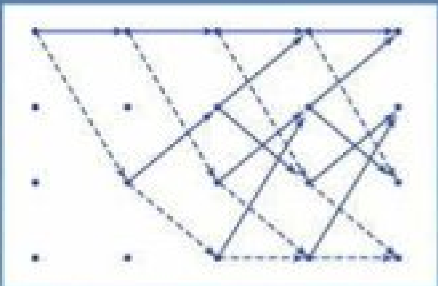
COMMUNICATION SYSTEMS ENGINEERING

SECOND EDITION



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SOLUTIONS MANUAL

Communication Systems Engineering

Second Edition

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Chapter 2

Problem 2.1
1)

$$\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \left(x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right) \left(x^*(t) - \sum_{j=1}^{N} \alpha_j^* \phi_j^*(t) \right) dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt \\
&\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t) \phi_j^* dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^{N} |\alpha_i|^2 - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt
\end{aligned}$$

Completing the square in terms of α_i we obtain

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 + \sum_{i=1}^{N} \left| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2$$

The first two terms are independent of α's and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt$$

which causes the last term to vanish.

2) With this choice of α_i's

$$\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2
\end{aligned}$$

Problem 2.2
1) The signal $x_1(t)$ is periodic with period $T_0 = 2$. Thus

$$\begin{aligned}
x_{1,n} &= \frac{1}{2} \int_{-1}^{1} \Lambda(t) e^{-j2\pi \frac{n}{2} t} dt = \frac{1}{2} \int_{-1}^{1} \Lambda(t) e^{-j\pi n t} dt \\
&= \frac{1}{2} \int_{-1}^{0} (t+1) e^{-j\pi n t} dt + \frac{1}{2} \int_{0}^{1} (-t+1) e^{-j\pi n t} dt \\
&= \frac{1}{2} \left(\frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} \right) \Bigg|_{-1}^{0} + \frac{j}{2\pi n} e^{-j\pi n t} \Bigg|_{-1}^{0} \\
&\quad - \frac{1}{2} \left(\frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} \right) \Bigg|_{0}^{1} + \frac{j}{2\pi n} e^{-j\pi n t} \Bigg|_{0}^{1} \\
&\quad \frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2} (e^{j\pi n} + e^{-j\pi n}) = \frac{1}{\pi^2 n^2} (1 - \cos(\pi n))
\end{aligned}$$

When $n=0$ then

$$x_{1,0} = \frac{1}{2}\int_{-1}^{1} \Lambda(t)dt = \frac{1}{2}$$

Thus

$$x_1(t) = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}(1-\cos(\pi n))\cos(\pi n t)$$

2) $x_2(t) = 1$. It follows then that $x_{2,0} = 1$ and $x_{2,n} = 0$, $\forall n \neq 0$.

3) The signal is periodic with period $T_0 = 1$. Thus

$$\begin{aligned}
x_{3,n} &= \frac{1}{T_0}\int_0^{T_0} e^t e^{-j2\pi n t}dt = \int_0^1 e^{(-j2\pi n+1)t}dt \\
&= \frac{1}{-j2\pi n+1}e^{(-j2\pi n+1)t}\bigg|_0^1 = \frac{e^{(-j2\pi n+1)}-1}{-j2\pi n+1} \\
&= \frac{e-1}{1-j2\pi n} = \frac{e-1}{\sqrt{1+4\pi^2 n^2}}(1+j2\pi n)
\end{aligned}$$

4) The signal $\cos(t)$ is periodic with period $T_1 = 2\pi$ whereas $\cos(2.5t)$ is periodic with period $T_2 = 0.8\pi$. It follows then that $\cos(t)+\cos(2.5t)$ is periodic with period $T = 4\pi$. The trigonometric Fourier series of the even signal $\cos(t)+\cos(2.5t)$ is

$$\begin{aligned}
\cos(t)+\cos(2.5t) &= \sum_{n=1}^{\infty} \alpha_n \cos(2\pi\frac{n}{T_0}t) \\
&= \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n}{2}t)
\end{aligned}$$

By equating the coefficients of $\cos(\frac{n}{2}t)$ of both sides we observe that $a_n = 0$ for all n unless $n = 2, 5$ in which case $a_2 = a_5 = 1$. Hence $x_{4,2} = x_{4,5} = \frac{1}{2}$ and $x_{4,n} = 0$ for all other values of n.

5) The signal $x_5(t)$ is periodic with period $T_0 = 1$. For $n = 0$

$$x_{5,0} = \int_0^1 (-t+1)dt = (-\frac{1}{2}t^2 + t)\bigg|_0^1 = \frac{1}{2}$$

For $n \neq 0$

$$\begin{aligned}
x_{5,n} &= \int_0^1 (-t+1)e^{-j2\pi n t}dt \\
&= -\left(\frac{j}{2\pi n}te^{-j2\pi n t} + \frac{1}{4\pi^2 n^2}e^{-j2\pi n t}\right)\bigg|_0^1 + \frac{j}{2\pi n}e^{-j2\pi n t}\bigg|_0^1 \\
&= -\frac{j}{2\pi n}
\end{aligned}$$

Thus,

$$x_5(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin 2\pi n t$$

6) The signal $x_6(t)$ is periodic with period $T_0 = 2T$. We can write $x_6(t)$ as

$$x_6(t) = \sum_{n=-\infty}^{\infty} \delta(t-n2T) - \sum_{n=-\infty}^{\infty} \delta(t-T-n2T)$$

$$
\begin{aligned}
&= \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{j\pi\frac{n}{T}t} - \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{j\pi\frac{n}{T}(t-T)} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2T}(1-e^{-j\pi n})e^{j2\pi\frac{n}{2T}t}
\end{aligned}
$$

However, this is the Fourier series expansion of $x_6(t)$ and we identify $x_{6,n}$ as

$$
x_{6,n} = \frac{1}{2T}(1-e^{-j\pi n}) = \frac{1}{2T}(1-(-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{T} & n \text{ odd} \end{cases}
$$

7) The signal is periodic with period T. Thus,

$$
\begin{aligned}
x_{7,n} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta'(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}(-1)\frac{d}{dt}e^{-j2\pi\frac{n}{T}t}\bigg|_{t=0} = \frac{j2\pi n}{T^2}
\end{aligned}
$$

8) The signal $x_8(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{8,n} = a_{8,n}/2$ or

$$
\begin{aligned}
x_{8,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2 f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\
&= \frac{1}{2\pi(1+2n)}\sin(2\pi f_0(1+2n)t)|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)}\sin(2\pi f_0(1-2n)t)|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]
\end{aligned}
$$

9) The signal $x_9(t) = \cos(2\pi f_0 t) + |\cos(2\pi f_0 t)|$ is even and periodic with period $T_0 = 1/f_0$. It is equal to $2\cos(2\pi f_0 t)$ in the interval $[-\frac{1}{4f_0}, \frac{1}{4f_0}]$ and zero in the interval $[\frac{1}{4f_0}, \frac{3}{4f_0}]$. Thus

$$
\begin{aligned}
x_{9,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-n)t)dt \\
&= \frac{1}{2\pi(1+n)}\sin(2\pi f_0(1+n)t)|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-n)}\sin(2\pi f_0(1-n)t)|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{1}{\pi(1+n)}\sin(\frac{\pi}{2}(1+n)) + \frac{1}{\pi(1-n)}\sin(\frac{\pi}{2}(1-n))
\end{aligned}
$$

Thus $x_{9,n}$ is zero for odd values of n unless $n = \pm 1$ in which case $x_{9,\pm 1} = \frac{1}{2}$. When n is even $(n = 2l)$ then

$$
x_{9,2l} = \frac{(-1)^l}{\pi}\left[\frac{1}{1+2l} + \frac{1}{1-2l}\right]
$$

Problem 2.3

It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(2\pi\frac{n}{T_0}t) + b_n \sin(2\pi\frac{n}{T_0}t)\right]$$

The even part of $x(t)$ is

$$\begin{aligned}
x_e(t) &= \frac{x(t)+x(-t)}{2} \\
&= \frac{1}{2}\left(a_0 + \sum_{n=1}^{\infty} a_n(\cos(2\pi\frac{n}{T_0}t) + \cos(-2\pi\frac{n}{T_0}t))\right. \\
&\quad \left. +b_n(\sin(2\pi\frac{n}{T_0}t) + \sin(-2\pi\frac{n}{T_0}t))\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi\frac{n}{T_0}t)
\end{aligned}$$

