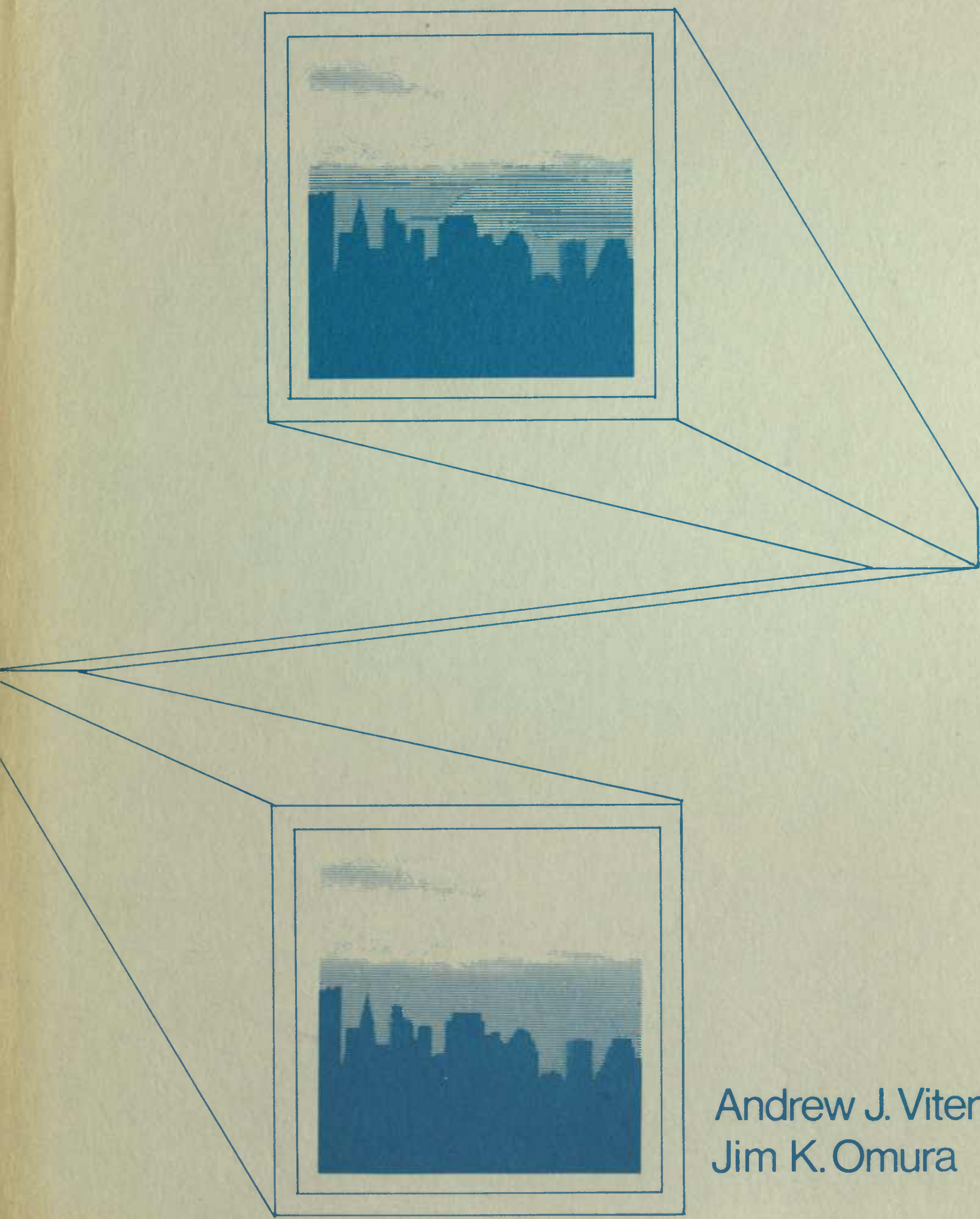
Solutions Manual to accompany

Principles of Digital Communication and Coding



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Solutions Manual to accompany

Principles of Digital Communication and Coding

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CHAPTER 1

1.1 (a) $\mathscr{H}(p) = -p\ell n p - (1-p)\,\ell n(1-p)$ nats.

$$\frac{d}{dp}\mathscr{H}(p) = \ell n\,\frac{1-p}{p} = 0 \Longrightarrow p = \frac{1}{2} \text{ and } \frac{d^2}{dp^2}\mathscr{H}(p) = \frac{1}{p(1-p)} < 0.$$

(b)

$\mathscr{U}_2$	P
a_1a_1	p^2
a_1a_2	$p(1-p)$
a_2a_1	$p(1-p)$
a_2a_2	$(1-p)^2$

$$\begin{aligned} H(\mathscr{U}_2) &= p^2\log\frac{1}{p^2} + p(1-p)\log\frac{1}{p(1-p)} \\ &\quad + p(1-p)\log\frac{1}{p(1-p)} + (1-p)^2\log\frac{1}{(1-p)^2} \\ &= 2\left[p\,\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}\right] \\ &= 2\,\mathscr{H}(p) \end{aligned}$$

(c) i) $H(\mathscr{U}) = \log 52 = 5.7$ bits

ii) $H(\mathscr{U}) = \log 13 = 3.7$ bits

iii) $H(\mathscr{U}) = \frac{3}{13}\log\frac{13}{3} + \frac{10}{13}\log\frac{13}{10} = .77$ bits

(d) i) $H(\text{fair die}) = \log 6 = 2.58$ bits

ii) $P(k) = kC \qquad k = 1,2,3,4,5,6$

$$\sum_{k=1}^{6} P(k) = C\sum_{k=1}^{6} k = 1 \Longrightarrow C = \frac{1}{21}$$

$$\Longrightarrow H(\mathscr{U}) = \sum_{k=1}^{6}\frac{k}{21}\log\frac{21}{k} = 2.39 \text{ bits}$$

1.2 Inequality (1.1.8) gives

$$\begin{aligned} H(\mathscr{U}^{(1)},\ldots,\mathscr{U}^{(N)}) &= \sum_{\underset{\sim}{u}} P_N(\underset{\sim}{u})\log\frac{1}{P_N(\underset{\sim}{u})} \\ &\le \sum_{\underset{\sim}{u}} P_N(\underset{\sim}{u})\log\frac{1}{Q_N(\underset{\sim}{u})} \cdot \;(= \text{ iff } Q_N(\cdot) = P_N(\cdot)) \end{aligned}$$

Choose $Q_N(\underset{\sim}{u}) = \prod_{n=1}^{N} P^{(n)}(u_n)$ where

$$P^{(n)}(u_n) = \sum_{i \neq n} \sum_{u_i} P_N(\underset{\sim}{u}) \qquad n = 1,2,\ldots,N.$$

are the marginal probability distributions. Then

$$H\left(\mathscr{U}^{(1)},\ldots,\mathscr{U}^{(N)}\right) \leq \sum_{\underset{\sim}{u}} P_N(u)\left(\sum_{n=1}^{N} \log \frac{1}{P^{(n)}(u_n)}\right)$$

$$= \sum_{n=1}^{N}\left(\sum_{\underset{\sim}{u}} P_N(\underset{\sim}{u}) \log \frac{1}{P^{(n)}(u_n)}\right)$$

$$= \sum_{n=1}^{N}\left(\sum_{u} P^{(n)}(u) \log \frac{1}{P^{(n)}(u)}\right)$$

$$= \sum_{n=1}^{N} H(\mathscr{U}^{(n)}).$$

with equality iff $P_N(\underset{\sim}{u}) = \prod_{n=1}^{N} P^{(n)}(u_n)$

<u>1.3</u> (a) Given discrete random variables $x,y \in \mathscr{X} \times \mathscr{Y}$ with joint probability $P(x,y)$, we can define marginal probabilities $P(x) = \sum_y P(x,y)$, $P(y) = \sum_x P(x,y)$ and conditional probability $P(y/x) = P(x,y)/P(x)$. Then using inequality (1.1.8),

