\cos t + 2\sin t)e^{2t} + c$, where we computed $\int e^{2t}\sin t$ by parts two times. Dividing by e^{2t} gives $y = \frac{1}{5}(2\sin t - \cos t) + ce^{-2t}$.

9. In this case $p(t) = -3$ and the integrating factor is $e^{\int -3\,dt} = e^{-3t}$. Now multiply to get $e^{-3t}y' + 2e^{-3t}y = 25e^{-3t}\cos 4t$, which simplifies to $(e^{-3t}y)' = 25e^{-3t}\cos 4t$. Now integrate both sides to get $e^{-3t}y = (4\sin 4t - 3\cos 4t)e^{-3t} + c$, where we computed $\int 25e^{-3t}\cos 4t$ by parts twice. Dividing by e^{-3t} gives $y = 4\sin 4t - 3\cos 4t + ce^{3t}$.

10. In standard form this equation becomes

$$y' - \frac{1}{t(t+1)}y = \frac{2}{t(t+1)}.$$

Using partial fractions we get $p(t) = \frac{-1}{t(t+1)} = \frac{1}{t+1} - \frac{1}{t}$, an antiderivative is $P(t) = \ln(t+1) - \ln t = \ln\left(\frac{t+1}{t}\right)$, and the integrating factor is $\mu(t) = \frac{t+1}{t}$. Now multiply by the integrating factor to get

$$\frac{t+1}{t}y' - \frac{1}{t^2}y = \frac{2}{t^2},$$

the left hand side of which is a perfect derivative $(\frac{t+1}{t}y)'$. Thus

$$(\frac{t+1}{t}y)' = \frac{2}{t^2}$$

and taking antiderivatives of both sides gives $\frac{t+1}{t}y = \frac{-2}{t} + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by $\frac{t}{t+1}$ to get $y = \frac{-2}{t+1} + \frac{ct}{t+1} = \frac{ct-2}{t+1}$ for the general solution.

11. In standard form we get $z' - 2tz = -2t^3$. An integrating factor is $e^{\int -2t\,dt} = e^{-t^2}$. Thus $(e^{-t^2}z)' = -2t^3e^{-t^2}$. Integrating both sides gives $e^{-t^2}z = (t^2+1)e^{-t^2} + c$, where the integral of the right hand side is done by parts. Now divide by the integrating factor e^{-t^2} to get $z = t^2 + 1 + ce^{t^2}$.

12. The given differential equation is in standard form, $p(t) = a$, an antiderivative is $P(t) = at$, and the integrating factor is $\mu(t) = e^{at}$. Now multiply by the integrating factor to get

$$e^{at}y' + ae^{at}y = be^{at},$$

the left hand side of which is a perfect derivative $((e^{at})y)'$. Thus

$$((e^{at})y)' = be^{at}.$$

If $a \neq 0$ then taking antiderivatives of both sides gives $e^{at}y = \frac{b}{a}e^{at} + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by e^{-at} to get $y = \frac{b}{a} + ce^{-at}$ for the general solution. In the case $a = 0$ then $y' = b$ and $y = bt + c$.