The last is true since $\cos(\theta)$ is even so that $\cos(\theta)+\cos(-\theta) = 2\cos\theta$ whereas the oddness of $\sin(\theta)$ provides $\sin(\theta) + \sin(-\theta) = \sin(\theta) - \sin(\theta) = 0$.
The odd part of $x(t)$ is

$$\begin{aligned}
x_o(t) &= \frac{x(t)-x(-t)}{2} \\
&- \sum_{n=1}^{\infty} b_n \sin(2\pi\frac{n}{T_0}t)
\end{aligned}$$

Problem 2.4

a) The signal is periodic with period T. Thus

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T e^{-t}e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^T e^{-(j2\pi\frac{n}{T}+1)t}dt \\
&= -\frac{1}{T\left(j2\pi\frac{n}{T}+1\right)}e^{-(j2\pi\frac{n}{T}+1)t}\Big|_0^T = -\frac{1}{j2\pi n+T}\left[e^{-(j2\pi n+T)}-1\right] \\
&= \frac{1}{j2\pi n+T}[1-e^{-T}] = \frac{T-j2\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]
\end{aligned}$$

If we write $x_n = \frac{a_n - jb_n}{2}$ we obtain the trigonometric Fourier series expansion coefficients as

$$a_n = \frac{2T}{T^2+4\pi^2n^2}[1-e^{-T}], \qquad b_n = \frac{4\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]$$

b) The signal is periodic with period $2T$. Since the signal is odd we obtain $x_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{2T}\int_{-T}^{T} x(t)e^{-j2\pi\frac{n}{2T}t}dt = \frac{1}{2T}\int_{-T}^{T}\frac{t}{T}e^{-j2\pi\frac{n}{2T}t}dt \\
&= \frac{1}{2T^2}\int_{-T}^{T} te^{-j\pi\frac{n}{T}t}dt \\
&= \frac{1}{2T^2}\left(\frac{jT}{\pi n}te^{-j\pi\frac{n}{T}t} + \frac{T^2}{\pi^2n^2}e^{-j\pi\frac{n}{T}t}\right)\Big|_{-T}^{T} \\
&= \frac{1}{2T^2}\left[\frac{jT^2}{\pi n}e^{-j\pi n} + \frac{T^2}{\pi^2n^2}e^{-j\pi n} + \frac{jT^2}{\pi n}e^{j\pi n} - \frac{T^2}{\pi^2n^2}e^{j\pi n}\right] \\
&= \frac{j}{\pi n}(-1)^n
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \qquad b_n = (-1)^{n+1}\frac{2}{\pi n}$$

c) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{3}{2}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&= \frac{j}{2\pi n}\left[e^{-j\pi n} - e^{j\pi n} + e^{-j\pi\frac{n}{2}} - e^{-j\pi\frac{n}{2}}\right] \\
&= \frac{1}{\pi n}\sin(\pi\frac{n}{2}) = \frac{1}{2}\text{sinc}(\frac{n}{2})
\end{aligned}$$

Note that $x_n = 0$ for n even and $x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$. The trigonometric Fourier series expansion coefficients are:

$$a_0 = 3, \quad , a_{2l} = 0, \quad , a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad , b_n = 0, \ \forall n$$

d) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{2}{3}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^{\frac{T}{3}} \frac{3}{T}te^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{2T}{3}}^{T} (-\frac{3}{T}t + 3)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2}e^{-j2\pi\frac{n}{T}t}\right)\Big|_0^{\frac{T}{3}} \\
&\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2}e^{-j2\pi\frac{n}{T}t}\right)\Big|_{\frac{2T}{3}}^{T} \\
&\quad + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{\frac{T}{3}}^{\frac{2T}{3}} + \frac{3}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{\frac{2T}{3}}^{T} \\
&= \frac{3}{2\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1]
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = \frac{4}{3}, \quad a_n = \frac{3}{\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1], \quad b_n = 0, \ \forall n$$

e) The signal is periodic with period T. Since the signal is odd $x_0 = a_0 = 0$. For $n \neq 0$

$$\begin{aligned} x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} -e^{-j2\pi\frac{n}{T}t}dt \\ &\quad + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4}{T} t e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt \\ &= \frac{4}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\ &\quad - \frac{1}{T}\left(\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\right)\Big|_{-\frac{T}{2}}^{-\frac{T}{4}} + \frac{1}{T}\left(\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\right)\Big|_{\frac{T}{4}}^{\frac{T}{2}} \\ &= \frac{j}{\pi n}\left[(-1)^n - \frac{2\sin(\frac{\pi n}{2})}{\pi n}\right] = \frac{j}{\pi n}\left[(-1)^n - \operatorname{sinc}(\frac{n}{2})\right] \end{aligned}$$

For n even, $\operatorname{sinc}(\frac{n}{2}) = 0$ and $x_n = \frac{j}{\pi n}$. The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \ \forall n, \quad b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l+1 \end{cases}$$

f) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{3}}^{\frac{T}{3}} x(t)dt = 1$$

For $n \neq 0$

$$\begin{aligned} x_n &= \frac{1}{T}\int_{-\frac{T}{3}}^{0} (\frac{3}{T}t + 2)e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{3}} (-\frac{3}{T}t + 2)e^{-j2\pi\frac{n}{T}t}dt \\ &= \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Big|_{-\frac{T}{3}}^{0} \\ &\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Big|_{0}^{\frac{T}{3}} \\ &\quad + \frac{2}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{3}}^{0} + \frac{2}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Big|_{0}^{\frac{T}{3}} \\ &= \frac{3}{\pi^2 n^2}\left[\frac{1}{2} - \cos(\frac{2\pi n}{3})\right] + \frac{1}{\pi n}\sin(\frac{2\pi n}{3}) \end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = 2, \quad a_n = 2\left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2} - \cos(\frac{2\pi n}{3})\right) + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right], \quad b_n = 0, \ \forall n$$

Problem 2.5

1) The signal $y(t) = x(t - t_0)$ is periodic with period $T = T_0$.

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t - t_0)e^{-j2\pi\frac{n}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}}(v + t_0)dv \\ &= e^{-j2\pi\frac{n}{T_0}t_0}\frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv \\ &= x_n e^{-j2\pi\frac{n}{T_0}t_0} \end{aligned}$$

where we used the change of variables $v = t - t_0$

2) For $y(t)$ to be periodic there must exist T such that $y(t+mT) = y(t)$. But $y(t+T) = x(t+T)e^{j2\pi f_0 t}e^{j2\pi f_0 T}$ so that $y(t)$ is periodic if $T = T_0$ (the period of $x(t)$) and $f_0 T = k$ for some k in $\mathcal{Z}$. In this case

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}e^{j2\pi f_0 t}dt \\ &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{(n-k)}{T_0}t}dt = x_{n-k} \end{aligned}$$

3) The signal $y(t)$ is periodic with period $T = T_0/\alpha$.

$$\begin{aligned} y_n &= \frac{1}{T}\int_{\beta}^{\beta+T} y(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{\alpha}{T_0}\int_{\beta}^{\beta+\frac{T_0}{\alpha}} x(\alpha t)e^{-j2\pi\frac{n\alpha}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\beta\alpha}^{\beta\alpha+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv = x_n \end{aligned}$$

where we used the change of variables $v = \alpha t$.

4)

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x'(t)e^{-j2\pi\frac{n}{T_0}t}dt \\ &= \frac{1}{T_0}x(t)e^{-j2\pi\frac{n}{T_0}t}\Big|_{\alpha}^{\alpha+T_0} - \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}(-j2\pi\frac{n}{T_0})e^{-j2\pi\frac{n}{T_0}t}dt \\ &= j2\pi\frac{n}{T_0}\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt = j2\pi\frac{n}{T_0}x_n \end{aligned}$$

Problem 2.6

$$\begin{aligned} \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}\sum_{n=-\infty}^{\infty} x_n e^{\frac{j2\pi n}{T_0}t}\sum_{m=-\infty}^{\infty} y_m^* e^{-\frac{j2\pi m}{T_0}t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} e^{\frac{j2\pi(n-m)}{T_0}t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \delta_{mn} = \sum_{n=-\infty}^{\infty} x_n y_n^* \end{aligned}$$

Problem 2.7
Using the results of Problem 2.6 we obtain

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)x^*(t)dt = \sum_{n=-\infty}^{\infty}|x_n|^2$$

Since the signal has finite power

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}|x(t)|^2 dt = K < \infty$$

Thus, $\sum_{n=-\infty}^{\infty}|x_n|^2 = K < \infty$. The last implies that $|x_n| \to 0$ as $n \to \infty$. To see this write

$$\sum_{n=-\infty}^{\infty}|x_n|^2 = \sum_{n=-\infty}^{-M}|x_n|^2 + \sum_{n=-M}^{M}|x_n|^2 + \sum_{n=M}^{\infty}|x_n|^2$$

Each of the previous terms is positive and bounded by K. Assume that $|x_n|^2$ does not converge to zero as n goes to infinity and choose $\epsilon = 1$. Then there exists a subsequence of x_n, x_{n_k}, such that

$$|x_{n_k}| > \epsilon = 1, \qquad \text{for } n_k > N \geq M$$

Then

$$\sum_{n=M}^{\infty} |x_n|^2 \geq \sum_{n=N}^{\infty} |x_n|^2 \geq \sum_{n_k} |x_{n_k}|^2 = \infty$$

This contradicts our assumption that $\sum_{n=M}^{\infty} |x_n|^2$ is finite. Thus $|x_n|$, and consequently x_n, should converge to zero as $n \to \infty$.