$$H(\mathscr{Y}\,|\,\mathscr{X}) \equiv \sum_x \sum_y P(x,y) \log \frac{1}{P(y|x)}$$

$$= \sum_x P(x)\left(\sum_y P(y|x) \log \frac{1}{P(y|x)}\right)$$

$$\leq \sum_x P(x)\left(\sum_y P(y|x) \log \frac{1}{P(y)}\right)$$

$$= \sum_y \left(\sum_x P(x)P(y|x)\right) \log \frac{1}{P(y)}$$

$$= H(\mathscr{Y}). \qquad (1)$$

Now fix N and consider sequences $\underset{\sim}{u} \in \mathscr{U}_N = \mathscr{U}^{(1)} \times \ldots \times \mathscr{U}^{(N)}$ where $\mathscr{U}^{(k)} = \mathscr{U}$ is the alphabet of the k^{th} term in the sequence. Using the relations

$$P_N(u_1,\dots,u_N) = P(u_N | u_1,\dots,u_{N-1})\, P_{N-1}(u_1,\dots,u_{N-1})$$
$$= \prod_{n=2}^{N} P(u_n | u_1,\dots,u_{n-1})\cdot P(u_1)$$

we have

$$H(\mathcal{U}_N) = H(\mathcal{U}_{N-1}) + H\left(\mathcal{U}^{(N)} \middle| \mathcal{U}^{(1)} x \dots x \mathcal{U}^{(N-1)}\right) \tag{2}$$

and

$$H(\mathcal{U}_N) = H\left(\mathcal{U}^{(1)}\right) + \sum_{n=2}^{N} H\left(\mathcal{U}^{(n)} \middle| \mathcal{U}^{(1)} x \dots x \mathcal{U}^{(n-1)}\right) \tag{3}$$

Note that from (1) we have

$$H\left(\mathcal{U}^{n)} \middle| \mathcal{U}^{(1)} x \dots x \mathcal{U}^{(n-1)}\right) \le H\left(\mathcal{U}^{(n)} \middle| \mathcal{U}^{(2)} x \dots x \mathcal{U}^{(n-1)}\right)$$
$$= H\left(\mathcal{U}^{(n-1)} \middle| \mathcal{U}^{(1)} x \dots x \mathcal{U}^{(n-2)}\right) \tag{4}$$

where the second equality comes from the stationary property. Hence (3) is bounded by

$$H(\mathcal{U}_N) \ge NH\left(\mathcal{U}^{(N)} \middle| \mathcal{U}^{(1)} x \dots x \mathcal{U}^{(N-1)}\right) \tag{5}$$

Using (5) in (2) gives

$$H(\mathcal{U}_N) \le H(\mathcal{U}_{N-1}) + \frac{1}{N} H(\mathcal{U}_N).$$

or

$$\frac{1}{N} H(\mathcal{U}_N) \le \frac{1}{N-1} H(\mathcal{U}_{N-1}) . \tag{6}$$

Then

$$\frac{H(\mathcal{U}_n)}{n} \le \frac{H(\mathcal{U}_k)}{k} \quad \text{for } k \le n.$$

(b) This follows directly from the proof of Theorem 1.1.1 when we define

$$S(N,\varepsilon) = \left\{ \underset{\sim}{u} : 2^{-N[\hat{H}+\varepsilon]} \le P_N(\underset{\sim}{u}_n) \le 2^{-N[\hat{H}-\varepsilon]} \right\}.$$

and use $\lim_{N\to 0} F_N = 0$ from (1.1.31) to (1.1.33).

1.4 (a) $\sigma^2 = E\{(x-m)^2\} = E\{(x-m)^2 \mid |x-m| \geq \varepsilon\} \Pr\{|x-m| \geq \varepsilon\}$

$$+ E\{(x-m)^2 \mid |x-m| < \varepsilon\} \Pr\{|x-m| < \varepsilon\}$$

$$\geq E\{(x-m)^2 \mid |x-m| \geq \varepsilon\} \Pr\{|x-m| \geq \varepsilon\} \geq \varepsilon^2 \Pr\{|x-m| \geq \varepsilon\}.$$

(b) $\Pr\{\underline{Z} : \frac{1}{N}\sum_{n=1}^{N} Z_n \geq \bar{Z} + \varepsilon\} \leq \Pr\{|\frac{1}{N}\sum_{n=1}^{N} Z_n - \bar{Z}| \geq \varepsilon\} \quad \frac{\mathrm{Var}\left\{\frac{1}{N}\sum_{n=1}^{N} Z_n\right\}}{\varepsilon^2}$

$$= \frac{\sigma^2}{N\varepsilon^2}.$$

1.5 For $\underset{\sim}{u} \varepsilon \overline{S(N,\varepsilon)_+}$ we have $1 \leq 2^{s\left[\sum_{n=1}^{N} Z_n - N(H+\varepsilon)\right]}$, $s > 0$.

Hence,

$$F_N^+ = \sum_{\underset{\sim}{u} \varepsilon \overline{S(N,\varepsilon)_+}} P_N(\underset{\sim}{u}) \leq \sum_{\underset{\sim}{u} \varepsilon \overline{S(N,\varepsilon)_+}} P_N(\underset{\sim}{u}) 2^{s\left[\sum_{n=1}^{N} Z_n - N(H+\varepsilon)\right]}$$

$$\leq \sum_{\underset{\sim}{u}} P_N(\underset{\sim}{u}) 2^{s\left[\sum_{n=1}^{N} Z_n - N(H+\varepsilon)\right]}$$

$$= E\left\{2^{s\left[\sum_{n=1}^{N} Z_n - N(H+\varepsilon)\right]}\right\} = \prod_{n=1}^{N} E\left\{2^{s\left[Z_n - (H+\varepsilon)\right]}\right\}$$

$$= 2^{-NG(\ )}$$

where $G(s) = s(H+\varepsilon) - \log E\left\{2^{sZ}\right\}$.

$$= s(H+\varepsilon) - \log\left\{\sum_{k=1}^{A} P(a_k)^{1-s}\right\}$$

and $G(0) = 0$.

Next note that

$$\frac{dG(s)}{ds} = H + \varepsilon - \frac{\sum_{k=1}^{A} P(a_k)^{1-s} \log \frac{1}{P(a_k)}}{\sum_{k=1}^{A} P(a_k)^{1-s}}$$

where

$$\left.\frac{dG(s)}{ds}\right|_{s=0} = \varepsilon > 0$$

and

$$\frac{d^2G(s)}{ds^2} = -\left[\sum_{k=1}^{A} Q_s(a_k)\left(\log \frac{1}{P(a_k)}\right)^2 - \left(\sum_{k=1}^{A} Q_s(a_k) \log \frac{1}{P(a_k)}\right)^2\right]$$

$$< 0$$

since this is negative of the variance of $\log \frac{1}{P(u)}$ with respect to distribution

$$Q_s(u) = \frac{P(u)^{1-s}}{\sum_{k=1}^{A} P(a_k)^{1-s}}$$

Thus, there is a unique maximum of $G(s)$ for some $s^* > 0$ where $G(s^*) > 0$. Then we have $F_N^+ \leq 2^{-NG(s^*)}$. Similarly we get $F_N^- \leq 2^{-NG^2(s^{**})}$ where $\tilde{G}(s^{**}) > 0$. Then

$$F_N = F_N^+ + F_N^- \leq 2^{-NG(s^*)} + 2^{-N\tilde{G}(s^{**})}$$

1.6 Multiply the inequalities

$$\log \frac{1}{P_N(\underset{\sim}{u})} \leq \ell(\underset{\sim}{u}) \leq \log \frac{1}{P_N(\underset{\sim}{u})} + 1$$