Problem 2.8

The power content of $x(t)$ is

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

But $|x(t)|^2$ is periodic with period $T_0/2 = 1$ so that

$$P_x = \frac{2}{T_0} \int_0^{T_0/2} |x(t)|^2 dt = \frac{2}{3T_0} t^3 \Big|_0^{T_0/2} = \frac{1}{3}$$

From Parseval's theorem

$$P_x = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

For the signal under consideration

$$a_n = \begin{cases} -\frac{4}{\pi^2 n^2} & \text{n odd} \\ 0 & \text{n even} \end{cases} \qquad b_n = \begin{cases} -\frac{2}{\pi n} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

Thus,

$$\begin{aligned} \frac{1}{3} &= \frac{1}{2} \sum_{n=1}^{\infty} a^2 + \frac{1}{2} \sum_{n=1}^{\infty} b^2 \\ &= \frac{8}{\pi^4} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} + \frac{2}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} \end{aligned}$$

But,

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8}$$

and by substituting this in the previous formula we obtain

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{\pi^4}{96}$$

Problem 2.9

1) Since $(a-b)^2 \geq 0$ we have that

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

with equality if $a = b$. Let

$$A = \left[\sum_{i=1}^{n} \alpha_i^2\right]^{\frac{1}{2}}, \qquad B = \left[\sum_{i=1}^{n} \beta_i^2\right]^{\frac{1}{2}}$$

Then substituting α_i/A for a and β_i/B for b in the previous inequality we obtain

$$\frac{\alpha_i}{A}\frac{\beta_i}{B} \leq \frac{1}{2}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\frac{\beta_i^2}{B^2}$$

with equality if $\frac{\alpha_i}{\beta_i} = \frac{A}{B} = k$ or $\alpha_i = k\beta_i$ for all i. Summing both sides from $i = 1$ to n we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\frac{\alpha_i\beta_i}{AB} &\leq \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\beta_i^2}{B^2} \\
&= \frac{1}{2A^2}\sum_{i=1}^{n}\alpha_i^2 + \frac{1}{2B^2}\sum_{i=1}^{n}\beta_i^2 = \frac{1}{2A^2}A^2 + \frac{1}{2B^2}B^2 = 1
\end{aligned}$$

Thus,

$$\frac{1}{AB}\sum_{i=1}^{n}\alpha_i\beta_i \leq 1 \Rightarrow \sum_{i=1}^{n}\alpha_i\beta_i \leq \left[\sum_{i=1}^{n}\alpha_i^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\beta_i^2\right]^{\frac{1}{2}}$$

Equality holds if $\alpha_i = k\beta_i$, for $i = 1, \ldots, n$.

2) The second equation is trivial since $|x_iy_i^*| = |x_i||y_i^*|$. To see this write x_i and y_i in polar coordinates as $x_i = \rho_{x_i}e^{j\theta_{x_i}}$ and $y_i = \rho_{y_i}e^{j\theta_{y_i}}$. Then, $|x_iy_i^*| = |\rho_{x_i}\rho_{y_i}e^{j(\theta_{x_i}-\theta_{y_i})}| = \rho_{x_i}\rho_{y_i} = |x_i||y_i| = |x_i||y_i^*|$. We turn now to prove the first inequality. Let z_i be any complex with real and imaginary components $z_{i,R}$ and $z_{i,I}$ respectively. Then,

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_i\right|^2 &= \left|\sum_{i=1}^{n} z_{i,R} + j\sum_{i=1}^{n} z_{i,I}\right|^2 = \left(\sum_{i=1}^{n} z_{i,R}\right)^2 + \left(\sum_{i=1}^{n} z_{i,I}\right)^2 \\
&= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})
\end{aligned}$$

Since $(z_{i,R}z_{m,I} - z_{m,R}z_{i,I})^2 \geq 0$ we obtain

$$(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})^2 \leq (z_{i,R}^2 + z_{i,I}^2)(z_{m,R}^2 + z_{m,I}^2)$$

Using this inequality in the previous equation we get

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_i\right|^2 &= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I}) \\
&\leq \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)\left(\sum_{m=1}^{n}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\right) = \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2
\end{aligned}$$

Thus

$$\left|\sum_{i=1}^{n} z_i\right|^2 \leq \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2 \quad \text{or} \quad \left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n}|z_i|$$

The inequality now follows if we substitute $z_i = x_iy_i^*$. Equality is obtained if $\frac{z_{i,R}}{z_{i,I}} = \frac{z_{m,R}}{z_{m,I}} = k_1$ or $\angle z_i = \angle z_m = \theta$.

3) From 2) we obtain

$$\left|\sum_{i=1}^{n} x_iy_i^*\right|^2 \leq \sum_{i=1}^{n}|x_i||y_i|$$

But $|x_i|$, $|y_i|$ are real positive numbers so from 1)

$$\sum_{i=1}^{n} |x_i||y_i| \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$$

Combining the two inequalities we get

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$$

From part 1) equality holds if $\alpha_i = k\beta_i$ or $|x_i| = k|y_i|$ and from part 2) $x_i y_i^* = |x_i y_i^*|e^{j\theta}$. Therefore, the two conditions are

$$\begin{cases} |x_i| = k|y_i| \\ \angle x_i - \angle y_i = \theta \end{cases}$$

which imply that for all i, $x_i = K y_i$ for some complex constant K.

3) The same procedure can be used to prove the Cauchy-Schwartz inequality for integrals. An easier approach is obtained if one considers the inequality

$$|x(t) + \alpha y(t)| \geq 0, \qquad \text{for all } \alpha$$

Then

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} |x(t) + \alpha y(t)|^2 dt = \int_{-\infty}^{\infty} (x(t) + \alpha y(t))(x^*(t) + \alpha^* y^*(t)) dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt + \alpha \int_{-\infty}^{\infty} x^*(t) y(t) dt + \alpha^* \int_{-\infty}^{\infty} x(t) y^*(t) dt + |a|^2 \int_{-\infty}^{\infty} |y(t)|^2 dt \end{aligned}$$

The inequality is true for $\int_{-\infty}^{\infty} x^*(t)y(t)dt = 0$. Suppose that $\int_{-\infty}^{\infty} x^*(t)y(t)dt \neq 0$ and set

$$\alpha = -\frac{\int_{-\infty}^{\infty} |x(t)|^2 dt}{\int_{-\infty}^{\infty} x^*(t) y(t) dt}$$

Then,

$$0 \leq -\int_{-\infty}^{\infty} |x(t)|^2 dt + \frac{[\int_{-\infty}^{\infty} |x(t)|^2 dt]^2 \int_{-\infty}^{\infty} |y(t)|^2 dt}{|\int_{-\infty}^{\infty} x(t) y^*(t) dt|^2}$$

and

$$\left|\int_{-\infty}^{\infty} x(t) y^*(t) dt\right| \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |y(t)|^2 dt\right]^{\frac{1}{2}}$$

Equality holds if $x(t) = -\alpha y(t)$ a.e. for some complex α.

Problem 2.10

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right) \mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3)+\Pi(t+3)] \\ &= \text{sinc}(f)e^{-j2\pi f3}+\text{sinc}(f)e^{j2\pi f3} \\ &= 2\text{sinc}(f)\cos(2\pi 3f)\end{aligned}$$

3)

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{F}[\Lambda(2t+3)+\Lambda(3t-2)] \\ &= \mathcal{F}[\Lambda(2(t+\frac{3}{2}))+\Lambda(3(t-\frac{2}{3})] \\ &= \frac{1}{2}\text{sinc}^2(\frac{f}{2})e^{j\pi f3}+\frac{1}{3}\text{sinc}^2(\frac{f}{3})e^{-j2\pi f\frac{2}{3}}\end{aligned}$$

4) $T(f)=\mathcal{F}[\text{sinc}^3(t)]=\mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)]=\Lambda(f)\star\Pi(f)$. But

$$\Pi(f)\star\Lambda(f)=\int_{-\infty}^{\infty}\Pi(\theta)\Lambda(f-\theta)d\theta=\int_{-\frac{1}{2}}^{\frac{1}{2}}\Lambda(f-\theta)d\theta=\int_{f-\frac{1}{2}}^{f+\frac{1}{2}}\Lambda(v)dv$$

$$\begin{aligned}
&\text{For} \quad f\leq -\frac{3}{2} \Longrightarrow T(f)=0 \\
&\text{For} \quad -\frac{3}{2}<f\leq-\frac{1}{2} \Longrightarrow T(f)=\int_{-1}^{f+\frac{1}{2}}(v+1)dv=(\frac{1}{2}v^2+v)\Big|_{-1}^{f+\frac{1}{2}}=\frac{1}{2}f^2+\frac{3}{2}f+\frac{9}{8} \\
&\text{For} \quad -\frac{1}{2}<f\leq\frac{1}{2} \Longrightarrow T(f)=\int_{f-\frac{1}{2}}^{0}(v+1)dv+\int_{0}^{f+\frac{1}{2}}(-v+1)dv \\
&\qquad\qquad =(\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{0}+(-\frac{1}{2}v^2+v)\Big|_{0}^{f+\frac{1}{2}}=-f^2+\frac{3}{4} \\
&\text{For} \quad \frac{1}{2}<f\leq\frac{3}{2} \Longrightarrow T(f)=\int_{f-\frac{1}{2}}^{1}(-v+1)dv=(-\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{1}=\frac{1}{2}f^2-\frac{3}{2}f+\frac{9}{8} \\
&\text{For} \quad \frac{3}{2}<f \Longrightarrow T(f)=0
\end{aligned}$$

Thus,

$$T(f)=\begin{cases}0 & f\leq-\frac{3}{2} \\ \frac{1}{2}f^2+\frac{3}{2}f+\frac{9}{8} & -\frac{3}{2}<f\leq-\frac{1}{2} \\ -f^2+\frac{3}{4} & -\frac{1}{2}<f\leq\frac{1}{2} \\ \frac{1}{2}f^2-\frac{3}{2}f+\frac{9}{8} & \frac{1}{2}<f\leq\frac{3}{2} \\ 0 & \frac{3}{2}<f\end{cases}$$

5)

$$\mathcal{F}[t\text{sinc}(t)]=\frac{1}{\pi}\mathcal{F}[\sin(\pi t)]=\frac{j}{2\pi}\left[\delta(f+\frac{1}{2})-\delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

$$\mathcal{F}[t\text{sinc}]=\frac{j}{2\pi}\frac{d}{df}\Pi(f)=\frac{j}{2\pi}\left[\delta(f+\frac{1}{2})-\delta(f-\frac{1}{2})\right]$$

6)

$$\begin{aligned}
\mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)\right)\\
&= \frac{j}{4\pi}\left(\delta'(f-f_0)+\delta'(f+f_0)\right)
\end{aligned}$$

7)

$$\mathcal{F}[e^{-\alpha|t|}\cos(\beta t)] = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2}+\frac{2\alpha}{\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2}\right]$$

8)

$$\begin{aligned}
\mathcal{F}[te^{-\alpha|t|}\cos(\beta t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{\alpha}{\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2}+\frac{\alpha}{\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2}\right)\\
&= -j\left[\frac{2\alpha\pi(f-\frac{\beta}{2\pi})}{\left(\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2\right)^2}+\frac{2\alpha\pi(f+\frac{\beta}{2\pi})}{\left(\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2\right)^2}\right]
\end{aligned}$$

Problem 2.11

$$\begin{aligned}
\mathcal{F}[\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))] &= \int_{-\infty}^{\infty}\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))e^{-j2\pi ft}dt\\
&= \frac{1}{2}(e^{-j\pi f}+e^{-j\pi f}) = \cos(\pi f)
\end{aligned}$$

Using the duality property of the Fourier transform:

$$X(f)=\mathcal{F}[x(t)] \Longrightarrow x(f)=\mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t)=\cos(\pi t+\frac{\pi}{2})$. Thus

$$\begin{aligned}
\mathcal{F}[\sin(\pi t)] &= \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))e^{j\pi f}\\
&= \frac{1}{2}e^{j\pi\frac{1}{2}}\delta(f+\frac{1}{2})+\frac{1}{2}e^{-j\pi\frac{1}{2}}\delta(f-\frac{1}{2})\\
&= \frac{j}{2}\delta(f+\frac{1}{2})-\frac{j}{2}\delta(f-\frac{1}{2})
\end{aligned}$$

Problem 2.12

a) We can write $x(t)$ as $x(t)=2\Pi(\frac{t}{4})-2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})]-\mathcal{F}[2\Lambda(\frac{t}{2})] = 8\text{sinc}(4f)-4\text{sinc}^2(2f)$$

b)

$$x(t)=2\Pi(\frac{t}{4})-\Lambda(t) \Longrightarrow \mathcal{F}[x(t)] = 8\text{sinc}(4f)-\text{sinc}^2(f)$$

c)

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-1}^{0}(t+1)e^{-j2\pi ft}dt + \int_{0}^{1}(t-1)e^{-j2\pi ft}dt \\
&= \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi ft}\Big|_{-1}^{0} + \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{-1}^{0} \\
&\quad + \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi ft}\Big|_{0}^{1} - \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{0}^{1} \\
&= \frac{j}{\pi f}(1-\sin(\pi f))
\end{aligned}$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \text{sinc}^2(f)e^{j2\pi f} - \text{sinc}^2(f)e^{-j2\pi f} = 2j\text{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \text{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \text{sinc}^2(f)(1 + 2\cos(2\pi f))$$

f) We can write $x(t)$ as

$$x(t) = \left[\Pi\left(2f_0(t - \frac{1}{4f_0})\right) - \Pi\left(2f_0(t - \frac{1}{4f_0})\right)\right]\sin(2\pi f_0 t)$$

Then

$$\begin{aligned}
X(f) &= \left[\frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0}\right)e^{-j2\pi\frac{1}{4f_0}f} - \frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0})\right)e^{j2\pi\frac{1}{4f_0}f}\right] \\
&\quad \star\frac{j}{2}(\delta(f+f_0) - \delta(f+f_0)) \\
&= \frac{1}{2f_0}\text{sinc}\left(\frac{f+f_0}{2f_0}\right)\sin\left(\pi\frac{f+f_0}{2f_0}\right) - \frac{1}{2f_0}\text{sinc}\left(\frac{f-f_0}{2f_0}\right)\sin\left(\pi\frac{f-f_0}{2f_0}\right)
\end{aligned}$$

Problem 2.13

We start with

$$\mathcal{F}[x(at)] = \int_{-\infty} \infty x(at)e^{-j2\pi ft}dt$$

and make the change in variable $u = at$, then,

$$\begin{aligned}
\mathcal{F}[x(at)] &= \frac{1}{|a|}\int_{-\infty} \infty x(u)e^{-j2\pi fu/a}du \\
&= \frac{1}{|a|}X\left(\frac{f}{a}\right)
\end{aligned}$$

where we have treated the cases $a > 0$ and $a < 0$ separately.

Note that in the above expression if $a > 1$, then $x(at)$ is a contracted form of $x(t)$ whereas if $a < 1$, $x(at)$ is an expanded version of $x(t)$. This means that if we expand a signal in the time domain its frequency domain representation (Fourier transform) contracts and if we contract a signal in the time domain its frequency domain representation expands. This is exactly what one expects since contracting a signal in the time domain makes the changes in the signal more abrupt, thus, increasing its frequency content.