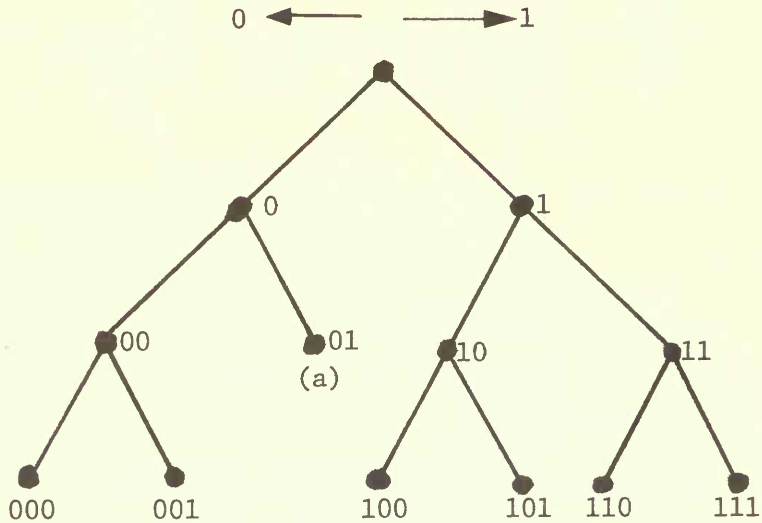
by $P_N(\underset{\sim}{u})$ and sum over all $\underset{\sim}{u} \varepsilon \mathcal{U}_N$. Then

$$H(\mathcal{U}_N) \leq \langle L_N \rangle < H(\mathcal{U}_N) + 1.$$

But from (1.1.14), $H(\mathcal{U}_N) = NH(\mathcal{U})$ giving us (a) and (b). To show (c) note that in (a) the inequality $\log \frac{1}{P(u)} \leq \ell(u) \iff 2^{-\ell(u)} \leq P(u)$ Hence we satisfy the Kraft-McMillan inequality,

$$\sum_u 2^{-\ell(u)} \leq \sum_u P(u) = 1.$$

The solution is best understood in terms of a tree diagram where left directed branches correspond to "0" and right to "1". Each node corresponds to a binary sequence so that code words can be represented as nodes in a tree such as



If node (a) is selected as a codeword of length 2, then in order that no other codeword have 01 as a prefix no nodes that branch out from node (a) can ever be selected as a codeword. Hence we can terminate the branching at this node. Thus uniquely decodable codes with the property that no codeword is a prefix of another codeword corresponds to nodes in a tree where no codeword node branches out from a shorter codeword node. In general, let $\ell_1, \ell_2, \ldots, \ell_A$ be the set of codeword lengths where $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_A$. If $\sum_{i=1}^{A} 2^{-\ell i} = 1$ then we can easily find such a code with these lengths where all branch paths of the code tree terminate in a codeword node. If $\sum_{i=1}^{A} 2^{-\ell i} < 1$ then some branch paths can continue forever without encountering a codeword node. Let $\underline{X}_1$ be any node sequence with ℓ_1

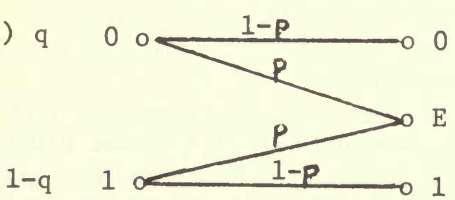
branches leading to it and denote it the first codeword. There remain $2^{\ell_1}-1$ unterminated nodes at the same level that can be a codeword or a prefix of a codeword. If $\ell_2 = \ell_1$ choose any one of these remaining nodes as the codeword node of sequence $\underline{X}_2$. If $\ell_2 > \ell_1$ then use any one of these as a prefix and proceed along any $\ell_2 - \ell_1$ additional branches to find a codeword node of $\underline{X}_2$. There now remains $(2^{\ell_1}-1)\,2^{\ell_2-\ell_1}-1$ nodes at the level ℓ_2 that can be a codeword or a prefix of a codeword. Continue in this manner until $\underline{X}_A$ is selected. If at any point this procedure cannot be completed because of no remaining nodes then $\sum_{i=1}^{A} 2^{-\ell_i} > 1$, which is a contradiction.

__1.7__ (a)
$$I(\mathscr{X};\mathscr{Y}) = \sum_y \sum_x P(y,x) \log \frac{P(y,x)}{P(y)q(x)}$$
$$= \underbrace{\sum_y \sum_x P(y,x)\log \frac{1}{q(x)}}_{\substack{\shortparallel \\ H(\mathscr{X})}} - \underbrace{\sum_y \sum_x P(y,x)\log \frac{1}{q(x|y)}}_{\substack{\shortparallel \\ H(\mathscr{X}|\mathscr{Y})}}$$

But $H(\mathscr{X}|\mathscr{Y}) \geq 0$ (see 1.1.9) so $I(\mathscr{X};\mathscr{Y}) \leq H(\mathscr{X})$ and by symmetry $I(\mathscr{X};\mathscr{Y}) \leq H(\mathscr{Y})$.

(b) Use $P(x,y) = P(y)q(x|y)$ in
$$H(\mathscr{X},\mathscr{Y}) = \sum_x \sum_y P(x,y)\log \frac{1}{P(x,y)}$$
$$= \sum_x \sum_y P(x,y)\log \frac{1}{P(y)} + \sum_x \sum_y P(x,y)\log \frac{1}{q(x|y)}$$
$$= H(\mathscr{Y}) + H(\mathscr{X}|\mathscr{Y}).$$

__1.8__ (a)



$P(0) = q(1-p)$

$P(E) = p$

$P(1) = (1-q)(1-p)$

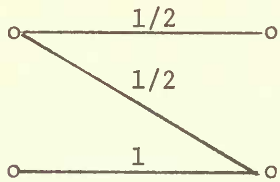
$$I(\mathscr{X};\mathscr{Y}) = q(1-p)\log\left(\frac{1-p}{q(1-p)}\right) + qp\log\left(\frac{p}{p}\right) + (1-q)p\log\left(\frac{p}{p}\right)$$

$$+ (1-q)(1-p)\log\left(\frac{1-p}{(1-q)(1-p)}\right)$$

$$= (1-p)\mathscr{H}(q)$$

$q = \frac{1}{2}$ maximizes $\mathscr{H}(q)$ so $C = 1-p$.

(b) q 0 … 0 $\quad P(0) = \frac{1}{2}q$



$1-q$ 1 … 1 $\quad P(1) = 1 - \frac{1}{2}q$

$$I(\mathscr{X};\mathscr{Y}) = \frac{1}{2}q\log\left(\frac{\frac{1}{2}}{\frac{1}{2}q}\right) + \frac{1}{2}q\log\left(\frac{\frac{1}{2}}{\frac{1}{2}q}\right) + (1-q)\log\left(\frac{1}{1-\frac{1}{2}q}\right)$$

$$= q\log\frac{1}{q} + (1-q)\log\left(\frac{2}{2-q}\right)$$

$$\frac{d}{dq}I(\mathscr{X};\mathscr{Y}) = \frac{1}{2}\log\left(\frac{2-q}{q}\right) - 1 = 0 \Longrightarrow q = \frac{2}{5}$$

Hence $C = \log\frac{5}{4}$.