Problem 2.14

We have

$$
\begin{aligned}
\mathcal{F}[x(t) \star y(t)] &= \int_{-\infty} \infty \left[\int_{-\infty} \infty x(\tau) y(t-\tau)\, d\tau \right] e^{-j2\pi f t} dt \\
&= \int_{-\infty} \infty x(\tau) \left[\int_{-\infty} \infty y(t-\tau) e^{-j2\pi f (t-\tau)}\, dt \right] e^{-j2\pi f \tau} d\tau
\end{aligned}
$$

Now with the change of variable $u = t - \tau$, we have

$$
\begin{aligned}
\int_{-\infty} \infty y(t-\tau) e^{-j2\pi f(t-\tau)} dt &= \int_{-\infty} \infty f y(u) e^{-j2\pi f u} du \\
&= \mathcal{F}[y(t)] \\
&= Y(f)
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
\mathcal{F}[x(t) \star y(t)] &= \int_{-\infty} \infty x(\tau) Y(f) e^{-j2\pi f \tau} d\tau \\
&= X(f) \cdot Y(f)
\end{aligned}
$$

Problem 2.15

We start with the Fourier transform of $x(t - t_0)$,

$$
\mathcal{F}[x(t-t_0)] = \int_{-\infty} \infty x(t-t_0) e^{-j2\pi f t} dt
$$

With a change of variable of $u = t - t_0$, we obtain

$$
\begin{aligned}
\mathcal{F}[x(t-t_0)] &= \int_{-\infty} \infty x(u) e^{-j2\pi f t_0} e^{-j2\pi f u} du \\
&= e^{-j2\pi f t_0} \int_{-\infty} \infty x(u) e^{-j2\pi f u} du \\
&= e^{-j2\pi f t_0} \mathcal{F}[x(t)]
\end{aligned}
$$

Problem 2.16

$$
\begin{aligned}
\int_{-\infty} \infty x(t) y^*(t)\, dt &= \int_{-\infty} \infty \left[\int_{-\infty} \infty X(f) e^{j2\pi f t} df \right] \left[\int_{-\infty} \infty Y(f') e^{j2\pi f' t} df' \right]^* dt \\
&= \int_{-\infty} \infty \left[\int_{-\infty} \infty X(f) e^{j2\pi f t} df \right] \left[\int_{-\infty} \infty Y^*(f') e^{-j2\pi f' t} df' \right] dt \\
&= \int_{-\infty} \infty X(f) \left[\int_{-\infty} \infty Y^*(f') \left[\int_{-\infty} \infty e^{j2\pi t(f-f')}\, dt \right] df' \right] df
\end{aligned}
$$

Now using properties of the impulse function.

$$
\int_{-\infty} \infty e^{j2\pi t(f-f')} dt = \delta(f - f')
$$

and therefore

$$
\begin{aligned}
\int_{-\infty} \infty x(t) y^*(t)\, dt &= \int_{-\infty} \infty X(f) \left[\int_{-\infty} \infty Y^*(f') \delta(f-f')\, df' \right] df \\
&= \int_{-\infty} \infty X(f) Y^*(f)\, df
\end{aligned}
$$

where we have employed the sifting property of the impulse signal in the last step.

Problem 2.17
(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned}
\text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\
&= \text{sinc}(t)
\end{aligned}$$

Problem 2.18

$$\begin{aligned}
\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta \right) y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} X(\theta) \left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt \right) d\theta \\
&= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f)
\end{aligned}$$

Problem 2.19
1) Clearly

$$\begin{aligned}
x_1(t+kT_0) &= \sum_{n=-\infty}^{\infty} x(t+kT_0-nT_0) = \sum_{n=-\infty}^{\infty} x(t-(n-k)T_0) \\
&= \sum_{m=-\infty}^{\infty} x(t-mT_0) = x_1(t)
\end{aligned}$$

where we used the change of variable $m = n - k$.

2)

$$x_1(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)$$

This is because

$$\int_{-\infty}^{\infty} x(\tau) \sum_{n=-\infty}^{\infty} \delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

3)

$$\begin{aligned}
\mathcal{F}[x_1(t)] &= \mathcal{F}[x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)] = \mathcal{F}[x(t)]\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t-nT_0)] \\
&= X(f)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T_0}) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f-\frac{n}{T_0})
\end{aligned}$$

Problem 2.20

1) By Parseval's theorem

$$\int_{-\infty}^{\infty} \text{sinc}^5(t)dt = \int_{-\infty}^{\infty} \text{sinc}^3(t)\text{sinc}^2(t)dt = \int_{-\infty}^{\infty} \Lambda(f)T(f)df$$

where

$$T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Pi(f) \star \Lambda(f)$$

But

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

$$\begin{aligned}
\text{For} \quad & f \leq -\frac{3}{2} \Longrightarrow T(f) = 0 \\
\text{For} \quad & -\frac{3}{2} < f \leq -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} \\
\text{For} \quad & -\frac{1}{2} < f \leq \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv \\
& \qquad = (\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4} \\
\text{For} \quad & \frac{1}{2} < f \leq \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} \\
\text{For} \quad & \frac{3}{2} < f \Longrightarrow T(f) = 0
\end{aligned}$$

Thus,

$$T(f) = \begin{cases} 0 & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Lambda(f)T(f)df &= \int_{-1}^{-\frac{1}{2}} (\frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8})(f+1)df + \int_{-\frac{1}{2}}^{0} (-f^2 + \frac{3}{4})(f+1)df \\
&\quad + \int_{0}^{\frac{1}{2}} (-f^2 + \frac{3}{4})(-f+1)df + \int_{\frac{1}{2}}^{1} (\frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8})(-f+1)df \\
&= \frac{41}{64}
\end{aligned}$$

2)

$$\begin{aligned}
\int_{0}^{\infty} e^{-\alpha t}\text{sinc}(t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u_{-1}(t)\text{sinc}(t)dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}\Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\alpha + j2\pi f}df \\
&= \frac{1}{j2\pi}\ln(\alpha + j2\pi f)|_{-1/2}^{1/2} = \frac{1}{j2\pi}\ln(\frac{\alpha + j\pi}{\alpha - j\pi}) = \frac{1}{\pi}\tan^{-1}\frac{\pi}{\alpha}
\end{aligned}$$

3)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\mathrm{sinc}^2(t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\mathrm{sinc}^2(t)dt \\
&= \int_{-\infty}^\infty \frac{1}{\alpha+j2\pi f}\Lambda(f)df df \\
&= \int_{-1}^0 \frac{f+1}{\alpha+j\pi f}df + \int_0^1 \frac{-f+1}{\alpha+j\pi f}df
\end{aligned}$$

But $\int \frac{x}{a+bx}dx = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$ so that

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\mathrm{sinc}^2(t)dt &= (\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha+j2\pi f))\Big|_{-1}^0 \\
&\quad -(\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha+j2\pi f))\Big|_0^1 + \frac{1}{j2\pi}\ln(\alpha+j2\pi f)\Big|_{-1}^1 \\
&= \frac{1}{\pi}\tan^{-1}(\frac{2\pi}{\alpha}) + \frac{\alpha}{2\pi^2}\ln(\frac{\alpha}{\sqrt{\alpha^2+4\pi^2}})
\end{aligned}$$

4)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\cos(\beta t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\cos(\beta t)dt \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\alpha+j2\pi f}(\delta(f-\frac{\beta}{2\pi}) + \delta(f+\frac{\beta}{2\pi}))dt \\
&= \frac{1}{2}[\frac{1}{\alpha+j\beta} + \frac{1}{\alpha-j\beta}] = \frac{\alpha}{\alpha^2+\beta^2}
\end{aligned}$$

Problem 2.21

Using the convolution theorem we obtain

$$\begin{aligned}
Y(f) &= X(f)H(f) = (\frac{1}{\alpha+j2\pi f})(\frac{1}{\beta+j2\pi f}) \\
&= \frac{1}{(\beta-\alpha)}\frac{1}{\alpha+j2\pi f} - \frac{1}{(\beta-\alpha)}\frac{1}{\beta+j2\pi f}
\end{aligned}$$

Thus

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \frac{1}{(\beta-\alpha)}[e^{-\alpha t} - e^{-\beta t}]u_{-1}(t)$$

If $\alpha = \beta$ then $X(f) = H(f) = \frac{1}{\alpha+j2\pi f}$. In this case

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \mathcal{F}^{-1}[(\frac{1}{\alpha+j2\pi f})^2] = te^{-\alpha t}u_{-1}(t)$$

The signal is of the energy-type with energy content

$$\begin{aligned}
E_y &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} |y(t)|^2 dt = \lim_{T\to\infty}\int_0^{\frac{T}{2}} \frac{1}{(\beta-\alpha)^2}(e^{-\alpha t} - e^{-\beta t})^2 dt \\
&= \lim_{T\to\infty}\frac{1}{(\beta-\alpha)^2}\left[-\frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{T/2} - \frac{1}{2\beta}e^{-2\beta t}\Big|_0^{T/2} + \frac{2}{(\alpha+\beta)}e^{-(\alpha+\beta)t}\Big|_0^{T/2}\right] \\
&= \frac{1}{(\beta-\alpha)^2}[\frac{1}{2\alpha} + \frac{1}{2\beta} - \frac{2}{\alpha+\beta}] = \frac{1}{2\alpha\beta(\alpha+\beta)}
\end{aligned}$$

Problem 2.22

$$x_\alpha(t) = \begin{cases} x(t) & \alpha \le t < \alpha + T_0 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$X_\alpha(f) = \int_{-\infty}^{\infty} x_\alpha(t) e^{-j2\pi f t} dt = \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi f t} dt$$

Evaluating $X_\alpha(f)$ for $f = \frac{n}{T_0}$ we obtain

$$X_\alpha(\frac{n}{T_0}) = \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi \frac{n}{T_0} t} dt = T_0 x_n$$

where x_n are the coefficients in the Fourier series expansion of $x(t)$. Thus $X_\alpha(\frac{n}{T_0})$ is independent of the choice of α.