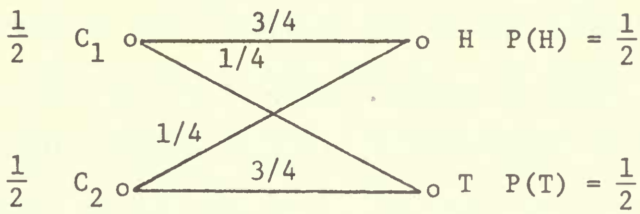
1.9 Since the encoder keeps sending the information symbol until an unerased channel output is achieved there is no error and $P_e = 0$. The probability that n channel symbols are transmitted for an information symbol is $P_n = (1-p)p^{n-1}$. The average length of a codeword is thus

$$L = \sum_{n=1}^{\infty} nP_n = \sum_{n-1}^{\infty} n(1-p)p^{n-1} = \frac{1}{1-p}$$

Thus $R = \frac{1}{L} = 1 - p$ bits per channel use is the rate which also equals the channel capacity (see problem 1.8a).

1.10 There are two coins C_1 and C_2 where $P(H|C_1) = \frac{3}{4}$ and $P(H|C_2) = \frac{1}{4}$. Here $q(C_1) = q(C_2) = \frac{1}{2}$ are the probabilities of selecting each coin.

We can interpret this as a BSC with $p = \frac{1}{4}$ as follows:



(a) $I(C_1;H) = \log \frac{3/4}{1/2} = \log \frac{3}{2}$ bit

$I(C_2;H) = \log \frac{1/4}{1/2} = \log \frac{1}{2} = -1$ bit

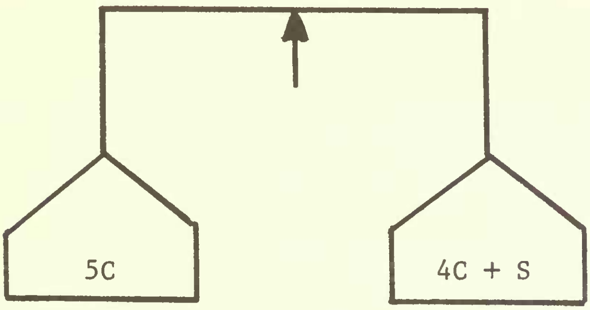
(b) $I(\mathscr{X};\mathscr{Y}) = 1 - \mathscr{H}\left(\frac{1}{4}\right)$ bits.

1.11 (a) There are 13 ways one coin may be heavier, 13 ways one coin may be lighter, and the possibility that all coins weigh the same. Thus we are attempting to determine one of 27 possible situations using a balance and a known standard coin. We are thus asked to obtain at most log 27 = 3 log 3 bits of information on the average. Each weighing has 3 possible outcomes (left, right, balance) and provides at most log 3 bits of average information. Two weighings has 9 possible outcomes and at most 2 log 3 bits of average information. Three weighings has 27 possible outcomes. Clearly two weighings cannot guarantee determining one of 27 possible situations while with three it may be possible.

(b) The maximum amount of average information from three weighings is log 27 which is achieved if all 27 weighing sequences are equally probable. This means that we must choose a weighing strategy where the outcomes of each weighing are equal probable and each weighing outcome is independent of other weighing outcomes. Clearly each weighing must reduce the number of possible cases by 1/3.

Strategy: Let S denote the standard coin, C denote a coin that can be heavy, light, or normal, h denote a coin that is heavy or normal, and ℓ denote a coin that is light or normal. We start with 13C and an S.

1st Weighing: Set aside 4C and place 5C on the left pan and 4C + S on the right pan.



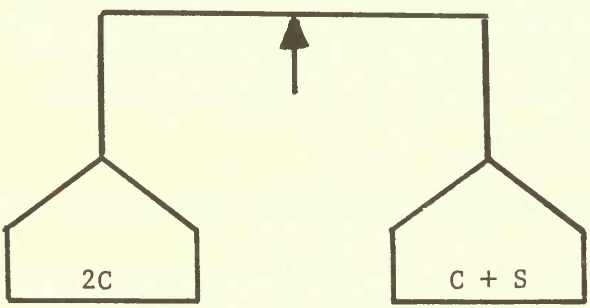
Balance ⟹ 4C remain

Left ⟹ 5h, 4ℓ remain

Right ⟹ 5ℓ, 4h remain

Note that for each of the three possible outcomes we have 9 remaining unknown possibilities to be resolved with two more weighings.

2nd Weighing When 1st Outcome is Balanced: Of the 4C set aside C and place 2C on the left and C + S on the right.



Balance ⟹ C remain

Left ⟹ 2h, ℓ remain

Right ⟹ 2ℓ, h remain

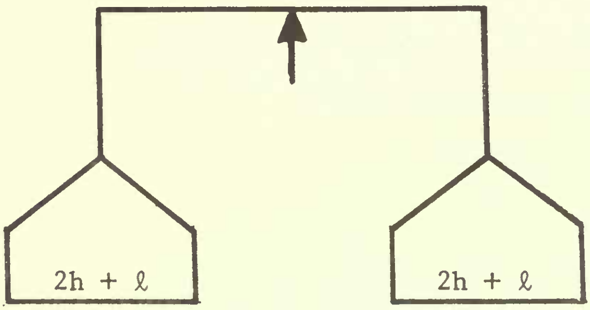
Note that here each outcome leaves only 3 remaining unknown possibilities to be resolved with one more weighing. The third weighing given the 2nd weighing outcome is as follows:

(i) Balance: Place C on the left and S on the right.

(ii) Left: Place h on the left and h on the right.

(iii) Right: Place ℓ on the left and ℓ on the right.

2nd Weighing When 1st Outcome is Left: Of the 5h and 4ℓ remaining set aside h + 2ℓ on the left and 2h + ℓ on the right.



Balance $\Longrightarrow$ h, 2ℓ remain

Left $\Longrightarrow$ 2h, ℓ remain

Right $\Longrightarrow$ 2h, ℓ remain

One more weighing easily resolves the 3 remaining unknown possibilities by placing the same type of coin on the balance. That is, if 2h, ℓ remain then place h on the left and h on the right.

(c) Without a standard coin we cannot always reduce the number of unknown possibilities by 1/3 with each weighing. This is a necessary requirement.

1.12 Using (1.1.8) we have

$$H_i(\mathcal{U}) = \sum_u P_i(u) \log \frac{1}{P_i(u)} \leq \sum_u P_i(u) \log \frac{1}{P_\lambda(u)}, \quad i = 1,2.$$

Hence $\lambda H_1(\mathcal{U}) + (1-\lambda)\, H_2(\mathcal{U}) \leq \sum_u \left[\lambda P_1(u) + (1-\lambda)\, P_2(u)\right] \log \frac{1}{P_\lambda(u)}$

$$= \sum_u P_\lambda(u) \log \frac{1}{P_\lambda(u)} = H_\lambda(\mathcal{U}).$$

1.13 Let $\hat{p}(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}}\, e^{-\frac{y^2}{2\sigma_y^2}}$. Then (1.1.8) gives

$$\int_{-\infty}^{\infty} p(y)\log\frac{1}{p(y)}\,dy \leq \int_{-\infty}^{\infty} p(y)\log\frac{1}{\hat{p}(y)}\,dy = \int_{-\infty}^{\infty} p(y)\left[\frac{y^2}{2\sigma_y^2}\log e + \frac{1}{2}\log\left(2\pi\sigma_y^2\right)\right]dy$$

$$= \frac{1}{2}\log e + \frac{1}{2}\log\left(2\pi\sigma_y^2\right) = \frac{1}{2}\log\left(2\pi e\sigma_y^2\right)$$

with equality when $p(y) = \hat{p}(y)$ or y is a Gaussian random variable.