Problem 2.23

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(t - nT_s) &= x(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} x(t) \star \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_s} t} \\
&= \frac{1}{T_s} \mathcal{F}^{-1}\left[X(f) \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \right] \\
&= \frac{1}{T_s} \mathcal{F}^{-1}\left[\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) \delta(f - \frac{n}{T_s}) \right] \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{j2\pi \frac{n}{T_s} t}
\end{aligned}$$

If we set $t = 0$ in the previous relation we obtain Poisson's sum formula

$$\sum_{n=-\infty}^{\infty} x(-nT_s) = \sum_{m=-\infty}^{\infty} x(mT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$

Problem 2.24

1) We know that

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

Applying Poisson's sum formula with $T_s = 1$ we obtain

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 n^2}$$

2) Use the Fourier transform pair $\Pi(t) \to \text{sinc}(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Pi(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{n}{K})$$

But $\Pi(nK) = 1$ for $n = 0$ and $\Pi(nK) = 0$ for $|n| \ge 1$ and $K \in \{1, 2, \ldots\}$. Thus the left side of the previous relation reduces to 1 and

$$K = \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{n}{K})$$

3) Use the Fourier transform pair $\Lambda(t) \to \text{sinc}^2(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Lambda(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}^2(\frac{n}{K})$$

Reasoning as before we see that $\sum_{n=-\infty}^{\infty} \Lambda(nK) = 1$ since for $K \in \{1, 2, \ldots\}$

$$\Lambda(nK) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $K = \sum_{n=-\infty}^{\infty} \text{sinc}^2(\frac{n}{K})$

Problem 2.25

Let $H(f)$ be the Fourier transform of $h(t)$. Then

$$H(f)\mathcal{F}[e^{-\alpha t}u_{-1}(t)] = \mathcal{F}[\delta(t)] \Longrightarrow H(f)\frac{1}{\alpha + j2\pi f} = 1 \Longrightarrow H(f) = \alpha + j2\pi f$$

The response of the system to $e^{-\alpha t}\cos(\beta t)u_{-1}(t)$ is

$$y(t) = \mathcal{F}^{-1}\left[H(f)\mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)]\right]$$

But

$$\begin{aligned} \mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)] &= \mathcal{F}[\frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{j\beta t} + \frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{-j\beta t}] \\ &= \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right] \end{aligned}$$

so that

$$Y(f) = \mathcal{F}[y(t)] = \frac{\alpha + j2\pi f}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right]$$

Using the linearity property of the Fourier transform, the Convolution theorem and the fact that $\delta'(t) \xrightarrow{\mathcal{F}} j2\pi f$ we obtain

$$\begin{aligned} y(t) &= \alpha e^{-\alpha t}\cos(\beta t)u_{-1}(t) + (e^{-\alpha t}\cos(\beta t)u_{-1}(t)) \star \delta'(t) \\ &= e^{-\alpha t}\cos(\beta t)\delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t) \\ &= \delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t) \end{aligned}$$

Problem 2.26

1)

$$\begin{aligned} y(t) &= x(t) \star h(t) = x(t) \star (\delta(t) + \delta'(t) \\ &= x(t) + \frac{d}{dt}x(t) \end{aligned}$$

With $x(t) = e^{-\alpha|t|}$ we obtain $y(t) = e^{-\alpha|t|} - \alpha e^{-\alpha|t|}\text{sgn}(t)$.

2)

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau \end{aligned}$$

$$\begin{aligned}
\text{If} \qquad & \alpha = \beta \Rightarrow y(t) = te^{-\alpha t}u_{-1}(t) \\
& \alpha \neq \beta \Rightarrow y(t) = e^{-\beta t}\frac{1}{\beta-\alpha}e^{-(\alpha-\beta)t}\Big|_0^t u_{-1}(t) = \frac{1}{\beta-\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]u_{-1}(t)
\end{aligned}$$

3)

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} e^{-\alpha\tau}\cos(\gamma\tau)u_{-1}(\tau)e^{-\beta(t-\tau)}u_{-1}(t-\tau)d\tau \\
&= \int_0^t e^{-\alpha\tau}\cos(\gamma\tau)e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{(\beta-\alpha)\tau}\cos(\gamma\tau)d\tau
\end{aligned}$$

$$\begin{aligned}
\text{If } \alpha = \beta \Rightarrow y(t) &= e^{-\beta t}\int_0^t \cos(\gamma\tau)d\tau u_{-1}(t) = \frac{e^{-\beta t}}{\gamma}\sin(\gamma t)u_{-1}(t) \\
\text{If } \alpha = \beta \Rightarrow y(t) &= e^{-\beta t}\int_0^t e^{(\beta-\alpha)\tau}\cos(\gamma\tau)d\tau u_{-1}(t) \\
&= \frac{e^{-\beta t}}{(\beta-\alpha)^2+\gamma^2}\left((\beta-\alpha)\cos(\gamma\tau)+\gamma\sin(\gamma\tau)\right)e^{(\beta-\alpha)\tau}\Big|_0^t u_{-1}(t) \\
&= \frac{e^{-\alpha t}}{(\beta-\alpha)^2+\gamma^2}\left((\beta-\alpha)\cos(\gamma t)+\gamma\sin(\gamma t)\right)u_{-1}(t) \\
&\quad -\frac{e^{-\beta t}(\beta-\alpha)}{(\beta-\alpha)^2+\gamma^2}u_{-1}(t)
\end{aligned}$$

4)

$$y(t) = \int_{-\infty}^{\infty} e^{-\alpha|\tau|}e^{-\beta(t-\tau)}u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} e^{-\alpha|\tau|}e^{-\beta(t-\tau)}d\tau$$

Consider first the case that $\alpha \neq \beta$. Then

$$\begin{aligned}
\text{If } t < 0 \Rightarrow y(t) &= e^{-\beta t}\int_{-\infty}^t e^{(\beta+\alpha)\tau}d\tau = \frac{1}{\alpha+\beta}e^{\alpha t} \\
\text{If } t < 0 \Rightarrow y(t) &= \int_{-\infty}^0 e^{\alpha\tau}e^{-\beta(t-\tau)}d\tau + \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau \\
&= \frac{e^{-\beta t}}{\alpha+\beta}e^{(\alpha+\beta)\tau}\Big|_{-\infty}^0 + \frac{e^{-\beta t}}{\beta-\alpha}e^{(\beta-\alpha)\tau}\Big|_0^t \\
&= -\frac{2\alpha e^{-\beta t}}{\beta^2-\alpha^2} + \frac{e^{-\alpha t}}{\beta-\alpha}
\end{aligned}$$

Thus

$$y(t) = \begin{cases} \frac{1}{\alpha+\beta}e^{\alpha t} & t \leq 0 \\ -\frac{2\alpha e^{-\beta t}}{\beta^2-\alpha^2} + \frac{e^{-\alpha t}}{\beta-\alpha} & t > 0 \end{cases}$$

In the case of $\alpha = \beta$

$$\begin{aligned}
\text{If } t < 0 \Rightarrow y(t) &= e^{-\alpha t}\int_{-\infty}^t e^{2\alpha\tau}d\tau = \frac{1}{2\alpha}e^{\alpha t} \\
\text{If } t < 0 \Rightarrow y(t) &= \int_{-\infty}^0 e^{-\alpha t}e^{2\alpha\tau}d\tau + \int_0^t e^{-\alpha t}d\tau \\
&= \frac{e^{-\alpha t}}{2\alpha}e^{2\alpha\tau}\Big|_{-\infty}^0 + te^{-\alpha t} \\
&= [\frac{1}{2\alpha}+t]e^{-\alpha t}
\end{aligned}$$

5) Using the convolution theorem we obtain

$$Y(f) = \Pi(f)\Lambda(f) = \begin{cases} 0 & \frac{1}{2} < |f| \\ f+1 & -\frac{1}{2} < f \leq 0 \\ -f+1 & 0 \leq f < \frac{1}{2} \end{cases}$$

Thus

$$\begin{aligned}
y(t) &= \mathcal{F}^{-1}[Y(f)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} Y(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{2}}^{0} (f+1)e^{j2\pi ft}df + \int_{0}^{\frac{1}{2}} (-f+1)e^{j2\pi ft}df \\
&= \left(\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2 t^2}e^{j2\pi ft}\right)\Big|_{-\frac{1}{2}}^{0} + \frac{1}{j2\pi t}e^{j2\pi ft}\Big|_{-\frac{1}{2}}^{0} \\
&\quad - \left(\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2 t^2}e^{j2\pi ft}\right)\Big|_{0}^{\frac{1}{2}} + \frac{1}{j2\pi t}e^{j2\pi ft}\Big|_{0}^{\frac{1}{2}} \\
&= \frac{1}{2\pi^2 t^2}[1-\cos(\pi t)] + \frac{1}{2\pi t}\sin(\pi t)
\end{aligned}$$

Problem 2.27

Let the response of the LTI system be $h(t)$ with Fourier transform $H(f)$. Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \Longrightarrow \Lambda(f) = \Pi(f)H(f)$$

However, this relation cannot hold since $\Pi(f) = 0$ for $\frac{1}{2} < |f|$ whereas $\Lambda(f) \neq 0$ for $1 < |f| \leq 1/2$.

Problem 2.28

1) No. The input $\Pi(t)$ has a spectrum with zeros at frequencies $f = k$, $(k \neq 0,\ k \in \mathcal{Z})$ and the information about the spectrum of the system at those frequencies will not be present at the output. The spectrum of the signal $\cos(2\pi t)$ consists of two impulses at $f = \pm 1$ but we do not know the response of the system at these frequencies.

2)

$$\begin{aligned}
h_1(t) \star \Pi(t) &= \Pi(t) \star \Pi(t) = \Lambda(t) \\
h_2(t) \star \Pi(t) &= (\Pi(t) + \cos(2\pi t)) \star \Pi(t) \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\mathrm{sinc}^2(f) + \delta(f+1)\mathrm{sinc}^2(f)\right] \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\mathrm{sinc}^2(1) + \delta(f+1)\mathrm{sinc}^2(-1)\right] \\
&= \Lambda(t)
\end{aligned}$$

Thus both signals are candidates for the impulse response of the system.

3) $\mathcal{F}[u_{-1}(t)] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$. Thus the system has a nonzero spectrum for every f and all the frequencies of the system will be excited by this input. $\mathcal{F}[e^{-at}u_{-1}(t)] = \frac{1}{a+j2\pi f}$. Again the spectrum is nonzero for all f and the response to this signal uniquely determines the system. In general the spectrum of the input must not vanish at any frequency. In this case the influence of the system will be present at the output for every frequency.

Problem 2.29

1)

$$\begin{aligned} E_{x_1} &= \int_{-\infty}^{\infty} x_1^2(t)\, dt \\ &= \int_0^{\infty} e^{-2t}\cos^2 t\, dt \\ &< \int_0^{\infty} e^{-2t}\, dt = \frac{1}{2} \end{aligned}$$

where we have used $\cos^2 t \leq 1$. Therefore, $x_1(t)$ is energy type. To find the energy we have

$$\begin{aligned} \int_0^{\infty} e^{-2t}\cos^2 t\, dt &= \frac{1}{2}\int_0^{\infty} e^{-2t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-2t}\cos 2t\, dt \\ &= \frac{1}{4} + \frac{1}{2}\left[-\frac{1}{4}e^{-2t}\cos(2t) + \frac{1}{4}e^{-2*t}\sin(2t)\right]_0^{\infty} \\ &= \frac{3}{8} \end{aligned}$$

2)

$$\begin{aligned} E_{x_2} = \int_{-\infty}^{\infty} x_2^2(t)\, dt &= \int_{-\infty}^{0} e^{-2t}\cos^2(t)\, dt + \int_0^{\infty} e^{-2t}\cos^2(t)\, dt \\ &= \int_0^{\infty} e^{2t}\cos^2(t)\, dt + \frac{3}{8} \\ &= \frac{3}{8} - \frac{3}{8} + \lim_{t\to\infty} \frac{e^{2t}}{8}\left(2\cos^2(t) + \sin(2t) + 1\right) \\ &= \lim_{t\to\infty} \frac{e^{2t}}{8} f(t) \end{aligned}$$

where $f(t) = 2\cos^2(t) + \sin(2t) + 1$. By taking the derivative and setting it equal to zero we can find the minimum of $f(t)$ and show that $f(t) > 0.5$. This shows that $\lim_{t\to\infty} \frac{e^{2t}}{8} f(t) \geq \lim_{t\to\infty} \frac{e^{2t}}{16} = \infty$. This shows that the signal is not energy-type.

To check if the signal is power type, we obviously have $\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-2t}\cos^2 t\, dt = 0$. Therefore

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{2t}\cos^2(t)\, dt \\ &= \lim_{T\to\infty} \frac{1/4\, e^{2\,T}\left(\cos(T)\right)^2 + 1/4\, e^{2\,T}\cos(T)\sin(T) + 1/8\left(e^{T}\right)^2 - 3/8}{T} \\ &= \infty \end{aligned}$$

Therefore $x_2(t)$ is neither power- nor energy-type.

3)

$$\begin{aligned} E_{x_3} = \int_{-\infty}^{\infty} (\mathrm{sgn}(t))^2\, dt &= \int_{-\infty}^{\infty} 1\, dt \\ &= \infty \end{aligned}$$

and hence the signal is not energy-type. To find the power

$$\begin{aligned} P_{x_3} &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (\mathrm{sgn}(t)))^2\, dt \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 1^2\, dt \\ &= \lim_{T\to\infty} \frac{1}{2T} 2T = 1 \end{aligned}$$

4) Since $x_4(t)$ is periodic (or almost periodic when f_1/f_2 is not rational) the signal is not energy type. To see whether it is power type, we have

$$\begin{aligned}
P_{x_4} &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (A\cos 2\pi f_1 t + B\cos 2\pi f_2 t)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(A^2\cos^2 2\pi f_1 t + B^2\cos^2 2\pi f_2 t + 2AB\cos 2\pi f_1 t\cos 2\pi f_2 t\right)\, dt \\
&= \frac{A^2+B^2}{2}
\end{aligned}$$

Problem 2.30

1)

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|Ae^{j(2\pi f_0 t+\theta)}\right|^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\, dt \\
&= A^2
\end{aligned}$$

2)

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} 1^2\, dt \\
&= \frac{1}{2}
\end{aligned}$$

3)

$$\begin{aligned}
E &= \lim_{T\to\infty} \int_{0}^{T} K^2/\sqrt{t}\, dt \\
&= \lim_{T\to\infty} \left[2K^2\sqrt{t}\right]_0^T \\
&= \lim_{T\to\infty} 2K^2\sqrt{T} \\
&= \infty
\end{aligned}$$

therefore, it is not energy-type. To find the power

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K^2/\sqrt{t}\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T} 2K^2\sqrt{T} \\
&= 0
\end{aligned}$$

and hence it is not power-type either.