For the additive Gaussian noise channel we have

$$p(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(y-x)^2}{2\sigma^2}}$$

With input probability density q(x), the average mutual information is

$$I(\mathscr{X};\mathscr{Y}) = \int_{-\infty}^{\infty} p(y)\log\frac{1}{p(y)}\,dy - \int_{-\infty}^{\infty} q(x)\left\{\int_{-\infty}^{\infty} p(y|x)\log\frac{1}{p(y|x)}\,dy\right\}dx$$

But

$$\int_{-\infty}^{\infty} p(y|x)\log\frac{1}{p(y|x)}\,dy = \int_{-\infty}^{\infty} p(y|x)\left[\frac{(y-x)^2}{2\sigma^2} + \frac{1}{2}\log\left(2\pi\sigma^2\right)\right]dy$$

$$= \frac{1}{2}\log\left(2\pi e\sigma^2\right).$$

Using this plus the above bound we have

$$I(\mathscr{X};\mathscr{Y}) \leq \frac{1}{2}\log\left(2\pi e\sigma_y^2\right) - \frac{1}{2}\log\left(2\pi e\sigma^2\right)$$

$$= \frac{1}{2}\log\frac{\sigma_y^2}{\sigma^2}$$

with equality if and only if y is a Gaussian random variable. It is Gaussian if

$$q(x) = \frac{1}{\sqrt{2\pi\mathscr{E}}}\, e^{-\frac{x^2}{2\mathscr{E}}}$$

with the result that $\sigma_y^2 = \mathscr{E} + \sigma^2$. Hence

$$\max I(\mathscr{X};\mathscr{Y}) = \frac{1}{2}\log\left(\frac{\mathscr{E}+\sigma^2}{\sigma^2}\right) = \frac{1}{2}\log\left(1+\frac{\mathscr{E}}{\sigma^2}\right)$$

which is channel capacity.

1.14 From problem 1.6(b) we have inequalities

$$-\log P_N(\underset{\sim}{u}) \le \ell(\underset{\sim}{u}) < -\log P_N(\underset{\sim}{u}) + 1.$$

Thus

$$\frac{-\log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})} \le 1$$

and

$$\frac{-\log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})} \ge \frac{\ell(\underset{\sim}{u})-1}{\ell(\underset{\sim}{u})} = 1 - \frac{1}{\ell(\underset{\sim}{u})} \ge 1 + \frac{1}{\log P_N(\underset{\sim}{u})}$$

But

$$P_N(\underset{\sim}{u}) = \prod_{n=1}^{N} P(u_n) \le \prod_{n=1}^{N} P(u^*) = P(u^*)^N$$

and

$$\log P_N(\underset{\sim}{u}) \le N \log P(u^*)$$

Hence

$$1 + \frac{1}{N\log P(u^*)} \le \frac{-\log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})} \le 1$$

and

$$1 + \frac{1}{N\log P(u^*)} \le H_N \le 1.$$

If $H_N = 1$ for all N then $\dfrac{-\log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})} = 1$ for all N.

Hence $P_N(\underset{\sim}{u}) = 2^{-\ell(\underset{\sim}{u})}$ for all N which is possible only if $P_N(\underset{\sim}{u}) = 2^{-N}$

and the source is a BSS.

1.15 For $\underset{\sim}{u} \varepsilon S(N,\varepsilon)$,

$$N[H(\mathscr{U}) + \varepsilon] \leq \ell(\underset{\sim}{u}) < N[H(\mathscr{U}) + \varepsilon] + 1$$

and

$$N[H(\mathscr{U}) - \varepsilon] \leq - \log P_N(\underset{\sim}{u}) \leq N[H(\mathscr{U}) + \varepsilon]$$

Thus for $\underset{\sim}{u} \varepsilon S(N,\varepsilon)$,

$$\frac{N[H(\mathscr{U}) - \varepsilon]}{1 + N[H(\mathscr{U}) + \varepsilon]} \leq \frac{- \log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})} \leq 1 \qquad (1)$$

For $\underset{\sim}{u} \varepsilon S(N,\varepsilon)$,

$$N \log A \leq \ell(\underset{\sim}{u}) \leq N \log A + 1$$

and

$$- N \log P(u^*) \leq - \log P_N(\underset{\sim}{u}) \leq - N \log P(u^{**})$$

where $P(u^{**}) = \min\limits_{u} P(u) > 0$. Thus for $\underset{\sim}{u} \varepsilon \overline{S(N,\varepsilon)}$,

$$\frac{- N \log P(u^*)}{N \log A + 1} \leq \frac{- \log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})} \leq \frac{- N \log P(u^{**})}{N \log A} \qquad (2)$$

Define $F_N = Pr \left\{\underset{\sim}{u} \varepsilon \overline{S(N,\varepsilon)}\right\}$ and using the form

$$H_N = \sum_{\underset{\sim}{u}} P_N(\underset{\sim}{u}) \left(\frac{- \log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})}\right)$$

$$= \sum_{\underset{\sim}{u} \in S(N,\varepsilon)} P_N(\underset{\sim}{u}) \left(\frac{- \log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})}\right) + \sum_{\underset{\sim}{u} \in \overline{S(N,\varepsilon)}} P_N(\underset{\sim}{u}) \left(\frac{- \log P_N(\underset{\sim}{u})}{\ell(\underset{\sim}{u})}\right)$$

we have from inequalities (1) and (2),

$$H_N \geq \frac{N[H-\varepsilon]}{1 + N[H+\varepsilon]} (1-F_N) + \frac{- N \log P(u^*)}{N \log A + 1} F_N$$

$$= \frac{H - \varepsilon}{H + \varepsilon + \frac{1}{N}} (1-F_N) + \frac{- \log P(u^*)}{\log A + \frac{1}{N}} F_N$$

$$\xrightarrow[N \to \infty]{} \frac{H - \varepsilon}{H + \varepsilon} \quad \text{since} \quad F_N \xrightarrow[N \to \infty]{} 0 ,$$

and

$$H_N \leq (1-F_N) + \frac{-N \log P(u^{**})}{N \log A} F_N$$

$$= 1 - F_N - \frac{\log P(u^{**})}{\log A} F_N \xrightarrow[N \to \infty]{} 1.$$

Hence for any $\varepsilon > 0$,

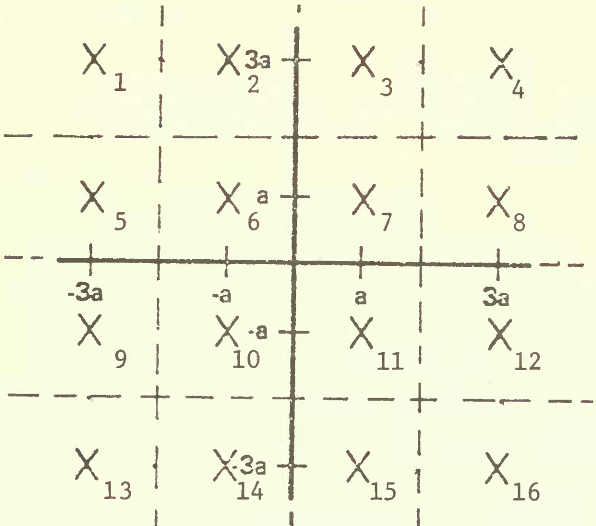
$$\frac{H - \varepsilon}{H + \varepsilon} \leq \lim_{N \to \infty} H_N \leq 1$$

or

$$\lim_{N \to \infty} H_N = 1.$$

—//—

2.1 (a)



(a)

The noise components in the perpendicular coordinate directions are independent with variance $N_0/2$.

Let $q = \Pr\{n>a\}$

$$= Q\left(\sqrt{\frac{2}{N_0}}\, a\right),$$

$$\mathscr{E}_{av} = \frac{1}{16}\sum_{i=1}^{16} \mathscr{E}_i$$

$$= 10a^2$$

$$P_{C_1} = P_{C_4} = P_{C_{13}} = P_{C_{16}} = \Pr\{n_1<a, n_2<a\} = (1-q)^2$$

$$P_{C_2} = P_{C_3} = P_{C_5} = P_{C_8} = P_{C_9} = P_{C_{12}} = P_{C_{14}} = P_{C_{15}} = \Pr\{n_1<a, -a<n_2<a\} = (1-q)(1-2q)$$

$$P_{C_6} = P_{C_7} = P_{C_{10}} = P_{C_{11}} = \Pr\{-a<n_1<a, -a<n_2<a\} = (1-2q)^2$$

Hence

$$P_C = \frac{1}{16}\left[4(1-q)^2 + 8(1-q)(1-2q) + 4(1-2q)^2\right]$$

$$= \frac{1}{4}\left[(1-q)^2 + 2(1-q)(1-2q) + (1-2q)^2\right]$$

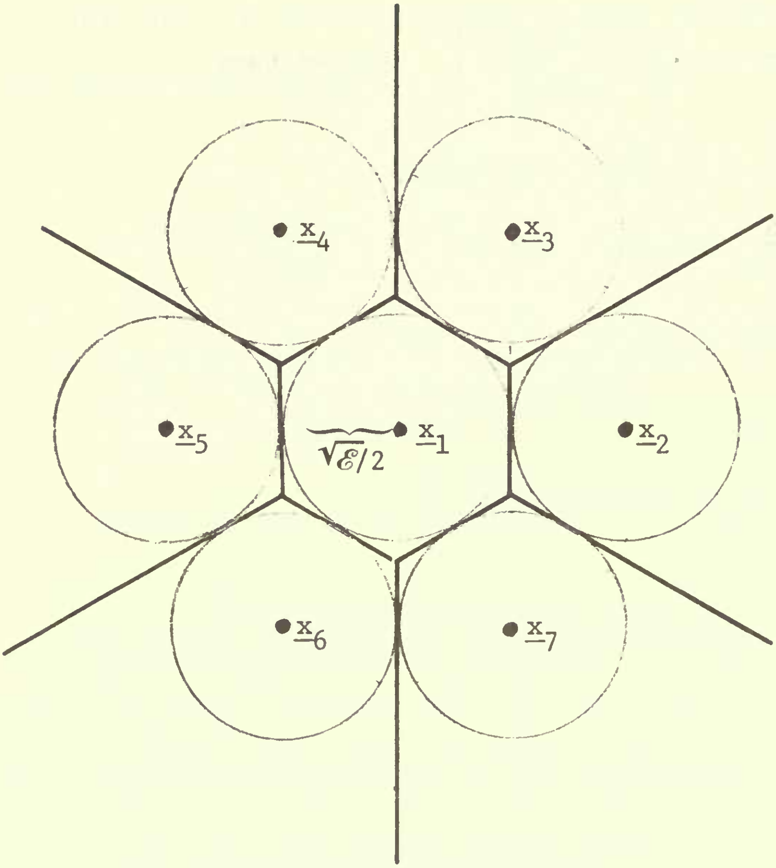
$$= \frac{1}{4}\left[(1-q) + (1-2q)\right]^2 = \left(\frac{2-3q}{2}\right)^2 = \left(1 - \frac{3}{2}q\right)^2$$

and

$$P_E = 3q - \frac{9}{4}q^2$$

(b) Rotation and translation does not change P_E.

2.2 (a)



(b) $\Lambda_m = \{\underline{y}: ||\underline{y}-\underline{x}_{m'}||^2 < ||\underline{y}-\underline{x}_m||$ for all $m' \neq m\}$ $m = 1,2,\ldots,7$ are the optimum decision boundaries. We have

$$S_m = \{\underline{y}: ||\underline{y}-\underline{x}_m||^2 \leq \sqrt{\mathscr{E}}/2\} \subset \Lambda_m \quad m = 1,2,\ldots,7.$$

Hence

$$P_{E_m} = \Pr\{\underline{y}\notin\Lambda_m|\underline{x}_m\} \leq \Pr\{\underline{y}\notin S_m|\underline{x}_m\}$$

$$= \Pr\left\{n_1^2+n_2^2 > \frac{\mathscr{E}}{4}\right\} = e^{-\frac{\mathscr{E}}{4N_0}} \quad m = 1,2,\ldots,7.$$

and

$$P_E = \frac{1}{7}\sum_{m=1} P_{E_m} \leq e^{-\frac{\mathscr{E}}{4N_0}}$$

$\underline{2.3}$ $P_{E_m} \le \sum_{m' \neq m} P_E(m \to m')$ where

$$P_E(m \to m') = Q\left(\frac{||\underline{s}_m - \underline{s}_{m'}||}{\sqrt{2N_0}}\right).$$ Here we have

$$||\underline{s}_1 - \underline{s}_{m'}|| = \sqrt{\mathcal{E}} \qquad m' = 2,3,\ldots,7$$

$$||\underline{s}_2 - \underline{s}_{m'}|| = \sqrt{\mathcal{E}} \qquad m' = 3,7$$

$$||\underline{s}_2 - \underline{s}_{m'}|| = \sqrt{3\mathcal{E}} \qquad m' = 4,6$$

$$||\underline{s}_2 - \underline{s}_{m'}|| = 2\sqrt{\mathcal{E}} \qquad m' = 5$$

$$P_{E_1} \le \sum_{m' \neq 1} P_E(1 \to m') = 6\, Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right)$$

$$P_{E_2} \le \sum_{m' \neq 2} P_E(2 \to m') = 2\, Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right) + 2\, Q\left(\sqrt{\frac{3\mathcal{E}}{2N_0}}\right) + Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right)$$

By symmetry $P_{E_2} = P_{E_3} = P_{E_4} = P_{E_5} = P_{E_6} = P_{E_7} < P_{E_1}$.

Hence

$$P_E \le P_{E_1} \le 6\, Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right).$$

From problem 2.2 we have $P_E \le e^{-\frac{\mathcal{E}}{4N_0}}$ which is "exponentially" the same since $Q(x)$ is bounded as shown in (2.3.18).