Problem 2.31

1) $x(t) = e^{-\alpha t}u_{-1}(t)$. The spectrum of the signal is $X(f) = \frac{1}{\alpha + j2\pi f}$ and the energy spectral density

$$\mathcal{G}_X(f) = |X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2}$$

Thus,

$$R_X(\tau) = \mathcal{F}^{-1}[\mathcal{G}_X(f)] = \frac{1}{2\alpha}e^{-\alpha|\tau|}$$

The energy content of the signal is

$$E_X = R_X(0) = \frac{1}{2\alpha}$$

2) $x(t) = \text{sinc}(t)$. Clearly $X(f) = \Pi(f)$ so that $\mathcal{G}_X(f) = |X(f)|^2 = \Pi^2(f) = \Pi(f)$. The energy content of the signal is

$$E_X = \int_{-\infty}^{\infty} \Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(f)df = 1$$

3) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$. The signal is periodic and thus it is not of the energy type. The power content of the signal is

$$\begin{aligned} P_x &= \frac{1}{2}\int_{-1}^{1} |x(t)|^2 dt = \frac{1}{2}\int_{-1}^{0} (t+1)^2 dt + \int_{0}^{1} (-t+1)^2 dt \\ &= \frac{1}{2}\left(\frac{1}{3}t^3 + t^2 + t\right)\Big|_{-1}^{0} + \frac{1}{2}\left(\frac{1}{3}t^3 - t^2 + t\right)\Big|_{0}^{1} \\ &= \frac{1}{3} \end{aligned}$$

The same result is obtain if we let

$$\mathcal{S}_X(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{2})$$

with $x_0 = \frac{1}{2}$, $x_{2l} = 0$ and $x_{2l+1} = \frac{2}{\pi(2l+1)}$ (see Problem 2.2). Then

$$\begin{aligned} P_X &= \sum_{n=-\infty}^{\infty} |x_n|^2 \\ &= \frac{1}{4} + \frac{8}{\pi^2}\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{1}{4} + \frac{8}{\pi^2}\frac{\pi^2}{96} = \frac{1}{3} \end{aligned}$$

4)

$$E_X = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty} \int_{0}^{\frac{T}{2}} dt = \lim_{T\to\infty} \frac{T}{2} = \infty$$

Thus, the signal is not of the energy type.

$$P_X = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Hence, the signal is of the power type and its power content is $\frac{1}{2}$. To find the power spectral density we find first the autocorrelation $R_X(\tau)$.

$$\begin{aligned} R_X(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}(t)u_{-1}(t-\tau)dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{\tau}^{\frac{T}{2}} dt \\ &= \lim_{T\to\infty} \frac{1}{T}(\frac{T}{2} - \tau) = \frac{1}{2} \end{aligned}$$

Thus, $\mathcal{S}_X(f) = \mathcal{F}[R_X(\tau)] = \frac{1}{2}\delta(f)$.

5) Clearly $|X(f)|^2 = \pi^2 \text{sgn}^2(f) = \pi^2$ and $E_X = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \pi^2 dt = \infty$. The signal is not of the energy type for the energy content is not bounded. Consider now the signal

$$x_T(t) = \frac{1}{t}\Pi(\frac{t}{T})$$

Then,

$$X_T(f) = -j\pi \text{sgn}(f) \star T\text{sinc}(fT)$$

and

$$\mathcal{S}_X(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} = \lim_{T\to\infty} \pi^2 T \left| \int_{-\infty}^{f} \text{sinc}(vT)dv - \int_{f}^{\infty} \text{sinc}(vT)dv \right|^2$$

However, the squared term on the right side is bounded away from zero so that $\mathcal{S}_X(f)$ is ∞. The signal is not of the power type either.

Problem 2.32

1)

a) If $\alpha \neq \gamma$,

$$\begin{aligned}
|Y(f)|^2 &= |X(f)|^2 |H(f)|^2 \\
&= \frac{1}{(\alpha^2 + 4\pi^2 f^2)(\beta^2 + 4\pi^2 f^2)} \\
&= \frac{1}{\beta^2 - \alpha^2} \left[\frac{1}{\alpha^2 + 4\pi^2 f^2} - \frac{1}{\beta^2 + 4\pi^2 f^2} \right]
\end{aligned}$$

From this, $R_Y(\tau) = \frac{1}{\beta^2-\alpha^2}\left[\frac{1}{2\alpha}e^{-\alpha|\tau|} - \frac{1}{2\beta}e^{-\beta|\tau|}\right]$ and $E_y = R_y(0) = \frac{1}{2\alpha\beta(\alpha+\beta)}$.

If $\alpha = \gamma$ then

$$\mathcal{G}_Y(f) = |Y(f)|^2 = |X(f)|^2|H(f)|^2 = \frac{1}{(\alpha^2 + 4\pi^2 f^2)^2}$$

The energy content of the signal is

$$\begin{aligned}
E_Y &= \int_{-\infty}^{\infty} \frac{1}{(\alpha^2 + 4\pi^2 f^2)^2} \\
&= \frac{1}{4\alpha^2} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} df \\
&= \frac{1}{4\alpha^2} \int_{-\infty}^{\infty} e^{-2\alpha|t|} dt = \frac{1}{4\alpha^2} 2 \int_0^{\infty} e^{-2\alpha t} dt \\
&= \frac{1}{2\alpha^2} - \frac{1}{2\alpha} e^{-2\alpha t}\Big|_0^{\infty} = \frac{1}{4\alpha^3}
\end{aligned}$$

b) $H(f) = \frac{1}{\gamma + j2\pi f} \Longrightarrow |H(f)|^2 = \frac{1}{\gamma^2 + 4\pi^2 f^2}$. The energy spectral density of the output is

$$\mathcal{G}_Y(f) = \mathcal{G}_X(f)|H(f)|^2 = \frac{1}{\gamma^2 + 4\pi^2 f^2}\Pi(f)$$

The energy content of the signal is

$$\begin{aligned}
E_Y &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\gamma^2 + 4\pi^2 f^2} df = \frac{1}{2\pi\gamma} \arctan \frac{f\gamma}{2\pi}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{1}{\pi\gamma} \arctan \frac{f\gamma}{4\pi}
\end{aligned}$$

c) The power spectral density of the output is

$$\begin{aligned}
\mathcal{S}_Y(f) &= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{2})|^2 \delta(f-\frac{n}{2}) \\
&= \frac{1}{4\gamma^2}\delta(f) + 2\sum_{l=0}^{\infty} \frac{|x_{2l+1}|^2}{\gamma^2+\pi^2(2l+1)^2}\delta(f-\frac{2l+1}{2}) \\
&= \frac{1}{4\gamma^2}\delta(f) + \frac{8}{\pi^2}\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4(\gamma^2+\pi^2(2l+1)^2)}\delta(f-\frac{2l+1}{2})
\end{aligned}$$

The power content of the output signal is

$$\begin{aligned}
P_Y &= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{2})|^2 \\
&= \frac{1}{4\gamma^2} + \frac{8}{\pi^2}\sum_{l=0}^{\infty}\left[\frac{1}{\gamma^2(2l+1)^4} + \frac{\pi^4}{\gamma^4(\gamma^2+\pi^2(2l+1)^2)} - \frac{\pi^2}{\gamma^4(2l+1)^2}\right] \\
&= \frac{1}{4\gamma^2} + \frac{8}{\pi^2}\left(\frac{\pi^2}{\gamma^2 96} - \frac{\pi^4}{8\gamma^4} + \frac{\pi^2}{\gamma^4}\sum_{l=0}^{\infty}\frac{1}{\frac{\gamma^2}{\pi^2}+(2l+1)^2}\right) \\
&= \frac{1}{3\gamma^2} - \frac{\pi^2}{\gamma^4} + \frac{2\pi^2}{\gamma^5}\tanh(\frac{\gamma}{2})
\end{aligned}$$

where we have used the fact

$$\tanh(\frac{\pi x}{2}) = \frac{4x}{\pi}\sum_{l=0}^{\infty}\frac{1}{x^2+(2l+1)^2}, \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

d) The power spectral density of the output signal is

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{1}{2}\frac{1}{\gamma^2+4\pi^2 f^2}\delta(f) = \frac{1}{2\gamma^2}\delta(f)$$

The power content of the signal is

$$P_Y = \int_{-\infty}^{\infty}\mathcal{S}_Y(f)df = \frac{1}{2\gamma^2}$$

e) $X(f) = -j\pi\mathrm{sgn}(f)$ so that $|X(f)|^2 = \pi^2$ for all f except $f=0$ for which $|X(f)|^2 = 0$. Thus, the energy spectral density of the output is

$$\mathcal{G}_Y(f) = |X(f)|^2|H(f)|^2 = \frac{\pi^2}{\gamma^2+4\pi^2 f^2}$$

and the energy content of the signal

$$E_Y = \pi^2\int_{-\infty}^{\infty}\frac{1}{\gamma^2+4\pi^2 f^2}df = \pi^2\frac{1}{2\pi\gamma}\arctan(\frac{f2\pi}{\gamma})\Big|_{-\infty}^{\infty} = \frac{\pi^2}{2\gamma}$$

2)
a) $h(t) = \mathrm{sinc}(6t) \Longrightarrow H(f) = \frac{1}{6}\Pi(\frac{f}{6})$ The energy spectral density of the output signal is $\mathcal{G}_Y(f) = \mathcal{G}_X(f)|H(f)|^2$ and with $\mathcal{G}_X(f) = \frac{1}{\alpha^2+4\pi^2 f^2}$ we obtain

$$\mathcal{G}_Y(f) = \frac{1}{\alpha^2+4\pi^2 f^2}\frac{1}{36}\Pi^2(\frac{f}{6}) = \frac{1}{36(\alpha^2+4\pi^2 f^2)}\Pi(\frac{f}{6})$$