$\underline{2.4}$ (a) Choose $\phi_m(t) = \frac{1}{\sqrt{\mathcal{E}}} x_m(t)$ $m = 1,2,\ldots, M$ as the orthonormal basis. Then $x_{mn} = \sqrt{\mathcal{E}}\delta_{mn}$ $m,n = 1,2,\ldots, M$ and $||\underline{x}_m||^2 = \mathcal{E}$ for all m. The decision boundries become

$$\Lambda_m = \{\underline{y}: ||\underline{y}-\underline{x}_m||^2 < ||\underline{y}-\underline{x}_{m'}||^2 \text{ for all } m' \neq m\}$$

$$= \{\underline{y}: (\underline{y},\underline{x}_m) > (\underline{y},\underline{x}_{m'}) \text{ for all } m' \neq m\}$$

$$= \{\underline{y}: y_m > y_{m'} \text{ for all } m' \neq m\} \quad m = 1,2,\ldots,M.$$

Hence by symmetry,

$$P_E = P_{E_1} = 1 - P_{C_1} = 1 - \Pr\{y_1 > y_m \text{ for all } m \neq 1 | \underline{x}_1\}$$

(b) Given $\underset{\sim}{x}_1$ was sent we have probability density functions for the independent random variables $y_1, y_2, \ldots, y_M$ given by

$$p_{y_1}(y) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(y-\sqrt{\mathscr{E}})^2}{N_0}}$$

$$p_{y_m}(y) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{y^2}{N_0}} \qquad m = 2,3,\ldots,M$$

Hence

$$\Pr\{y_m < y_1 \text{ for all } m \neq 1 | \underset{\sim}{x}_1\}$$

$$= \int_{-\infty}^{\infty} \Pr\{y_m < \alpha \text{ for all } m \neq 1 | \underset{\sim}{x}_1, y_1 = \alpha\}\, p_{y_1}(\alpha)\, d\alpha$$

(c) Using $\mathscr{E} = \mathscr{E}_b \log_2 M$, the fact that as ε gets small $\ln(1+\varepsilon)$ behaves as ε, and $Q(x)$ behaves like $e^{-x^2/2}$ for large x,

$$\lim_{M\to\infty} \ln\left[1- Q\left(x+\sqrt{2\,\mathscr{E}_b \log M/N_0}\right)\right]^{M-1}$$

$$= \lim_{M\to\infty} (M-1) \ln\left[1- Q\left(x+\sqrt{2\,\mathscr{E}_b \log M/N_0}\right)\right]$$

$$= -\lim_{M\to\infty} M\, Q\left(x+\sqrt{2\,\mathscr{E}_b \log M/N_0}\right)$$

$$= -\lim_{M\to\infty} M \exp\left\{-\frac{1}{2}\left(x+\sqrt{2\,\mathscr{E}_b \log M/N_0}\right)^2\right\}$$

$$= -\lim_{M\to\infty} M \exp\left\{-\frac{\mathscr{E}_b \log M}{N_0}\right\}$$

$$= -\lim_{M\to\infty} M^{1-\frac{\mathscr{E}_b}{N_0 \ln 2}}$$

$$= \begin{cases} -\infty & ; \quad \dfrac{\mathscr{E}_b}{N_0} < \ln 2 \\ 0 & ; \quad \dfrac{\mathscr{E}_b}{N_0} > \ln 2 \end{cases}$$

Hence

$$\lim_{M\to\infty}\left[1-Q\left(x+\sqrt{2\mathcal{E}/N_0}\right)\right]^{M-1}=\begin{cases}0 & ; \quad \dfrac{\mathcal{E}_b}{N_0}<\ln 2\\ 1 & ; \quad \dfrac{\mathcal{E}_b}{N_0}>\ln 2\end{cases}$$

But

$$\Pr\{y_m<\alpha \text{ for all } m\neq 1 \mid \underset{\sim}{x}_1, y_1=\alpha\}$$

$$=\prod_{m=2}^{M}\Pr\{y_m<\alpha \mid \underline{x}_1\}$$

$$=\left[1-Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^{M-1}$$

Hence

$$P_E = 1-\int_{-\infty}^{\infty}\left[1-Q\left(\frac{\alpha}{\sqrt{N_0/2}}\right)\right]^{M-1}\frac{1}{\sqrt{\pi N_0}}e^{-\frac{(\alpha-\sqrt{\mathcal{E}})^2}{N_0}}d\alpha$$

$$=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}\left[1-Q\left(x+\sqrt{\frac{2\mathcal{E}}{N_0}}\right)\right]^{M-1}dx$$

2.5 (a) $K=1 \Longrightarrow \underset{\sim}{x}_1=\sqrt{\mathcal{E}/2}\,[1\ 1]$, $\underset{\sim}{x}_2=\sqrt{\mathcal{E}/2}\,[1\ -1]$

and $(\underset{\sim}{x}_1,\underset{\sim}{x}_2)=(\mathcal{E}/2)\,[1\ 1]\begin{bmatrix}1\\-1\end{bmatrix}=0.$

Let $\underset{\sim}{h}_{K-1}^{(i)}$ denote the i^{th} row of H_{K-1} and $\underset{\sim}{h}_{K}^{(i)}$ the i^{th} row of H_K.

The codewords of length $M=2^K$ are given by

$$\underset{\sim}{h}_K^{(i)}=\left[\underset{\sim}{h}_{K-1}^{(i)}\ \underset{\sim}{h}_{K-1}^{(i)}\right] \text{ and } \underset{\sim}{h}_K^{(i+2^{K-1})}=\left[\underset{\sim}{h}_{K-1}^{(i)}\ \ -\underset{\sim}{h}_{K-1}^{(i)}\right],\ i=1,\ldots,2^{K-1}$$

Suppose $\left(\underset{\sim}{h}_{K-1}^{(i)},\underset{\sim}{h}_{K-1}^{(i)}\right)=0$ for all $i\neq j<2^{K-1}$

Then

$$\left(\underset{\sim}{h}_{K-1}^{(i)}\ \underset{\sim}{h}_{K-1}^{(i)},\ \underset{\sim}{h}_{K-1}^{(j)}\ \underset{\sim}{h}_{K-1}^{(j)}\right)=0 \text{ for all } i\neq j\leq 2^{K-1}$$

and

$$\left(\underset{\sim}{h}_{K-1}^{(i)}\ \underset{\sim}{h}_{K-1}^{(i)},\ \underset{\sim}{h}_{K-1}^{(j)}\ -\underset{\sim}{h}_{K-1}^{(j)}\right)=0 \text{ for all } i,\ j\leq 2^{K-1}$$

yields

$$\left(\underset{\sim}{h}_K^{(i)}, \underset{\sim}{h}_K^{(j)}\right) = 0 \quad \text{for all } i \neq j \leq 2^K .$$

(b) Note that $\left(\underset{\sim}{h}_K^{(i)}, \underset{\sim}{h}_K^{(j)}\right) = \mathscr{E}\delta_{ij}$. Hence if we subtract the 1st product term of this inner product we have

$$(\hat{\underset{\sim}{x}}_j, \hat{\underset{\sim}{x}}_k) = \mathscr{E}\delta_{jk} - \mathscr{E}/M = \begin{cases} \mathscr{E}\left(\frac{M-1}{M}\right) = \mathscr{E}' , & j = k \\ -\frac{\mathscr{E}}{M} = -\frac{\mathscr{E}'}{M-1} , & j \neq k . \end{cases}$$

(c) Let $\underset{\sim}{a} = \sqrt{\mathscr{E}/M}\,[1\ 0\ 0\ \ldots\ 0]$. Then the orthogonal signal set $\left\{\underset{\sim}{h}_K^{(i)}\right\}$ and the simplex signal set are essentially related by a simple translation of the signal set given by,

$$\hat{\underset{\sim}{x}}_i = \underset{\sim}{h}_K^{(i)} - \underset{\sim}{a} \qquad i = 1,2,\ldots,2^K .$$

(Since the 1st component of $\hat{\underset{\sim}{x}}_i$ is always "0" we can ignore it.) This means the error probability of the orthogonal signal set of energy $\mathscr{E} = \mathscr{E}'\left(\frac{M}{M-1}\right)$ is the same as the error probability of the simplex signal set of energy $\mathscr{E}'$.

(d) Let $\underset{\sim}{W} = \sum_{i=1}^{M} \underset{\sim}{x}_i$. Then,

$$0 \leq (\underset{\sim}{W}, \underset{\sim}{W}) = \sum_i \sum_j (\underset{\sim}{x}_i, \underset{\sim}{x}_j)$$

$$= \sum_i \sum_{j=i} (x_i, x_j) + \sum_{i \neq j} (\underset{\sim}{x}_i, \underset{\sim}{x}_j)$$

$$= M\mathscr{E} + \sum_{i \neq j}\sum (\underset{\sim}{x}_i, \underset{\sim}{x}_j)$$

or

$$\sum_{i \neq j}\sum (\underset{\sim}{x}_i \underset{\sim}{x}_j) \geq - M\mathscr{E}$$

or

$$\rho_{AV} \equiv \frac{1}{\mathscr{E} M(M-1)} \sum_{i \neq j}\sum (\underset{\sim}{x}_i, \underset{\sim}{x}_j) \geq -\frac{1}{M-1} .$$

(e) Let $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_M$ be an orthogonal signal set of energy $\hat{\mathscr{E}}$. Then $(\underset{\sim}{z}_i, \underset{\sim}{z}_j) = \hat{\mathscr{E}} \delta_{ij}$.

Let

$$\underset{\sim}{a} = \alpha \frac{1}{M} \sum_{i=1}^{M} \underset{\sim}{z}_i$$

and consider the translation set formed by

$$\underset{\sim}{y}_i = \underset{\sim}{z}_i - \underset{\sim}{a}$$

Then

$$\begin{aligned}
(\underset{\sim}{y}_i, \underset{\sim}{y}_j) &= (\underset{\sim}{z}_i - \underset{\sim}{a}, \underset{\sim}{z}_j - \underset{\sim}{a}) \\
&= (\underset{\sim}{z}_i, \underset{\sim}{z}_j) - (\underset{\sim}{z}_i, \underset{\sim}{a}) - (\underset{\sim}{z}_j \underset{\sim}{a}) + (\underset{\sim}{a}, \underset{\sim}{a}) \\
&= \hat{\mathscr{E}} \delta_{ij} - \alpha \hat{\mathscr{E}}/M - \alpha \hat{\mathscr{E}}/M + \alpha^2 \hat{\mathscr{E}}/M \\
&= \begin{cases} \hat{\mathscr{E}}(1-2\alpha/M+\alpha^2/M) \\ \hat{\mathscr{E}}(\alpha^2-2\alpha)/M \end{cases}
\end{aligned}$$

Choosing

$\mathscr{E}$ and α to satisfy the equations,

$$\mathscr{E} = \hat{\mathscr{E}}(1-2\alpha/M+\alpha^2/M)$$

and

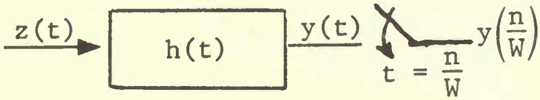
$$\mathscr{E}\rho = \hat{\mathscr{E}}(\alpha^2-2\alpha)/M ,$$

yields a signal set $\{\underset{\sim}{y}_i\}$ with the same energy and inner products as the signal set $\{\underset{\sim}{x}_j\}$. It then has the same error probability and

$$P_E\left(\frac{\mathscr{E}}{N_0}, \rho\right) = P_E\left(\frac{\mathscr{E}(1-\rho)}{N_0}, 0\right).$$

2.6 (a) $h(t) = \frac{1}{2\pi} \int_{-\pi W}^{\pi W} \frac{1}{W} e^{jwt} \, dw = \frac{1}{2\pi W}\left[\frac{e^{j\pi Wt} - e^{-j\pi Wt}}{jt}\right] = \frac{\sin \pi Wt}{\pi Wt}$

(b)



$$y(t) = \int_0^\infty h(t-\tau)z(\tau)\,d\tau = \int_0^\infty \frac{\sin[\pi W(t-\tau)]}{\pi W(t-\tau)}\, z(\tau)\,d\tau$$

and
$$y\left(\frac{n}{W}\right) = \int_0^\infty \frac{\sin\left[\pi W\left(\frac{n}{W} - \tau\right)\right]}{\pi W\left(\frac{n}{W} - \tau\right)}\, z(\tau)\,d\tau = \int_0^\infty z(t)\, \frac{\sin\left[\pi W\left(t - \frac{n}{W}\right)\right]}{\pi W\left(t - \frac{n}{W}\right)}\,dt$$

(c) In Figure 2.9, $y(t)\Phi_{2n}(t)$ is integrated over $[(n-1)T/N, nT/N]$

where
$$\Phi_{2n}(t) = \frac{\sqrt{2}\,\sin\left[\pi W\left(t - \frac{n}{W}\right)\right]}{\pi W\left(t - \frac{n}{W}\right)}\,\sin \omega_0 T\ .$$

Thus
$$y_{2n} = \int_{(n-1)T/N}^{nT/N} y(t)\,\sqrt{2}\,\sin\omega_0 t\, \frac{\sin\left[\pi W\left(t - \frac{n}{W}\right)\right]}{\pi W\left(t - \frac{n}{W}\right)}\,dt\ .$$

Replace the integration over $[(n-1)T/N, nT/N$ to $[0,\infty)$ we obtain
$$y_{zn} = \int_0^\infty y(t)\,\sqrt{2}\,\sin\omega_0 t\, \frac{\sin\left[\pi W\left(t - \frac{n}{W}\right)\right]}{\pi W\left(t - \frac{n}{W}\right)}\,dt$$

which is the process in Figure 2.11.

2.7 (a) Here $y(t) = y_n\phi_n(t)$ where $\phi_n(t) = \sqrt{2N/T}\,\sin\omega_0 t$ for $(n-1)T/N \le t \le nT/N$ whereas we compute
$$y'_n = \int_{(n-1)T/N}^{nT/N} y(t)\,\phi'_n(t)\,dt$$

where
$$\phi'_n(t) = \sqrt{2N/T}\,\sin[\omega_0 + \Delta\omega)(t-(n-1)T/N) + \phi],$$

$$(n-1)T/N \le t \le nT/N.$$

Using $\int_0^{T/N} \cos[(2\omega_0+\Delta\omega)t+\phi]dt \cong 0$ since $\frac{\omega_0 T}{N} >> 1$

we have
$$y_n' = y_n \int_{(n-1)T/N}^{nT/N} \phi_n(t)\ \phi_n'(t)dt$$

$$= y_n \int_{(n-1)T/N}^{nT/N} (2N/T)\sin\ \omega_0 t\ \sin[(\omega_0+\Delta\omega)(t-(n-1)T/N)+\phi]$$

$$= y_n \int_0^{T/N} (2N/T)\sin\ \omega_0 t\ \sin[(\omega_0+\Delta\omega)t+\phi]dt$$

$$= y_n(N/T) \int_0^{T/N} \{\cos(\Delta\omega t+\phi) - \cos[(2\omega_0+\Delta\omega)t+\phi]\}dt$$

$$= y_n(N/T)\int_0^{T/N} \cos(\Delta\omega t+\phi)dt$$

$$= y_n(N/T)\int_0^{T/N} \{\cos\ \Delta\omega t\ \cos\phi - \sin\ \Delta\omega t\ \sin\phi\}dt\ .$$

We assumed here that $\omega_0 T/N$ is a multiple of π. Also since $\Delta\omega T/N <<$ |

we have $\sin\ \Delta\omega t \cong 0$ for $0 \leq t \leq T/N$.

Hence
$$y_n' = y_n(N/T)\cos\phi \int_0^{T/N} \cos\ \Delta\omega t\ dt$$

$$= y_n\ \cos\phi \left(\frac{\sin(\Delta\omega T/N)}{(\Delta\omega T/N)}\right)$$

(b) Here
$$y_{2n}' = \int_{(n-1)T/N}^{nT/N} y(t)\ \sqrt{2N/T}\ \sin(\omega_0 t+\phi)dt$$